Analysis of certain random operators related to solid state physics

vorgelegt von
Maxim Drabkin
aus Fergana
"...The cynic might feel that I have finally sunk to my proper level. I started with quantum field theory, analysis in infinitely many variables. That was too hard so I switched to the N-body Schrödinger equation; but that was too hard so I switched to one-body, then one-dimensional, then discrete one-dimensional. Finally to rank one perturbations – may be something so easy that I can say something useful! Alas, we’ll see even this is hard and exceedingly rich. ..."  

Barry Simon

"...A man grows stale if he works all the time on insoluble problems, and a trip to the beautiful world of one dimension will refresh his imagination better than a dose of LSD..."  

Freeman Dyson
Aknowledgement

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Analyse gewisser zufälliger Operatoren mit Bezug zur Festkörperphysik.

Zusammenfassung


Im ersten Teil der Arbeit werden die bekannten Tatsachen zur Theorie zufälliger Schrödinger Operatoren zusammengestellt. Als eines der wichtigsten technischen Werkzeuge wird dabei die Beschreibung des Eigenwertproblems mit Hilfe von Transfermatrizen verwendet.

Für einen eindimensionalen zufälligen Schrödinger Operator $H_\omega$ auf $\ell^2(\mathbb{Z})$, der nur Wechselwirkungen zwischen nächsten Nachbarn berücksichtigt, wie es zum Beispiel beim Andersonmodell der Fall ist, mit dem Operator

$$ (H_\omega \psi)(n) = -\psi(n-1) - \psi(n+1) + v_n \psi(n), \quad (v_n)_{n \in \mathbb{Z}} \text{ u.i.v.} $$

kann die Eigenwertgleichung $H_\omega \psi = E \psi$ in Matrizenschreibweise umformuliert werden:

$$ \begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \begin{pmatrix} v_n - E & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix} = T_\omega^{E}(n+1,n) \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}. $$

Die Familie von zufälligen $\text{SL}(2, \mathbb{R})$ Matrizen spielt also bei der Analyse die entscheidende Rolle. Das asymptotische Verhalten der (verallgemeinerten) Eigenfunktionen kann an dem Lyapunovexponenten abgelesen werden. Der Letzte ist mit Hilfe der Fundamentallösung $T_\omega^{E}(x,y) = T_\omega^{E}(y,y-1) \ldots T_\omega^{E}(x+1,x) \in \text{SL}(2, \mathbb{R}), \ x, y \in \mathbb{Z}, \ x < y,$ definiert durch

$$ \gamma^E = \lim_{N \to \infty} \frac{1}{N} \log(\|T_\omega^{E}(N,0) e\|). $$

Dabei kann für den Vektor $e$ ein beliebiger Einheitsvektor in $\mathbb{R}^2$ gewählt werden. Es gilt die fast sichere Konvergenz (siehe z.B. [BL], bzw [PF]). Mit Hilfe einiger später skizzierten Techniken lässt sich unter relativ allgemeinen Annahmen nachweisen, dass solche eindimensionalen Operatoren reines Punktspektrum besitzen.

Zusammenfassung

Im dritten Kapitel wird das eindimensionale zufällige Kronig-Penney Modell studiert. Dieses beschreibt die Bewegung der Elektronen in einem unendlich langen dünnen Draht, der aus Atomen mit unterschiedlichem Potential besteht. Da die Anordnung dieser als unbekannt vorausgesetzt wird, wird zur Beschreibung dieses Vorgangs ein zufälliges Potential gewählt. Es wird jedoch angenommen, dass die Atome sich an festen Gitterpunkten aufhalten. Der zugehörige zufällige Hamiltonoperator lässt sich formal wie folgt definieren:

$$ H_\omega = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} v_n \delta_n, $$

wobei \((v_n)_{n \in \mathbb{Z}}\) eine Familie der u.i.v. Zufallsvariablen ist. Um die Schwierigkeiten technischer Natur zu umgehen, soll die Verteilung \(p\) von \(v_n\) 'seinen kompakten Träger besitzen und nicht entartet sein, was auch eine physikalisch sinnvolle Einschränkung darstellt. Die mathematisch präzisere formale Beschreibung kann mit Hilfe der Neumannschen Theorie der selbstadjungierten Erweiterungen erfolgen. Eine detaillierte funktionalanalytische Abhandlung, welche sich mit deterministischen Modellen solcher Art auseinandersetzt, ist [Alb].

Das reelle Energiespektrum des deterministischen Modells \(v_n = v > 0\) besitzt eine Bandstruktur. Die Bandränder \(E\) sind implizit als Lösungen folgender Gleichung mit \(E = k^2\) gegeben:

$$ \left| 2 \cos(k) + \frac{v}{k} \sin(k) \right| = 2. $$

Die oberen Bandränder hängen nicht vom Potential \(v\) ab und sind gleich \(E_l = (l\pi)^2\). Für das zufällige Modell existieren Konstanten \(c_l(v) \in \mathbb{R}^+\) mit \(v = \inf \text{supp}(p) > 0\) so, dass für das fast sichere Spektrum \(\sigma(H_\omega)\) gilt

$$ \sigma(H_\omega) = \sigma(H_v) = \bigcup_{l \in \mathbb{N}} [E_l - c_l(v), E_l]. $$

Falls der Wert 0 im Träger von \(p\) liegt, gilt für das fast sichere Spektrum \([0, \infty) \subset \sigma(H_\omega)\).

Um die Transporteigenschaften des Hamiltonoperators zu beschreiben, betrachtet man das Zeitmittel des \(q\)-ten Moments vom Ortsoperator \(X\) auf \(L^2(\mathbb{R})\):

$$ M_q(T) = \frac{2}{T} \int_0^\infty dt \ e^{-\frac{t}{2}} \ e^{iH_\omega t} |X|^q e^{-iH_\omega t}, \quad q > 0. $$

Dies ist ein Integraloperator mit Integralkern \(\langle x|M_q(T)|y\rangle\) in der Dirac-Notation. Die Werte \(\langle a|M_q(T)|a\rangle\) können als Zeitmittel eines ursprünglich im Zustand \(a \in \mathbb{R}\) lokalisierten Wellenpakets interpretiert werden. Dieser ist allerdings kein Element des Hilbertraumes. Man kann zeigen, dass für ausreichend große Werte von \(q\) diese Momente mit der Zeit \(T\) wachsen, was kein typisches Verhalten innerhalb des Lokalisierungs-Regimes ist.

Falls \(E(v) \neq 0\) und \(a \in \mathbb{R} \setminus \mathbb{Z}\) gilt, existiert für jedes \(\alpha > 0\) eine positive Konstante \(C_\alpha\), so dass für \(T > 1\) gilt

$$ \langle a|M_q(T)|a\rangle \geq C_\alpha T^{q(\frac{5}{2} - \frac{5}{2} - \alpha)}, \quad q > \frac{5}{2}. \quad (1) $$

2
Zusammenfassung

Dieses Verhalten kann heuristisch wie folgt erklärt werden. Sowohl die Lokalisierungslänge, die gleich dem Inversen des Lyapunovexponenten ist, als auch die Zustandsdichte divergieren an den kritischen Energien $E_l$. Dies beruht auf dem verschwindenden Lyapunovexponenten $\gamma_{E_l}$ an den kritischen Energien $E_l = (\pi l)^2$, $l \in \mathbb{N}$, was spätestens bereits seit der Arbeit [Ish] bekannt ist.

Die Beweisstrategie und die heuristischen Überlegungen sind ähnlich zu denen im zufälligen Polymermodell wie im Artikel [JSS]. Der wesentliche Unterschied besteht darin, dass die Transfermatrizen an den kritischen Energien $E_l$ Jordanblockstruktur im zufälligen Kronig-Penney Modell haben, während es sich an den kritischen Energien des zufälligen Polymermodells um Rotationsmatrizen handelt. Dies führt zu einem völlig anderen Verhalten des Lyapunovexponenten und folglich auch der gemittelten Momente in Gleichung (1). Für das zufällige Polymermodell gilt $\langle a| M_q(T)|a \rangle \sim T^{q-\frac{1}{2}}$, wobei obere und untere Schranken in den Publikationen [JSS] und [JS] gezeigt wurden. Es wird kein Anspruch darauf erhoben, dass die in Ungleichung (1) gegebene untere Schranke optimal ist.

In der Umgebung der kritischen Energien können der Lyapunovexponent und die integrierte Zustandsdichte perturbativ berechnet werden.

Sei $E(v) \neq 0$. Mit den positiven Konstanten

$$D_- = \frac{E(v^2 - E(v)^2)}{16E(v)E_l}, \quad D_+ = \left(\frac{E(v)}{2E_l}\right)^{\frac{1}{2}},$$

und $\varepsilon \geq 0$ gilt für den Lyapunovexponenten

$$\gamma_{E_l-\varepsilon} = D_- \varepsilon + O(\varepsilon^\frac{3}{2}), \quad \gamma_{E_l+\varepsilon} = D_+ \varepsilon^{\frac{1}{2}} + O(\varepsilon),$$

sowie für die integrierte Zustandsdichte

$$N_{E_l-\varepsilon} = l - \frac{1}{\pi} D_+ \varepsilon^{\frac{1}{2}} + O(\varepsilon).$$

Der Beweis dieser Formeln basiert auf der Störungstheorie für Produkte zufälliger Matrizen um einen Jordanblock. Es werden Techniken aus [SS] verwendet. Bekanntlich ist $-\gamma_{E_l} + i \pi N_{E_l}$ der Randwert einer Herglotz-Funktion $w^z$, die durch die später definierte Weyl-Titchmarsh Funktion $m_{\omega, \pm}^z$ gegeben ist mit $w^z = E(m_{\omega, \pm}^z)$.

Es gilt

$$E(m_{\omega, \pm}^{E_l+z}) = i\pi l - D_+ z^{\frac{1}{2}} + (D_- + iB)z + O(z^{\frac{3}{2}}),$$

mit $B$ reell. Dann gilt $N_{E_l+z} = l + \frac{1}{\pi} B \varepsilon + O(\varepsilon^{\frac{3}{2}})$. Die Konstante $B$ verschwindet, falls der Träger von $p$ strikt positiv ist, da in diesem Falle $E_l$ der obere Bandrand ist. Die Berechnung von $B$ ist möglich, aber mit erheblichem technischen Aufwand verbunden, da der dafür benötigte Fokker-Planck Operator auf $S^1_{\pi}$ singulär wird.

Die für deterministische eindimensionale Modelle charakteristische van Hove Singularität der Zustandsdichte an den kritischen Energien wird beibehalten, obwohl das Modell zufällig ist. Dies kann damit begründet werden, dass eine zufällige Störung
Zusammenfassung


Im vierten Kapitel beschäftige ich mich mit dem asymptotischen Verhalten von Produkten bestimmter zufälliger $\text{Sl}(2, \mathbb{R})$ Matrizen. Sei $(T_{\lambda,n})_{n\geq 1}$ eine Familie zufälliger $\text{Sl}(2, \mathbb{R})$ Matrizen von der Form

$$T_{\lambda,n} = \exp\left(\lambda P_n + \lambda^2 Q_{\lambda,n}\right),$$

mit $\lambda \geq 0$ der sog. Kopplungsparameter und $P_n, Q_{\lambda,n} \in \text{sl}(2, \mathbb{R})$. Sei $Q_{\lambda,n}$ analytisch in $\lambda$, und die Matrizen $P_n$ und $Q_{\lambda,n}$ unabhängig identisch verteilt auf einem Kompaktum. Dann sind die Matrizen $T_{\lambda,n}$ in einer $\lambda$ Umgebung um die Einheitsmatrix verteilt. Um zu gewährleisten, dass die zufällige Störung genügend Einfluss auf die Dynamik hat, nehmen wir an, dass eine bestimmte Linearkombination der Einträge von $P_n$ positive Varianz hat. In dem Kontext eindimensionaler zufälliger Schrödinger Operatoren bezeichnet man eine solche Situation als Anomalie, siehe [KW, DG, CK, SS]. Ausgehend von der Klassifikation der speziellen linearen Gruppe, können die Anomalien wie in [SS] klassifiziert werden:

Eine Anomalie ist von erster Ordnung, falls $E[P_n] \neq 0$. Eine Anomalie erster Ordnung ist elliptisch, hyperbolisch oder parabolisch wenn jeweils $\det(E[P_n]) > 0$, $\det(E[P_n]) < 0$ oder $\det(E[P_n]) = 0$ gilt. Falls $E[P_n] = 0$, so hat man eine Anomalie zweiter Ordnung.

