Partial Regularity for Bounded Solutions of a Class of Cross-Diffusion Systems

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Overview

Cross-diffusion systems

• We are interested in systems of the form

$$\partial_t u_i - \sum_{j=1}^{n} \nabla \cdot A_{ij}(u) \nabla u_j = f_i(u) \quad \text{in} \quad \Lambda := \Omega \times (0, T),$$

for $i = 1, \ldots, n$, with the boundary/initial data

$$A_{ij}(u) \nabla u_j \cdot \nu = 0, \qquad u(\cdot, 0) = u_0 \text{ a.e. in } \Omega.$$

Here, $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ is a bounded domain and $T > 0$.

• Systems of this form can be used to model: gaseous or fluid mixtures (Maxwell-Stefan model, 1866 and 1871), population dynamics (SKT model, '79), semiconductors with electron-hole scattering (Reznik, '95), tumour growth (Jackson and Bryne, 2002), ...

$\rightarrow$ The components u_i are chemical or population densities, with interactions governed by the diffusion coefficients $A_{ij}(u)$ and the reaction terms $f_i(u)$.

Heuristic description of Main Results

• The existence, uniqueness, and long-time behavior of weak solutions of cross-diffusion systems have been quite well-studied in recent years.

→ But there are few works addressing the regularity of weak solutions.

Our main result is of the following form: Let $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be uniformly continuous and $f : \mathbb{R}^n \to \mathbb{R}$ continuous.

- We derive sufficient conditions on A such that bounded weak solutions of the cross-diffusion system satisfy a partial $C^{0,\alpha}$-regularity result.

  → These conditions are on an *entropy structure* for the system.

- If A is also Hölder continuous with exponent $\sigma \in (0, 1)$, then we find that the gradient of a bounded weak solution also satisfies a partial $C^{0,\sigma}$-regularity result.

- We motivate our techniques with the 2-component Shigesada-Kawasaki-Teramoto model for population dynamics.

⋆ The only previous works that we are aware of are (Le, '98), (Le, 2005), and (Le and Nguyen, 2006). These do not take advantage of an entropy structure.

Shigesada-Kawasaki-Teramoto Model

We will use the following as an example of a prototypical cross-diffusion system:

Example: (SKT model for population dynamics) This model is for the dynamics of interacting subpopulations –here we give it for $n = 2$. The model is defined by the diffusion matrix

$$A_{\mathrm{SKT}}(u) = \begin{bmatrix} \alpha_{10} + 2\alpha_{11}u_1 + \alpha_{12}u_2 & \alpha_{12}u_1 \\ \alpha_{21}u_2 & \alpha_{20} + \alpha_{21}u_1 + 2\alpha_{22}u_2 \end{bmatrix},$$

where we assume that each $\alpha_{ij} > 0$. One usually considers this model with Lotka-Volterra type source terms

$$f_i(u) = (\beta_{i0} - \beta_{i1}u_1 - \beta_{i2}u_2)u_i \qquad \text{for} \quad i = 1, 2,$$

where the $\beta_{ij} \geq 0$.

Some observations:

- $A_{\mathrm{SKT}}(u)$ is not bounded, unless u is bounded.
- $A_{\mathrm{SKT}}(u)$ is not symmetric.
- $A_{\mathrm{SKT}}(u)$ is not strictly positive-definite.

→ Obtaining *a priori* estimates is going to be a problem...

A priori estimates for cross-diffusion systems

An illustrative example: Let $A = \mathrm{Id}$, then the coupling of the components only enters in the reaction terms on the right-hand side.

Two equivalent notions:

Energy estimate Testing the system with u, we obtain:

$$\int_\Lambda \partial_t u^2 = -\int_\Lambda \nabla u : A(u)\nabla u + \int_\Lambda u \cdot f(u).$$

Since $\nabla u : A(u)\nabla u = |\nabla u|^2$, we then find that:

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega))} \lesssim \int_\Lambda |f(u)|^2.$$

Entropy estimate Let $h(u) = u^2$ and take the time derivative:

$$\partial_t \int_\Omega h(u) = -\int_\Omega \nabla u : h''(u)A(u)\nabla u + \int_\Omega u \cdot f(u).$$

Since $\nabla u : h''(u)A(u)\nabla u = 2|\nabla u|^2$, we then find that:

$$\partial_t \int_\Omega h(u) \leq -2\int_\Omega |\nabla u|^2 + \int_\Omega u \cdot f(u).$$

If we assume that $\int_\Omega u \cdot f(u) < 0$, then $\mathcal{H}(\cdot) := \int_\Omega h(\cdot)$ is a Lyapunov functional.

