Finite volume calculation of $K$-theory invariants

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Erlangen January, 2017
Plan of the talk

- Classical topological invariants and index theorem
- Construction of associated Bott operator (matrix)
- Main result: invariant as signature of Bott operator
- Connection to $\eta$-invariant
- Elements of proof based on $K$-theory
- Implementation of symmetries
- Application to topological insulators
- Even dimensional case
Motivating example: higher winding numbers

$\mathbb{T}^d$ torus of odd dimension $d$

Given: smooth function $k \in \mathbb{T}^d \mapsto A(k) \in \text{Gl}(N, \mathbb{C})$

Higher winding number (also called odd Chern number):

$$\text{Ch}_d(A) = \frac{1}{d!} \left( \frac{i}{2\pi} \right)^{d+1} \int_{\mathbb{T}^d} \text{Tr} \left( (A^{-1}dA)^d \right)$$

Faithful irrep $\Gamma_1, \ldots, \Gamma_d$ of complex Clifford $\mathbb{C}_d$ on $\mathbb{C}^N$

(possibly given only after augmenting $N$)

Selfadjoint Dirac operator on $L^2(\mathbb{T}^d, \mathbb{C}^N)$:

$$D = \sum_{j=1}^{d} \Gamma_j \partial_{k_j}$$

Positive spectral (Hardy) projection $\Pi = \chi(D \geq 0)$
**Theorem**

*Viewing $A$ as multiplication operator on $L^2(\mathbb{T}^d, \mathbb{C}^N)$, the operator $\Pi A \Pi + (1 - \Pi)$ is Fredholm and:

$$
\text{Ch}_d(A) = \text{Ind}(\Pi A \Pi + (1 - \Pi))
$$

Case $d = 1$: Fritz Noether 1921 and Gohberg-Krein 1960

Case $d \geq 3$: probably follows from Atiyah-Singer 1960’s and 1970’s

Extension to covariant operators with Prodan 2016

**Aim:** express $\text{Ch}_d(A)$ as signature of a finite dimensional matrix

Also extend to situations where no differential calculus available

This makes invariants numerically calculable
Extension to local operators on lattice

After Fourier transform \( \mathcal{F} : L^2(\mathbb{T}^d, \mathbb{C}^N) \rightarrow \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \)

\[
(\mathcal{F}\psi)(x) = \int_{\mathbb{T}^d} \frac{dk}{(2\pi)^d} e^{-i k \cdot x} \psi(k)
\]

Dirac \( \hat{D} = \mathcal{F} D \mathcal{F}^* = \sum_{j=1}^d X_j \Gamma_j \) with position operators \( X_j \)

\( \hat{A} = \mathcal{F} A \mathcal{F}^* \) convolution operator

Differentiability satisfied if locality condition holds:

\[
\| [\hat{A}, X_j] \| \leq C \quad \forall j = 1, \ldots, d \quad \iff \quad \| [\hat{A}, D] \| \leq C'
\]

From now on only local operators on \( \ell^2(\mathbb{Z}, \mathbb{C}^N) \), so let’s drop hats

**Fact:** If \( A \) invertible local operator, \( \Pi A \Pi + 1 - \Pi \) is Fredholm

**Fact:** If \( A \) covariant, index is still given by a Chern number

**Aim:** calculate index as signature of finite matrix
For tuning parameter $\kappa > 0$ and invertible local $A$:

$$B_\kappa = \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix} = \kappa D \otimes \sigma_3 + H$$

where $H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$. Clearly $B_\kappa$ selfadjoint

$D$ unbounded with discrete spectrum, $A$ viewed as perturbation $A$ may lead to spectral asymmetry of $B_\kappa$, but not for $A = 1$

Measured by signature, already on finite volume approximation!