Die Produkte $\prod_{n=1}^N T_{\lambda,n} = T_{\lambda,N} \cdots T_{\lambda,1}$ solcher Familien zufälliger Matrizen genügen asymptotisch einem 0-1 Gesetz, das mit dem Lyapunovexponenten $\gamma_\lambda$ beschrieben werden kann.

Ferner gilt ein Analogon des zentralen Grenzwertsatzes [Tut, BL]. Die Verteilung von

$$\frac{1}{\sqrt{N}} \left( \log \left( \prod_{n=1}^N T_{\lambda,n} e \right) - N \gamma_\lambda \right)$$

Zusammenfassung

Berechnung bzw. Kontrolle der Standardabweichungen in den jeweiligen Fällen ist neu.

- Für eine elliptische Anomalie erster Ordnung existiert eine Konstante $C_e$, die explizit aus den Varianzen der Einträge von $P_n$ berechnet werden kann, so dass gilt
  $$\gamma_\lambda = C_e \lambda^2 + \mathcal{O}(\lambda^3), \quad \sigma_\lambda = C_e \lambda^2 + \mathcal{O}(\lambda^3).$$

- Für eine hyperbolische Anomalie erster Ordnung gilt mit $C_h = \sqrt{-\det(E(P_n))}$,
  $$\gamma_\lambda = C_h \lambda + \mathcal{O}(\lambda^{{\frac{3}{2}}}), \quad \sigma_\lambda = \mathcal{O}(\lambda^{{\frac{3}{2}}}).$$

- Und für eine Anomalie zweiter Ordnung existieren nichtnegative Konstanten $C_s$ und $C'_s$ mit
  $$\gamma_\lambda = C_s \lambda^2 + \mathcal{O}(\lambda^3), \quad \sigma_\lambda = C'_s \lambda^2 + \mathcal{O}(\lambda^3).$$

Da $\sigma_\lambda = \gamma_\lambda + \mathcal{O}(\lambda^2)$ im Falle einer elliptischen Anomalie erster Ordnung gilt, kann man die asymptotische Verteilung vom Produkt zufälliger Matrizen bis auf Fehler höherer Ordnung mit einem einzigen Parameter beschreiben. Dies wird als single parameter scaling für zufällige Schrödinger Operatoren bezeichnet. In [SSS] wurde gezeigt, dass dies außerhalb von Anomalien, aber nur in niedrigster Ordnung perturbativ gilt. Dies wurde nun für das elliptische Regime erster Ordnung nachgewiesen. Bezogen auf das Anderson Modell deckt dies Energien innerhalb des Energiebandes, aber nahe eines Bandrandes, ab.


Bei Anomalien zweiter Ordnung sind der Lyapunovexponent und die Standardabweichung von gleicher Größenordnung, allerdings kann die Gleichheit der entsprechenden Konstanten eher als Zufall betrachtet werden. Streng genommen heißt das, dass das single parameter scaling auch in diesem Fall nicht gilt.

Als Anwendung dieser Erkenntnisse werden zwei weitere Modelle in Kürze diskutiert, nämlich eine Kette harmonischer Oszillatoren und das Kronig-Penney Modell.

Im fünften Kapitel betrachten wir eine spezielle Klasse höherdimensionaler zufälliger Schrödinger Operatoren, die sogenannten Bogoliubov-de Gennes Operatoren. Diese beschreiben das Verhalten des Elektronengases in einem Supraleiter und haben eine Blockstruktur

$$H_\mu = \frac{1}{2} \begin{pmatrix} \hbar - \mu & \Delta \\ -\Delta & -(\hbar - \mu) \end{pmatrix}$$
mit dem Operator $h = h^*$, der auf dem komplexen Ein-Teilchen Hilbertraum $\mathcal{H}$ mit der komplexen Involution $\mathcal{C}$ wirkt und ein Elektron beschreibt, $\mu \in \mathbb{R}$ ist das chemische Potential, und $\Delta$, auch ein Operator auf $\mathcal{H}$, das sogenannte Elektronenpaarpotential oder Paarerzeugungspotential. Der komplex konjugierte Operator eines Operators $A$ auf $\mathcal{H}$ ist definiert durch $\overline{A} = \mathcal{C}A\mathcal{C}$. Das Paarerzeugungspotential erfüllt die BdG-Bedingung

$$\Delta^* = -\overline{\Delta},$$

um die Selbstadjungiertheit von $H_\mu$ zu gewährleisten. Einfachheitshalber betrachten wir nur beschränkte Operatoren $h$ und $\Delta$. Dann ist $H_\mu$ ein beschränkter selbstadjungierter Operator auf dem Teilchen-Loch Hilbertraum $\mathcal{H}_{ph} = \mathcal{H} \otimes \mathbb{C}^2_{ph}$. Den Faktor $\mathbb{C}^2_{ph}$ bezeichnet man als Teilchen-Loch-Faser. Der BdG Hamiltonoperator besitzt die Teilchen-Loch-Symmetrie (TLS)

$$K^* \mathcal{T}_\mu K = -H_\mu, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Der BdG Hamilton-Operator wird aus der Bardeen-Cooper-Schrieffer-Theorie durch eine Mean-Field-Näherung hergeleitet, siehe dazu [dG]. In der zweiten Quantisierung vom Hamiltonoperator auf dem zugehörigen Fockraum sind die Nebendiagonaleinträge $\Delta$ and $-\overline{\Delta}$ jeweils für die Vernichtung und Erzeugung der Cooper-Paare verantwortlich.

Um es konkreter zu gestalten, werden verschiedene Standardbeispiele für $\Delta$ untersucht. Dabei verwenden man eine Nähe rung für stark gebundene Elektronen.

Wie in [AZ] dargelegt, gibt es vielerlei Gründe dafür, dass man die Einträge von $H_\mu$ zufällig wählt. Ein sogenannter verschmutzter Supraleiter kann mit Hilfe eines zufälligen Potentials beschrieben werden, wie jede Legierung oder Halbleiter. Ferner ist es sinnvoll die Mean-Field-Näherung durch einen zufälligen Prozess zu modellieren. Daher können sämtliche Einträge des oben angegebenen Blockoperators zufällige Operatoren sein. Es ist für mesoskopische Systeme sogar zweckmäßig zufällige Matrizen zu verwenden, siehe wieder [AZ]. Allerdings beschäftigen wir uns ausschließlich mit Operatoren mit endlicher Reichweite, das heißt sowohl $h$ als auch $\Delta$ sind Polynome in Verschiebungsoperatoren auf $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^r$. Dabei ist $r$ die Anzahl der inneren Freiheitsgrade. Die komplexe Struktur auf dem Hilbertraum wird durch die komplexe Konjugation induziert.

BdG Hamiltonoperatoren, die nur die oben bereits erwähnte Teilchen-Loch-Symmetrie besitzen, sind in der Klasse D nach der Altland-Zirnbauer (AZ) Klassifikation. Wenn außerdem die Zeitumkehrsymmetrie vorhanden ist, werden sie den Klassen BDI bzw DIII zugeordnet, je nachdem, ob der Spin gerade oder ungerade ist. Von besonderem Interesse sind auch Modelle, die eine $\text{SU}(2)$ Spinrotation-Invarianz aufweisen. Der Hamiltonoperator mit ungeradem Spin kann dann laut [AZ, DS] eine direkte Summe von spinlosen Hamiltonoperatoren $H_\mu^{\text{red}}$ geschrieben werden mit

$$I^* H_\mu^{\text{red}} I = -H_\mu^{\text{red}}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
Zusammenfassung


Eine der wichtigsten Auswirkungen der Teilchen-Loch-Symmetrie ist die spektrale Symmetrie. Für das Spektrum $\sigma(H)$ eines selbstadjungierten Operators $H = H^*$ mit Teilchen-Loch-Symmetrie gilt

$$\sigma(H) = -\sigma(H).$$

Daher ist die Energie 0 ein besonderer Punkt für das Spektrum des Operators. Wir zeigen, dass die integrierte Zustandsdichte für kovariante BdG Hamiltonianoperatoren punktsymmetrisch um die 0 ist, und das in den meisten Fällen die Energie 0 entweder in der spektralen Lücke oder in der Pseudolücke liegt. Insbesondere ist der in [KMM] betrachtete Fall eine Ausnahme.


Zusammenfassung


Summary

The presented thesis summarizes my research on quantum mechanical models in recent years. The main objects in consideration are random mostly one dimensional and discrete Schrödinger operators and their spectral properties. From the physical point of view the most interesting objects in these models are conductivity and charge transport in disordered solid media. These show different behavior than the ordered systems. This was discovered first in the work of Anderson [And]. The mathematical framework was delivered especially by the works of [PF], [BL], as well as the numerous articles of Simon [Sim1],[Sim2].

The first part of this thesis is devoted to recollection of basic facts from the theory of random Schrödinger operators. The most powerful tool for our purposes is the reformulation of the problem by transfer matrices.

Let us consider a one dimensional random Schrödinger operator $H_\omega^\prime$ on $\ell^2(\mathbb{Z})$, which describes only nearest neighbor interactions, for example the Anderson operator

$$(H_\omega^\prime \psi)(n) = -\psi(n - 1) - \psi(n + 1) + v_n \psi(n), \quad (v_n)_{n \in \mathbb{Z}} \text{ u.i.v.}$$

Then one can rewrite the eigenvalue equation $H_\omega^\prime \psi = E \psi$ using transfer matrices:

$$
\begin{pmatrix}
\psi(n + 1) \\
\psi(n)
\end{pmatrix} =
\begin{pmatrix}
v_n - E & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\psi(n) \\
\psi(n - 1)
\end{pmatrix} = T_{E}(n + 1, n) \begin{pmatrix}
\psi(n) \\
\psi(n - 1)
\end{pmatrix}.
$$

The family of random $\text{Sl}(2, \mathbb{R})$ plays a crucial role in the whole analysis. The asymptotic behavior could be read off the Lyapunov exponent. The latter is defined in terms of the fundamental solution $T_{E}(x, y) = T_{E}(y, y-1) \ldots T_{E}(x+1, x) \in \text{Sl}(2, \mathbb{R})$, $x, y \in \mathbb{Z}$, $x < y$, by

$$
\gamma^E = \lim_{N \to \infty} \frac{1}{N} \log (\|T_{E}(N, 0) e\|).
$$

Where $e$ is any unity vector in $\mathbb{R}^2$. The convergence holds almost surely ([BL], bzw [PF]). Later on we will briefly sketch the techniques, using which one is able to prove under rather general assumptions, the pure point spectrum of this type of operators almost surely.

The following chapters contain slightly revisited publications [DKS], [DraS] and [DDS]. These were written in collaboration with coauthors mentioned in the bibliography.

In the third chapter we study the random one dimensional Kronig-Penney model. It describes the motion of a particle in a one-dimensional system with singular potentials. We consider the case of a random $\delta$-potential on the points of the lattice $\mathbb{Z}$. The Hamiltonian is given by

$$
H_\omega = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} v_n \delta_n,
$$

where $\omega = (v_n)_{n \in \mathbb{Z}}$ are i.i.d. random variables. We assume that the distribution of the $v_n$’s has compact support on $\mathbb{R}^+$ and is nondegenerate, i.e. is not concentrated.
in a single point. The formal mathematical description of the \( \delta \)-potential could be made using the extension theory for self-adjoint operators, a detailed treatment for deterministic models with \( \delta \)-potential is [Alb].

The real energy spectrum of the deterministic model with \( v_n = v > 0 \) has a band structure. The band edges \( E \) are implicitly given as the solutions of the following equation with \( E = k^2 \):

\[
\left| 2 \cos(k) + \frac{v}{k} \sin(k) \right| = 2.
\]

The upper band edges are independent of the potential \( v \) and are equal to \( E_l = (l\pi)^2 \).

