Topological Insulators: From K-theory to Physics

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What is a topological insulator?

- d-dimensional disordered system of independent Fermions with a combination of basic symmetries

TRS, PHS, CHS = time reversal, particle hole, chiral symmetry

- Fermi level in a Gap or Anderson localization regime
- Topology of bulk (in Bloch bundles over Brillouin torus):

winding numbers, Chern numbers, $\mathbb{Z}_2$-invariants, higher invariants

- Delocalized edge modes with non-trivial topology
- Bulk-edge correspondence
- Topological bound states at defects (zero modes)
- Toy models: tight-binding

Here: K-theory, index theory and non-commutative geometry

Start with concrete model in dimension $d = 1$

Su-Schrieffer-Heeger (1980, conducting polyacetelyn polymer)

$$H \;=\; \tfrac{1}{2}(\sigma_1 + i\sigma_2) \otimes S + \tfrac{1}{2}(\sigma_1 - i\sigma_2) \otimes S^* + m\,\sigma_2 \otimes \mathbf{1}$$

where S bilateral shift on $\ell^2(\mathbb{Z})$, $m \in \mathbb{R}$ mass and Pauli matrices. In their grading

$$H \;=\; \begin{pmatrix} 0 & S - im \\ S^* + im & 0 \end{pmatrix} \qquad \text{on } \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

Off-diagonal $\cong$ chiral symmetry $\sigma_3^* H \sigma_3 = -H$. In Fourier space:

$$H = \int^{\oplus} dk\, H_k \qquad H_k = \begin{pmatrix} 0 & e^{-ik} - im \\ e^{ik} + im & 0 \end{pmatrix}$$

Topological invariant for $m \neq -1, 1$

$$\mathrm{Wind}(k \mapsto e^{ik} + im) \;=\; \delta(m \in (-1,1))$$

Chiral bound states

Half-space Hamiltonian

$$\widehat{H} \;=\; \begin{pmatrix} 0 & \widehat{S} - im \\ \widehat{S}^* + im & 0 \end{pmatrix} \qquad \text{on } \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$$

where $\widehat{S}$ unilateral right shift on $\ell^2(\mathbb{N})$

Still chiral symmetry $\sigma_3^* \widehat{H} \sigma_3 = -\widehat{H}$

If $m = 0$, simple bound state at $E = 0$ with eigenvector $\psi_0 = \binom{|0\rangle}{0}$.

Perturbations, *e.g.* in m, cannot move or lift this bound state ψ_m!

Positive chirality conserved: $\sigma_3 \psi_m = \psi_m$

Theorem (Basic bulk-boundary correspondence)

If $\widehat{P}$ projection on bound states of $\widehat{H}$, then

$$\mathrm{Wind}(k \mapsto e^{ik} + im) \;=\; \mathrm{Tr}(\widehat{P}\sigma_3)$$

Disordered model

Add i.i.d. random mass term $\omega = (m_n)_{n\in\mathbb{Z}}$:

$$H_\omega \;=\; H \;+\; \sum_{n\in\mathbb{Z}} m_n\,\sigma_2 \otimes |n\rangle\langle n|$$

Still chiral symmetry $\sigma_3^* H_\omega \sigma_3 = -H_\omega$ so

$$H_\omega \;=\; \begin{pmatrix} 0 & A_\omega^* \\ A_\omega & 0 \end{pmatrix}$$

Bulk gap at $E = 0 \Longrightarrow A_\omega$ invertible

Non-commutative winding number, also called first Chern number:

$$\mathrm{Wind} \;=\; \mathrm{Ch}_1(A) \;=\; i\,\mathbf{E}_\omega\,\mathrm{Tr}\,\langle 0|A_\omega^{-1}\,i[X, A_\omega]|0\rangle$$

where $\mathbf{E}_\omega$ is average over probability measure $\mathbb{P}$ on i.i.d. masses

Index theorem and bulk-boundary correspondence

Theorem (Disordered Noether-Gohberg-Krein Theorem)

If Π is Hardy projection on positive half-space, then $\mathbb{P}$-almost surely

$$\mathrm{Wind} \;=\; \mathrm{Ch}_1(A) \;=\; -\,\mathrm{Ind}(\Pi A_\omega \Pi)$$

For periodic model as above, $A_\omega = e^{ik} \in C(\mathbb{S}^1)$

Fredholm operator is then standard Toeplitz operator

Theorem (Disoreded bulk-boundary correspondence)

