Index theory for disordered insulators and semimetals

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Plan of the talk

- Review of role of disorder in quantum Hall effect (QHE)
- Review of signatures of Anderson localization
- Disorder in topological insulators
- Weak invariants and delocalization of surface states
- Bulk-boundary correspondence in chiral semimetals
- Analytical tool: Besov spaces for $\mathbb{R}^n$-action on semifinite W^*'s and non-commutative Peller criteria for associated Hankel operators

References

[BES] J. Bellissard, A. van Elst, H. Schulz-Baldes,
The Non-Commutative Geometry of the Quantum Hall Effect,
J. Math. Phys. **35**, 5373-5451 (1994).

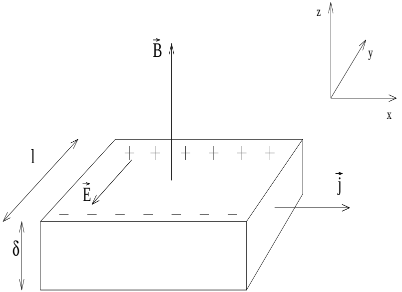
[PS] E. Prodan, H. Schulz-Baldes,
Bulk and Boundary Invariants for Complex Topological Insulators: From K-Theory to Physics,
(Springer International, Cham, 2016).

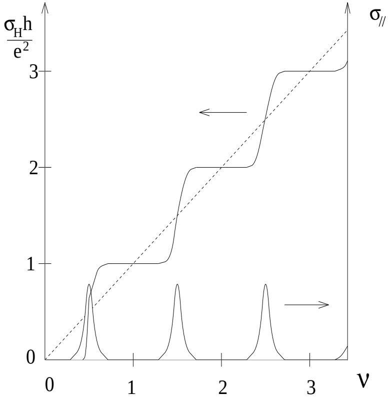
[PS2] E. Prodan, H. Schulz-Baldes,
Generalized Connes-Chern characters in KK-theory with an application to weak invariants of topological insulators,
Rev. Math. Phys. **28**, 1650024 (2016).

[SS] H. Schulz-Baldes, Tom Stoiber,
Harmonic analysis in operator algebras and its applications to index theory,
180 pages, submitted 2020.

Integer quantum Hall effect

effectively $2d$ electron gas in strong magnetic field B at $T = 0$





Hall conductance = bulk Hall conductivity = edge Hall conductivity

$$\sigma_H = \frac{I}{V} = \frac{j_{\text{bulk}}}{E} = \frac{j_{\text{edge}}}{\delta\mu} \in \frac{e^2}{h}\mathbb{Z} \quad \text{(or } \mathbb{Q}\text{)}$$

Modeling disorder in dimension d

Description within one-particle framework by random Hamiltonians

$$H_\omega \;=\; H_B + \lambda\, V_{\text{dis}} \qquad\qquad \text{on } \ell^2(\mathbb{Z}^d,\mathbb{C}^L) \text{ or } L^2(\mathbb{R}^d,\mathbb{C}^L)$$

with coupling constant λ and disordered potential

$$V_{\text{dis}} \;=\; \sum_{n\in\mathbb{Z}^d} v_n\, W_n$$

where $\omega = (v_n)_{n\in\mathbb{Z}^d}$ i.i.d. from $[-1,1]$ and W_n some (matrix) bump

Disorder configs $(\Omega,\mathbb{Z}^d)$ C^*-dyn. system with invariant ergodic $\mathbb{P}$

Basic quantity for random $(H_\omega)_{\omega\in\Omega}$: integrated density of states (IDOS)