For the random model there are constants \( c_l(v) \in \mathbb{R}^+ \) with \( v = \inf \text{supp}(p) \geq 0 \) such that the almost surely spectrum \( \sigma(H_\omega) \) is given by

\[
\sigma(H_\omega) = \sigma(H_v) = \bigcup_{l \in \mathbb{N}} [E_l - c_l(v), E_l].
\]

If \( 0 \) is in the support of \( p \), then for the almost surely spectrum holds \( [0, \infty) \subset \sigma(H_\omega) \).

The main focus here is on the transport properties of the random Kronig-Penney Hamiltonian. We analyze the growth of the time-averaged \( q^\text{th} \) moment of the position operator \( X \) on \( L^2(\mathbb{R}) \):

\[
M_q(T) = \frac{2}{T} \int_0^\infty dt \, e^{-\frac{T}{2}} e^{iH_\omega t} |X|^q e^{-iH_\omega t}, \quad q > 0.
\]

This operator is an integral operator and its integral kernel is denoted by \( \langle x|M_q(T)|y \rangle \) in the bra-ket notation. The diagonal entries \( \langle a|M_q(T)|a \rangle \) can be interpreted as the time-averaged moments of a wave packet initially localized in a Dirac state at \( a \in \mathbb{R} \) (which, of course, is not an element of Hilbert space). The following result shows that for sufficiently large \( q \) these moments grow with time \( T \), which is certainly not the typical behavior inside the localization regime.

Suppose that \( E(v) \neq 0 \) and let \( a \in \mathbb{R} \setminus \mathbb{Z} \). For every \( \alpha > 0 \) there is a positive constant \( C_\alpha \) such that for \( T > 1 \)

\[
\langle a|M_q(T)|a \rangle \geq C_\alpha T^{q\left(\frac{3}{4} - \frac{\alpha}{2}\right) - \alpha}, \quad q > \frac{5}{2}.
\]

This atypical behavior can be explained as follows. Both the localization length (given by the inverse of the Lyapunov exponent) and the density of states diverge at certain critical energies \( E_l \). The divergence of the latter means, that there are many such states. The vanishing of the Lyapunov exponent \( \gamma_E \) at the critical energies \( E_l = (\pi l)^2, l \in \mathbb{N} \), is known at least since the work of Ishii [Ish].

The quantum motion in these states is more or less ballistic until it reaches the localization length. This decreasing fraction of delocalized states allows to prove the lower bound. For the heuristics and the implementation of the idea in a proof we use quite similar techniques as in the case of the random polymer model [JSS]. There is, however, a crucial difference, which will be studied carefully also in the
next chapter. The transfer matrices at the critical energies $E_i$ are all of Jordan block structure in the random Kronig-Penney model, while they are random rotations at the critical energies of a random polymer model. This leads to a completely different quantitative behavior of the Lyapunov exponent and ultimately also for the moments in (2). Indeed, in the random polymer model the behavior is $\langle a|M_q(T)|a\rangle \sim T^{q-\frac{1}{2}}$ with lower and upper bounds proved in [JSS] and [JS] respectively. Yet we do not claim that the lower bound (2) is optimal.

Let $E(v) \neq 0$. With the positive constants

$$D_- = \frac{E(v^2 - E(v)^2)}{16 E(v) E_l}, \quad D_+ = \left(\frac{E(v)}{2E_l}\right)^{\frac{1}{2}},$$

one has for $\varepsilon \geq 0$

$$\gamma^{E_l-\varepsilon} = D_- \varepsilon + O(\varepsilon \frac{1}{2}), \quad \gamma^{E_l+\varepsilon} = D_+ \varepsilon^{\frac{1}{2}} + O(\varepsilon),$$

and

$$N^{E_l-\varepsilon} = l - \frac{1}{\pi} D_+ \varepsilon^{\frac{1}{2}} + O(\varepsilon).$$

The proof of these formulas is based on perturbation theory for products of random matrices around random Jordan blocks, and uses the techniques of [SS]. It is well-known that $-\gamma^E + i\pi N^E$ are the boundary value of a Herglotz function $w^z$ which in terms of the Weyl-Titchmarsh function $m^z_{\omega,\pm}$ defined later is given by $w^z = E(m^z_{\omega,\pm})$.

One can deduce from the above formulas, that

$$E(m^{E_l+\varepsilon}_{\omega,\pm}) = i\pi l - D_+ \varepsilon^{\frac{1}{2}} + (D_- + iB) z + O(z^{\frac{1}{2}}),$$

where $B$ is some real constant. Then $N^{E_l+\varepsilon} = l + \frac{1}{\pi} B \varepsilon + O(\varepsilon^{\frac{3}{2}})$. This constant $B$ vanishes if the support of the distribution of the $v_n$ is strictly positive because in this case $E_l$ is an upper band edge. We believe, that the calculation of $B$ is in principle possible by analogous techniques, however with considerable effort because the Fokker-Planck operators on $S^1$ used in [SS] become singular.

Formula (9) shows that the density of states conserves a one-dimensional van Hove singularity at $E_l$ (even though the model is random). This is due to the fact that the random potential cannot move any eigenvalue from below $E_l$ to above $E_l$ and vice versa [KN] (this follows from a basic Sturm-Liouville argument, see Section 3.3 for details). The existence of such singularities was already proved in [KN], but the perturbative expansion of the integrated density of states is new, as are the results on the Lyapunov exponent. For the above random Kronig-Penney model with positive potentials Gredeskul and Pastur [GP] have analyzed the density of states at the lower band edges (this is a Lifshitz tail type regime), see also [KN].

In the fourth chapter we study the asymptotic behavior of the products of random $S^1(2, \mathbb{R})$ matrices. Let be given a family $(\mathcal{T}_{\lambda,n})_{n \geq 1}$ of random matrices in $S^1(2, \mathbb{R})$ which are of the form

$$\mathcal{T}_{\lambda,n} = \exp(\lambda P_n + \lambda^2 Q_{\lambda,n}),$$
where $\lambda \geq 0$ is a small coupling parameter and $P_n, Q_{\lambda,n} \in \text{sl}(2, \mathbb{R})$. The matrix $Q_{\lambda,n}$ is supposed to be analytic in $\lambda$, and both $P_n$ and $Q_{\lambda,n}$ are independent and identically distributed on a bounded set. The distribution of $T_{\lambda,n}$ is thus supported in a neighborhood of size $\lambda$ around the identity. Averaging over this measure is denoted by $E$. In the context of one-dimensional random Schrödinger operators, this situation was termed an anomaly in [KW, DG, CK, SS]. Such anomalies can be further classified as in [SS]:

An anomaly is said to be of the first order, if $E[P_n] \neq 0$. A first order anomaly is said to be elliptic, hyperbolic or parabolic whenever $\det(E[P_n]) > 0$, $\det(E[P_n]) < 0$ or $\det(E[P_n]) = 0$ respectively. If $E[P_n] = 0$, the anomaly is said to be of the second order.

Associated to any random sequence of matrices are products $\prod_{n=1}^{N} T_{\lambda,n} = T_{\lambda,N} \cdots T_{\lambda,1}$ and their assymptotics satisfy a 0-1 law characterized by the Lyapunov exponent $\gamma_\lambda$ [BL]. It can be calculated by

$$
\gamma_\lambda = \lim_{N \to \infty} \frac{1}{N} E \left[ \log \left( \prod_{n=1}^{N} T_{\lambda,n} e \right) \right],
$$

with some initial condition $e \in \mathbb{R}^2$, $\|e\| = 1$. Furthermore, a central limit theorem [Tut, BL] states that the expression

$$
\frac{1}{\sqrt{N}} \left( \log \left( \prod_{n=1}^{N} T_{\lambda,n} e \right) - N \gamma_\lambda \right)
$$

converges to a centered Gaussian law with a variance denoted by $\sigma_\lambda$. Both the variance and the Lyapunov exponent are independent of the initial condition $e$. The Lyapunov exponent at a second order anomaly was calculated perturbatively in [KW, DG, CK] and a rigorous proof of this expansion was given in [SB], while such a perturbative formula at a first order elliptic anomaly was proved in [SS]. The case of a first order hyperbolic anomaly was only sketched in [SS] and here a detailed proof is presented. These facts about the Lyapunov exponent are complemented by new results on the variances in the following theorem.

Let the i.i.d. random matrices $T_{\lambda,n}$ be of the form (47). Furthermore, it is supposed that in each of the cases below a certain linear combination of the entries of $P_n$ specified below has a strictly positive variance.

- If the anomaly is first order elliptic, then there is a constant $C_e$, which is calculated explicitly from the variances of the entries of $P_n$ in Section 4.3, such that

$$
\gamma_\lambda = C_e \lambda^2 + \mathcal{O}(\lambda^3), \quad \sigma_\lambda = C_e \lambda^2 + \mathcal{O}(\lambda^3).
$$

- If the anomaly is first order hyperbolic, then with $C_h = \sqrt{-\det(E[P_n])}$,

$$
\gamma_\lambda = C_h \lambda + \mathcal{O}(\lambda^{3/2}), \quad \sigma_\lambda = \mathcal{O}(\lambda^{3/2}).
$$
• If the anomaly is second order, then there exists constants $C_s$ and $C'_s$ such that

\[ \gamma_\lambda = C_s \lambda^2 + O(\lambda^3), \quad \sigma_\lambda = C'_s \lambda^2 + O(\lambda^3). \]

In the first order elliptic case, the result implies the equality $\sigma_\lambda = \gamma_\lambda + O(\lambda^2)$. Hence the asymptotic distribution of the product of random matrices is described by a single parameter, up to errors of higher order. In the framework of random Schrödinger operators this is referred to as single parameter scaling. It was already shown to hold away from anomalies in [SSS], but only to lowest order in perturbation theory. This is now extended to the first order elliptic regime. For an Anderson model, this covers energies inside of the band, but close to a band edge (see Section 4.6).

On the other hand, energies outside of the band, but close to a band edge, correspond to a first order hyperbolic anomaly. In this situation, item (ii) of the theorem shows that the Lyapunov exponent is given by its deterministic value, up to fluctuations which are of much smaller order as expected (because the system is a random perturbation around a hyperbolic one, albeit a very weakly hyperbolic one). In particular, there is no single parameter scaling in this regime.

At anomalies of second order the Lyapunov exponent and variance are of the same order of magnitude by item (iii), however an equality $C_s = C'_s$ is to be considered a coincidence so that strictly speaking, single parameter scaling does not hold in this case.

As an application of the above calculations, we briefly discuss two relevant models, namely a harmonic chain and the Kronig-Penny model. Both have already been considered in the work of Ishi [Ish], but there has been continuous and also recent interest in them (e.g. [AH] and [DKS] respectively).

In the fifth chapter we consider a special class of higher dimensional random Schrödinger operators, the so called random Bogoliubov-de Gennes (BdG) model operators and the associated physics. BdG Hamiltonians are used for describing the electron gas in a superconductor. They have the following block form

\[ H_\mu = \frac{1}{2} \begin{pmatrix} h - \mu & \Delta \\ -\Delta & -(h - \mu) \end{pmatrix} \]  

where the operator $h = h^*$ acting on a one-particle complex Hilbert space $\mathcal{H}$ with complex structure $\mathcal{C}$ describes a single electron, $\mu \in \mathbb{R}$ is the chemical potential, and $\Delta$, also an operator on $\mathcal{H}$, is called the pairing potential or pair creation potential. The complex conjugate of an operator $A$ on $\mathcal{H}$ is defined by $\overline{A} = \mathcal{C} A \mathcal{C}$. The pairing potential satisfies the so-called BdG equation

\[ \Delta^* = -\overline{\Delta}, \]

assuring the self-adjointness of $H_\mu$. For the sake of simplicity, we choose both $h$ and $\Delta$ to be bounded operators. Hence $H_\mu$ is a bounded self-adjoint operator on the
particle-hole Hilbert space $\mathcal{H}_{\text{ph}} = \mathcal{H} \otimes \mathbb{C}^2_{\text{ph}}$. The factor $\mathbb{C}^2_{\text{ph}}$ is called the particle-hole fiber. In the associated grading, the BdG Hamiltonian has the particle hole symmetry (PHS)

$$K^* \mathcal{P}_\mu K = -H_\mu, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

The physical background of BdG Hamiltonian is the BCS model. By a self-consistent mean-field approximation one gets a BdG Hamiltonian from it [dG]. In the associated second quantized operator on Fock space (quadratic in the creation and annihilation operators), the off-diagonal entries $\Delta$ and $-\Delta$ lead to annihilation and creation of Cooper pairs respectively. We describe various standard tight-binding models for $\Delta$.