If $\widehat{P}_\omega$ projection on bound states of $\widehat{H}_\omega$, then

$$\mathrm{Wind} \;=\; \mathrm{Ch}_1(A) \;=\; \mathrm{Ch}_0(\widehat{P}_\omega) \;=\; \mathrm{Tr}(\widehat{P}_\omega \sigma_3)$$

Structural robust result:

holds for chiral Hamiltonians with larger fiber, other disorder, etc.

Structure: Toeplitz extension (no disorder)

S bilateral shift on $\ell^2(\mathbb{Z})$, then $\mathrm{C}^*(S) \cong C(\mathbb{S}^1)$

$\widehat{S}$ unilateral shift on $\ell^2(\mathbb{N})$, only partial isometry with a defect:

$$\widehat{S}^*\widehat{S} = \mathbf{1} \qquad \widehat{S}\,\widehat{S}^* = \mathbf{1} - |0\rangle\langle 0|$$

Then $\mathrm{C}^*(\widehat{S}) = \mathcal{T}$ Toeplitz algebra with exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{S}^1) \longrightarrow 0$$

K-groups for any C^*-algebra $\mathcal{A}$ (only rough definition):

$$\begin{aligned} K_0(\mathcal{A}) &= \{[P]-[Q] \;:\; \text{projections in some } M_n(\mathcal{A})\} \\ K_1(\mathcal{A}) &= \{[U] \;:\; \text{unitary in some } M_n(\mathcal{A})\} \end{aligned}$$

Abelian group operation: Whitney sum

Example: $K_0(\mathbb{C}) = \mathbb{Z} = K_0(\mathcal{K})$ with invariant $\dim(P)$

Example: $K_1(C(\mathbb{S}^1)) = \mathbb{Z}$ with invariant given by winding number

6-term exact sequence for Toeplitz extension

C*-algebra short exact sequence $\Longrightarrow$ K-theory 6-term sequence

$$\begin{array}{ccccc}
K_0(\mathcal{K}) = \mathbb{Z} & \longrightarrow & K_0(\mathcal{T}) = \mathbb{Z} & \longrightarrow & K_0(C(\mathbb{S}^1)) = \mathbb{Z} \\
\mathrm{Ind} \ \uparrow & & & & \downarrow \ \mathrm{Exp} \\
K_1(C(\mathbb{S}^1)) = \mathbb{Z} & \longleftarrow & K_1(\mathcal{T}) = 0 & \longleftarrow & K_1(\mathcal{K}) = 0
\end{array}$$

Here: $[A]_1 \in K_1(C(\mathbb{S}^1))$ and $[\widehat{P}\sigma_3]_0 = [\widehat{P}_+]_0 - [\widehat{P}_-]_0 \in K_0(\mathcal{K})$

$$\mathrm{Ind}([A]_1) \ = \ [\widehat{P}_+]_0 - [\widehat{P}_-]_0 \qquad \text{(bulk-boundary for } K\text{-theory)}$$

$$\mathrm{Ch}_0(\mathrm{Ind}(A)) \ = \ \mathrm{Ch}_1(A) \qquad \text{(bulk-boundary for invariants)}$$

Disordered case: analogous

Tight-binding toy models in dimension d

One-particle Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L$

Fiber $\mathbb{C}^L = \mathbb{C}^{2s+1} \otimes \mathbb{C}^r$ with spin s and r internal degrees

e.g. $\mathbb{C}^r = \mathbb{C}^2_{\mathrm{ph}} \otimes \mathbb{C}^2_{\mathrm{sl}}$ particle-hole space and sublattice space

Typical Hamiltonian

$$H_\omega \;=\; \Delta^B \;+\; W_\omega \;=\; \sum_{i=1}^{d} \left(t_i^* S_i^B + t_i (S_i^B)^*\right) \;+\; W_\omega$$