$\rightarrow$ In this example, these two estimate are pretty much the same thing.

- But what happens if things become more complicated?

A priori estimates for cross-diffusion systems: An entropy structure

• For the cross-diffusion systems that we are interested in A is neither symmetric nor positive definite → We cannot use energy estimates.

• To make some *apriori* estimates available, we restrict ourselves to cross-diffusion systems with a *strict entropy structure*.

→ This means that... there exists a convex function h such that

$$h''(y)A(y) \geq \lambda |y|^2 \text{ for some } \lambda > 0 \text{ and } y \in \mathcal{D},$$

where we assume that the range of u is contained in $\mathcal{D}$.

• The function $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is the *entropy density* and the functional $\mathcal{H}(\cdot) = \int_\Omega h(\cdot)$ is the *entropy*.

• When there is a strict entropy structure, we have access to the entropy estimate shown on the previous slide.

Intuitive take-away: If we would like to obtain partial regularity results similarly to the traditional way via energy methods, we will have to replace energy estimates by entropy estimates.

Existence of weak solutions via entropy methods.

• The concept of replacing the use of energy estimates by entropy estimates is by no means new.

→ This can, *e.g.*, be seen is in the existence theory for weak solutions of cross-diffusion systems with entropy structure via the *boundedness by entropy method* (Jüngel, 2015).

Steps of the boundedness by entropy method:

- Regularize: Discretize the time derivative with a first-order implicit Euler scheme and add vanishing viscosity and massive terms.
- Solve: The regularized systems (which are elliptic) can be solved using a Lax-Milgram argument.
- Pass to the limit: Using uniform estimates for the regularized solutions obtained via entropy methods, one can then pass to the limit in the regularization.

Example of an entropy structure : SKT Model

A simple calculation shows that the SKT model has an entropy structure with

$$h(u) = \sum_{i=1}^{2} h_i(u_i) = \frac{u_1}{\alpha_{12}}(\log(u_1) - 1) + \frac{u_2}{\alpha_{21}}(\log(u_2) - 1).$$

In particular, we find that

$$h''(u)A(u) = \begin{bmatrix} \frac{1}{\alpha_{12}u_1}(\alpha_{10} + 2\alpha_{11}u_1 + \alpha_{12}u_2) & 1 \\ 1 & \frac{1}{\alpha_{21}u_2}(\alpha_{20} + 2\alpha_{21}u_1 + 2\alpha_{22}u_2) \end{bmatrix}.$$

→ $h''(\cdot)A(\cdot)$ is positive-definite on $\mathcal{D} = \mathbb{R}^n_+$, which contains the range of u, and is symmetric.

- Non-negative weak solutions of the SKT system are now provided by the boundedness by entropy method.

→ But since the system is not volume-filling, the boundedness of the solutions is not free. While there are partial results available –*e.g.* in the case that self-diffusion dominates (Le, 2006) or $A(u)$ is triangular (Choi, Liu, and Yamada, 2003)– showing the boundedness of solutions of the SKT system remains an open problem.

A notion of distance between two functions: Relative entropy

• Let h be an entropy density. Then for $v \in H^1(\Lambda, \mathbb{R}^n)$ we define the *relative entropy density with respect to v* as

$$\mathcal{H}[\,\cdot\,|v] := \int_\Lambda h(\cdot\,|v) = \int_\Lambda h(\cdot) - \int_\Lambda h(v) - \left\langle \int_\Lambda h'(v)\,, \cdot - v \right\rangle .$$

• The above expression is obtained by looking for an affine functional, ℓ, such that $\mathcal{H}[\,\cdot\,] - \mathcal{H}[v] - \ell(\cdot)$ is positive and takes its minimum value of 0 at v

• The relative entropy also satisfies an entropy estimate.