$A_\rho$ restriction of $A$ (Dirichlet b.c.) to $\mathbb{D}_\rho = \{ x \in \mathbb{Z}^d : |x| \leq \rho \}$

$$B_{\kappa, \rho} = \begin{pmatrix} \kappa D_\rho & A_\rho \\ A_\rho^* & -\kappa D_\rho \end{pmatrix}$$
Main Result

**Theorem**

Let $g = \|A^{-1}\|^{-1}$ be the invertibility gap. Provided that

$$\| [D, A] \| \leq \frac{g^3}{18 \| A \| \kappa} \quad (*)$$

and

$$\frac{2g}{\kappa} \leq \rho \quad (**)$$

the matrix $B_{\kappa, \rho}$ is invertible and

$$\frac{1}{2} \operatorname{Sig}(B_{\kappa, \rho}) = \operatorname{Ind}(\Pi A \Pi + (1 - \Pi))$$

**How to use:** form (*) infer $\kappa$, then $\rho$ from (**) 

If $A$ unitary, $g = \| A \| = 1$ and $\kappa = (18\| [D, A] \|)^{-1}$ and $\rho = 2/\kappa$

Hence small matrix of size $\leq 100$ sufficient! Great for numerics!
Finite volume calculation of $K$-theory invariants

Why it can work:

**Proposition**

If (*) and (**) hold,

$$B^2_{\kappa, \rho} \geq \frac{g^2}{2}$$

**Proof:**

$$B^2_{\kappa, \rho} = \begin{pmatrix} A^*_\rho A_\rho & 0 \\ 0 & A_\rho A^*_\rho \end{pmatrix} + \kappa^2 \begin{pmatrix} D^2_\rho & 0 \\ 0 & D^2_\rho \end{pmatrix} + \kappa \begin{pmatrix} 0 & [D_\rho, A_\rho]^* \\ [D_\rho, A_\rho] & 0 \end{pmatrix}$$

Last term is a perturbation controlled by (*)

First two terms positive (indeed: close to origin and away from it)

Now $A^* A \geq g^2$, but $(A^* A)_\rho \neq A^*_\rho A_\rho$

This issue can be dealt with by tapering argument:
Proposition (Bratelli-Robinson)

For $f : \mathbb{R} \rightarrow \mathbb{R}$ with Fourier transform defined without $\sqrt{2\pi}$,

$$\|[f(D), A]\| \leq \|\hat{f}'\|_1 \|[D, A]\|$$

Lemma

$\exists$ even function $f : \mathbb{R} \rightarrow [0, 1]$ with $f(x) = 0$ for $|x| \geq \rho$

and $f(x) = 1$ for $|x| \leq \frac{\rho}{2}$ such that $\|\hat{f}'\|_1 = \frac{8}{\rho}$

With this, $f = f(D) = f(|D|)$ and $1_{\rho} = \chi(|D| \leq \rho)$:

$$A_{\rho}^*A_{\rho} = 1_{\rho}A^*1_{\rho}A1_{\rho} \geq 1_{\rho}A^*f^2A1_{\rho}$$

$$= 1_{\rho}fA^*Af1_{\rho} + 1_{\rho}([A^*, f]fA + fA^*[f, A])1_{\rho}$$

$$\geq g^2 f^2 + 1_{\rho}([A^*, f]fA + fA^*[f, A])1_{\rho}$$

So indeed $A_{\rho}^*A_{\rho}$ positive close to origin
Then one can conclude... but TEDIOUS
**η-invariant (Atiyah-Patodi-Singer 1977)**

**Definition**

\[ B = B^* \text{ invertible operator on } \mathcal{H} \text{ with compact resolvent. Then} \]

\[ \eta(B) = \text{Tr}(B|B|^{-s-1})|_{s=0} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty dt \ t^{\frac{s-1}{2}} \ \text{Tr}(B e^{-tB^2}) \bigg|_{s=0} \]

provided it exists!