Magnetic translations $S_j^B S_i^B = e^{iB_{i,j}} S_i^B S_j^B$ in Laudau gauge:

$$S_1^B = S_1 \qquad S_2^B = e^{iB_{1,2}X_1} S_2 \qquad S_3^B = e^{iB_{1,3}X_1 + iB_{2,3}X_2} S_3$$

t_i matrices $L \times L$, e.g. spin orbit coupling, (anti)particle creation

matrix potential $W_\omega = W_\omega^* = \sum_{n\in\mathbb{Z}^d} |n\rangle \omega_n \langle n|$ with matrices ω_n

Observable algebra

Configurations $\omega = (\omega_n)_{n\in\mathbb{Z}^d} \in \Omega$ compact probability space $(\Omega, \mathbb{P})$

$\mathbb{P}$ invariant and ergodic w.r.t. $T : \mathbb{Z}^d \times \Omega \to \Omega$

Covariance w.r.t. to dual magnetic translations $V_a = S_j^B V_a (S_j^B)^*$

$$V_a H_\omega V_a^* \;=\; H_{T_a\omega} \qquad a \in \mathbb{Z}^d$$

$\|A\| = \sup_\omega \|A_\omega\|$ is C*-norm on

$$\mathcal{A}_d \;=\; \mathrm{C}^*\left\{A = (A_\omega)_{\omega\in\Omega} \text{ finite range covariant operators}\right\}$$
$$\cong\; \text{twisted crossed product } C(\Omega) \rtimes_B \mathbb{Z}^d$$

Fact: Suppose Ω contractible

$\Longrightarrow$ rotation algebra C$^*(S_j^B)$ is deformation retract of $\mathcal{A}_d$

In particular: K-groups of C$^*(S_j^B)$ and $\mathcal{A}_d$ coincide

Pimsner-Voiculescu (1980)

Theorem

$K_0(\mathcal{A}_d) = \mathbb{Z}^{2^{d-1}}$ *and* $K_1(\mathcal{A}_d) = \mathbb{Z}^{2^{d-1}}$

Explicit generators $[G_I]$ of K-groups labelled by $I \subset \{1, \ldots, d\}$

Top generator $I = \{1, \ldots, d\}$ identified with $K_j(C(\mathbb{S}^d)) = \mathbb{Z}$

Example $G_{\{1,2\}}$ Powers-Rieffel projection and Bott projection

In general, any projection $P \in M_n(\mathcal{A}_d)$ can be decomposed as

$$[P] = \sum_{I \subset \{1,\ldots,d\}} n_I \, [G_I] \qquad\qquad n_I \in \mathbb{Z}, \ |I| \text{ even}$$

Invariants n_I, top invariant $n_{\{1,\ldots,d\}} \in \mathbb{Z}$ called *strong* , others weak

Questions: calculate $n_I = c_I \, \mathrm{Ch}_I(P)$, physical significance?

K-group elements of physical interest

Fermi level $\mu \in \mathbb{R}$ in spectral gap of H_ω

$$P_\omega \;=\; \chi(H_\omega \leq \mu) \qquad \text{covariant Fermi projection}$$

Hence: $P = (P_\omega)_{\omega\in\Omega} \in \mathcal{A}_d$ fixes element in $K_0(\mathcal{A}_d)$

If chiral symmetry present: Fermi invertible (or unitary)

$$H_\omega \;=\; -\,J_{\rm ch}^* H_\omega J_{\rm ch} \;=\; \begin{pmatrix} 0 & A_\omega \\ A_\omega^* & 0 \end{pmatrix} \qquad J_{\rm ch} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

If $\mu = 0$ in gap, $A = (A_\omega)_{\omega\in\Omega} \in \mathcal{A}_d$ invertible and $[A]_1 \in K_1(\mathcal{A}_d)$

Remark Sufficient to have an approximate chiral symmetry

$$H_\omega \;=\; \begin{pmatrix} B_\omega & A_\omega \\ A_\omega^* & C_\omega \end{pmatrix} \qquad \text{with invertible } A_\omega$$