For various reasons, which will be discussed later, the operator entries of $H_\mu$ can be seen as random [AZ]. In the models considered in this paper a spacial structure is conserved by supposing that both $h$ and $\Delta$ only contain finite range hopping operators on $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^r$ where $r$ is the number of internal degrees of freedom and the complex structure is induced by complex conjugation.

BdG Hamiltonians having only the particle-hole symmetry are said to be in Class D of the Altland-Zirnbauer (AZ) classification. If moreover other symmetries are imposed, one obtains also other classes. The direct consequence of the PHS is the following fact:

If $H = H^*$ has a PHS, then the spectrum satisfies

$$\sigma(H) = -\sigma(H).$$

The energy 0 is special since it is a reflection point of the spectrum. We show that the integrated density of states for covariant BdG Hamiltonians is symmetric around 0 and that it is generic that 0 either lies in a gap or in a pseudo gap. In particular, the situation in [KMM] is non-generic.

Furthermore we concentrate on the two-dimensional case $d = 2$ and study the topological properties of BdG Hamiltonians which can be read of the Fermi projection $P_\mu = \chi(H_\mu \leq 0)$ on particle-hole Hilbert space $\mathcal{H}_{\text{ph}}$, namely the spectral projection on all negative energy states of $H_\mu$. Then the Fermi projection can have non-vanishing Chern numbers. For disordered systems, these invariants are defined as in [BES] and enjoy stability properties. Non-triviality of the Chern number makes the system into a so-called topological insulator [SRFL] and leads to a number of interesting physical phenomena, which are discussed in [DS]. For periodic models, the Chern numbers can be calculated using the transfer matrices or the Bloch functions. Subsequently we discuss and apply this two techniques to carry out these calculations.

Our main result is to provide a mathematical proof of Anderson localization in the weakly disordered regime for BdG models of topologic insulators, and to show
that the topological invariants are indeed stable. By an adequate choice of the parameters one can produce localized states near zero energy. Let us point out that the localization proof transposes directly to yet other classes of models of interest.

The techniques used for the proof are mainly a combination of known results. We closely follow the outline of the Aizenman-Molchanov method [AM] in its weak disorder version [Aiz]. With the help of the publication [ESS] we adjust the method due to the fact that for our purposes the potentials and hopping amplitudes are matrix valued. The stability of the Chern numbers under disordered perturbation is then obtained just as in [RS].
1 Physical background and basic definitions

The main subject of this work is the analysis of some models related to solid state physics. Our main interest lies in the transport of electron waves in different types of solids. In order to make the discussion possible and to obtain a physically reasonable yet still manageable model one has to make some simplifying assumptions. The motion of the big particles such as atoms is neglected, since it happens on a much bigger time scale. The contribution of other electrons is seen as the potential. The reason for this assumption is the rather big amount of electrons, which would make the problem hardly computable. Furthermore we assume, that atoms are placed on the grid points of an ideal lattice. This type of simplifying assumptions is called the tight-binding approximation.

1.1 Basic quantum mechanical setting

The \(d\)-dimensional motion of a quantum mechanical particle, e.g. an electron can be described by a normalized vector \(\psi\), \(|\psi| = 1\) in the Hilbert space \(\mathcal{H}\), which is the \(L^2(\mathbb{R}^d)\) or for discrete tight-binding models simply \(\ell^2(\mathbb{Z}^d)\). The time evolution of the state \(\psi\) is determined by the time-dependent Schrödinger equation of motion

\[
\imath \frac{\partial}{\partial t} \psi_t = H \psi_t.
\]

Where the Hamiltonian \(H\) acting on a subset of the corresponding Hilbert space is a self-adjoint operator, which by analogy with classical mechanics, is commonly expressed as the sum of two operators \(H_0\) and \(V\).

The first one \(H_0\) is the free operator, it represents the kinetic energy and is mostly given as the self-adjoint extension of the \(d\)-dimensional Laplacian \(\Delta = \sum_{i=1}^d \frac{\partial}{\partial x_i}\) defined on a subset of \(C^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\) or by the discrete Laplacian \((\Delta_d \psi)(n) = \sum_{|n-m|=1} (\psi(n) - \psi(m))\) on \(\ell^2(\mathbb{Z}^d)\).

The second operator \(V\) corresponds to the potential energy of a system and is mostly a multiplication operator. Different types of potentials will be described later.

Using the spectral calculus, the above equation can be solved for an initial state \(\psi_0\) by

\[
\psi_t = \exp(-\imath t H) \psi_0.
\]

In order to define the unitary operator \(\exp(-\imath t H)\) and to obtain the information about the quantum mechanical system, one needs to know the spectral properties of the Hamiltonian.

Let us first recollect some basic definitions and facts of the spectral theory for self-adjoint operators. More detailed considerations can be found in a monography [Teschl], which we closely follow. Good references on this topic are [DK] and, of course, [RS1]. For a self-adjoint operator \(H\) and any vector \(\psi \in \mathcal{H}\) one defines

\[
F_\psi(z) = \langle \psi | (H - z)^{-1} \psi \rangle.
\]
For $z \in \mathbb{C}$ in a resolvent set, where the expression $(H - z)^{-1}$ is defined, this is a holomorphic function, which for $\Im m(z) > 0$ further satisfies

$$|F_\psi(z)| \leq \frac{\|\psi\|^2}{\Im m(z)} \quad F_\psi(z) = \overline{F_\psi(z)}.$$ 

The above function maps the upper half-plane into itself.

**Definition 1.** Holomorphic functions with non-negative imaginary part, mapping the upper half-plane into itself are called Herglotz functions.

Using the Stieltjes inversion formula, one can reconstruct from any Herglotz function a unique Borel measure $\mu_\psi$, which is in our case called the spectral measure of the operator $H$ for $F_\psi$.

$$\mu_\psi(\lambda) = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\lambda+\delta} \Im m F_\psi(t + i \epsilon) \, dt.$$ 

Thus $F_\psi(z)$ can also be written as a Borel transform of the measure $\mu_\psi$:

$$F_\psi(z) = \int \frac{1}{t - z} \, d\mu_\psi(t).$$

By a similar construction, one can also define for any two vectors $\psi$ and $\phi$ a complex-valued measure $\mu_{\psi, \phi}$ by

$$\langle \psi \mid (H - z)^{-1} \phi \rangle = \int \frac{1}{t - z} \, d\mu_{\psi, \phi}(t).$$

There is a corresponding family of spectral projections $E_H$, which is uniquely defined by the above measures. Namely for every Borel set $A$ with characteristic function $\chi_A$ holds

$$\langle \psi \mid E_H(A)\phi \rangle = \int \chi_A(t) \, d\mu_{\psi, \phi}(t).$$

Since one can identify any Borel set with a corresponding spectral projection, one thus has a projection valued measure $\mu_H$. It is called the spectral measure of $H$. For $\mu_H$ holds

$$H = \int t \, d\mu_H(t).$$

One can also obtain the spectral measure by the Stone’s formula

$$\mu_H((a, b)) = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} [(H - t - i \epsilon)^{-1} - (H - t + i \epsilon)^{-1}] \, dt.$$ 

In the above equation the limits are taken in the strong operator topology.

The spectrum of the operator $H$ can be defined as a set, where the spectral measure is growing

$$\sigma(H) = \left\{ \lambda \in \mathbb{R} \mid E_H(\lambda + \epsilon) - E_H(\lambda - \epsilon) \neq 0 \text{ for all } \epsilon > 0 \right\}.$$
By the Lebesgue decomposition theorem, any measure $\mu$ can be decomposed in absolutely continuous, singularly continuous and atomic part. This can be used to decompose the underlying Hilbert space

$$
\mathcal{H}^{ac} = \{ \psi \mid \mu_\psi \text{ is absolutely continuous} \}, \\
\mathcal{H}^{sc} = \{ \psi \mid \mu_\psi \text{ is singularly continuous} \}, \\
\mathcal{H}^{pp} = \{ \psi \mid \mu_\psi \text{ is pure point} \}
$$

For this decomposition one has

$$\mathcal{H} = \mathcal{H}^{ac} \bigoplus \mathcal{H}^{sc} \bigoplus \mathcal{H}^{pp}.$$ 

Then one can define the absolutely continuous, singularly continuous pure point spectrum of $H$ by restricting $\mathcal{H}$ on those sets. $\sigma_{ac}(H) = \sigma(H |_{\mathcal{H}^{ac}})$, $\sigma_{sc}(H) = \sigma(H |_{\mathcal{H}^{sc}})$ and $\sigma_{pp}(H) = \sigma(H |_{\mathcal{H}^{pp}})$. Note that these parts of spectrum are not necessarily disjoint. One should also mention that in most cases the pure point spectrum $\sigma_{pp}(H)$ is bigger than the set of the eigenvalues $\sigma_p(H)$, so only $\sigma_{pp}(H) = \sigma_p(H)$ holds.

For any self-adjoint operator $H$ its associated strongly continuous unitary group is everywhere defined and maps its domain $\mathcal{D}(H)$ into itself and it holds for any $\psi \in \mathcal{D}(H)$

$$\exp(-itH)H\psi = H \exp(-itH)\psi.$$ 

It is quite a commonplace that the evolution of eigenvectors is only a change of phase, thus if $H\psi = E\psi$ for some energy $E \in \mathbb{R}$,

$$\exp(-itH)\psi = \exp(-itE)\psi,$$

and at any time $t \in \mathbb{R}$

$$|\langle \psi \mid \exp(-itH)\psi \rangle|^2 = \|\psi\|^2.$$ 

The time evolution of states in the absolutely continuous subspace $\mathcal{H}^{ac}$ and the continuous subspace $\mathcal{H}^c = \mathcal{H}^{ac} \bigoplus \mathcal{H}^{sc}$ can be characterized as follows.

**Proposition 2.** The subspace $\mathcal{H}^{ac}$ is spanned by

$$\left\{ \psi \in \mathcal{H} \mid \int_{\mathbb{R}} |\langle \psi \mid \exp(-itH)\psi \rangle|^2 dt < \infty \right\}.$$ 

Generally for the states in continuous subspace one can observe some sort of decay.

**Proposition 3.** For any state $\psi \in \mathcal{H}^c$ and $\phi \in \mathcal{H}$ holds

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \phi \mid \exp(-itH)\psi \rangle|^2 dt = 0.$$ 

Thus, it is natural to introduce the states decaying in time

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Definition 4. Define the subspace $\mathcal{H}^w$ by

$$\mathcal{H}^w = \left\{ \psi \in \mathcal{H} \left| \lim_{t \to \infty} \exp(-itH)\psi = 0 \right. \right\} = \left\{ \psi \in \mathcal{H} \left| \lim_{t \to -\infty} \exp(-itH)\psi = 0 \right. \right\},$$

where the limits are taken in the weak operator topology.

The definition above is Definition 2.7.4 in [DK].

Now one can define bound and scattering states and connect them with the above defined subspaces.

Definition 5. We call $\psi \in \mathcal{H}$ a bound state of a self-adjoint operator $H$, if for any $\epsilon > 0$ there is a compact set $K \subset \mathbb{R}^d$, such that for all $t \in \mathbb{R}$ the time evolution of $\psi$ remains up to a small contribution in $K$.

$$\|\chi_{\mathbb{R}^d \setminus K} \exp(-itH)\psi\| < \epsilon.$$

The subset of all bound states is closed. In contrast, scattering states are tending to leave any bounded region.

Definition 6. We call $\psi \in \mathcal{H}$ a scattering state of a self-adjoint operator $H$, if for any compact set $K \subset \mathbb{R}^d$ holds

$$\lim_{t \to \pm \infty} \|\chi_K \exp(-itH)\psi\| = 0.$$

There are also states, which behave like scattering states, but only in the mean.