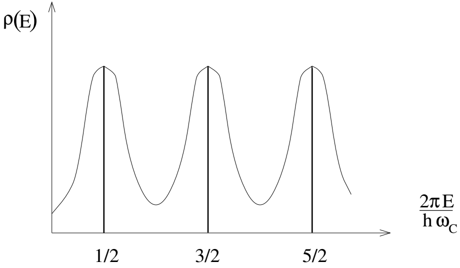
$$\mathcal{N}(E) \;=\; \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\,\#\{\text{eigenvalues of } H_\omega|_\Lambda \;\leqslant\; E\} \qquad \mathbb{P}\text{-a.s.}$$

and DOS (if it exists)

$$\rho(E) \;=\; \frac{d\mathcal{N}}{dE}(E)$$

Importance of disorder for QHE & topo. insulators

$d = 2$ and $H_B = H_L = \frac{1}{2}(P + eA)^2$ Landau operator, λ large, DOS like:



In interval Δ between Landau bands: Anderson localization

- $\mathbb{P}$-a.s. the spectrum of H_ω in Δ is dense pure-point (DOS smooth)
- eigenfunctions exponentially localized in space $\mathbb{Z}^d$
- Fractional moments of resolvent are bounded (next slide)
- Decaying eigenfunction correllators (later)

Fact: these eigenstates do not carry currents and lead to plateaux

Adding disorder: stability of topological phases with closed bulk gap

Proofs of Anderson localization $d \geqslant 2$

First method (Fröhlich, Spencer 1983, ...): multiscale analysis

Second method (Aizenman, Molcanov 1993, ...): fractional moments

Definition (Mobility gap regime MGR)

H has mobility gap in interval Δ, if for some $s \in (0, 1)$ there are A_s and $\beta_s > 0$ such that

$$\int_\Omega \left\| \langle 0 | \frac{1}{H_\omega - z} | x \rangle \right\|^s \mathbb{P}(d\omega) \;\leqslant\; A_s \, e^{-\beta_s |x|}$$

holds uniformly for all $x \in \mathbb{Z}^d$ and $z \in \mathbb{C} \backslash \mathbb{R}$ with $\Re e(z) \in \Delta$

Two regimes can be attained (actually with short proofs):

- $\Delta = \mathbb{R}$ and strong disorder (perturbation theory in kinetic term)
- small λ and Δ at band edges of periodic op. (perturbation in λ)

Open: between Landau bands & topological insulators & pseudogaps

Fermi projection in non-commut. Sobolev space

Suppose mobility gap given: What can one say?

C*-alg. of observables twisted crossed product $\mathcal{A} = M_L(C(\Omega) \rtimes_B \mathbb{Z}^d)$

represented as covariant operator families $A = (A_\omega)_{\omega \in \Omega}$ on $\ell^2(\mathbb{Z}^d, \mathbb{C}^L)$

Supports unbounded derivations $\partial_1, \ldots, \partial_d$ given by $\partial_j A_\omega = i[X_j, A_\omega]$

n-times differentiable elements form $C^n(\mathcal{A})$

$\mathbb{P}$ provides (finite) tracial state $\mathcal{T}(A_\omega) = \int_\Omega \mathbb{P}(d\omega)\, \mathrm{Tr}_L \langle 0|A_\omega|0\rangle$

W*-algebra of observables $\mathcal{M} = L^\infty(\Omega, \mathbb{P}) \rtimes_B \mathbb{Z}^d$ with semifinite $\mathcal{T}$

$L^p(\mathcal{M})$ closure of $\mathcal{M}$ w.r.t. $\|A\|_p = \mathcal{T}(|A|^p)^{\frac{1}{p}}$

Sobolev space $W_p^n(\mathcal{M})$ closure of $C^n(\mathcal{A})$ w.r.t. $\|A\|_{W_p^n} = \sum_{0 \leqslant |j| \leqslant n} \|\partial^j A\|_p$

Proposition ([BES], [PS], [SS])

H has mobility gap in Δ and $E_F \in \Delta$

Fermi projection $P_F = \chi(H \leqslant E_F) \in \mathcal{M}$ lies in $W_p^n(\mathcal{M})$ for all $n, p \geqslant 1$