Definition of topological invariants

For invertible $A \in \mathcal{A}_d$ and odd $|I|$, with $\rho : \{1, \dots, |I|\} \to I$:

$$\mathrm{Ch}_I(A) \;=\; \frac{i(i\pi)^{\frac{|I|-1}{2}}}{|I|!!} \sum_{\rho \in S_I} (-1)^\rho \, \mathcal{T}\left(\prod_{j=1}^{|I|} A^{-1} \nabla_{\rho_j} A \right) \;\in\; \mathbb{R}$$

where $\quad \mathcal{T}(A) = \mathbf{E}_{\mathbb{P}} \, \mathrm{Tr}_L \, \langle 0 | A_\omega | 0 \rangle \quad$ and $\quad \nabla_j A_\omega = i[X_j, A_\omega]$

For even $|I|$ and projection $P \in \mathcal{A}_d$:

$$\mathrm{Ch}_I(P) \;=\; \frac{(2i\pi)^{\frac{|I|}{2}}}{\frac{|I|}{2}!} \sum_{\rho \in S_I} (-1)^\rho \, \mathcal{T}\left(P \prod_{j=1}^{|I|} \nabla_{\rho_j} P \right) \;\in\; \mathbb{R}$$

Theorem (Connes 1985)

$\mathrm{Ch}_I(A)$ and $\mathrm{Ch}_I(P)$ homotopy invariants; pairings with $K(\mathcal{A}_d)$

Bulk-boundary via Toeplitz extension

$$\begin{array}{ccccccccc} & & \text{edge} & & \text{half-space} & & \text{bulk} & & \\ 0 & \to & \mathcal{E}_d & \to & \mathcal{T}(\mathcal{A}_d) & \to & \mathcal{A}_d & \to & 0 \end{array}$$

Moreover: $$\mathcal{E}_d \cong \mathcal{A}_{d-1} \otimes \mathcal{K}(\ell^2(\mathbb{N}))$$

$$\begin{array}{ccccc} K_0(\mathcal{A}_{d-1}) & \longrightarrow & K_0(\mathcal{T}(\mathcal{A}_d)) & \longrightarrow & K_0(\mathcal{A}_d) \\ \mathrm{Ind} \uparrow & & & & \downarrow \mathrm{Exp} \\ K_1(\mathcal{A}_d) & \longleftarrow & K_1(\mathcal{T}(\mathcal{A}_d)) & \longleftarrow & K_1(\mathcal{A}_{d-1}) \end{array}$$

Theorem

$$\mathrm{Ch}_{I\cup\{d\}}(A) = \mathrm{Ch}_I(\mathrm{Ind}(A)) \qquad |I| \text{ even}\,,\ [A] \in K_1(\mathcal{A}_d)$$

$$\mathrm{Ch}_{I\cup\{d\}}(P) = \mathrm{Ch}_I(\mathrm{Exp}(P)) \qquad |I| \text{ odd}\,,\ [P] \in K_0(\mathcal{A}_d)$$

Physical implication in $d = 2$: QHE

P Fermi projection below a bulk gap $\Delta \subset \mathbb{R}$. Kubo formula:

$$\text{Hall conductance} \; = \; \mathrm{Ch}_{\{1,2\}}(P)$$

Bulk-boundary:

$$\mathrm{Ch}_{\{1,2\}}(P) \; = \; \mathrm{Ch}_{\{1\}}(\mathrm{Exp}(P)) \; = \; \mathrm{Wind}(\mathrm{Exp}(P))$$

With continuous $g(E) = 1$ for $E < \Delta$ and $g(E) = 0$ for $E > \Delta$:

$$\mathrm{Exp}(P) \; = \; \exp(-2\pi i\, g(\widehat{H}))$$

Theorem (Quantization of boundary currents)

$$\mathrm{Ch}_{\{1,2\}}(P) \; = \; \mathbb{E} \sum_{n_2 \geq 0} \langle 0, n_2 | g'(\widehat{H}) i[X_2, \widehat{H}] | 0, n_2 \rangle$$

Chiral system in $d = 3$: anomalous surface QHE

Chiral Fermi projection P (off-diagonal) $\Longrightarrow$ Fermi unitary A

$$\mathrm{Ch}_{\{1,2,3\}}(A) \;=\; \mathrm{Ch}_{\{1,2\}}(\mathrm{Ind}(A))$$

Magnetic field perpendicular to surface opens gap in surface spec.