$\rightarrow$ So, it is a convenient notion of distance between two functions that lends itself to obtaining estimates.

Intuitive take-away: *If we would like to obtain partial regularity results similarly to the traditional way via energy methods,* we will have to replace the squared L^2-distance by the relative entropy.

• This intuition has already been used to study the uniqueness (Chen and Jüngel, 2018; Fischer, 2017) and the long-time asymptotics (Carrillo, Jüngel Markowich, Toscani, and Unterreiter, 2001) of solutions.

Campanato spaces and Hölder regularity

- To prove regularity results, it is helpful to characterize Hölder spaces in terms of Campanato norms –*i.e.* their approximability properties with respect to polynomials.

- For $\alpha \in (0,1)$, we define the *Campanato space*

$$\mathcal{L}^{2,d+2+2\alpha}(\Lambda) := \Big\{ u \in L^2(\Lambda) : \sup_{z_0 \in \Lambda, R>0} R^{-\alpha} \int_{\mathcal{C}_R(z_0)} |u - (u)_{z_0,R}|^2 \mathrm{d}z < \infty \Big\},$$

where $\mathcal{C}_R(z_0) = B_R(x_0) \times (t_0, t_0 - R^2)$ with $z_0 = (x_0, t_0) \in \Omega \times (0,T)$.

- It is a classical observation that $\mathcal{L}^{2,d+2+2\alpha}(\Lambda) \cong C^{0,\alpha}(\Lambda)$, where both spaces are defined in terms of the *parabolic metric*

$$\delta(z_1, z_2) := \max(|x_1 - x_2|, |t_1 - t_2|^{\frac{1}{2}}).$$

- We will use the following local version of this equivalence: If there exists a neighborhood of $z_0 \in \Lambda$ in which the estimate

$$R^{-d-2-2\alpha} \int_{\mathcal{C}_R(z)} |u - (u)_{z,R}|^2 \mathrm{d}z < C \qquad (\star)$$

holds uniformly in z for any $R > 0$ small enough, then u is locally Hölder continuous with exponent α around z_0.

Campanato iteration: An excess-decay

• To show the local condition $(\star)$ around a point z_0, one usually proceeds by showing an *iterative decay of the tilt excess*.

• For $z_0 \in \Lambda$ and $R > 0$, we define the tilt excess as

$$\phi(z_0, R) := \fint_{\mathcal{C}_R(z_0)} \left|u - (u)_{z,R}\right|^2 \mathrm{d}z.$$

• The condition $(\star)$ is equivalent to the following excess decay: For $R_0 > 0$ chosen sufficiently small, there exists a neighborhood of z_0 with the property that

$$\phi(z, r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} \phi(z, R) + R^{2\alpha}$$

holds uniformly in z for any $0 < r \le R < R_0$.

$\rightarrow$ To summarize: The above excess decay is equivalent to the local $C^{0,\alpha}$-regularity of u around the point z_0.

Partial regularity for nonlinear parabolic systems: (Giaquinta and Struwe, 1982)

• To obtain our partial regularity result for cross-diffusion systems, we emulate the previous work of Giaquinta and Struwe.

Giaquinta and Struwe: Consider weak solutions of the nonlinear parabolic system

$$\partial_t u_i - \sum_{j=1}^{n} \nabla \cdot (A(z,u)\nabla u_j) = f_i(z,u,\nabla u) \quad \text{in} \quad \Omega \times (0,T),$$

for $i = 1, \ldots, n$. The assumptions on the system are:

- A is uniformly bounded and satisfies an ellipticity condition.
- $f(z,u,p) \leq a|p|^2 + b$ for some $a, b \in \mathbb{R}_+$.

