Anderson localization inhibited by topology

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Phase-Averaged Transport for Quasi-Periodic Hamiltonians

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Abstract: For a class of discrete quasi-periodic Schrödinger operators defined by covariant representations of the rotation algebra, a lower bound on phase-averaged transport in terms of the multifractal dimensions of the density of states is proven. This result is established under a Diophantine condition on the incommensuration parameter. The relevant class of operators is distinguished by invariance with respect to symmetry automorphisms of the rotation algebra. It includes the critical Harper (almost-Mathieu) operator. As a by-product, a new solution of the frame problem associated with Weyl–Heisenberg–Gabor lattices of coherent states is given.

Random Dirac operator with time-reversal symmetry

Hamiltonian on $L^2(\mathbb{R}) \otimes \mathbb{C}^{2N}$

$$H \;=\; I\,\partial_x \;+\; V \qquad\qquad I = \begin{pmatrix} 0 & -\mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix}$$

with random $2N \times 2N$ matrix potential with TRS

$$V \;=\; V^* \;=\; I^*\overline{V}I \;=\; \sum_n V_n \delta_n \qquad \Longrightarrow \qquad I^*\overline{H}I \;=\; H$$

Hypothesis: distribution of i.i.d. V_n's absolutely continuous

Theorem (with Sadel, 2010) $\mathbb{Z}_2$ dichotomy (in N mod 2):

N odd $\Longrightarrow$ almost surely pure a.c. spectrum of multiplicity 2

N even $\Longrightarrow$ no a.c. spectrum (Only pure point?)

Physical interpretation

- for odd N no Anderson localization,

  even though quasi-one-dimensional random model
- Exactly 1 double channels survives (left and right mover)

  others "dissolve"
- Why should one care about a.c. spectrum?

  Guarneri bound in $d = 1$ implies ballistic transport
- Anderson localization for even number of channels N
- Is this of physical relevance for anything?

Effective model for edge states in spin quantum Hall systems

Why is the theorem true?

Solve Schrödinger at energy $E \in \mathbb{R}$ using transfer matrices

$$T^E(n, n-1) \;=\; e^{IV_n} e^{\partial_x - E\,I}$$

Lies in the group

$$\mathrm{SO}^*(2N) \;=\; \left\{ T \in \mathrm{GL}(2N, \mathbb{C}) \,\middle|\, T^*\,I\,T = I\,, \;\; I^*\overline{T}I = T \right\}$$

For such T one has Kramers' degeneracy:

$$T^*T\,v \;=\; \lambda v \qquad \Longrightarrow \qquad T^*T\,I\overline{v} \;=\; \lambda\,I\overline{v}$$

Implies double degeneracy of Lyapunov spectrum $\gamma_n \geq \gamma_{n+1}$

Moreover, usual symmetry $\gamma_n = -\gamma_{2N-n}$

Together for N odd: $\gamma_N = \gamma_{N+1} = 0$ open channel

Now the work starts (for a mathematician):

- Show that all other Lyapunov exponents are non-vanishing

  Apply Goldsheid-Margulis theory for to the group $\mathrm{SO}^*(2N)$

  For even N there are no vanishing Lyapunov exponents

- Adapt Kotani-Simon (magical) theory for ergodic Dirac operator

  mult. of a.c. spectrum $=$ $\#$ of vanishing Lyapunov exponents

  Proves existence of a.c. spectrum

- Almost sure absence of singular spectrum

  Adapt Jaksic-Last theory (purity of a.c. spectrum in Anderson)

Is all this tightly linked to the group $\mathrm{SO}^*(2N)$?

Transfer matrices in $\mathrm{SO}^*(2N)$ for H in CAZ AII (odd TRS)

- 2 vanishing γ's in groups $\mathrm{O}(N, \mathbb{C})$ with N odd (H Class DIII)
- $|N - M|$ vanishing in $\mathrm{U}(N, M)$, $\mathrm{O}(N, M)$, $\mathrm{SP}(N, M)$

  Corresponds to Hamiltonians of CAZ classes A, D and C

Effective model for edge states in QHE on $L^2(\mathbb{R}) \otimes \mathbb{C}^{N+M}$

$$H \;=\; J\, i\, \partial_x \;+\; V \qquad\qquad J = \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_M \end{pmatrix}$$

Random matrix potential $V = V^* = \sum_n V_n \delta_n$ with coupling hyp.