(Weak) topological invariants: preparations

Derivations $\partial = (\partial_1, \ldots, \partial_d)$ lead to action $k \in \mathbb{T}^d \mapsto \theta_k = e^{k \cdot \partial}$ on $\mathcal{M}$

On magnetic translation u^x by $x \in \mathbb{Z}^d$, one has $\theta_k(u^x) = e^{ik \cdot x} u^x$

$n \leqslant d$ and orthonormal system $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{R}^d$ give action on $\mathcal{M}$

$$t \in \mathbb{R}^n \;\mapsto\; \alpha_t \;=\; \theta_{t \cdot \xi} \qquad\qquad t \cdot \xi = \sum_{j=1}^{n} t_j \xi_j \in \mathbb{R}^d$$

Associated derivations $\nabla = (\nabla_1, \ldots, \nabla_n)$

Definition ((Weak) Chern cocyles)

For $A_0, \ldots, A_n \in W_n^1(\mathcal{M}, \alpha) \cap \mathcal{M}$

$$\mathrm{Ch}_{\mathcal{T},\alpha}(A_0, \ldots, A_n) \;=\; c_n \sum_{\rho \in S_n} (-1)^\rho \, \mathcal{T}\big(A_0 \nabla_{\rho(1)} A_1 \ldots \nabla_{\rho(n)} A_n\big)$$

with normalization constants $c_n = \begin{cases} \frac{(2\pi i)^k}{k!}\,, & \text{for } n = 2k \\ \frac{i\,(\pi i)^k}{(2k+1)!!}\,, & \text{for } n = 2k+1 \end{cases}$

(Weak) topological invariants

Not restricted to C^*-pairings, but sufficiently smooth elements of $\mathcal{M}$

Definition (Even and odd Chern numbers)

Let $P_F \in W^1_n(\mathcal{M}, \alpha) \cap \mathcal{M}$ (e.g. mobility gap regime). For n even,

$$\mathrm{Ch}_{\mathcal{T},\alpha}(P_F) \;=\; \mathrm{Ch}_{\mathcal{T},\alpha}(P_F, \dots, P_F)$$

If H has chiral symmetry and n odd,

$$\mathrm{Ch}_{\mathcal{T},\alpha}(U_F) \;=\; \mathrm{Ch}_{\mathcal{T},\alpha}(U_F^* - 1, U_F, U_F^*, \dots, U_F) \quad , \quad P_F \;=\; \begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix}$$

As usual: constant on norm-continuous paths in $W^1_n(\mathcal{M}, \alpha) \cap \mathcal{M}$

Example: For $n = 1$ (dual) translation $t \mapsto e^{t\nabla_\xi}$ in direction $\xi \in \mathbb{R}^d$

If ξ irrational, connected to half-spaces $\xi \cdot X > 0$ with "irrational" edges

For chiral system $\mathrm{Ch}_{\mathcal{T},\xi}(U_F) = i\,\mathcal{T}(U_F^* \nabla_\xi U_F)$ weak winding number

Constructions for index theorem (C^* in [PS]):

Crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}^n$ with semifinite trace $\widehat{\mathcal{T}}_\alpha$ (via Hilbert alg.)

W^*-crossed product defined in regular representation on $L^2(\mathbb{R}^n, \mathcal{H})$

$$\mathcal{N} \;=\; L^\infty(\mathcal{M} \rtimes_\alpha \mathbb{R}^n, \widehat{\mathcal{T}}_\alpha) \;=\; \mathcal{M} \rtimes_\alpha \mathbb{R}^n \;\subset\; \mathcal{B}(L^2(\mathbb{R}^n, \mathcal{H}))$$

Contains bd. Borel functions of $D = (D_1, \ldots, D_n) = i\partial_t$ on $L^2(\mathbb{R}^n, \mathcal{H})$

Furthermore: L^p-spaces $L^p(\mathcal{N}, \widehat{\mathcal{T}}_\alpha)$ for $p \geqslant 1$