If \( \dim(\mathcal{H}) < \infty \), then \( \eta(B) = \text{Sig}(B) \)

Usually existence of \( \eta \)-invariant for \( \psi \)-Diffs difficult issue

**Proposition**

*If (*) holds, \( B_{\bar{\kappa}} \) has well-defined \( \eta \)-invariant*

**Proof.** Integral for large \( t \) controlled by gap (Proposition above)
For small $t$ appeal to Dyson series (iteration of DuHamel):

$$e^{-tB_{\kappa}^2} = e^{-t\Delta} + t \int_0^1 dr \, e^{-(1-r)t\Delta} \text{Re}^{-rtB_{\kappa}^2}$$

where $B_{\kappa}^2 = \Delta + R$ with

$$\Delta = \kappa^2 \begin{pmatrix} D^2 & 0 \\ 0 & D^2 \end{pmatrix}, \quad R = \begin{pmatrix} AA^* & \kappa[D, A] \\ \kappa[D, A]^* & A^*A \end{pmatrix}$$

Now replacing $B_{\kappa} = \kappa D \otimes \sigma_3 + H$

$$\text{Tr}(B_{\kappa}e^{-t\Delta}) = \kappa \text{Tr} \left( \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} e^{-t\Delta} \right) + \text{Tr} \left( \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} e^{-t\Delta} \right) = 0$$

Second term has supplementary factor $t$
Finite volume calculation of $K$-theory invariants

Theorem (follows from Getzler 1993, Carey-Phillips 2004)

Suppose (*) so that $B_\kappa$ has well-defined $\eta$-invariant

For path $\lambda \in [0, 1] \mapsto B_\kappa(\lambda) = \kappa D \otimes \sigma_3 + \lambda H$ of selfadjoints

\[ 2 \text{SF}(\lambda \in [0, 1] \mapsto B_\kappa(\lambda)) = \eta(B_\kappa(1)) - \eta(B_\kappa(0)) = \eta(B_\kappa) \]

Consequence: As spectral flow homotopy invariant, so is $\eta(B_\kappa)$

Using this, first proof of Main Result for dimension $d = 1$:

By homotopy invariance sufficient: $A = S^n$ for $n \in \mathbb{Z}$ and $S$ shift

Then calculate spectrum of $B_\kappa(\lambda)$ explicity using $XS = (X + 1)S$:

\[ \sigma(B_\kappa(\lambda)) = \left\{ \frac{\kappa}{2} \left( n \pm \left( (n - 2k)^2 + \frac{4\lambda^2}{\kappa^2} \right)^{\frac{1}{2}} \right) : k \in \mathbb{Z} \right\} \]

Now carefully follow eigenvalues to calculate spectral flow \(\Box\)
Preparations for $K$-theoretic argument for other $d$

Unitization $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ of $\mathbb{C}^*$-algebra $\mathcal{A}$ by

$$(A, t)(B, s) = (AB + As + Bt, ts), \quad (A, t)^* = (A^*, \bar{t})$$

Natural $\mathbb{C}^*$-norm $\|(A, t)\| = \max\{\|A\|, |t|\}$. Unit $1 = (0, 1) \in \mathcal{A}^+$

Exact sequence of $\mathbb{C}^*$-algebras $0 \to \mathcal{A} \xrightarrow{i} \mathcal{A}^+ \xrightarrow{\rho} \mathbb{C} \to 0$

$\rho$ has inverse $i'(t) = (0, t)$, then $s = i' \circ \rho : \mathcal{A}^+ \to \mathcal{A}^+$ scalar part

$\mathcal{V}_0(\mathcal{A}) = \{ V \in \bigcup_{n \geq 1} M_{2n}(\mathcal{A}^+) : V^* = V, \ V^2 = 1, \ s(V) \sim_0 E_{2n} \}$

where homotopic to $E_{2n} = E_2^\oplus n$ with $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Equivalence relation $\sim_0$ on $\mathcal{V}_0(\mathcal{A})$ by homotopy and $V \sim_0 \begin{pmatrix} V & 0 \\ 0 & E_2 \end{pmatrix}$

Then $K_0(\mathcal{A}) = \mathcal{V}_0(\mathcal{A})/ \sim_0$ abelian group via $[V] + [V'] = [\begin{pmatrix} V & 0 \\ 0 & V' \end{pmatrix}]$
Definition of $K_0(\mathcal{A})$ is equivalent standard one via $V = 2P - 1$:

$$K_0(\mathcal{A}) = \{ [P] - [s(P)] : \text{projections in some } M_n(\mathcal{A}^+) \}$$