With $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$ projection on central surface band, as in SSH:

$$\mathrm{Ind}(A) \;=\; [\widehat{P}_+] - [\widehat{P}_-]$$

Theorem

Suppose either $\widehat{P}_+ = 0$ *or* $\widehat{P}_- = 0$ (conjectured to hold). *Then:*

$\mathrm{Ch}_{\{1,2,3\}}(A) \neq 0 \Longrightarrow$ *surface QHE, Hall cond. imposed by bulk*

Experiment? No (approximate) chiral topological material known

Generalized Streda formulæ

In QHE: integrated density of states grows linearly in magnetic field

integrated density of states: $\mathbf{E}\,\langle 0|P|0\rangle \;=\; \mathrm{Ch}_{\emptyset}(P)$

$$\partial_{B_{1,2}}\,\mathrm{Ch}_{\emptyset}(P) \;=\; \frac{1}{2\pi}\,\mathrm{Ch}_{\{1,2\}}(P)$$

Theorem

$$\partial_{B_{i,j}}\,\mathrm{Ch}_{I}(P) \;=\; \frac{1}{2\pi}\,\mathrm{Ch}_{I\cup\{i,j\}}(P) \qquad |I|\,\mathrm{even},\;\; i,j\notin I$$

$$\partial_{B_{i,j}}\,\mathrm{Ch}_{I}(A) \;=\; \frac{1}{2\pi}\,\mathrm{Ch}_{I\cup\{i,j\}}(A) \qquad |I|\;\mathrm{odd}\;,\;\; i,j\notin I$$

Application: magneto-electric effects in $d = 3$

Time is 4th direction needed for calculation of polarization

Non-linear response is derivative w.r.t. B given by $\mathrm{Ch}_{\{1,2,3,4\}}(P)$

Link to Volovik-Essin-Gurarie invariants

Express the invariants in terms of the Green function/resolvent

Consider path $z : [0,1] \to \mathbb{C} \setminus \sigma(H)$ encircling $(-\infty, \mu] \cap \sigma(H)$

Set

$$G(t) \;=\; (H - z(t))^{-1}$$

Theorem

For $|I|$ even and with $\nabla_0 = \partial_t$,

$$\mathrm{Ch}_I(P_\mu) \;=\; \frac{(i\pi)^{\frac{|I|}{2}}}{i(|I|-1)!!} \sum_{\rho \in S_{I \cup \{0\}}} (-1)^\rho \int_0^1 dt\, \mathcal{T}\left(\prod_{j=1}^{|I|} G(t)^{-1} \nabla_{\rho_j} G(t) \right)$$

Proof by suspension. Similar formula for odd pairings.

Index theorem for strong invariants and odd d

$\gamma_1, \ldots, \gamma_d$ irrep of Clifford C_d on $\mathbb{C}^{2^{(d-1)/2}}$

$$D \;=\; \sum_{j=1}^{d} X_j \otimes \mathbf{1} \otimes \gamma_j \qquad \text{Dirac operator on } \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L \otimes \mathbb{C}^{2^{(d-1)/2}}$$

Dirac phase $F = \frac{D}{|D|}$ provides odd Fredholm module on $\mathcal{A}_d$:

$$F^2 = \mathbf{1} \qquad [F, A_\omega] \text{ compact and in } \mathcal{L}^{d+\epsilon} \;\text{ für } A = (A_\omega)_{\omega \in \Omega} \in \mathcal{A}_d$$

Theorem (Local index = generalizes Noether-Gohberg-Krein)

Let $E = \frac{1}{2}(F + \mathbf{1})$ be Hardy Projektion for F. For invertible A_ω

$$\mathrm{Ch}_{\{1,\ldots,d\}}(A) \;=\; \mathrm{Ind}(E\, A_\omega E)$$

The index is $\mathbb{P}$-almost surely constant.