Irrep of complex Clifford algebra generated by $\Gamma_1, \ldots, \Gamma_n \in M_{2N}$ with

$$\{\Gamma_i, \Gamma_j\} \;=\; 0 \qquad , \qquad \Gamma_j^2 = \mathbf{1}$$

Introduce Dirac operator affiliated with $M_{2N}(\mathcal{N})$ and Hardy projection

$$\mathbf{D} \;=\; \sum_{j=1}^{n} \Gamma_j \otimes D_j \qquad\qquad \Pi \;=\; \chi(\mathbf{D} > 0) \;\in\; M_{2N}(\mathcal{N})$$

Case $n = 1$, $\alpha \cong \xi$: $\mathcal{M} \rtimes_\xi \mathbb{R}$ edge alg., $\widehat{\mathcal{T}}_\xi$ boundary trace per unit vol.

$\Pi = \Pi_\xi = \chi(\xi \cdot X > 0)$ in physical representation

Theorem (Besov index theorem, with Tom Stoiber [SS])

$(\mathcal{M},\mathcal{T})$ semifinite von Neumann with $\mathbb{R}^n$-action α leaving $\mathcal{T}$ invariant

Generators of α on $\mathcal{M}$ denoted by $\nabla_1,\ldots,\nabla_n$

Let n be odd and unitary $U_F \in \mathcal{M} \cap W^1_n(\mathcal{M},\alpha) \cap W^1_{n+\epsilon}(\mathcal{M},\alpha)$, then

$$\mathrm{Ch}_{\mathcal{T},\alpha}(U_F) \;=\; \widehat{\mathcal{T}}_\alpha\text{-}\mathrm{Ind}\big(\Pi\, U_F\, \Pi + (\mathbf{1}-\Pi)\big) \;\in\; \mathbb{R}$$

where semifinite index of $\widehat{\mathcal{T}}$-Breuer-Fredholm $T \in M_{2N}(\mathcal{N})$ is

$$\widehat{\mathcal{T}}_\alpha\text{-}\mathrm{Ind}(T) \;=\; \widehat{\mathcal{T}}_\alpha(\mathrm{Ker}(T)) \;-\; \widehat{\mathcal{T}}_\alpha(\mathrm{Ker}(T^*))$$

Similar results for n even

Weaker hypothesis on symbol: $U_F \in \mathcal{M} \cap W^1_n \cap B^{n/(n+1)}_{n+1,n+1}$ Besov space

Important: no differentiability assumption (as Lesch, Wahl for $n=1$)

If U_F in C^*-algebra $\mathcal{A} \subset \mathcal{M}$, values in discrete set (see [PS2])

Proof: non-commutative Peller criteria for $[\Pi, U_F] \in L^p(\mathcal{N},\mathcal{T}_\alpha)$

Stability properties:

Stability for $\hat{\mathcal{T}}$-compact perturbations and on norm continuous paths

But: in MGR usually only strong continuity under parameter change

Proposition (with Tom Stoiber [SS])

Let $s \mapsto H_s \in \mathcal{M}$ strongly continuous path with E_F not eigenvalue and uniformly $\|P_{F,s}\|_{W^1_{n+\epsilon}} < C$ for some $\epsilon > 0$

Then $s \mapsto \mathrm{Ch}_{\mathcal{T},\alpha}(P_{F,s})$ continuous for even n

Similarly, $s \mapsto \mathrm{Ch}_{\mathcal{T},\alpha}(U_{F,s})$ continuous for odd n and chiral H_s

Corollary (already in [BES,PS])

If $n = d$ integer-valued strong invariants are constant

For QHE: explains plateaux for bulk invariants [BES]

Question: weak invariants useless?