For definition of $K_1(\mathcal{A})$ set

$$\mathcal{V}_1(\mathcal{A}) = \{ U \in \bigcup_{n \geq 1} M_n(\mathcal{A}^+) : U^{-1} = U^* \}$$

Equivalence relation $\sim_1$ by homotopy and $[U] = [(U_{01})]$  

Then $K_1(\mathcal{A}) = \mathcal{V}_1(\mathcal{A})/ \sim_1$ with addition $[U] + [U'] = [U \oplus U']$

If $\mathcal{A}$ unital, one can work with $M_n(\mathcal{A})$ instead of $M_n(\mathcal{A}^+)$ in $\mathcal{V}_1(\mathcal{A})$

**Example 1:** $K_0(\mathbb{C}) = \mathbb{Z}$ with invariant $\dim(P)$

**Example 2:** $K_1(C(S^1)) = \mathbb{Z}$ with invariant "winding number"
Index map

Example 3: Calkin’s exact sequence over a Hilbert space:

$$0 \to \mathcal{K} \to \mathcal{B} \xrightarrow{\pi} \mathcal{Q} = \mathcal{B}/\mathcal{K} \to 0$$

For Calkin algebra $K_1(\mathcal{Q}) = \mathbb{Z}$ with invariant = index of Fredholm

Also $K_0(\mathcal{B}) = K_1(\mathcal{B}) = 0$ and $K_0(\mathcal{K}) = \mathbb{Z}$

Isomorphism $K_1(\mathcal{Q}) \cong K_0(\mathcal{K})$ given by index map (Rordam et. al.):

Unitary $U = \pi(B) \in \mathcal{V}_1(\mathcal{Q})$, with contraction lift $B \in \mathcal{B}$,

$$\text{Ind}[U]_1 = \begin{bmatrix}
2BB^* - 1 & 2B(1 - B^*B)^{\frac{1}{2}} \\
2(1 - B^*B)^{\frac{1}{2}}B^* & 1 - 2B^*B
\end{bmatrix}_0$$

where for r.h.s. $V \in \mathcal{K}^+$: $V^2 = 1$ and $s(V) \sim_0 E_2$ up to compact
Index map versus index of Fredholm operator

$B$ unitary up to compact $\iff 1 - B^*B \in \mathcal{K}$ and $1 - BB^* \in \mathcal{K}$

$\implies B$ Fredholm operator and $U = \pi(B) \in \mathcal{Q}$ unitary

Fedosov formula if $1 - B^*B$ and $1 - BB^*$ are traceclass:

$$\text{Ind}(B) = \dim(\text{Ker}(B)) - \dim(\text{Ker}(B^*))$$

$$= \text{Tr}(1 - B^*B) - \text{Tr}(1 - BB^*)$$

$$= \text{Tr} \left( \begin{pmatrix} BB^* - 1 & B(1 - B^*B)^{1/2} \\ (1 - B^*B)^{1/2}B^* & 1 - B^*B \end{pmatrix} \right)$$

$$= \text{Tr} \left( \frac{1}{2}(V - 1) \right)$$

$$= \frac{1}{2} \text{Sig}(V) \quad \text{if } 1 - B^*B, 1 - BB^* \text{ projections}$$

$$= \text{Tr} \left( \frac{1}{2}(\text{Ind}[U] - 1) \right)$$

$$= \text{Tr}(\text{Ind}^\sim[U])$$

if $\text{Ind}^\sim[U]$ is the projection-valued version of index map
Localizing index map for index pairings

Suppose now \( U = \pi(\Pi A \Pi + (1 - \Pi)) \in \mathcal{Q} \) as in Main Theorem but first \( A \) unitary. Then contraction lift \( B = \Pi A \Pi + (1 - \Pi) \)

Modify \( \Pi \) and \( 1 - \Pi \) to \( p = p(D) \) smooth and \( n = n(D) \) where

\[
p(x) = \begin{cases} 
0, & x \leq -\rho \\
p(x), & |x| \leq \rho \\
1, & x \geq \rho
\end{cases}
\]

\[
n(x) = \begin{cases} 
1, & x \leq -\rho \\
0, & x \geq -\rho
\end{cases}
\]