Definition 7. The state $\psi \in \mathcal{H}$ is called a scattering state in mean of a self-adjoint operator $H$, if for any compact set $K \subset \mathbb{R}^d$ holds

$$\lim_{T \to \infty} 1/2T \int_{-T}^{T} \|\chi_K \exp(-itH)\psi\|^2 dt = 0.$$

There is a famous theorem by Ruelle, Amrein, Georgescu, Enss, which helps in classifying the states in $\mathcal{H}$.

Theorem 8. RAGE For a self-adjoint operator $H$ on a Hilbert space $\mathcal{H}$ let $S$ be a bounded operator, such that $S(H + i)^{-1}$ is compact. Then for all states in the continuous subspace $\psi \in \mathcal{H}^c$ holds

$$\lim_{T \to \infty} 1/2T \int_{-T}^{T} \|S \exp(-itH)\psi\|^2 dt = 0.$$
subspace $H^c$. If the singular continuous part of the Hilbert space $L^2(\mathbb{R}^d)$ is empty, which is for example the case for the Laplacian, then the subspace of all scattering states coincides with the absolutely continuous subspace.

$$H^{ac} = \left\{ \psi \left| \lim_{t \to \pm \infty} \| \chi_K \exp(-itH)\psi \| = 0 \text{ for any compact } K \subset \mathbb{R}^d \right. \right\},$$

$$H^{pp} = \left\{ \psi \left| \lim_{|K| \to \infty} \sup_{t \geq 0} \| \chi_{\mathbb{R}^d \setminus K} \exp(-itH)\psi \| = 0 \text{ for compact balls } K \subset \mathbb{R}^d \right. \right\}.$$

Now after briefly sketching the basic physical properties of the initial states in the Hilbert space and their connection to its spectral decomposition, one can consider a special class of deterministic models. These should describe the motion of an electron in an ideal crystal or an ideal lead, where the atoms are supposed to build an ideal lattice. For the sake of simplicity we consider only $\mathbb{Z}^d$, assuming that the distance between the atom is equal to 1, but one can also consider $a\mathbb{Z}^d$ for any positive $a$.

The potential operator for those models is given by a multiplication with the following function

$$V(x) = \lambda \sum_{i \in \mathbb{Z}^d} f(x - i),$$

where the single cite potential function $f$ is supposed to satisfy continuity and boundedness conditions. For most cases, $f$ is a nonnegative continuous function, supported on some bounded interval. The potential is then periodic with respect to the lattice $\mathbb{Z}^d$, since $V(x - i) = V(x)$ for all $x \in \mathbb{R}^d$ and all sites, $\lambda$ is called a coupling constant. By changing the coupling constant, one can regulate the influence of the potential on the motion of a particle. Next step in the analysis are alloy-type models. Instead of one single site potential function one can take a collection of different single site potential functions $(f_i)_{i \in \mathbb{Z}^d}$. A well elaborated source on the mathematical theory of periodic potentials is [RS1].

Furthermore, it is also possible to consider degenerated potentials. By degeneracy we mean the $\delta$ or $\delta'$-functionals. The most prominent example is the one-dimensional deterministic Kronig-Penney model. The Schrödinger operator of this model is given by a self-adjoint extension of the operator

$$H^{KP} = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(H^{KP}) = \left\{ \psi \in H^{2,2}(\mathbb{R}) \left| \psi(n) = 0, n \in \mathbb{Z} \right. \right\},$$

where $H^{2,2}(\mathbb{R})$ is the Sobolev space of weakly twice differentiable square integrable functions. Its adjoint operator is

$$(H^{KP})^* = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}((H^{KP})^*) = H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus \mathbb{Z}).$$

The deficiency subspaces of $(H^{KP})^*$ have infinite dimension and are spanned by the functions $\psi_n(x) = \exp(ik|x - n|)$, where $k$ are complex square roots of $i$ and $-i$. 
The particular class of self-adjoint extensions of $H^{KP}$ is $\hat{H}^{KP} = -\frac{\partial^2}{\partial x^2}$ with 
\[ \mathcal{D}(\hat{H}^{KP}) = \{ \psi \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus \mathbb{Z}) \mid \psi'(n+) - \psi'(n-) = w_n \psi(n), n \in \mathbb{Z} \} , \]

Here $(w_n)_{n \in \mathbb{Z}}$ is a bounded countable set of points in $\mathbb{R}$, which describes the strength of $\delta$-interactions on any site $n \in \mathbb{Z}$. More details could be found in [Alb]. The most important property of this is that, if all $w_n$'s are either equal to $w \in \mathbb{R}$ or periodically ordered, the spectrum is absolutely continuous. Furthermore, for non-zero potentials it has a band structure which reflects the theory of conduction bands. In the dimension one this can be calculated explicitly. This is Theorem 2.3.3 in [Alb].

**Theorem 9.** The spectrum of the deterministic one-dimensional Kronig-Penney model with constant potential $w \in \mathbb{R}$ is absolutely continuous. 
\[ \sigma(\hat{H}^{KP}) = \sigma_{ac}(\hat{H}^{KP}) = \bigcup_{n \in \mathbb{N}} [a_n, b_n] . \]

Where $a_n < b_n \leq a_{n+1}$ and for $w > 0$ one has $a_1 > 0$.

One should mention, that a big variety of deterministic models shows similar behavior.

### 1.2 Random potentials

In the reality it is quite difficult or nearly impossible to make an ideal wire. Most of them have impurities, sometimes conductors are purposely made with impurities. It is quite impossible to describe all the deviations of the potential deterministically. Thus one rather has to deal with a random family of operators, actually a family of random potentials, which is also the main focus of this work. Since the nobel price winning work of Anderson it is known, that random models can show different conducting behavior than the deterministic ones. Due to the impurities some electrons with certain energies can be caught in specific regions of a wire. Mathematically this means a presence of pure point spectrum and localized states. This phenomenon is called Anderson or spectral localization.

There are some assumptions to be made on the randomness of the potentials. All the used concepts are common knowledge, for the sake of completeness we start with some basic definitions, which also can be found in [PF] or briefly in [K],[KM1].

First of all, one starts with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 10.** A measurable mapping $T : \Omega \rightarrow \Omega$ is called a measure preserving transformation if $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ holds for all $A \in \mathcal{F}$.

Since most of the physical models show complicated behaviour, we consider a countable family of measure preserving transformations $(T_i)_{i \in \mathbb{Z}^d}$.

**Definition 11.** A set $A \in \mathcal{F}$ is called invariant under the action of a countable family of measure preserving transformations $(T_i)_{i \in \mathbb{Z}^d}$, if $T_i^{-1}A = A$ for all $i \in \mathbb{Z}^d$. 

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We are interested in dynamics for which the invariant sets have either full or zero measure.

**Definition 12.** A countable family of measure preserving transformations \((T_i)_{i \in \mathbb{Z}^d}\) is ergodic, if any invariant set \(A \in \mathcal{F}\) has probability \(P(A) = 0\) or \(1\).

In order to consider simple physically interesting models one can choose a random potential on a lattice, this means a family of i.i.d. random variables \(v_{n, \omega}\) with common distribution \(p\). Here one usually takes \(\Omega = \mathbb{R}^{(\mathbb{Z}^d)}\) and the \(\sigma\)-algebra \(\mathcal{F}\) generated by cylinder sets of the form

\[
\{ \omega | \omega_{i_1} \in A_1, \ldots, \omega_{i_k} \in A_k \},
\]

Where \(A_1, \ldots, A_k\) are elements of the Borel \(\sigma\)-algebra in \(\mathbb{R}\). The most prominent and natural example of an ergodic family is the family of shifts \(T_i \omega_j = \omega_{i+j}\) for the product measure \(p^{(\mathbb{Z}^d)}\).

**Definition 13.** A random operator \(H_\omega\) with a domain \(\mathcal{D}\), where \(\mathcal{D}\) is a dense linear subspace of the Hilbert space \(\mathcal{H}\), is a map from the probability space \(\Omega\) on the set of the linear operators on \(\mathcal{H}\), such that \(\mathcal{D}\) lies in the domain of \(H_\omega\) with probability 1 and that \(H_\omega v\) is a random vector in \(\mathcal{H}\) for all \(v \in \mathcal{D}\).

By the above definition we mean that \(\langle w | H_\omega v \rangle\) is a random variable for any \(w \in \mathcal{H}\) for a fixed arbitrary \(v \in \mathcal{D}\).

Now we have all the components in order to define a random ergodic operator, this is a random family of operators on a Hilbert space \(\mathcal{H}\).

**Definition 14.** A random operator \(H_\omega\) on a Hilbert space \(\mathcal{H}\) is called ergodic, if there exists a homomorphism from an ergodic family \((T_i)_{i \in \mathbb{Z}^d}\) into a group \(\mathcal{U} = (U_{T_i})_{i \in \mathbb{Z}^d}\) of unitary operators on \(\mathcal{H}\), such that

\[
H_{T_i \omega} = U_{T_i} H_\omega U_{T_i}^*.
\]

If, moreover, the operator \(H\) is symmetric, its domain \(\mathcal{D}(H)\) remains invariant under the action of the group \(\mathcal{U}\).

The standard example for \(\mathcal{U}\) is the unitary group of shifts on \(\ell^2(\mathbb{Z}^d)\) with \(U_{T_i} |n\rangle = |n+i\rangle\) for the canonical basis of \(\ell^2(\mathbb{Z}^d)\) and the simplest nontrivial example of a random ergodic operator is the discrete one-dimensional Anderson operator \(h_A\) on \(\ell^2(\mathbb{Z})\) with

\[
(h_A \psi)_n = -\Delta \psi_n + \lambda v_{n, \omega} \psi_n, \quad (v_{n, \omega})_{n \in \mathbb{Z}} \text{ i.i.d. random variables},
\]

with \(\psi \in \ell^2(\mathbb{Z})\) and \(\Delta\) a discrete Laplacian, \(\lambda \in \mathbb{R}^+\) a coupling constant.

One should mention that one also has to define and check the measurability properties of an ergodic operator \(H_\omega\), the precise formulation is given in [K] and [KM].
In addition, the ergodic family of measure preserving transformations need not be countable, for a more general definition see [PF], where instead of ergodic the term metrically transitive is used.

The main advantage of ergodic operators is that due to Bikhoff ergodic theorem, some of their important properties are deterministic, since every invariant function is almost surely constant. Let us require two quite general assumptions, which hold for a lot of random models considered in the literature and also for all examples presented in this work.

- The group $\mathcal{U}$ associated with $H_\omega$ has no non-trivial finite-dimensional invariant subspaces in $\mathcal{H}$.
- There exists a countable set $D_1 \subset D(H)$ dense in $\mathcal{H}$, such that with probability one for any $\epsilon > 0$ and any $x \in D(H)$ there exists an $y \in D_1$ with $\|x - y\| + \|H_\omega(x - y)\| < \epsilon$.

These conditions are not very restrictive in our cases, since the first one obviously holds for the unitary shifts and the second is always given in a separable Hilbert space. These are the conditions from the Theorem 2.4 in [PF]. They guarantee that the deficiency indices of the operator are nonrandom and equal to either 0 or $\infty$. Under the above conditions on the operator, Pastur and later Kirsch have shown that the spectrum of such operators is almost surely independent of the randomness.

**Theorem 15.** There is a deterministic set $\Sigma \subset \mathbb{R}$, such that for the spectrum of an ergodic essentially self-adjoint operator $H_\omega$ holds almost surely $\sigma(H_\omega) = \Sigma$.

A more delicate result of Kunz and Soulliard is that also the parts of the spectrum are almost surely deterministic.

**Theorem 16.** There are nonrandom sets $\Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subset \mathbb{R}$, such that for the spectrum of an ergodic essentially self-adjoint operator $H_\omega$ holds almost surely $\sigma_{pp}(H_\omega) = \Sigma_{pp}, \sigma_{ac}(H_\omega) = \Sigma_{ac} \text{ and } \sigma_{sc}(H_\omega) = \Sigma_{sc}$.

A modern extensive review on ergodic operators is [Jit].