Weak invariants prohibit localization of edge states

Set-up: $\xi \in \mathbb{R}^d$ perpendicular to boundary of codimension 1

$\widehat{H} = \Pi H \Pi + K$ with $\Pi = \chi(X \cdot \xi > 0)$ and K boundary term

$\mathbb{R}^n$-action α generated by $(\xi_1, \ldots, \xi_n)$ all $\perp\ \xi$, so $\alpha \times \xi$ action of $\mathbb{R}^{n+1}$

Theorem (Delocalization of surface states [SS], $n = d - 1$ [PS])

Suppose E_F in bulk gap Δ

If bulk invariant $\mathrm{Ch}_{\mathcal{T},\alpha\times\xi}(P_F) \neq 0$, no Anderson localization of $\widehat{H}$ in Δ

Same holds if $\widehat{H}$ chiral, $\mathrm{Ch}_{\mathcal{T},\alpha\times\xi}(U_F) \neq 0$ and $0 \in \Delta$

No Anderson localization means: no bounded eigenfunction correlator

$$\sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} (1+|x-y|)^k \int_0^R dr \int_\Omega \mathbb{P}(d\omega) \sup_{f\in\mathcal{B}(I)\,,\,\|f\|_\infty\leqslant 1} \big\|\langle x|f(\widehat{H}_{\omega,r})|y\rangle\big\|_2$$

Proof: contains bulk-boundary correspondence for all ξ

Surface states for chiral system via index theorem

H chiral Hamiltonian and $\widehat{H} = \Pi H \Pi$ with $\Pi = \chi(X \cdot \xi > 0)$ have polars

$$\mathrm{sgn}(H) = \begin{pmatrix} 0 & U_F \\ U_F^* & 0 \end{pmatrix} \quad , \quad \mathrm{sgn}(\widehat{H}) = \begin{pmatrix} 0 & \widehat{U}_F \\ \widehat{U}_F^* & 0 \end{pmatrix}$$

R.h.s. of Besov index theorem for $n = 1$ contains $\Pi U_F \Pi$ and **not** $\widehat{U}_F$:

$$\mathrm{Ch}_{\mathcal{T},\xi}(U_F) = \widehat{\mathcal{T}}\text{-Ind}\big(\Pi\, U_F\, \Pi\big)$$

Lemma

$\widehat{U}_F - \Pi U_F \Pi$ is $\widehat{\mathcal{T}}$-compact

Thus:

$$\mathrm{Ch}_{\mathcal{T},\xi}(U_F) = \widehat{\mathcal{T}}\text{-Ind}\big(\widehat{U}_F\big) = \widehat{\mathcal{T}}(\sigma_3\,\mathrm{Ker}(\widehat{H})) = \widehat{\mathcal{T}}(\widehat{P}_+ - \widehat{P}_-)$$

where $\widehat{P}_\pm$ pos/neg chiral sector of flat band projection $\widehat{P} = \chi(\widehat{H} = 0)$

Flat band of edge states

Sobolev (or Besov) condition holds in MGR, but also for pseudogap:

Theorem ([SS] with Tom Stoiber, $d = 1$ Graf-Shapiro)

H with chiral symmetry $\sigma_3 H \sigma_3 = -H$

Suppose that either there is pseudo-gap at 0, namely $\gamma > 1$ with

$$\mathcal{N}([-\epsilon,\epsilon]) \;=\; \mathcal{T}\big(\chi(|H| \leqslant \epsilon)\big) \;\leqslant\; C_\gamma\, \epsilon^\gamma$$

or there is mobility gap in $(-\epsilon_0, \epsilon_0)$

Then, for Fermi unitary U_F and kernel projection $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$,

$$i\,\mathcal{T}(U_F^{-1}\nabla_\xi U_F) \;=\; \widehat{\mathcal{T}}(\widehat{P}_+) - \widehat{\mathcal{T}}(\widehat{P}_-)$$

Generically: all in one chiral sector, namely $\widehat{P}_+$ or $\widehat{P}_-$ vanishes

Corollary

Periodic chiral Hamiltonians in $d = 2$ have edge states for irrat. edges

Moreover: stable w.r.t. boundary disorder

Open: localization properties in pseudogap, fate of pseudogap

Most prominent example: Graphene, for which

$$i\,\mathcal{T}(U_F^{-1}\nabla_\xi U_F) \;=\; i\,\mathcal{T}(U_F^{-1}\nabla_1 U_F)\,\xi_1 \;+\; i\,\mathcal{T}(U_F^{-1}\nabla_2 U_F)\,\xi_2 \;=\; \frac{1}{3}\,\xi_2$$

most edge states for zigzag $\xi_2 = 1$, none for armchair $\xi_2 = 0$

Value $\frac{1}{3}$ is **not** topological !

