Continuum limits of interfacial energies on (sparse and) dense graphs

Andrea Braides (Roma Tor Vergata)

One-World Seminar
Mathematical Methods for Arbitrary Data Sources (MADS)

June 15, 2020
Minimal-cut problems on large graphs

**Graph** $G; \mathcal{V}(G) =$ vertices of $G; \mathcal{E}(G) =$ edges of $G$ (describing “interacting vertices”)

**Minimal-cut Problem:** subdivide $\mathcal{V}(G)$ into families of given cardinality by minimizing an **edge energy** (in the simplest case: the number of edges between vertices of two different families)

**Asymptotic Analysis:** describe the behaviour of minimal-cut problems on **sequences of graphs** with a **diverging number of vertices**.
Minimal-cut problems on large graphs

Graph $G$; $\mathcal{V}(G)$ = vertices of $G$; $\mathcal{E}(G)$ = edges of $G$ (describing “interacting vertices”)

Minimal-cut Problem: subdivide $\mathcal{V}(G)$ into families of given cardinality by minimizing an edge energy (in the simplest case: the number of edges between vertices of two different families)

Asymptotic Analysis: describe the behaviour of minimal-cut problems on sequences of graphs with a diverging number of vertices.
Plan of the seminar

1. Introduction: asymptotic analysis of minimal-cut problems on graphs via variational methods
2. A class of sparse graphs (i.e., with $\#\mathcal{E}(G) \ll (\mathcal{V}(G))^2$) parameterized on lattices
3. Intermission: topological issues for general sparse graphs
4. Dense graphs (i.e., with $\#\mathcal{E}(G)$ of the order of $(\mathcal{V}(G))^2$)
1. Introduction: asymptotic analysis of minimal-cut problems on graphs
Introduction of an order parameter.
For simplicity, we only treat the case of subdivisions in two families, so that we may choose $1$ and $-1$ as parameters. Subdivisions are described by functions $u : \mathcal{V}(G) \rightarrow \{-1, 1\}$ (spin functions) (Constraints on the cardinality of the subdivisions can be imposed separately).

Edge energies.
We may describe edge energies as defined on such $u$:

$$E_G(u) = \frac{1}{4} \sum_{i,j \in \mathcal{V}(G)} a_{ij} |u_i - u_j| = \frac{1}{8} \sum_{i,j \in \mathcal{V}(G)} a_{ij} |u_i - u_j|^2,$$

where $a_{ij} \geq 0$ is the “strength” of the bond with the vertices $i$ and $j$ as endpoints ($a_{ij} = 0$ if this bond is not an edge of $\mathcal{E}(G)$).
Asymptotic analysis of families of edge energies by variational methods.

Given $G_n$ with $\#\mathcal{V}(G_n) \to +\infty$ define a limit energy $F$ for which solutions to minimal-cut problems provide an approximate description of minimal-cut problems for $G_n$ ($\Gamma$-convergence).

The crucial issue here is to embed all graphs $G_n$ is a common environment, in which we have compactness properties of minimizing sequences $\{u_n\}$.

The notion of converge of functions $u_n$ defined on $\mathcal{V}(G_n)$, and correspondingly the definition of $F$ and the type of approximate description will be different in the case of sparse (lattice) graphs and dense graphs.
A class of lattice sparse graphs

(in collaboration with Alicandro, Causin, Cicalese, Piatnitski and Solci)
A model lattice energy: the (nearest-neighbour) Ising system
(Caffarelli-De la Llave 2004, etc.)

In this case the graph \( G \) is \( \mathbb{Z}^d \), or (in its ‘localized’ version) \( D \cap \mathbb{Z}^d \) with \( D \) a bounded domain in \( \mathbb{R}^d \), with edges only the bonds between nearest neighbors.

- **Parameter**: (scalar) spin function \( i \mapsto u_i \in \{-1, 1\} \) defined on \( \mathbb{Z}^d \)

- **Ferromagnetic energy**: \( E(u) = \frac{1}{4} \sum_{|i-j|=1} |u_i - u_j| \)

- **Piecewise-constant interpolation and identification with a set**: \( u \sim \{u = 1\} =: A(u) \)

- **Energy as a perimeter functional** \( E(u) = \text{Per}(A(u)) \) in a 1-periodic environment (the perimeter is the same as the number of edges in \( \partial A(u) \))
Discrete-to-continuum analysis

**Fundamental compactness result:** families of sets of equibounded perimeter are precompact in the class of sets of finite perimeter
(for our purposes, we may regard such sets as sets $A$ with piecewise $C^1$-boundary, whose inner normal at $\mathcal{H}^{d-1}$-a.a. points of $\partial A$ is denoted by $\nu$)

We introduce a small parameter $\varepsilon > 0$ and consider the graphs $G_\varepsilon$ as $\varepsilon G$, or, in the “local” version, $\mathcal{V}(G_\varepsilon) = D \cap \varepsilon \mathbb{Z}^d$ so that $\# \mathcal{V}(G_\varepsilon)$ is of order $\varepsilon^{-d}$.