### 1.3 Density of states

One of the important quantities used to characterize quantum mechanical systems is the density of states. It measures the "number of energy levels per unit volume" near (resp. below) a given energy. Typical systems do not have discrete spectrum, therefore one can not immediately apply this idea directly. There are several approaches to define the density of states. One of them is the exhaustion procedure. One can take an increasing sequence $\Lambda_n$ of open sets with finite volume, such that $\bigcup_{n=1}^{\infty} \Lambda_n = \mathbb{R}^d$. Then consider a (possibly random) sequence $H_\omega|_{\Lambda_n}$ of self-adjoint
operators which are restriction of the operator $H_\omega$ on $\Lambda_n$ with Dirichlet boundary conditions. The spectrum of each $H_\omega|_{\Lambda_n}$ consists of a finite number of real eigenvalues which increases with growing $n$. For this sequence of operators one can define normalized eigenvalue counting function or finite volume integrated density of states $N^E_{\omega,\Lambda_n}$ at the energy $E$ by just counting the number of eigenvalues below $E$ per volume $|\Lambda_n|$.

$$N^E_{\omega,\Lambda_n} = \frac{\# \{ k | \lambda_{k,n} \in \sigma(H_\omega|_{\Lambda_n}), \lambda_{k,n} < E \}}{|\Lambda_n|}.$$ 

The numerator can also be expressed as the trace of the spectral projection of the operator $H_\omega|_{\Lambda_n}$ on the energy interval $(-\infty, E]$. Now letting $n \to \infty$, one can consider the limit function $N^E$ which is called the integrated density of states. This construction is called the thermodynamic limit.

For the integrated density of states of an ergodic operator $H_\omega$ the following properties hold:

- For almost all $\omega \in \Omega$ and all continuity points $E$ the function $N^E$ is well-defined, that means, the limit $\lim_{n \to \infty} N^E_{\omega,\Lambda_n}$ exists almost surely.
- As already suggested by the notation, $N^E$ is an ergodic quantity, so it is almost surely independent of $\omega$.
- $N^E$ is a limit of distribution functions of point measures and also itself a distribution function. Thus it is non-negative, left-continuous and non-decreasing. In particular, it has at most countably many points of discontinuity.
- If the derivative of the integrated density of states $\frac{dN^E}{dE}$ exists, it is supported on the almost surely given spectral set $\Sigma$, $\Sigma = \sigma(H_\omega)$ for almost every $\omega$.

One should also mention that there are also other approaches towards the construction of the integrated density of states. First of all, one could also use Neumann boundary conditions instead of the Dirichlet boundary conditions in the above construction. Furthermore, at least for ergodic random operators on $\ell^2(\mathbb{Z}^d)$ of the Anderson type and for any bounded measurable function $f$ one can observe the following equality

$$\int f(E) dN^E = E \langle 0 \mid f(H_\omega) \mid 0 \rangle.$$

The right hand side of the above equation defines a positive measure, which is called the density of states measure and its distribution function is the integrated density of states. The above statements are proven for a big variety of special models on $\ell^2(\mathbb{Z}^d)$ and also for some on $L^2(\mathbb{R}^d)$, see [PF],[CFKS],[K] and [KM1], the latter one is a good review on the topic.

Since the (integrated) density of states is one of the crucial quantities which characterizes the corresponding operator, one can observe a completely different behavior for random and non-random periodic models. This is especially clear near the bottom $E_0$ of the spectrum.
For a deterministic model with periodic potential, the integrated density of states behaves near the bottom $E_0$ of the spectrum like

$$N^E = C(E - E_0)^{d/2}(1 + o(1)).$$

This type of behaviour was proved in [KS] for general periodic models, but was already known in the physical literature long before.

Quite different asymptotics for the integrated density of states for systems with disorder was predicted by Lifshitz on the basis of physical arguments.

$$\lim_{E \downarrow E_0} \frac{\ln(-\ln(N^E))}{\ln(E - E_0)} = -d/2$$

This behaviour is called Lifshitz tails regime. One can consider models with the following random potential:

$$V_\omega(x) = \sum_{n \in \mathbb{Z}^d} v_{n,\omega} f(x - n)$$

$v_{n,\omega}$ are i.i.d.

Due to some results of Pastur, Lifshitz tails regime occurs if the single site potential function $f$ has short range and single site distributions are defined on a bounded set with massive weight on the infimum of the support of the random distribution. This means $f$ is nonnegative, bounded and decays at infinity as fast as $|x|^{-(d+1)}$ and the support of the distribution $p$ of $v_n$ is compact and contains at least two points. Furthermore one needs for example two conditions:

- For the distribution $p$ holds $p([v, v + \epsilon]) \geq C\epsilon^N$ with $v = \inf \text{supp}(p)$, some $C$, $N$, and $\epsilon > 0$ small,

- For the potential function $f$ holds $f(x) \leq \tilde{C}(1 + |x|)^{-d+2}$ with some constant $\tilde{C}$.

More details as well as the proofs are given in [PF], [KM1].

### 1.4 Lyapunov exponent

For the sake of simplicity we concentrate from now on on one-dimensional models. In order to investigate the spectral properties of second order difference or differential operators on $L^2(\mathbb{Z})$ or $L^2(\mathbb{R})$ respectively, one has to use some techniques from the Sturm-Liouville theory. Let us briefly define an ansatz to determine the existence of eigenfunctions for the operators restricted on an interval or on the half-line.

Let us consider the random one-dimensional self-adjoint Schrödinger operator $(H_\omega)_{\omega \in \Omega}$ of the type

$$(H_\omega \psi)(n) = \psi(n + 1) + \psi(n - 1) + v_\omega(n)\psi(n).$$

in the discrete case or in the continuous case

$$(H_\omega \psi)(x) = -\psi''(x) + v_\omega(x)\psi(x).$$
where $v_\omega(\cdot)$ is an ergodic random potential which will be specified later. It is possible to determine the solution of the eigenvalue equation

$$H_\omega \psi_\omega = E \psi_\omega, \ E \in \mathbb{R}$$

for a fixed $\omega$ uniquely by fixing two values successively for the discrete case and by fixing $\psi$ and $\psi'$ at some point for the continuous case. For the discrete case one can determine the next pair of values by multiplication with a $2 \times 2$ transfer matrix $T_\omega^E(n + 1, n)$ given by

$$\begin{pmatrix} \psi(n + 1) \\ \psi(n) \end{pmatrix} = T_\omega^E(n + 1, n) \begin{pmatrix} \psi(n) \\ \psi(n - 1) \end{pmatrix} = \begin{pmatrix} E - v_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ \psi(n - 1) \end{pmatrix}.$$ 

Thus one obtains recursively for the discrete case

$$\begin{pmatrix} \psi(n + 1) \\ \psi(n) \end{pmatrix} = T_\omega^E(n + 1, 0) \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix} = T_\omega^E(n + 1, 0) \cdots T_\omega^E(2, 1) \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix}.$$ 

One should also mention that given the Dirichlet and Neumann solutions $\psi_\omega^D$ and $\psi_\omega^N$ of the discrete problem with the boundary conditions $\begin{pmatrix} \psi_\omega^D(1) \\ \psi_\omega^D(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \psi_\omega^N(1) \\ \psi_\omega^N(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, one can also write

$$T_\omega^E(n + 1, 0) = \begin{pmatrix} \psi_\omega^D(n + 1) & \psi_\omega^N(n + 1) \\ \psi_\omega^D(n) & \psi_\omega^N(n) \end{pmatrix}.$$ 

The construction for the continuous case is analogous. Given the Dirichlet and Neumann solutions $\psi_\omega^D$ and $\psi_\omega^N$ with the boundary conditions $\psi_\omega^D(0) = 1, (\psi_\omega^D)'(0) = 0$ and $\psi_\omega^N(0) = 0, (\psi_\omega^N)'(0) = 1$ respectively, one can obtain any solution $\psi$ by

$$\begin{pmatrix} \psi'(x) \\ \psi(x) \end{pmatrix} = T_\omega^E(x, 0) \begin{pmatrix} \psi'(0) \\ \psi(0) \end{pmatrix},$$

where the transfer matrix for the continuous system $T_\omega^E(x, 0)$ is given by

$$T_\omega^E(x, 0) = \begin{pmatrix} (\psi_\omega^D)' & (\psi_\omega^N)' \\ \psi_\omega^D & \psi_\omega^N \end{pmatrix}.$$ 

One can also rewrite the eigenvalue equation and obtain

$$\begin{pmatrix} \psi'(x) \\ \psi(x) \end{pmatrix}' = \begin{pmatrix} 0 & v_\omega(x) - E \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi'(0) \\ \psi(0) \end{pmatrix},$$

which leads to the following differential equation for the family of the transfer matrices

$$\frac{\partial}{\partial x} T_\omega^E(x, 0) = \begin{pmatrix} 0 & v_\omega(x) - E \\ 1 & 0 \end{pmatrix} T_\omega^E(x, 0).$$
Remark. Note that by the above construction one obtains a random family of transfer matrices. Since for every realization $\omega$ the determinant of any transfer matrix is equal one, the random family $(T^E_\omega(x,0))_{\omega \in \Omega, x \in \mathbb{R}}$ or $(T^E_\omega(n,0))_{\omega \in \Omega, n \in \mathbb{Z}}$ is a subset of special linear group $\text{Sl}(2, \mathbb{R})$.

In the localized regime, one expects the eigenfunctions to be exponentially decaying in space for some energies. In order to characterize this decay one can define the following quantity:

$$
\gamma^E_\omega = \lim_{n \to \infty} \frac{1}{n} \log \left( \| T^E_\omega(n,0) \| \right).
$$

The Lyapunov exponent exists almost surely and is not negative. This is a consequence of ergodicity and follows directly from Kingman's subharmonic argument, see [BL] or [PF] for more details and proofs. Furthermore, again due to the ergodicity, the Lyapunov exponent is a deterministic quantity and does not depend on $\omega$. Therefore from now on, we denote it just by $\gamma^E$. Also the limit for the negative half-axis $\lim_{n \to -\infty} \frac{1}{n} \log \left( \| T^E_\omega(0, -n) \| \right)$ exists and is equal to $\gamma^E$. Note that we nowhere used the fact that $E \in \mathbb{R}$, thus one can also define the Lyapunov exponent for complex energies $z \in \mathbb{C}$, usually specifying the condition $\text{Im}(z) \geq 0$.

The concept of the Lyapunov exponent could be also generalized for multidimensional models, see for example the monography [BL].

The connection between the Lyapunov exponent $\gamma^E$ and the integrated density of states $N^E$ is given by the Thouless formula

$$
\gamma^z = \int \ln |z - E'| dN(E'),
$$

for the discrete case of $\ell^2(\mathbb{Z})$ models, or by its continuous version with $\text{Im}(z_0) > 0$

$$
\gamma^z = \gamma^{z_0} + \int \ln \left| \frac{z - z_0}{z_0 - E'} \right| dN(E'),
$$

The above formulas hold for rather general conditions, see [PF], Theorem 11.8 and the references within.

Regarding the existence of eigenfunctions it is relevant to consider whether the Lyapunov exponent is vanishing or not. Heuristically, if the Lyapunov exponent is positive, there is an initial condition $v$, such that the eigenfunctions of the operator restricted to the positive half-line are exponentially decaying in $n$. This is due to the following theorem.

**Theorem 17. Oseledec-Ruelle** Suppose $A_n \in \text{Sl}(2, \mathbb{R})$ obey

$$
\lim_{n \to \infty} \frac{1}{n} \| A_n \| = 0,
$$

and

$$
\lim_{n \to \infty} \frac{1}{n} \log(\| A_n \cdots A_0 \|) = \gamma > 0.
$$
Then there exists a one-dimensional subspace $V \in \mathbb{R}^2$ such that
\[
\lim_{n \to \infty} \frac{1}{n} \log(\|A_n \cdots A_0 v\|) = -\gamma, \quad \text{for} \quad v \in V \setminus \{0\},
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \log(\|A_n \cdots A_0 v\|) = \gamma \quad \text{for} \quad v \notin V.
\]

In order to find the set of energies where the Lyapunov exponent is nonvanishing, one can use the Fürstenberg theorem. Define first an equivalence relation on $\mathbb{R}^2$. Two non-zero vectors $v, w \in \mathbb{R}^2$ are equivalent if $v = \lambda w$ for some $\lambda \in \mathbb{R}$. We denote the projective line, the set of equivalence classes by $\mathbb{P}^1$.