Pairing $\langle[\xi \cdot X], [U_F]_1\rangle = i\,\mathcal{T}(U_F^{-1}\nabla_\xi U_F)$ over huge algebra $B^{1/2}_{2,2} \cap L^\infty$

Thus values **not** in discrete range of $[U]_1 \in K_1(\mathcal{A}) \mapsto \langle[\xi \cdot X], [U]_1\rangle$

As chiral H changes continuously, so does $\mathrm{Ch}_{\mathcal{T},\xi}(U_F) = i\,\mathcal{T}(U_F^{-1}\nabla_\xi U_F)$

Only BBC equality always holds and is hence topological

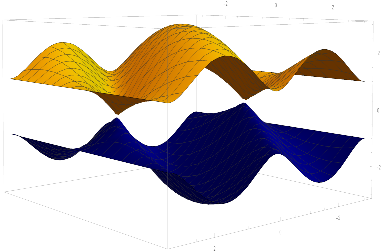
Model for graphene

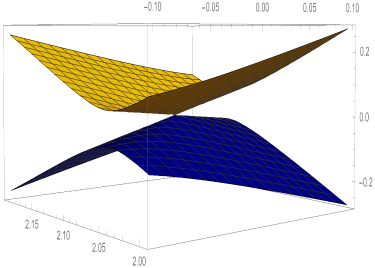
On honeycomb lattice = decorated triangular lattice, so on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$

$$H \;=\; \begin{pmatrix} 0 & S_1 + S_1^* S_2 + 1 \\ S_1^* + S_2^* S_1 + 1 & 0 \end{pmatrix}$$

where S_1, S_2 shifts on $\ell^2(\mathbb{Z}^2)$. Clearly chiral $\sigma_3 H \sigma_3 = -H$. Fourier:

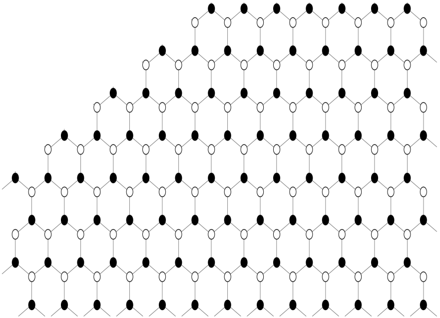
$$H \;\cong\; \int_{\mathbb{T}^2}^{\oplus} dk \;\begin{pmatrix} 0 & e^{ik_1} + e^{i(k_2-k_1)} + 1 \\ e^{-ik_1} + e^{-i(k_2-k_1)} + 1 & 0 \end{pmatrix}$$

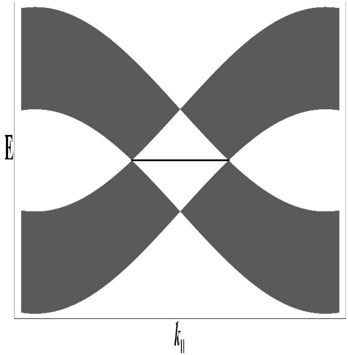




Dirac points $k_{\pm} = (\frac{(3\pm 1)\pi}{3}, 0)$ DOS vanishes at $E = 0$ (pseudogap)