They show a partial regularity result:

▶ Let $\alpha \in (0,1)$, then there exists an open set $\Lambda_0 \subset \Lambda$ such that $u \in C^{0,\alpha}_{\text{loc}}(\Lambda_0)$. The set Λ_0 satisfies the condition

$$\Lambda \setminus \Lambda_0 \subset \left\{ z \in \Lambda : \liminf_{R\to 0} \fint_{\mathcal{C}_R(z)} |u - (u)_{z,R}|^2 > \epsilon_0 \right\}.$$

▶ If $A \in C^{0,\sigma}_{\text{loc}}(\Lambda)$, then $\nabla u \in C^{0,\sigma}_{\text{loc}}(\Lambda_0)$.

▶ There exists $\gamma > 0$ so that $\mathcal{H}^{d-\gamma}(\Lambda \setminus \Lambda_0) = 0$.

Main ideas of the proof: (Giaquinta and Struwe, 1982)

For simplicity, let $z_0 = 0$, fix $0 < r < R/4 < R < R_0$, and assume that $f \equiv 0$.

Goal: Prove a $C^{0,\alpha}$-excess decay.

Main idea: Compare the weak solution u to $\bar{u}$, which solves the frozen system

$$\begin{aligned} \partial_t \bar{u} - \nabla A(0, (u)_{0,R}) \nabla \bar{u} = 0 \qquad &\text{in} \qquad \mathcal{C}_{R/4}(0) \\ \bar{u} = u \qquad &\text{on} \qquad \partial^P \mathcal{C}_{R/4}(0). \end{aligned}$$

Constant-coefficient regularity theory gives an estimate of the form

$$\int_{\mathcal{C}_r(0)} |\nabla \bar{u}|^2 \lesssim \left(\frac{r}{R}\right)^{d+2} \int_{\mathcal{C}_{R/4}(0)} |\nabla \bar{u}|^2 \lesssim \left(\frac{r}{R}\right)^{d+2} \int_{\mathcal{C}_{R/4}(0)} |\nabla u|^2,$$

which implies the local $C^{0,\alpha}$-regularity of $\bar{u}$ around the point 0.

$\to$ This regularity estimate follows from the Sobolev embedding and an iterative use of the Caccioppoli estimate satisfied by $\bar{u}$.

Main ideas of the proof: (Giaquinta and Struwe, 1982), cont'd

Now we want to transfer the regularity from $\bar{u}$ onto u.

- In particular, towards obtaining the desired excess decay for u, we write

$$\int_{\mathcal{C}_r(0)} |\nabla u|^2 \lesssim \int_{\mathcal{C}_r(0)} |\nabla \bar{u}|^2 \quad + \quad \int_{\mathcal{C}_r(0)} |\nabla (u - \bar{u})|^2.$$

regularity of $\bar{u}$

approximation error; $\nabla v = \nabla(\bar{u} - u)$

Treating the approximation error: After taking the energy estimate for the equation solved by v, for $p > 2$ one obtains

$$\int_{\mathcal{C}_{R/4}(0)} |\nabla \bar{v}|^2 \, \mathrm{d}z \lesssim \Big(\int_{\mathcal{C}_{R/4}(0)} |\nabla u|^p \, \mathrm{d}z \Big)^{\frac{2}{p}} \Big(\int_{\mathcal{C}_{R/4}(0)} |A((0,u)_{0,R}) - A(u)|^{\frac{2p}{p-2}} . \Big)^{\frac{p-2}{p}} .$$

- To treat the term on the right-hand side one requires a reverse Hölder inequality. In particular, we must know that for some $p > 2$

$$\Big(\fint_{\mathcal{C}_{R/4}(0)} |\nabla u|^p \, \mathrm{d}z \Big)^{\frac{1}{p}} \lesssim \Big(\fint_{\mathcal{C}_{R}(0)} |\nabla u|^2 \, \mathrm{d}z \Big)^{\frac{1}{2}} .$$

$\rightarrow$ Such a reverse Hölder estimate can be proved via Caccioppoli-type and Poincaré-Wirtinger type estimates for u.

Adapting the classical technique to cross-diffusions systems

To adapt the strategy of Giaquinta and Struwe, we need:

- A Caccioppoli-type estimate satisfied by $\bar{u}$.
- A reverse-Hölder estimate satisfied by u, which requires access to a Caccioppoli-type estimate and a Poincaré-Wirtinger type estimate.

• Since we have not assumed an ellipticity condition, each of these estimates will be a problem.