**Theorem 18. (Fürstenberg)** Let $\nu$ be a probability measure on $\text{Sl}(2, \mathbb{R})$ which satisfies
\[
\int \log \|A\| \, d\nu(A) < \infty.
\]
Denote by $G_\nu$ the smallest closed subgroup of $\text{Sl}(2, \mathbb{R})$ which contains the support of $\nu$. Assume that $G_\nu$ is not compact, and one of the following conditions holds:

- There is no finite non-empty set $L \in \mathbb{P}^1$ such that $T(L) = L$ for all $T \in G_\nu$.
- There no set $L \in \mathbb{P}^1$ of cardinality 1 or 2 such that $T(L) = L$ for all $T \in G_\nu$.

Then, $\gamma > 0$.

In order to consider some concrete models, one can rather use a corollary of the Fürstenberg theorem.

**Theorem 19.** If $G_\nu$ contains at least two elements of $\text{Sl}(2, \mathbb{R})$ with no common eigenvectors, then the conditions of the Fürstenberg theorem are satisfied and the Lyapunov exponent is positive.

This is especially the case if $G_\nu$ contains the elements $A_x = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ and $A_y = \begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix}$ with $x \neq y$. But in this case one can also verify the conditions of the Fürstenberg theorem directly. Indeed $A_x A_y^{-1} = \begin{pmatrix} 1 & x-y \\ 1 & 0 \end{pmatrix} \in G_\nu$ and by calculating the powers of this matrix one sees that $G_\nu$ is not compact. The invariant direction for the powers of $A_x A_y^{-1}$ is the equivalence class of the first vector of the standard basis, however, it is not invariant for the matrix $A_x^{-1} A_y$. We will use matrices of this type in the next chapter.
We closely follow [Sim2], [Sto] and Chapters 13 to 15 of [PF]. There are different possibilities to prove the spectral localization. The theory for one-dimensional discrete models is based on the theory of rank one perturbations. We consider first the discrete operator on the half-line and show that it has almost surely pure point spectrum. The operator on the whole space differs only by a rank one perturbation of the operators on the positive and negative axis glued together.

2.1 Rank one perturbations

We present a very brief sketch of the Aronszajn-Donoghue theory of rank one perturbations of self-adjoint operators. Our main source is [Sim2], but also [DK] was useful. Consider a self-adjoint operator $H$ on a Hilbert space $H$ and suppose $\phi$ is a normalized vector on $H$. We consider the operators $H_v$, which are rank one perturbations of $H$. They are given by

$$H_v = H + v \mid \phi \rangle \langle \phi \mid .$$

The Borel transform of the spectral measure of the perturbed operator with $\phi$ is given by

$$F_v(z) = \langle \phi \mid (H_v - z)^{-1} \phi \rangle = \int \frac{1}{x - z} d\mu_v(x).$$

By the second resolvent equation

$$(H_v - z)^{-1} = (H - z)^{-1} - v(H_v - z)^{-1} | \phi \rangle \langle \phi | (H - z)^{-1},$$

it is connected with the Borel transform $F$ of the spectral measure $\mu$ of the unperturbed operator by

$$F_v(z) = \frac{F(z) + vF(z)}{1 + vF(z)}.$$

Define the function $G(x) = \lim_{t \to 0} \int \frac{1}{(x - y)^2 + t} d\mu_0(y)$. Let us introduce following sets

$$S_1 = \{ x \in \mathbb{R} \mid G(x) < \infty, F(x + i0) = -v^{-1} \},$$

$$S_2 = \{ x \in \mathbb{R} \mid G(x) = \infty, F(x + i0) = -v^{-1} \},$$

$$S_3 = \{ x \in \mathbb{R} \mid 0 < \Im F(x + i0) < \infty \} .$$

Then for the spectrum of the perturbed operator the following holds:

**Theorem 20.** The spectrum of the operator $H_v$ can be characterized as follows

- The pure point part $\sigma_{pp}(H_v)$ of the spectral measure $\mu_{H_v, \phi}^H$ is supported on the set $S_1$ and is given by

$$\mu_{H_v, \phi}^H(\{x\}) = \sum_{y \in S_1} \frac{1}{v^2G(y)} \delta(x - y).$$
The singular continuous part of the spectral measure is supported on the set $S_2 = \sigma_{sc}(H_v)$. 

The absolutely continuous part of the spectral measure is supported on the set $S_3 = \sigma_{ac}(H_v)$ for all $v$.

**Proof.** For a measure that integrates the function $\frac{1}{1+x^2}$ (this is always given for measures constructed from Herglotz functions) holds

$$\Im m F_v(x + i\epsilon) = \int \frac{\epsilon}{(t-x)^2 + \epsilon^2} d\mu^H_v(t) = \frac{1}{\epsilon} \mu^H_v(\{x\}) + \frac{1}{\epsilon} \int_{\mathbb{R} \setminus \{x\}} \frac{1}{(t-x)^2 + \epsilon^2} d\mu^H_v(t).$$

Then the weight of $\mu^{H_v}_{pp,\phi}$ at the point $x$ is given by the following limit:

$$\mu^{H_v}_{pp,\phi}(\{x\}) = \lim_{\epsilon \downarrow 0} \epsilon \Im m F_v(x + i\epsilon).$$

The imaginary part of $F_v$ in $z$ is given by $\frac{\Im m F_v(z)}{|1+vF_v(z)|^2}$. Note, that if $G(x) < \infty$ holds, by dominated convergence theorem one has

$$F(x + i\epsilon) = F(x + i0) + i\epsilon G(x) + i\epsilon o(x,\epsilon).$$

Since $v\Re F(x) = -1$, one has

$$\lim_{\epsilon \downarrow 0} \epsilon \Im m F_v(x + i\epsilon) = \frac{1}{v^2 G(x)}.$$

On the other hand, if $G(x)$ is infinite, then there is a sequence $\epsilon_n$, such that $\Im m F(x + i\epsilon_n) \uparrow \infty$. Thus $\lim inf_{\epsilon_n \downarrow 0} \epsilon_n \Im m F_v(x + i\epsilon_n) = 0$, so there is no atom at $x$.

The singular part is supported on the set $\Im m F_v(x + i0) = \infty$, but not on the support of $\mu^{H_v}_{pp,\phi}(\{x\})$. Hence it is supported in $S_2$.

Since $F_v(z) = \frac{F(z)}{1+vF(z)}$ holds, one can follow that $0 < \Im m F_v(x + i0) < \infty$ whenever $0 < \Im m F(x + i0) < \infty$. Both are supports of the absolutely continuous parts of the spectra. More details could be found in Theorem 3.1.3 of [DK] or Theorem II.2 in [Sim2].

A very useful criterion to exhibit the point spectrum of random operators is given by spectral averaging. One can show that the averaged spectral measure $\int \mu^H_v dv$ is just the Lebesgue measure, see for example Proposition 3.1.4 in [DK] or Theorem I.8 in [Sim2].

**Theorem 21.** Let $\mu^H_v$, $v \in \mathbb{R}$ be the family of spectral measures, associated with the operators $H_v$, then for every integrable function $f$ holds

$$\int \left( \int f(x) d\mu^H_v(x) \right) dv = \int f(x) dx.$$
Proof. For any \( v \), \( \mu_{\phi}^{H_v} \) is the spectral measure of \( H_v \), associated with \( \phi \) and one can see, that

\[
\int dv \int d\mu_{\phi}^{H_v}(x) \frac{1}{1 + x^2} = \int dv \int d\mu_{\phi}^{H_v}(x) \frac{1}{2v} \left( \frac{1}{x - i} - \frac{1}{x + i} \right)
\]

\[
= \int dv \frac{1}{2v} \left( \frac{1}{v + F(i)^{-1}} - \frac{1}{v + F(-i)^{-1}} \right)
\]

\[
= \int dv \frac{F(i)^{-1} - F(-i)^{-1}}{2v} \frac{1}{|v + F(i)^{-1}|^2} < \infty
\]

Note that \( F(i) = F(-i) \) holds. Thus, the above measure can integrate the function \( \frac{1}{1 + x^2} \), so its Borel transform is defined and unique. Hence it is enough to show, that its Borel transform and the Borel transform of the Lebesgue measure are equal up to a real number. We will just subtract the function \( \frac{1}{x+i} \) for regularization purposes.

By contour integration one obtains for the Lebesgue measure

\[
\int dx \left( \frac{1}{x - z} - \frac{1}{x + i} \right) = 2\pi i, \quad \text{for} \quad \Im m(z) > 0.
\]

As for the above defined averaged spectral measure one has first

\[
\int d\mu_{\phi}^{H_v}(x) \left( \frac{1}{x - z} - \frac{1}{x + i} \right) = F_v(z) - F_v(-i) = \frac{1}{v + (F(z))^{-1}} - \frac{1}{v + (F(-i))^{-1}}
\]

The above function has for \( \Im m(z) > 0 \) one pole in the upper half plane, since \( F \) and consequently \( -F^{-1} \) maps an upper half plane into itself. Thus the contour integration gives again

\[
\int d\mu_{\phi}^{H_v}(x) \left( \frac{1}{x - z} - \frac{1}{x + i} \right) = 2\pi i, \quad \text{for} \quad \Im m(z) > 0.
\]

Since the finite linear combinations of the functions \( \frac{1}{x-z} - \frac{1}{x+i} \) are dense, the statement follows.

From this result one can investigate the connection between the spectral measure and the Lebesgue measure, see Corollary 3.1.6 in [DK].

**Proposition 22.** For any fixed set \( B \) of Lebesgue measure zero one also has \( \mu_{\phi}^{H_v}(B) = 0 \) for almost every \( v \). Also for almost all pairs \((v, v')\) with respect to the Lebesgue measure, their singular parts \( \mu_{\phi,s}^{H_v} \) and \( \mu_{\phi,s}^{H_{v'}} \) are mutually singular.

**Proof** By the spectral averaging and the fact that \( B \) has no measure with respect to the Lebesgue measure we obtain using Fubini

\[
0 = \int_B dx = \int \int_B d\mu_{\phi}^{H_v} \ dv \geq \int_a^b \mu_{\phi}^{H_v}(B) \ dv \geq 0,
\]

for any \( a, b \in \mathbb{R} \). This proves the first statement. The second part follows from the matter that both \( \mu_{\phi,s}^{H_v} \) and \( \mu_{\phi,s}^{H_{v'}} \) are singular with respect to the Lebesgue measure.
This means, that they are supported on sets of zero Lebesgue measure. The existence of a fixed zero measure set $A$ such that $\{v \mid \mu_{H_v}^\phi(A) > 0\}$ has positive Lebesgue measure stands in contradiction to the previous theorem.

A very important fact is that any fixed set $S$ of zero Lebesgue measure cannot be given non-zero measure by the measures $\mu_{H_v}^\phi$ for a set of $v$ having positive Lebesgue measure. Together with the classification of the measures associated to the rank one perturbation one obtains a theorem of Simon-Wolff, see Theorem 3.1.7 in [DK], Theorem II.5 in [Sim2] and references therein.

**Theorem 23.** Suppose that for every $v$, $\mu_{H_v}^\phi([a, b]) \neq 0$. Then the following statements are equivalent:

- For almost all $v$, $\mu_{H_v}^\phi$ is pure point in $[a, b]$.
- For almost every $x$ in $[a, b]$ holds $G(x) < \infty$.