- **Surface scaling:** we scale $E$ so that it maintains the dimension of a perimeter energy, and define

$$E_\varepsilon(u) = \frac{1}{4} \sum_{|i-j|=1} \varepsilon^{d-1} |u_{\varepsilon i} - u_{\varepsilon j}| \text{ for } u : \varepsilon \mathbb{Z}^d \to \{-1, 1\}$$

If $A_\varepsilon(u) (= \varepsilon A(u_{\varepsilon I}))$ is the union of the $\varepsilon$-cubes such that $u_{\varepsilon i} = 1$, still $E_\varepsilon(u) = \text{Per}(A_\varepsilon(u))$ in an $\varepsilon$-periodic environment
Convergence of lattice functions to sets: we say that $u^\varepsilon \rightarrow A$ if $A_\varepsilon(u^\varepsilon)$ tends to $A$ locally in $\mathbb{R}^2$

With this convergence we may compute the $\Gamma$-limit of $E_\varepsilon$ (homogenization) within energies on sets of finite perimeter (cf. Ambrosio-Braides JMPA 1991)

The $\Gamma$-limit of $E_\varepsilon$ is $F(A) = \int_{\partial A} \|\nu\|_1 \, d\mathcal{H}^{d-1} = \int_{\partial A} \sum_{j=1}^{d} |\nu_j| \, d\mathcal{H}^{d-1}$ for $A$ set of finite perimeter (anisotropic crystalline perimeter).

This can be read as: minimal-cut problems for $E_\varepsilon$ are approximated by constrained least-(crystalline) perimeter problems for $F$. 

\begin{align*}
\nu &= -1 \\
\nu &= +1
\end{align*}
Generality of this approach

- NN graphs can be substituted by graphs $G_\varepsilon$ with vertices $\varepsilon \mathbb{Z}^d$ but depending on $\varepsilon$ not only by scaling, and correspondingly

$$E_\varepsilon(u) = \frac{1}{4} \sum_{i,j} \varepsilon^{d-1} a_{i,j}^\varepsilon |u_{\varepsilon i} - u_{\varepsilon j}|,$$

with $a_{i,j}^\varepsilon$ “uniformly decaying” as $|i - j| \to +\infty$

(the model is finite-range interactions – we may also have $a_{i,j}^\varepsilon > 0$ for all $i,j$, in which case this is a dense graph family seen as a “uniform limit of sparse graphs”).

The limit then exists up to subsequences, and is of the form

$$F(A) = \int_{\partial A} \varphi(x, \nu) d\mathcal{H}^{d-1}$$

with $\varphi$ one-homogeneous and convex in $\nu$, and $\varphi(x, \nu) = \varphi(\nu)$ if $a_{i,j}^\varepsilon$ depend on $i - j$ (in which case, we also have that the whole sequence converges).

$\varphi$ is computed via discrete optimal-interface problems; the longer the range of (effective) interactions the more $\varphi$ is symmetric.

(e.g. in 2D NN give $\varphi$ with a square symmetry, NNN with an octagonal symmetry, etc.)
• $\mathbb{Z}^d$ may be substituted by any Bravais lattice or multi-lattice (also a-periodic lattices, as Penrose lattices, can be used), or a random subset of $\mathbb{Z}^d$ (with sufficiently high probability) according to an i.i.d. random variable.

• some dynamical problems can be treated in terms of crystalline-curvature flows (using the connection between perimeter and mean-curvature flow).