Then transfer matrices in $\mathrm{U}(N, M)$

Theorem (with Ludwig, Stolz 2013)

Almost surely pure a.c. spectrum of multiplicity $|N - M|$

Quantum spin Hall system (odd TRS, Class AII)

Disordered Kane-Mele model on hexagon lattice and with $s = \frac{1}{2}$

$$H \;=\; \Delta_{\rm hexagon} + H_{\rm SO} + H_{\rm Ra} + \lambda_{\rm dis} V$$

Pseudo-gap at Dirac point opens non-trivially due to

$$H_{\rm SO} \;=\; i\,\lambda_{\rm SO} \sum_{i=1,2,3} \left(S_i^{\rm nn} - (S_i^{\rm nn})^*\right) s^z$$

No s^z-conservation due to Rashba term $H_{\rm Ra}$, but odd TRS

$$H \;=\; I^*\,\overline{H}\,I \qquad\qquad I \;=\; e^{i\pi s^y}$$

Non-trivial topology:

Kane-Mele (2005): $\mathbb{Z}_2$ invariant for periodic system from Pfaffians

Haldane et al. (2005): spin Chern numbers for s^z invariant systems

Prodan (2009): spin Chern number from $P_{\rm s} = \chi(|Ps^zP - \frac{1}{2}| < \frac{1}{2})$

with Avila, Villegas (2012): $\mathbb{Z}_2$ invariant for edge states

Here: $\mathbb{Z}_2$ invariant for disordered system as index of Fredholm

$\mathbb{Z}_2$ index for odd TRS and $d = 2$

QHE: $P = \chi(H \leq \mu)$ Fermi projection and $F = \frac{X_1 + iX_2}{|X_1 + iX_2|} = F^t$

Then: $T = PFP$ Fredholm operator, namely $\dim(\mathrm{Ker}(T^{(*)})) < \infty$

And: Hall conductance $= \mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*))$

Here: $I^*\overline{H}I = H = I^*H^tI$ with $H^t = (\overline{H})^* \implies I^*P^tI = P$

Definition T odd symmetric $\iff I^*T^tI = T$ with $I^2 = -\mathbf{1}$

Theorem (Atiyah-Singer 1969, S-B 2013)

$\mathbb{F}_2(\mathcal{H}) = \{$*odd symmetric Fredholm operators*$\}$ *has 2 connected components labelled by compactly stable homotopy invariant*

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2 \in \mathbb{Z}_2$$

Application: $\mathbb{Z}_2$ phase label for Kane-Mele model if dyn. localized

Proof via Kramers degeneracy:

First of all: $\mathrm{Ind}(T) = 0$ because $\mathrm{Ker}(T^*) = I\,\overline{\mathrm{Ker}(T)}$

Idea: $\mathrm{Ker}(T) = \mathrm{Ker}(T^* T)$

and positive eigenvalues of $T^* T$ have even multiplicity

Let $T^* T v = \lambda v$ and $w = I\,\overline{Tv}$ (N.B. $\lambda \neq 0$). Then

$$\begin{aligned} T^* T\, w \;&=\; I\,(I^* T^* I)\,(I^* T I)\,\overline{Tv} \\ &=\; I\;\overline{T}\;\overline{T^*\,T\,v} \;=\; \lambda\, I\,\overline{T}\,\overline{v} \;=\; \lambda\, w\,. \end{aligned}$$

Suppose now $\mu \in \mathbb{C}$ with $v = \mu\, w$. Then

$$v \;=\; \mu\, I\,\overline{T}\,\overline{v} \;=\; \mu\, I\,\overline{T}\,\overline{\mu}\, I\, T\, v \;=\; -|\mu|^2\, T^* T\, v \;=\; -|\mu|^2\, \lambda\, v$$

Contradiction to $v \neq 0$.

Now $\mathrm{span}\{v, w\}$ invariant subspace of $T^* T$, so orth. complement

Connectedness statement complicated to prove!

Spin filtered helical edge channels for QSH

Theorem (S-B 2013)

$\mathrm{Ind}_2(PFP) = 1 \implies$ spin Chern numbers $\mathrm{SCh}(P) \neq 0$

Remark Non-trivial topology $\mathrm{SCh}(P)$ persists TRS breaking!

Theorem (S-B 2012) $\widehat{H}$ Kane-Mele on half-space $\mathbb{Z} \times \mathbb{N}$

If $\mathrm{SCh}(P) \neq 0$, dissipationless spin filtered edge currents are stable w.r.t. perturbations by magnetic field and disorder:

$$\widehat{\mathcal{T}}\big(g(\widehat{H})\,\tfrac{1}{2}\{i[\widehat{H},X_1],s^z\}\big) \;=\; \mathrm{SCh}(P) \;+\; \text{controlled corrections}$$

where $g \geq 0$ supported in bulk gap and $\int g = 1$

Resumé: $\mathrm{Ind}_2(PFP) = 1 \Longrightarrow$ no Anderson loc. for edge states

Rice group: Du, Knez, et al since 2011 in InAs/GaSb Bilayers

Four-terminal conductance plateaux stable w.r.t. magnetic field

No And. loc. for other edge states in $d = 2$?

Class A: QHE with quantized edge currents

Class C (BdG, odd PHS): spin quantum Hall effect (with De Nittis)

Class D and DIII (even PHS): thermal quantum Hall effect (???)