$\rightarrow$ Substitute for lacking ellipticity condition with the entropy structure.

• However, since the Hessian of the entropy often blows up as $u_i \to 0$, *e.g.* for the SKT system $h_i''(u_i) = 1/u_i$, it is necessary to modify the entropy structure to avoid this problem.

$\rightarrow$ For this purpose we introduce the "glued entropy."

Intuition behind our strategy

Consider the SKT model with

$$A_{\mathrm{SKT}}(u) = \begin{bmatrix} \alpha_{10} + 2\alpha_{11} u_1 + \alpha_{12} u_2 & \alpha_{12} u_1 \\ \alpha_{21} u_2 & \alpha_{20} + \alpha_{21} u_1 + 2\alpha_{22} u_2 \end{bmatrix}$$

and $h_1''(u_1) = \alpha_{12}/u_1$ and $h_2''(u_2) = \alpha_{21}/u_2$.

Our main observation: As $u \to 0$, we have that

$$A_{\mathrm{SKT}}(u) \to \begin{bmatrix} \alpha_{10} & 0 \\ 0 & \alpha_{20}. \end{bmatrix}$$

This motivates the main strategy of our paper, which is to for $z_0 \in \Lambda$ consider two cases:

Case 1: $(u(z_0) \geq \epsilon)$ We use the entropy structure to obtain the required *a priori* estimates.

Case 2: $(u(z_0) < \epsilon)$ We view the system as a perturbation of a system of heat equations (that are only coupled through the reaction terms) and obtain the required *a priori* estimates as energy estimates.

$\rightarrow$ This intuition is formalized by the "glued entropy."

Conditions that the glued entropy must satisfy: A Caccioppoli-type estimate for u

Motivation for the glued entropy: Comes from the proof of the Caccioppoli-type estimate satisfied by u.

A Caccioppoli-type estimate for u: Assuming that there exists a glued entropy and u is bounded, we find that

Lemma: *For $z_0 \in \Lambda$ and $0 < r < R$ such that $\mathcal{C}_{2R}(z_0) \subset \Lambda$ we have that*

$$\int_{\mathcal{C}_R(z_0)} |\nabla u|^2 \, dz \lesssim \frac{1}{R^2} \int_{\mathcal{C}_{2R}(z_0)} |u - (\tilde{u})_{x_0,R}(t)|^2 \, dz + R^{d+4} \|f(u)\|^2_{L^\infty(\mathcal{C}_{2R}(z_0))}.$$

Here, it is practical to use the weighted, time-dependent average

$$(\tilde{u})_{x_0,R})(t) := \frac{\int_{B_{2R}(x_0)} u(x,t) \chi^2_{x_0,R} \, \mathrm{d}x}{\int_{B_{2R}(x_0)} \chi^2_{x_0,R} \, \mathrm{d}x},$$

where $\chi_{x_0,R}$ is a smooth spatial-cutoff of $B_R(x_0)$ in $B_{2R}(x_0)$ such that $|\nabla \chi_{x_0,R}| \lesssim 1/R$.

Conditions that the glued entropy must satisfy: A Caccioppoli-type estimate, cont'd

Main idea of the proof: In the classical proof of the Caccioppoli estimate, replace energy estimates by entropy estimates and the squared L^2-distance by the relative entropy.

• Letting $\eta = \chi_{x_0,R}\tau$, where $\tau \equiv 1$ on $(t_0, t_0 - R^2)$, $\tau \equiv 0$ for $t \leq t_0 - (2R)^2$, and $|\nabla\tau| \lesssim 1/R^2$, we calculate

$$\begin{aligned}\partial_t\left(h(u|(\tilde{u})_{z_0,R})\eta^2\right) =& \eta^2(h'(u) - h'((\tilde{u})_{z_0,R}))\cdot\partial_t u + 2h(u|(\tilde{u})_{z_0,R})\eta\partial_t\eta \\ &- \eta^2 h''((\tilde{u})_{z_0,R})(u - (\tilde{u})_{z_0,R})\cdot\partial_t(\tilde{u})_{z_0,R}.\end{aligned}$$