Now \( p - \Pi, n - (1 - \Pi) \) compact, \( np = pn = 0 \) and \( n + p|\mathbb{D}_\rho^c = 1_{\mathbb{D}_\rho^c} \)

With notation \( A_p = pAp \) acting only on \( \ell^2(\mathbb{D}_\rho) \otimes \mathbb{C}^N \):

\[
\text{Ind}[U] = \text{Ind}[pAp + n] = \text{Ind}[A_p + n] = \begin{pmatrix} 
2A_p A_p^* - 1 & 2A_p (1 - A_p^* A_p)^{\frac{1}{2}} \\
2(1 - A_p^* A_p)^{\frac{1}{2}} A_p^* & 1 - 2A_p^* A_p
\end{pmatrix} \oplus \begin{pmatrix} 
1_{\mathbb{D}_\rho^c} & 0 \\
0 & -1_{\mathbb{D}_\rho^c}
\end{pmatrix}
\]
Summand on $\mathbb{D}_\rho^c$ trivial (as equal to $E_2$). Thus:

$$\text{Ind}[U] = \begin{pmatrix} 2A_pA_p^* - 1 & 2A_p(1 - A_p^*A_p)^{1/2} \\ 2(1 - A_p^*A_p)^{1/2}A_p^* & 1 - 2A_p^*A_p \end{pmatrix}$$

Numerical index is signature of this finite-dimensional matrix!

Modify to self-adjoint matrix without spoiling invertibility

$$\|A_pA_p^* - p^4\| = \|pAp^2A^*p - p^3AA^*p\| \leq \|[p^2, A]\|$$

$$\leq \frac{C}{\rho}\|[D, A]\| < \frac{1}{4}$$

by the smoothness of $p$ and for $\rho$ sufficiently large. Similarly

$$\|A_p(1 - A_p^*A_p)^{1/2} - (1 - p^4)^{1/4}pAp(1 - p^4)^{1/4}\| \leq \frac{C}{\rho}\|[D, A]\| < \frac{1}{4}$$

Thus just replace matrix entries without changing signature!
Proposition

If (*) and (**) hold,

\[
\text{Ind}(\Pi A \Pi + (1 - \Pi)) = \text{Sig}\left(\begin{array}{cc}
2p^4 - 1 & 2(1 - p^4)^{1/4} p A p (1 - p^4)^{1/4} \\
2(1 - p^4)^{1/4} p A^* p (1 - p^4)^{1/4} & 1 - 2p^4
\end{array}\right)
\]

Last tasks:

1) replace \(2p^4 - 1\) by \(\kappa D_\rho\)

2) replace \(\sqrt{2}(1 - p^4)^{1/4} p\) by \(1_\rho\) indicator on \(\mathbb{D}_\rho\). Then \(1_\rho A 1_\rho = A_\rho\)

Both follows again by a tapering argument
Implementation of real symmetries

Fix a real structure on complex Hilbert space, denoted by \( \overline{\cdot} \). There is irrep \( \Gamma_1, \ldots, \Gamma_d \) and real unitary matrix \( \Sigma \)

\[
\begin{array}{c | c c c c}
  d \text{ mod 8} & 1 & 3 & 5 & 7 \\
\hline
  \Sigma^* \overline{D} \Sigma & D & -D & D & -D \\
  \Sigma^2 & 1 & -1 & -1 & 1 \\
  \Sigma^* \overline{\Pi} \Sigma & \Pi & 1 - \Pi & \Pi & 1 - \Pi \\
\end{array}
\]

For \( d = 3 \): \( D = X_1 \sigma_1 + X_2 \sigma_2 + X_3 \sigma_3 \) and \( \Sigma = i \sigma_2 \)

Furthermore given real unitary \( S \) with \([S, \Sigma] = [S, D] = 0\):

\[
\begin{array}{c | c c c c}
  j \text{ mod 8} & 2 & 4 & 6 & 8 \\
\hline
  S^* \overline{A} S & A^* & A & A^* & A \\
  S^2 & 1 & -1 & -1 & 1 \\
\end{array}
\]
Symmetries of $T = \Pi A \Pi + (1 - \Pi)$ such that index pairings are:

<table>
<thead>
<tr>
<th>$\text{Ind}_{(2)}(T)$</th>
<th>$j = 2$</th>
<th>$j = 4$</th>
<th>$j = 6$</th>
<th>$j = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 1$</td>
<td>0</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$d = 3$</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>$d = 5$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>$2\mathbb{Z}$</td>
</tr>
<tr>
<td>$d = 7$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

where $\text{Ind}_{2}(T) = \dim(\text{Ker}(T)) \mod 2 \in \mathbb{Z}_2$

For Bott operator follows $R^* \overline{B_{\kappa}} R = s B_{\kappa}$ and $R^2 = s'1$ with

<table>
<thead>
<tr>
<th>$s = , s' =$</th>
<th>$j = 2$</th>
<th>$j = 4$</th>
<th>$j = 6$</th>
<th>$j = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 1$</td>
<td>$-1, -1$</td>
<td>$1, -1$</td>
<td>$-1, 1$</td>
<td>$1, 1$</td>
</tr>
<tr>
<td>$d = 3$</td>
<td>$1, -1$</td>
<td>$-1, 1$</td>
<td>$1, 1$</td>
<td>$-1, -1$</td>
</tr>
<tr>
<td>$d = 5$</td>
<td>$-1, 1$</td>
<td>$1, 1$</td>
<td>$-1, -1$</td>
<td>$1, -1$</td>
</tr>
<tr>
<td>$d = 7$</td>
<td>$1, 1$</td>
<td>$-1, -1$</td>
<td>$1, -1$</td>
<td>$-1, 1$</td>
</tr>
</tbody>
</table>
Same pattern!

Thus $\text{Ind}$ and $\text{Ind}_2$ can be calculated from Bott operator using:

**Proposition**

$B = B^*$ invertible complex matrix. $R = \overline{R}$ real unitary such

$$R^* \overline{B} R = s B, \quad R^2 = s' 1$$

(i) If $s = 1$ and $s' = 1$, then $\text{Sig}(B) \in \mathbb{Z}$ arbitrary

(ii) If $s = 1$ and $s' = -1$, then $\text{Sig}(B) \in 2\mathbb{Z}$ arbitrary

(iii) If $s = -1$ and $s' = 1$, then $\text{Sig}(B) = 0$, but setting $M = R^{\frac{1}{2}}$

one obtains real antisymmetric matrix $iMBM^*$ with

invariant $\text{sgn}(\text{Pf}(iMBM^*)) \in \mathbb{Z}_2$

(iv) If $s = -1$ and $s' = -1$, then $\text{Sig}(B) = 0$
Application to topological insulators

\[ B_\kappa = \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix} = \kappa D \otimes \sigma_3 + H \quad , \quad H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \]

Data: \( H = -J^* H J \) chiral quantum Hamiltonian where \( J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

Invertibility of \( H \) (and hence \( A \)) means: \( H \) describes insulator

Non-trivial higher winding numbers make it a topological insulator

Main Theorem allows to efficiently calculate this topology

As calculation local, one can determine quantum phase transitions

Implementation of physical symmetries on \( H \) (like TRS and PHS) lead to symmetries of \( A \) \( \Rightarrow \mathbb{Z}_2 \) invariants calculable

Now: not every \( H \) is chiral & dimension not always even...
Even dimensional pairings

Consider projection $P$ on $\ell^2(\mathbb{Z}^d, \mathbb{C}^{2N})$ with $d$ even.

Even-dimensional Dirac operator has grading $\Gamma_{d+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Dirac phase $F$ is unitary operator in $D|D|^{-1} = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}$

Fredholm operator $PFP + (1 - P)$ has index equal to $\mathrm{Ch}_d(P)$

Associated Bott operator

$$B_\kappa = \kappa D + (2P - 1)\Gamma_{d+1}$$

Theorem

Suppose $\|[P, D]\| < \infty$ and that $\kappa$ is sufficiently small.

For $\rho$ sufficiently large,

$$\mathrm{Ind}(PFP + (1 - P)) = \mathrm{Sig}(B_\kappa, \rho)$$
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