Edges





Zigzag boundary $\cong$ replace S_1 by unilateral shift $\widehat{S}_1$

Armchair boundary $\cong$ replace S_2 by unilateral shift $\widehat{S}_2$

Fact (Saito, Dresselhaus *et al.* 1988): edge states only for Zigzag

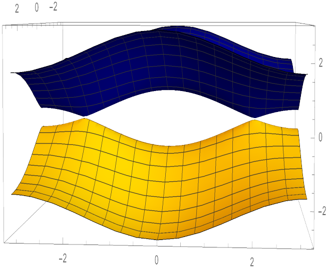
Stacked SSH as chiral $2d$ toy model

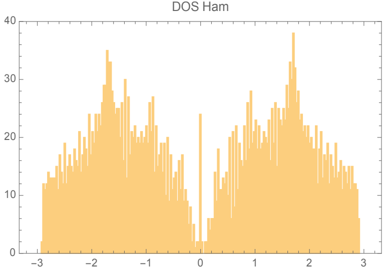
SSH in direction 1 with coupling in direction 2 and chiral randomness

$$H = \begin{pmatrix} 0 & S_1 - \mu \\ S_1^* - \mu & 0 \end{pmatrix} - \delta \begin{pmatrix} 0 & S_2 + S_2^* \\ S_2 + S_2^* & 0 \end{pmatrix} + \lambda \sum_{n \in \mathbb{Z}^2} v_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where v_n i.i.d. random variables with uniform distribution in $[-\frac{1}{2}, \frac{1}{2}]$

(2 or 4) Dirac points for periodic model if $k_1 = 0, \pi$, $2\delta \cos(k_2) + \mu = \pm 1$

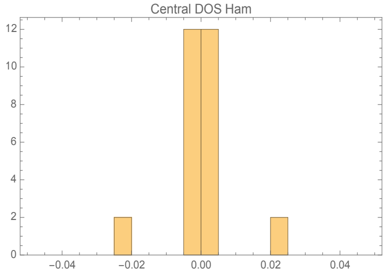


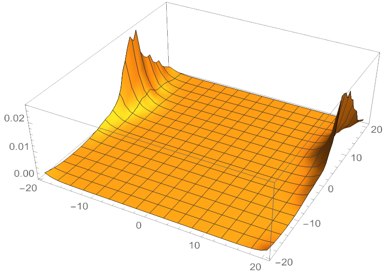


$\lambda = 0.2$, $\mu = 1.3$, $\delta = 0.3$ and volume $[-\rho, \rho]^2$ with $\rho = 20$

Central DOS and one of the edge states

Zoom into the central DOS   Same parameters as above





There are $28 = 2 \cdot 14$ (approximate) zero modes of H

Corresponding eigenstates only on two opposite edges

(edges weakly coupled, edge states vanish on other edges!)

Edge state dens. $= \frac{14}{41} \approx i\mathcal{T}(U^{-1}\nabla_1 U) = \int \frac{dk_2}{2\pi}\, \chi(\mu + 2\delta\cos(k_2) < 1) \approx \frac{1}{3}$

Here first $\approx$ is precisely the equality in the theorem (1 chiral sector)

Constructions for definition of Besov spaces:

Semifinite trace $\mathcal{T}$ gives von Neumann algebra $\mathcal{M}$

Non-commutative spaces $X = L^p(\mathcal{M})$, $p \geqslant 1$, Banach spaces

$L^2(\mathcal{M})$ is GNS-Hilbert space of $\mathcal{T}$

$\mathbb{R}^n$-action α on $\mathcal{M}$ which leaves $\mathcal{T}$ invariant

$\mathcal{T}$-invariance $\implies$ α extends isometrically to action β on $X = L^p(\mathcal{M})$

For $f \in L^1(\mathbb{R}^n)$ and $x \in X$ define $\beta_f(x)$ as Riemann integral

$$\beta_f(x) \;=\; \int_{\mathbb{R}^n} f(-t)\, \beta_t(x)\, dt$$

Then for $f \in FA(\mathbb{R}^n) = \mathcal{F}L^1(\mathbb{R}^n)$ define Fourier multiplier $\widehat{f} * \in \mathcal{B}(X)$ by

$$\widehat{f} * x \;=\; \beta_{\mathcal{F}^{-1} f}(x)$$

$\sigma(x) = \text{Arveson spectrum} = \{\lambda \in \hat{\mathbb{R}}^n : f(\lambda) = 0 \;\text{ if }\; \widehat{f} * x = 0,\; f \in \mathcal{F}L^1\}$