Integrating this and then using the equation in conjunction with the entropy structure and the boundedness of $A(u)$ yields

$$\begin{aligned}\int_{\mathcal{C}_{2R}(z_0)} \eta^2|\nabla u|^2\,\mathrm{d}z \lesssim& \int_{\mathcal{C}_{2R}(z_0)} \eta^2\nabla u : h''(u)A(u)\nabla u\,\mathrm{d}z \\ \lesssim& \frac{1}{R^2}\int_{\mathcal{C}_{2R}(z_0)}\left(h(u\,|(\tilde{u})_{z_0,R}) + \sup_{y\in\mathcal{D}}|h''(y)|^2|u - (\tilde{u})_{z_0,R}|^2\right)\mathrm{d}z \\ &+ R^{d+4}\|f\|_{L^\infty(\mathcal{C}_{2R}(z_0))}.\end{aligned}$$

⋆ Conditions on the glued entropy: To obtain the Caccioppoli-type estimate it would suffice that

$$h(u|b) \lesssim |u-b|^2 \quad \text{for } b\in\mathbb{R}^n \qquad \text{and} \qquad \sup_{y\in\mathcal{D}} h''(y) \lesssim 1,$$

which are actually equivalent conditions when h is C^2.

→ These are the conditions that we enforce on the glued entropy.

An ansatz for the glued entropy

Motivation via the SKT model: Recall that when $\alpha_{21} = \alpha_{12} = 1$, we have that $h_i''(u_i) = \frac{1}{u_i}$.

• To make sure that $h_\epsilon'' \lesssim 1$, the most naive ansatz for the glued entropy h_ϵ would be

$$h_{\epsilon,i}(x) \text{``}=\text{''} \int_0^x \int_0^z h_i''\big(\max\{y\,, \epsilon\}\big)\,\mathrm{d}y\,\mathrm{d}z.$$

• Notice that $\tilde{h}_\epsilon(u) = \epsilon^{-1} \sum_{i=1}^n u_i^2$ is an entropy for a system of n heat equations. So, the naive ansatz corresponds to gluing $\tilde{h}_{\epsilon,i}''$ to h_i'' and integrating up the result –this, of course, guarantees the boundedness of the Hessian.

$\rightarrow$ This naive ansatz coincides with our intuition as when $u_i < \epsilon$, the entropy used is that of the heat equation.

• Going from the naive ansatz to the actual definition of the glued entropy, is a simple matter of replacing "$\max\{\cdot, \epsilon\}$" by a smooth gluing via a partition of unity.

Sufficient conditions for the existence of a glued entropy

For the existence of a C^2- glued entropy h_ϵ, it suffices that the cross-diffusion system has an entropy structure such that

(H1) The entropy $h : \mathcal{D} \to [0, \infty)$ has the form

$$h(y) := \sum_{i=1}^{n} h_i(y_i)$$

for $y \in \mathcal{D}$ and $h_i \in C^2(\mathbb{R}_+; [0, \infty))$. We assume that each $h_i''(y_i) \to \infty$ monotonically as $y_i \to 0$ in such a way that there exists $C \in \mathbb{R}$ for which $h_i''(\epsilon) \leq C h_i''(2\epsilon)$ holds for any $\epsilon > 0$.

(H2) There exists $\beta' > 0$ such that for any $y \in \mathcal{D}$ and $\rho \in \mathbb{R}^n$ we have that

$$\rho \cdot h''(y) A(y) \rho \geq \beta' |\rho|^2.$$

Furthermore, $h''(y)A(y)$ is symmetric.

(H3) There exist functions $a_1, \ldots, a_n \in C^0(\bar{\mathcal{D}})$ such that

$$\mu := \min_{i=1,\ldots,n} \inf_{\mathcal{D}} a_i > 0$$

and the relation

$$\max_{i,j=1,\ldots,n} |A_{ij}(y) - a_i(y)\delta_{ij}||h_i''(y_i)| \lesssim 1$$

holds for any $y \in \mathcal{D}$.

$\rightarrow$ Checking that these conditions are sufficient is one of the novel contributions of our work.