• we may study energies depending on more than two parameters, in which case we define a convergence $u_\varepsilon \to (A_1, \ldots, A_M)$. The $\Gamma$-limit is then of the form (in the homogeneous case)

$$F(A_1, \ldots, A_M) = \sum_{k,l} \int_{\partial A_l \cap \partial A_k} \varphi_{kl}(\nu) d\mathcal{H}^{d-1}$$

(where $\nu$ is the common normal (up to a sign) of $\partial A_l$ and $\partial A_k$), and the limits of minimal-cut problems are optimal-partition problems.
Towards more arbitrary data sets-1: coarse-graining

(B-Solci, preprint 2019)

• This approach generalizes to “regular sparse graphs”. In this case we suppose that there exist length scales $\varepsilon$ and $R_\varepsilon$ such that
• in any cube of side length $\varepsilon R_\varepsilon$ we have order-$(\varepsilon R_\varepsilon)^d$ vertices of $G_\varepsilon$
• the graph $G_\varepsilon$ contains all edges of length not larger than $R_\varepsilon \varepsilon$

We may then show a compactness result by using a coarse-grained variable $U$ defined on an Ising system on a lattice $\varepsilon R_\varepsilon \mathbb{Z}^d$

(we set $U_i = 1$ if all values of $u$ are equal to 1 in the cube of centre $i$ and side-length $\varepsilon R_\varepsilon$ and $U_i = -1$ otherwise – and show that the cubes where $u$ is not constant are asymptotically negligible)

Note: we may use large $R_\varepsilon$ to obtain isotropic $\varphi$; cf. work by Garcia-Trillos and Slepčev
The graphs $G_\varepsilon = \varepsilon G$ is defined by scaling a graph $G$ in $\mathbb{R}^2$ defined as follows.

- The set of **vertices of** $G$ are given by $\mathcal{L}$, where $\mathcal{L}$ is a Poisson random set with intensity $\lambda$, characterized by the properties
  (a) for any bounded Borel set $B \subset \mathbb{R}^2$ the number of points in $B \cap \mathcal{L}$ has a Poisson law with parameter $\lambda|B|

\[ P(\{ \#(B \cap \mathcal{L}) = n \}) = e^{-\lambda|B|} \frac{(\lambda|B|)^n}{n!}; \]

(b) for any collection of bounded disjoint Borel subsets in $\mathbb{R}^2$ the random variables defined as the number of points of $\mathcal{L}$ in these subsets are independent.
the set of **edges of** $G$ are given by bonds connecting nearest neighbours, i.e., vertices whose Voronoi cells

$C_i = \{ x \in \mathbb{R}^2 : |x - i| \leq |x - j| \text{ for all } j \in \mathcal{L} \}$ have a common edge

Terminology: bonds between nearest neighbors give the **Delaunay triangulation** of $\mathcal{L}$

**Note:**

- $\mathcal{L}$ is not “regular”: we have pairs of points of $\mathcal{L}$ arbitrarily close, and cubes of arbitrary size not containing points of $\mathcal{L}$
- $\mathcal{L}$ is isotropic since the properties of Poisson random sets are invariant under (translations and) rotations
Discrete-to-continuum approach

In the same way as for the Ising system, we define the energy in terms of nearest neighbors for the Delaunay triangulation and scale them.

**Ferromagnetic energies:** We consider

\[
E_\varepsilon(u) = \sum_{\langle i,j \rangle} \varepsilon |u_{i\varepsilon} - u_{j\varepsilon}| \text{ for } u : \varepsilon \mathcal{L} \to \{-1, 1\}
\]

(\langle i, j \rangle denotes summation over nearest neighbours)

and \( V_\varepsilon(u) \) for the union of the \( \varepsilon \)-Voronoi cells \( \varepsilon C_i \) such that \( u_{\varepsilon i} = 1 \)

**Issue:** we cannot estimate \( \operatorname{Per} V_\varepsilon(u_\varepsilon) \) in terms of \( E_\varepsilon(u_\varepsilon) \).

(Even for a single Voronoi cell: we have large \( C_i \) with few edges or small \( C_i \) with many edges).
A compactness lemma

**Lemma**

Let \( u^\varepsilon \) be such that \( \sup_{\varepsilon} E_{\varepsilon}(u^\varepsilon) < +\infty \). Then we can write

\[
V_{\varepsilon}(u^\varepsilon) = (A_{\varepsilon} \cup B'_\varepsilon) \setminus B''_\varepsilon,
\]

where \( |B'_\varepsilon| + |B''_\varepsilon| \to 0 \) and the sets of the family \( \{A_{\varepsilon}\} \) have equi-bounded perimeter. Hence, (up to subsequences) there exists a set of finite perimeter \( A \) such that \( \chi_{V_{\varepsilon}(u^\varepsilon)} \) converges to \( \chi_A \) in \( L^1_{\text{loc}}(\mathbb{R}^2) \).
Coarse-graining by percolation

Given $\alpha > 0$ we define the set of $\alpha$-regular points of $\mathcal{L}$ as

$$\left\{ i \in \mathcal{L} : C_i \text{ contains a ball of radius } \alpha, \text{ diam } C_i \leq \frac{1}{\alpha}, \text{ #edges } \leq \frac{1}{\alpha} \right\}$$

If $\alpha$ is small enough, there exists $L > 0$ such that the event that $k, k' \in \mathbb{Z}^2$ with $|k - k'| = 1$ are such that the segment $[Lk, Lk']$ intersects only $\alpha$-regular sets has probability $p > 1/2$.