Non-commutative Besov spaces:

X Banach space with isometric $\mathbb{R}^n$-action β on X (above $X = L^p(\mathcal{M})$)

Given smooth $\varphi : \mathbb{R} \to [0,1]$ supported by $[-2,-2^{-1}] \cup [2^{-1},2]$ and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) \;=\; 1$$

Littlewood-Payley dyadic decomposition $(W_j)_{j\in\mathbb{N}}$ by

$$W_j \;=\; \varphi(|2^{-j}\cdot|) \quad \text{for } j>0\,, \qquad W_0 \;=\; 1-\sum_{j>0} W_j$$

Now:

$$B_q^s(X) \;=\; \Big\{x\in X \;:\; \|x\|_{B_q^s(X)} = \Big(\sum_{j\geqslant 0} 2^{qsj}\,\|\widehat{W_j}*x\|_X^q\Big)^{\frac{1}{q}} < \infty\Big\}$$

Set

$$B_{p,q}^s(\mathcal{M}) \;=\; B_q^s(L^p(\mathcal{M}))$$

Properties of Besov spaces:

Proposition

Definition of $B^s_q(X)$ *independent of choice of* φ

$(B^s_q(X), \| \,.\, \|_{B^s_q(X)})$ *Banach space for* $s \in \mathbb{R}$ *and* $q \in [1, \infty)$

An equivalent norm is given by

$$\|x\|_{\tilde{B}^s_q(X)} \;=\; \|x\|_X \;+\; \left(\int_{[0,1]} t^{-sq} \, \omega^N_X(x,t)^q \, \frac{dt}{t} \right)^{\frac{1}{q}}$$

where

$$\omega^N_X(x,t) \;=\; \sup_{|r|\leqslant t} \|\Delta^N_r(x)\|_X \qquad , \qquad N \geqslant s$$

with finite difference operator $\Delta_t : X \to X$ *given by*

$$\Delta_t(x) \;=\; \beta_t(x) - x$$

Corollary

For $B^s_{p,q}(\mathcal{M}) = B^s_q(L^p(\mathcal{M}))$ *and* $s \in [0,1]$, $B^s_{p,q}(\mathcal{M}) \cap \mathcal{M}$ *is a* $*$*-algebra*

Peller criterion for Hankel operators:

Hardy projection $\Pi = \chi(\mathbf{D} > 0)$ in $M_{2N}(\mathcal{N})$, but not $L^p(M_{2N}(\mathcal{N}), \mathrm{Tr} \otimes \widehat{\mathcal{T}})$

Now for "symbol" $A \in \mathcal{M}$, Toeplitz and Hankel operators in $M_{2N}(\mathcal{N})$ are

$$T_A = \Pi A \Pi \qquad , \qquad H_A = \Pi A (\mathbf{1} - \Pi)$$

Theorem

For all $p > n$ and $A \in \mathcal{M} \cap B^{n/p}_{p,p}(\mathcal{M})$, one has $H_A \in L^p(M_{2N}(\mathcal{N}), \mathrm{Tr} \otimes \widehat{\mathcal{T}})$

For $n = 1$, also $p = 1$ is sufficient

Proof: explicit calculations for $p = 1$

L^2-estimates for weighted Hankels with symbol $B^{p/2}_{2,2}$ for $p > 2$

Involved estimates on weighted Hankels for $p = \infty$

Intricate application *à la Peller* of analytic interpolation (*e.g.* Lunardi) □

Classical case is $n = 1$ and $\mathcal{M} = L^\infty(\mathbb{R})$ with $\alpha_t(f)(y) = f(y + t)$