Local index theorem for even dimension d

As above $\gamma_1, \ldots, \gamma_d$ Clifford, grading $\Gamma = -\imath^{-d/2}\gamma_1 \cdots \gamma_d$

Dirac $D = -\Gamma D \Gamma = |D| \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}$ even Fredholm module

Theorem (Connes $d = 2$, Prodan, Leung, Bellissard 2013)

Almost sure index $\mathrm{Ind}(P_\omega F P_\omega)$ *equal to* $\mathrm{Ch}_{\{1,\ldots,d\}}(P)$

Special case $d = 2$: $F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$ and

$$\mathrm{Ind}(P_\omega F P_\omega) \;=\; 2\pi\imath\, \mathcal{T}(P[[X_1, P], [X_2, P]])$$

Proofs: geometric identity of high-dimensional simplexes

Advantages: phase label also for dynamical localized regime

implementation of discrete symmetries (CPT)

Delocalization of boundary states

Hypothesis: bulk gap at Fermi level μ

Disorder: in arbitrary finite strip along boundary hypersurface

Theorem

For even d, if strong invariant $\mathrm{Ch}_{\{1,\ldots,d\}}(P) \neq 0$,

then no Anderson localization of boundary states in bulk gap.

Technically: Aizenman-Molcanov bound for no energy in bulk gap.

Theorem

For odd $d \geq 3$, if strong invariant $\mathrm{Ch}_{\{1,\ldots,d\}}(A) \neq 0$,

then no Anderson localization at $\mu = 0$.

Discrete symmetries (invoking real structure)

$$\begin{aligned} &\text{chiral symmetry (CHS)}: && J_{\rm ch}^* \, H \, J_{\rm ch} \;=\; -H \\ &\text{time reversal symmetry (TRS)}: && S_{\rm tr}^* \, \overline{H} \, S_{\rm tr} \;=\; H \\ &\text{particle-hole symmetry (PHS)}: && S_{\rm ph}^* \, \overline{H} \, S_{\rm ph} \;=\; -H \end{aligned}$$

$S_{\rm tr} = e^{i\pi s^y}$ orthogonal on $\mathbb{C}^{2s+1}$ with $S_{\rm tr}^2 = \pm\mathbf{1}$ even or odd

$S_{\rm ph}$ orthogonal on $\mathbb{C}^2_{\rm ph}$ with $S_{\rm ph}^2 = \pm\mathbf{1}$ even or odd

Note: TRS + PHS $\Longrightarrow$ CHS with $J_{\rm ch} = S_{\rm tr} S_{\rm ph}$

10 combinations of symmetries: none (1), one (5), three (4)

10 Cartan-Altland-Zirnbauer classes (CAZ): 2 complex, 8 real

Further distinction in each of the 10 classes: topological insulators

Periodic table of topological insulators

Schnyder-Ryu-Furusaki-Ludwig,Kitaev 2008: just strong invariants

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
1	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Periodic table: real classes only

64 pairings = 8 KR-cycles paired with 8 KR-groups

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Focus on system in $d = 2$ with odd TRS $S = S_{\mathrm{tr}}$:

$$S^2 = -\mathbf{1} \qquad S^* \overline{H} S = H$$

$\mathbb{Z}_2$ index for odd TRS and $d = 2$

Rewrite $S^*\overline{H}S = H = S^*H^tS$ with $H^t = (\overline{H})^*$

$\Longrightarrow \quad S^*(H^n)^tS = H^n$ for $n \in \mathbb{N} \quad \Longrightarrow \quad S^*P^tS = P$

For $d = 2$, Dirac phase $F = \frac{X_1+iX_2}{|X_1+iX_2|} = F^t$ and $[S, F] = 0$

Hence Fredholm operator $T = PFP$ of following type

Definition T odd symmetric $\Longleftrightarrow S^*T^tS = T \Longleftrightarrow (TS)^t = -TS$

Theorem (Atiyah-Singer 1969)

$\mathbb{F}_2(\mathcal{H}) = \{$*odd symmetric Fredholm operators*$\}$ *has 2 connected components labelled by compactly stable homotopy invariant*

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2 \;\in\; \mathbb{Z}_2$$

Application: $\mathbb{Z}_2$ phase label for Kane-Mele model if dyn. localized

Symmetries of the Dirac operator

$$D \;=\; \sum_{j=1}^{d} X_j \otimes \mathbf{1} \otimes \gamma_j$$

$\gamma_1, \ldots, \gamma_d$ irrep of C_d with $\gamma_{2j} = -\overline{\gamma_{2j}}$ and $\gamma_{2j+1} = \overline{\gamma_{2j+1}}$

In even d exists grading $\Gamma = \Gamma^*$ with $D = -\Gamma D \Gamma$ and $\Gamma^2 = \mathbf{1}$