We can then apply Bernoulli bond-percolation theory to the bonds $[k, k']$ of nearest neighbors in $\mathbb{Z}^2$ (Ising systems where edges are accounted for with probability $p$)

(This is the only 2D-argument in the proof)
The existence of infinite clusters for supercritical Bernoulli percolation allows to prove a formula for the surface tension ($\mathcal{D}$ the Delaunay triangulation).

$$
\tau = \lim_{t \to +\infty} \frac{1}{t} \min \{ \# \text{(segments of paths in $\mathcal{D}$ 'almost' joining $(0, 0)$ and $(t, 0)$)} \}
$$

A subadditive argument allows to show that this limit exists a.s. and is deterministic.

A scaling argument shows that $\tau = \tau_0 \sqrt{\lambda}$ ($\lambda$ the intensity of the Poisson random set). Eventually, we have:

**Theorem (B-Piatnitski 2020)**

*Almost surely the functionals $E_\varepsilon \Gamma$-converge to $\tau_0 \sqrt{\lambda} H^1(\partial A)$ with respect to the $L^1$-loc convergence of $V_\varepsilon(u^\varepsilon)$ to $A$.*

In this case the isotropy of the limit is due to the isotropy of the Poisson random set and not on the “regularizing effect” of long-range interactions (coarse-grained case with $R_\varepsilon \to +\infty$).
Intermission
Back to one-dimensional (trivial) toy models

\[ u_i \in \{ -1, 1 \} \text{ (spin function), } i \in \mathbb{Z} \]

Nearest-neighbour interaction chain (NN Ising system)

Energy = \#\{i : u_i \neq u_{i-1}\} \quad (= \sum_i (u_i - u_{i-1})^2)

Finite-range interaction chain

Energy = \sum_{|i-j| \leq T} c_{ij} (u_i - u_j)^2

In both cases we have a sharp-interface limit.

**Question:** is this always the case when the range of interactions is \( \ll 1 \) ?
A counterexample to sharp interfaces

Example: (B-Causin-Solci AMPA 2017) each point in $\varepsilon\mathbb{Z}$ is connected with its NN and its $1/\sqrt{\varepsilon}$-neighbours (So that the range is $\sqrt{\varepsilon}$)

- optimal sequences exhibits a diffuse interface

- we lose the constraint $u \in \{-1, 1\}$
- the $\Gamma$-limit is the total variation of $u \in BV_{\text{loc}}(\mathbb{R}; [-1, 1])$

Note: the behaviour is due to the topology of the graph of connections, which is “higher dimensional”
Dense graphs

(in collaboration with Cermelli and Dovetta)
Energies on abstract graphs

Abstract graph $G$; $\mathcal{V}(G)$ = vertices of $G$; $\mathcal{E}(G)$ = edges of $G$

If $n = \#(\mathcal{V}(G))$ then $G$ is parameterized on $\{1, \ldots, n\}$
(1D parameterization)

Adjacency matrix $A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{E}(G) \\ 0 & \text{otherwise} \end{cases}$

Energy $E(u) = \sum_{i,j} A_{ij}(u_i - u_j)^2$ for $u : \{1, \ldots, n\} \to \{-1, 1\}$

Example:

Note: no embedding in a fixed $\mathbb{R}^d$, no fixed geometry of graph
Left convergence for graphs

**Issue:** define a meaningful convergence of abstract graphs

A convergence maintaining topological characteristics of graphs
For $F$ and $G$ graphs, let $\text{hom}(F, G)$ be the set of homomorphisms $\phi : \mathcal{V}(F) \rightarrow \mathcal{V}(G)$. The homomorphism density of $F$ and $G$ is

$$ t(F, G) = \frac{\#\text{hom}(F, G)}{\#(\mathcal{V}(G)^{\mathcal{V}(F)})} $$

**Definition** A sequence of graphs $\{G_n\}_{n \in \mathbb{N}}$ is said to be left convergent if, for every simple graph $F$, the quantities $t(F, G_n)$ converge for $n \rightarrow +\infty$

…but converge to what?