Checking that the SKT model satisfies the conditions

- The conditions **(H1)** and **(H2)** have already been checked.
- Recall the condition **(H3)**: There exist functions $a_1, \dots, a_n \in C^0(\bar{\mathcal{D}})$ such that

$$\mu := \min_{i=1,\dots,n} \inf_{\overline{\mathcal{D}}} a_i > 0$$

and the relation

$$\max_{i,j=1,\dots,n} |A_{ij}(y) - a_i(y)\delta_{ij}||h_i''(y_i)| \lesssim 1$$

holds for any $y \in \mathcal{D}$.

- Recall that for the SKT model $h''(y_i) = 1/y_i$ and

$$A_{\mathrm{SKT}}(u) = \begin{bmatrix} \alpha_{10} + 2\alpha_{11}u_1 + \alpha_{12}u_2 & \alpha_{12}u_1 \\ \alpha_{21}u_2 & \alpha_{20} + \alpha_{21}u_1 + 2\alpha_{22}u_2 \end{bmatrix}$$

Checking the condition: We see that **(H3)** holds with

$$a_1 = \alpha_{10} + \alpha_{12}u_2 \quad \text{and} \quad a_2 = \alpha_{20} + \alpha_{21}u_1,$$

where we require that $\alpha_{10}, \alpha_{20} > 0$ and we remark that the components of u are non-negative.

Results and Summary of main ideas

Inserting the use of the glued entropy into the framework of Giaquinta and Struwe we obtain:

Theorem: Let u be a bounded weak solution and assume that the conditions **(H1)** - **(H3)** hold.

- Then there exists an open set $\Lambda_0 \subset \Lambda$ such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Lambda_0)$ for any $\alpha \in (0,1)$. Furthermore, we have that

$$\Lambda \setminus \Lambda_0 \subset \left\{ z_0 \in \Lambda \,\middle|\, \liminf_{R \to 0} \fint_{\mathcal{C}_R(z_0)} |u - (u)_{z_0,R}|^2 \,\mathrm{d}z > \epsilon \right\}.$$

- If $A \in C^{0,\sigma}_{\mathrm{loc}}(\Lambda)$, then $\nabla u \in C^{0,\sigma}_{\mathrm{loc}}(\Lambda_0)$.
- There exists $\gamma > 0$ such that $\mathcal{H}^{d-\gamma}(\Lambda \setminus \Lambda_0) = 0$.

Applications: SKT model and the semiconductor model with electron-hole scattering; see *e.g.* (Jüngel, 2015).

Main idea: Conditions **(H1)** - **(H3)** $\to$ glued entropy $\to$ partial Hölder regularity via classical techniques.

Future directions: Volume filling systems?

- Recall that part of condition **(H1)** was that the entropy $h : \mathcal{D} \to [0, \infty)$ has the form

$$h(y) := \sum_{i=1}^{n} h_i(y_i)$$

for $y \in \mathcal{D}$ and $h_i \in C^2(\mathbb{R}_+; [0, \infty))$.

$\to$ This mean that the construction of the glued entropy does not apply to volume-filling systems, for which the entropy has an addition term that enforces the volume constraint.

Example: The volume-filling model of Burgers –transport of ions through narrow channels– is given by

$$A(u) = \begin{bmatrix} D_1(1-\rho+u_1) & D_1 u_1 \\ D_2 u_2 & D_2(1-\rho+u_2) \end{bmatrix},$$

where $\rho = \sum_{i=1}^{2} u_i$ and $D_i > 0$. This system has the entropy

$$h(u) = u_1(\log(u_1) - 1) + u_2(\log(u_2) - 1) + (1-\rho)(\log(1-\rho) - 1),$$

the Hessian of which blows up when $u_i \to 0$ and when $\rho \to 1$.

- Notice that when $\rho \to 1$, we have that

$$A(u) \longrightarrow \begin{bmatrix} D_1 u_1 & D_1 u_1 \\ D_2 u_2 & D_2 u_2 \end{bmatrix}.$$

Conclusion: By taking advantage of the regularity theory for the porous medium equation, it may be possible to get a partial regularity result also for volume-filling systems.

Thanks for your attention!