Moreover, exists real unitary Σ (essentially unique) with

$d = 8 - i$	8	7	6	5	4	3	2	1
Σ^2	$\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\Sigma^* \overline{D} \Sigma$	D	$-D$	D	D	D	$-D$	D	D
$\Gamma \Sigma \Gamma$	Σ		$-\Sigma$		Σ		$-\Sigma$	

(D, Γ, Σ) defines a KR^i-cycle (spectral triple with real structure)

(Kasparov 1981, Connes 1995, Gracia-Varilly-Figueroa 2000)

Index theorems for periodic table

Symmetries of KR-cycles **and** Fermi projection/unitary lead to:

Theorem

Index theorems for all strong invariants in periodic table

Remarks:

Result holds also in the regime of strong Anderson localization

$2\,\mathbb{Z}$ entries result from quaternionic Fredholm (even Ker, CoKer)

Links to Atiyah-Singer classifying spaces

Formulation as Clifford valued index theorem possible

Physical implications: case by case study necessary!

Example: focus on TRS $d = 2$ quantum spin Hall system (QSH)

Spin filtered helical edge channels for QSH

Approximate spin conservation $\implies$ spin Chern numbers $\mathrm{SCh}(P)$

Theorem

$\mathrm{Ind}_2(PFP) = \mathrm{SCh}(P) \bmod 2$

Remark Non-trivial topology $\mathrm{SCh}(P)$ persists TRS breaking!

Theorem

If $\mathrm{SCh}(P) \neq 0$, spin filtered edge currents in $\Delta \subset$ gap are stable w.r.t. perturbations by magnetic field and disorder:

$$\mathbf{E}\ \mathrm{Tr}\ \langle 0|\chi_\Delta(\widehat{H})\,\tfrac{1}{2}\{i[\widehat{H},X_1],s^z\}|0\rangle \;=\; |\Delta|\ \mathrm{SCh}(P)\;+\;\textit{correct.}$$

Resumé: $\mathrm{Ind}_2(PFP) = 1 \implies$ no Anderson loc. for edge states

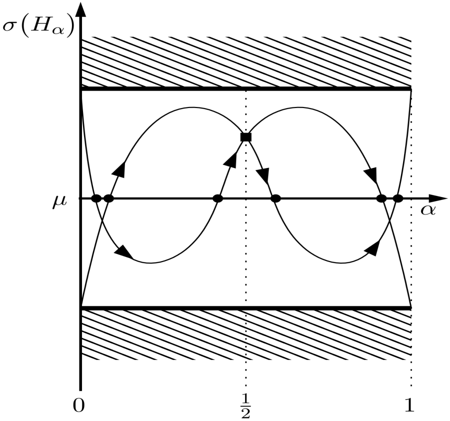
Rice group of Du (since 2011): QSH stable w.r.t. magnetic field

$\mathbb{Z}_2$ invariant and $\mathbb{Z}_2$ spectral flow for QSH

Theorem

$\alpha \in [0,1] \mapsto H(\alpha)$ *inserted flux through* 1 *lattice cell* (defect)

$\mathrm{Ind}_2(PFP) = 1 \implies H(\alpha = \frac{1}{2})$ *has* TRS + *Kramers pair in gap*



Résumé

- invariants for bulk and boundary
- bulk-boundary correspondence
- index theorems for strong invariants in complex classes
- index theorems with symmetries for **every** entry of periodic table

Current aims:

- Index theory for weak invariants via KK-theory
- bulk-edge correspondence in real cases
- further investigation of physical implications of invariants
- stability of invariants w.r.t. interactions

Related works and references

Other groups (each with personal point of view):

- Carey, Rennie, Bourne, Kellendonk
- Mathai, Thiang, Hanabus
- Zirnbauer, Kennedy
- Loring, Hastings, Boersema
- Graf, Porta
- Li, Kaufmann, Kaufmann
- many theoretical physics groups

Prodan, Schulz-Baldes: *Bulk and Boundary Invariants for complex topological insulators* (Springer Monograph 2016, see arXiv)

Grossmann, Schulz-Baldes: *Fredholm operators with symmetries* (CMP 2015, see arXiv)