**Note:** this convergence is meaningful only if $G_n$ are dense: if $G_n$ are sparse we have $t(F, G_n) \rightarrow 0$ for all $F$
Graphons (cf. Lovasz et al.)
(embedding graphs in a set of functions we obtain a functional version of left convergence)

To each finite graph $G$ with $\#V(G) = n$ we associate the piecewise-constant function $W_G : [0,1]^2 \to \{0,1\}$

$$W_G(x,y) = A_{ij} \quad \text{if} \quad (x,y) \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(\frac{j-1}{n}, \frac{j}{n}\right)$$

Example (continued): (black region = set where $W = 1$)

![Graph Example](image)

Definition A graphon is a symmetric bounded measurable function $W : [0,1]^2 \to \mathbb{R}$. We denote $\mathcal{W}$ the set of all graphons.

Heuristically, a graphon is a graph with $[0,1]$ as the set of vertices and with adjacency matrix given by $W(x,y)$. 
More examples

(a) the complete graph, (b) weakly connected subgraphs
(c) a complete bipartite graph, (d) the half graph
Fancy examples
Adjacency matrices for “real-life” large networks

(from F. Caron and E.B. Fox Sparse graphs using exchangeable random measures. Journal of the Royal Statistical Society 79 (2017))
The cut norm

If $W \in \mathcal{W}$ we define the **cut norm** of $W \in \mathcal{W}$ as

\[ ||W||_\square = \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} W(x, y) dx dy \right| \]

with the supremum taken over all measurable subsets $S, T$ of $[0,1]$.

**Equivalent definition**

\[ ||W||_\square = \sup_{f,g:[0,1] \rightarrow [0,1]} \left| \int_{[0,1]^2} W(x, y) f(x) g(y) dx dy \right|, \]

where the supremum in the last definition is taken over all measurable functions.

**Note:** $||W||_\square$ is a sort of “uniform weak-$L^1$ norm”

**Note:** $||W_G - W_F||_\square$ **depends on the parametrization** of $F$ and $G.$
The cut metric

Example: (two different parametrization of the same $G_n$)

parametrizations of bipartite graphs (in particular $\|W_{G_n} - W'_{G_n}\| □ \neq 0$). Note that the right-hand side one does not converge to its weak $L^1$ limit.

We define the cut distance between graphons

$$\delta □(W, W') = \inf_{\psi, \phi} \|W_{\psi} - W'_{\phi}\| □,$$

where $W_{\psi}$ is the rearrangement of $W$ through the measure-preserving map $\psi : [0, 1] \rightarrow [0, 1]$ $W_{\psi}(x, y) = W(\psi(x), \psi(y))$.

Note: (trivially) $\delta □(W_G, W_F)$ is independent on the parametrization of $F$ and $G$. 
Graph sequences

We denote by $\mathcal{W}_0$ the set of graphons on $[0, 1]^2$ taking values in the interval $[0, 1]$.

**Theorem** Let $G_n$ be a sequence of graphs and $W_n = W_{G_n}$. Then the following are equivalent:

- $\{G_n\}$ is left convergent;
- $\{W_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\delta_{\square}$;
- there exists a $W \in \mathcal{W}_0$ such that $t(F, G_n) \to t(F, W)$ as $n \to \infty$, for every simple graph $F$, where

$$t(F, W) = \int_{[0,1]^k} \prod_{ij \in E(F)} W(x_i, x_j) dx_1 \ldots dx_k$$

for any simple graph $F$ with $\mathcal{V}(F) = \{1, \ldots, k\}$

(graphon version of the homeomorphism density)

The limit is unique up to identification of graphons with cut distance equal to 0, that is if both $G_n \to W$ and $G_n \to W'$, then $\delta_{\square}(W, W') = 0$. Moreover, every bounded graphon $W \in \mathcal{W}_0$ arises as the limit in the cut distance of a convergent sequence of graphs $\{G_n\}$. 