Resuming: exactly CAZ classes as in quasi-1d above

Structuring: Topological insulators

Disordered Fermion systems with (mobility) gap and basic sym.

chiral sym. (CHS) and/or even/odd time reversal (TRS)

and/or even/odd particle-hole (PHS)

Ludwig *et al.* (2008): non-trivial $\Longleftrightarrow$ surface states don't localize

Here: topological invariants and Fredholm indices

Then prove bulk-edge correspondence and delocalized edge states

Periodic table of topological insulators

Schnyder-Ryu-Furusaki-Ludwig, Kitaev 2008

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
1	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Real K-theory (8-periodic) $\quad \mathrm{Inv}(j,d) = KR_j(\mathbb{R}^d_\tau) \cong \pi_{j-1-d}(O)$

Focus on chiral system in $d = 3$ (with Prodan)

Hamiltonian on $\ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^4$ first without disorder:

$$H \;=\; \sum_{j=1}^{3} \frac{1}{2i}(S_j - S_j^*) \otimes \gamma_j \;+\; \left(m + \sum_{j=1}^{3} \frac{1}{2}(S_j + S_j^*) \right) \otimes \gamma_4$$

where $\gamma_0, \ldots, \gamma_4$ irrep of Clifford C_5 such that $\gamma_0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Chiral sym: $H = -\gamma_0 \, H \, \gamma_0 = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ with invertible A in gap

Invariant (also with disorder!):

$$\mathrm{Ch}_d(A) \;=\; \frac{(-i\pi)^{\frac{d-1}{2}}}{i\, d!!} \sum_{\rho \in S_d} (-1)^\rho \, \mathcal{T} \left(\prod_{j=1}^{d} A^{-1} i[X_{\rho_j}, A] \right) \;\in\; \mathbb{Z}$$

where $\mathcal{T}(A) = \mathbf{E}_{\mathbb{P}} \, \mathrm{Tr} \, \langle 0 | A_\omega | 0 \rangle$ trace per unit volume

Gap for $m \neq -3, -1, 1, 3$ with $\mathrm{Ch}_3(A) = 0, -1, 2, -1, 0$

Why is invariant an integer?

Periodic system: differential geometric invariant (Schnyder et al)

$$\mathrm{Ch}_d(A) = \frac{(\frac{1}{2}(d-1))!}{d!} \left(\frac{i}{2\pi}\right)^{\frac{d+1}{2}} \int_{\mathbb{T}^d} \mathrm{Tr}\left(\left[A^{-1}dA\right]^d\right)$$

Disordered system: index theorem

$$D \;=\; \sum_{j=1}^{d} X_j \otimes \mathbf{1} \otimes \sigma_j \qquad \text{Dirac operator on } \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^{N'}$$

Dirac phase $F = \frac{D}{|D|}$ satisfies $F^2 = \mathbf{1}$ and $[F, A]$ compact

Theorem (with Prodan, 2014)

Let $E = \frac{1}{2}(F + \mathbf{1})$ be Hardy Projektion from D and A invertible.
Almost surely:

$$\mathrm{Ind}(E\,A_\omega E) \;=\; \mathrm{Ch}_d(A) \;\in \mathbb{Z}$$

QHE for surface states (with Prodan)

Restriction $\widehat{H}$ to half-space $\mathbb{Z}^2 \times \mathbb{N}$ has surface state bands

Adding magnetic field perpendicular to surface opens gaps

Decompose projection on central band

$$\widehat{P} \;=\; \widehat{P}_+ + \widehat{P}_- \qquad \gamma_0\, \widehat{P}_\pm \;=\; \pm\widehat{P}_\pm$$

Theorem

Bulk-edge correpondence

$$\mathrm{Ch}_3(A) \;=\; \mathrm{Ch}_2(\widehat{P}_+) \;-\; \mathrm{Ch}_2(\widehat{P}_-)$$

If $\mathrm{Ch}_3(A)$ *odd, surface QHE:* $\mathrm{Ch}_2(\widehat{P}) = \mathrm{Ch}_2(\widehat{P}_+) + \mathrm{Ch}_2(\widehat{P}_-) \neq 0$

Hence somewhere divergence of localization length in surface states

Everything stable under weak breaking of chiral symmetry.

Again: non-trivial topology $\Longrightarrow$ no Anderson localization

Resumé

- Index theorems guarantee stability of invariants
- Odd d invariants persist under weak breaking of CHS
- Non-trivial topology may survive weak breaking of TRS, PHS
- Bulk-edge correspondence establishes link of topologies
- Surface states are not exposed to Anderson localization (rigorous proofs)
- Physical effects have to be studied case by case

$\mathbb{Z}_2$ invariant and spin-charge separation

Other physical effect linked to non-trivial $\mathbb{Z}_2$ invariant:

Theorem (with De Nittis, 2014)

$\mathrm{Ind}_2(PFP) = 1 \implies H(\alpha = \frac{1}{2})$ has TRS + Kramers pair in gap