Theorem (relabeling) Let $\{G_n\}_{n \in \mathbb{N}}$ be any sequence of simple graphs. If
\[ \delta_{\square}(W_{G_n}, W) \to 0 \]
for some $W \in \mathcal{W}_0$, then there exists a relabeling of the vertices such that the resulting sequence $\{G'_n\}_{n \in \mathbb{N}}$ of labeled graphs converges in the cut norm to $W$:
\[ ||W_{G'_n} - W||_{\square} \to 0. \]
Moreover, this convergence is compact.
Variational theory of cut functionals

B-Cermelli-Dovetta, ESAIM: COCV 2020

We consider a sequence of finite graphs \( \{G_n\} \), and the related (normalized) energies

\[
E_n(u) = \frac{1}{N_n^2} \sum_{i,j} A_{ij}^n (u_i - u_j)^2,
\]

where \( N_n = \#\mathcal{V}(G_n) \).

Spin functions: cut functionals as energies on \( L^\infty \) functions

We consider spin functions \( u : G_n \to \{-1, 1\} \) as defined on \([0, 1]\) with the identification of \( u \) with the piecewise-constant function \( \tilde{u} : [0, 1] \to \{-1, 1\} \) satisfying

\[
\tilde{u}(x) = u(i) \quad \text{if} \ x \in \left( \frac{i - 1}{N_n}, \frac{i}{N_n} \right],
\]

(i.e., \( G_n \) is parameterized with \( \left\{ \frac{1}{N_n}, \ldots, \frac{N_n}{N_n} \right\} \) instead of \( \{1, \ldots, N_n\} \); i.e. we go back to the discrete-to-continuum argument with \( \varepsilon = 1/N_n \))

We write \( X_n \) for the space of such piecewise-constant spin functions. The energies \( E_n \) are regarded as defined on \( L^\infty(0, 1) \) with domain \( X_n \) (extended to \( +\infty \) elsewhere).
Theorem (Γ-convergence of cut functionals)

Let \( \{G_n\} \) be a dense graph sequence; i.e., \( \limsup_{n \to \infty} \frac{\#V(G_n)^2}{\#E(G_n)} < +\infty \), and let the graphons \( W_n = W_{G_n} \) satisfy

\[ \|W_n - W\|_\square \to 0 \]

for some \( W \in \mathcal{W}_0 \). Then the \( \Gamma \)-limit of \( E_n \) with respect to the weak* convergence in \( L^\infty(0, 1) \) is

\[ E(u) = 2 \int_{[0,1]^2} W(x, y)(1 - u(x)u(y)) \, dx \, dy \]

with domain \( X = \{ u \in L^\infty(0, 1) : \|u\|_\infty \leq 1 \} \).

Moreover the \( \Gamma \)-convergence of \( E_n \) is compatible with integral constraints (and in particular applies to minimal-cut problems), and the result can be extended to \( u : G_n \to K \), with \( K \) finite (optimal-partition cuts).
**Proof**

i) For $u \in X_n$ write

$$E_n(u) = \int_{[0,1]^2} W_n(x, y)(u(x) - u(y))^2 dxdy$$

ii) note that $(u(x) - u(y))^2 = 2(1 - u(x)u(y))$ since $|u| = 1$;

iii) note that we may replace $W_n$ by $W$ since $\|W_n - W\| \to 0$;

iv) note that $u \mapsto \int_{[0,1]^2} W(x, y)u(x)u(y) dxdy$ is continuous with respect to the $L^\infty$ weak* convergence;

v) by the closure of the constraint the liminf inequality holds.

The construction of recovery sequence is a more-or-less standard construction for the Young-measure

$$\nu_x = \frac{1}{2}(u(x) + 1)\delta_1 + \frac{1}{2}(1 - u(x))\delta_{-1}$$

and the use of the continuity property (iv) above. □
Conclusions

- We have examined Geometric Measure Theory methods for discrete-to-continuum limits of a class of lattice systems producing interfacial energies, for which minimal-cut problems on graphs may be approximated by minimal-perimeter problems on the continuum. These methods extend to various types of random graphs via coarse-graining and percolation techniques.
- Simple examples show that even for simple sparse graphs diffuse interfaces may appear. These interfaces hint at the dependence on some intrinsic topological features of the graphs.
- For dense graphs the asymptotic behaviour of topological features can be described by left convergence. This convergence can be set in an analytic form, so that it is possible to define the limit of graphs (graphons) and the limit of minimal-cut problems on graphs as minimal-cut problems on graphons, remarking that convergence of graphs to graphons implies the $\Gamma$-convergence of cut functionals in a weak$^\ast$-$L^\infty$ setting.
A list of references on lattice surface energies up to 2014 can be found in


Recent papers.


For graphons see the reference list of the paper A. Braides, P. Cermelli, and S. Dovetta
Thank you for you attention!