**Proof.** First note that when $G(x) < \infty$, the imaginary part of $F(x + i0)$ is zero, and also the imaginary part of $F_v(x + i0)$, since $\Im F_v(x + i0) = \frac{\Im m F(x + i0)}{1 + v F(x + i0)} = 0$. Therefore from $G(x) < \infty$ for almost every $x$ follows, that the absolutely continuous parts of $\mu_{H_v}^\phi$ give measure zero for every $v$ by the above given classification. For the set $S = \{x \in [a, b] \mid G(x) = \infty\}$ of Lebesgue measure zero one has by the spectral averaging

$$\int \mu_{H_v}^\phi(S) \, dv = \int_S dx = 0$$

Which implies $\mu_{H_v}^\phi(S) = 0$ for almost every $v$. Since the singular continuous part of $\mu_{H_v}^\phi$ is supported on a subset of $S \cap [a, b]$, then $\mu_{H_v}^{H_v,sc} = 0$ for almost every $v$, thus $\mu_{H_v}^{H_v}$ is almost surely pure point.

On the other hand, if the measure has only point masses in $[a, b]$ for almost all $v$, then almost everywhere $F(x + i0)$ is real for $x \in [a, b]$, therefore $\mu_{H_v}^{H_v}$ is supported in the complement of $S$ for almost every $v$. Again, by the spectral averaging one follows that $S$ is a zero measure set with respect to the Lebesgue measure.

An important criterion for a concrete one-dimensional case gives the following theorem. This is theorem II.3 in [Sim2].

**Theorem 24.** For a Schrödinger operator $H$ on a positive half-line with Neumann boundary conditions at zero, $G(x) < \infty$ holds if and only if both of the following hold

- The point $x$ is not an eigenvalue of the operator.
- The differential eigenvalue equation $H \psi = E \psi$ has an $L^2$ solution at $\infty$.

There is also a counterpart of the above theorem for the discrete case, Theorem II.4 in [Sim2].
Theorem 25. For a Schrödinger operator $h$ on $\ell^2(\mathbb{Z})$, $G(x) < \infty$ holds if and only if

- The point $x$ is not an eigenvalue of the operator.
- One of the following holds:
  - The difference eigenvalue equation $h\psi = E\psi$ has an $\ell^2$ solution on $(0, \infty)$ with $\psi(0) = 0$.
  - The above equation has an $\ell^2$ solution on $(-\infty, 0)$ with $\psi(0) = 0$
  - The above equation has $\ell^2$ solutions $\psi_{\pm}$ on both $(-\infty, 0)$ and $(0, \infty)$ with $\psi_+(0) \neq 0$ and $\psi_-(0) \neq 0$.

2.2 Spectral localization for some models

Using the positivity of the Lyapunov exponent together with the theory of rank one perturbations, one is able to prove the spectral localization for a big variety of random one-dimensional models. We just give some of them, but more details could be found in [Sim2]. The crucial point in our consideration is that the dependence on the randomness is "smooth", that means, that the distribution of the site potential has a density with respect to the Lebesgue measure. There are also some proofs for localization for models with singular potentials, but they are more complicated on the technical level. For more details on this non-smooth or singular models one can refer to the review article [Sto] and further articles cited there.

The most famous one-dimensional example is the Anderson model with an operator $H_A$ on $L^2(\mathbb{R})$

$$H_A = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} v_n f(x - n),$$

as well as its discrete analogue the discrete one-dimensional Anderson model with an operator $h_A$ on $\ell^2(\mathbb{Z})$ given by

$$(h_A\psi)_n = -\psi_{n+1} - \psi_{n-1} + v_n \psi_n.$$

One can assume the site potentials $(v_n)_{n \in \mathbb{Z}}$ to be i.i.d random variables distributed on a compact set with a continuous distribution function. The single site potential function $f \neq 0$ is non-negative, bounded and is supported on a subset of $[0, 1]$. These restrictions are rather artificial and could be relaxed.

With the above given techniques it is also possible to prove the existence of the almost sure pure point spectrum for the random operator formally given by

$$H_\omega = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} v_n \delta_n.$$
2 Strategies to prove the localization

where \( \omega = (v_n)_{n \in \mathbb{Z}} \) are i.i.d. random variables with non-degenerate compactly supported distribution. More detailed discussion of the physical meaning and mathematical properties of this operator is given in the next chapter, which closely follows our article [DKS].

One should also mention, that one can consider this model as a special case of the following model with external electrical force \( F \geq 0 \) studied in [DSS].

\[
H^F_\omega = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} v_n \delta_n - Fx.
\]

The assumptions made in [DSS] are slightly different, namely the distribution of the random variables \( v_n \) is assumed to be centered at zero and have bounded non-negative moments. With very extensive techniques it was possible to proof the spectral localization for \( H^F_\omega \) in [DSS]. We cite the Theorem I.3.

**Theorem 26.** If \( F = 0 \), the spectrum of \( H^F_\omega \) and the eigenfunctions are exponentially decaying.

The proof of this statement will be sketched on the following pages. We use the fact that the theory has evolved over the time.

Independently of the distribution of the random variables for a fixed realization \( \omega \), using the Fourier transform one can write the formal solution of the eigenvalue equation \( H_\omega \psi_\omega = E \psi_\omega \) for a real energy \( E \in \mathbb{R} \) in the spectrum by

\[
\psi_\omega = -\frac{i}{2\sqrt{E}} \sum_{n \in \mathbb{Z}} v_n \psi_\omega(n) \exp(i\sqrt{E}|x-n|).
\]

Now this equation can be solved for \( \psi_n = \psi_\omega(n) \), which is

\[
(1 + \frac{i}{2v_n\sqrt{E}}) = -\frac{i}{2\sqrt{E}} \sum_{m \neq n} v_n \psi_\omega(n) \exp(i\sqrt{E}|n-m|).
\]

By some simple calculations one obtains a finite difference equation for our problem

\[-\psi_{n+1} - \psi_{n-1} + \frac{v_n}{\sqrt{E}} \sin(\sqrt{E})\psi_n = -2\cos(\sqrt{E})\psi_n.
\]

There is a one-to-one correspondence between the eigenfunction \( \psi_\omega \) and the solutions of the above equation, in the sense, that if one constrains the above problems on a fixed interval \([a, b]\) or a half-line \([a, \infty)\) with appropriate boundary conditions, the discrete and continuous solutions correspond. Indeed for almost every energy \( E \), on every interval \((n, n+1]\) one can calculate the solution \( \psi_\omega \) with given boundary values \( \psi(n) \) and \( \psi'(n) \). These are given by

\[
\psi'(n) = \frac{\sqrt{E}}{\sin(\sqrt{E})}\psi_{n+1} - \frac{\sqrt{E}}{\tan(\sqrt{E})}\psi_n, \quad \psi(n) = \psi_n.
\]

Thus we have reduced our model to a discrete problem. Note that the \( L^2 \)-norm of \( \psi_\omega \) is bounded by \( \ell^2 \)-norm of the discrete solution and vice versa.
The transfer matrix for this difference equation is given by
\[ T_{E,K,P}^{\omega,n} = \begin{pmatrix} 2 \cos(\sqrt{E}) + \frac{v_n}{\sqrt{E}} \sin(\sqrt{E}) & -1 \\ 1 & 0 \end{pmatrix}, \]
and for the discrete Anderson model \( h_A \) is
\[ T_{E,A}^{\omega,n} = \begin{pmatrix} v_n -1 \\ 1 & 0 \end{pmatrix}. \]

These reformulations can be used to obtain concrete results for the random discrete Anderson model and the random Kronig-Penney model. The latter one is due to Ishii. If the site potentials \( v_n \) are bounded i.i.d. random variables, the set of energies, where the Lyapunov exponent vanishes is known explicitly. We cite Theorem 14.3 of [PF].

**Theorem 27. (Ishii)** The Lyapunov exponent \( \gamma^E \) of the one-dimensional random Kronig-Penney model vanishes only at points with critical energies \( E_n = (\pi n)^2 \), \( n \in \mathbb{Z} \).

This is just a consequence of the Fürstenberg theorem for transfer matrices of the above special type. The Lyapunov exponent of the discrete one-dimensional random Anderson model is everywhere positive by the same argument.

For a subset \( S \subset \mathbb{R} \), the essential closure of \( S \) is given by
\[ \overline{S}^{ess} = \left\{ E \in \mathbb{R} \left| \int_{S \cap (E-\epsilon,E+\epsilon)} dx > 0 \text{ for every } \epsilon > 0 \right. \right\}. \]

A general result of Kotani theory is, see for example the review article [D], that the almost sure absolutely continuous spectrum is given by the essential closure of the set of energies for which the Lyapunov exponent vanishes. Thus, the absolutely continuous spectrum of the operators \( H_A \), \( h_A \) and \( H_\omega \) is almost surely empty.

We briefly sketch the proof that also the singular continuous spectrum of the corresponding models is almost surely empty. For more details see Theorem 13.4, Theorem 15.8 and Example 15.9 b in [PF], Theorem II.6 and II.7 in [Sim2] and Theorem 3.3 in [Dam].

**Theorem 28.** The operators \( h_A \) and \( H_\omega \) defined above have pure point spectrum for almost every realization \( \omega \).

**Proof.** By the above cited Fürstenberg Theorem, the Oseledec-Ruelle Theorem, and Fubini, we have that the Lebesgue measure of the set \( \mathbb{R} \setminus A_\omega \) is 0 for almost every \( \omega \), where
\[ A_\omega = \left\{ E \in \mathbb{R} : \gamma^E > 0, \text{there are eigenfunctions } \psi_{n,\pm} \sim exp^{-|n|^\gamma} \text{ as } n \to \pm \infty \right\}. \]
By the existence of eigenfunctions we mean that there are boundary conditions, such that the solutions of the eigenvalue equation on \((0, +\infty)\) and \((-\infty, 0)\) decay
exponentially. The above sets $A_s$ are invariant with respect to a modification of the potential on a finite set. Let us perform such a modification on the set $\{0, 1\}$. Let us mention that the pair $\delta_0$ and $\delta_1$ is cyclic for each operator $h_A$ as well as for the discrete version of $H_\omega$. Denote the "good" set of $\omega$’s for which the Lebesgue measure of $\mathbb{R} \setminus A_\omega$ is 0 by $\Omega_0$. As we know, $\Omega_0$ has full measure.

On this set for $\omega \in \Omega_0$, consider the perturbed operators

$$h_{A,v_0,v_1} = H_A + v_0|0\rangle\langle 0| + v_1|1\rangle\langle 1|.$$  

and $H_{\omega,v_0,v_1}$ analogously defined, with $v_0, v_1 \in \mathbb{R}$. For every fixed $v_0$, it follows from the spectral averaging and the above argumentation that the spectral measure of the operator $h_{A,v_0,v_1}$ and the vector $\delta_1$ gives zero weight to the set $\mathbb{R} \setminus A_\omega$ for Lebesgue almost every $v_1$. A similar situation occurs if one fixes $v_1$ instead. As a consequence, for Lebesgue almost every pair $(v_0, v_1)$ the sum of the spectral measures of $\delta_0$ and $\delta_1$ gives no contribution to the set $\mathbb{R} \setminus A_\omega$. Write $B_\omega$ for this set of pairs $(v_0, v_1)$, so that the Lebesgue measure of $\mathbb{R}^2 \setminus B_\omega$ is zero. Take the set $\Omega_1$ by adding good pairs to $\Omega_0$.

$$\Omega_1 = \{ \omega + v_0\delta_0 + v_1\delta_1 \mid \omega \in \Omega_1, (v_0, v_1) \in B_\omega \}$$

Since the distribution of the random variables is absolutely continuous with respect to the Lebesgue measure, it follows that $\Omega_1$ has positive measure.

Thus, by assumption on the distribution of the single site potential, with positive probability, it follows that the whole-line spectral measure (corresponding to the sum of the $\delta_0$ and $\delta_1$ spectral measures, see the equation (12.35) in [PF] and the discussion above) assigns no weight to $\mathbb{R} \setminus A_\omega$ and hence, with positive probability, the operator $h_A$ has pure point spectrum by rank one perturbation theory and subordinacy theory, see [Sim2] and [GP1].

Since localization is a shift-invariant event, the operators $h_A$ and $H_\omega$ must in fact have pure point spectrum for almost every realization $\omega$.

A reasonable question is the justification that we are actually dealing with rank one perturbations even for the operator $H_\omega$. While for the discrete case $h_A$, it is quite obvious, that the projections on the first sites $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ are rank one perturbations in the usual sense, as well as $|\phi\rangle\langle \phi|$ for a bounded function $\phi \in L^2(\mathbb{R})$ with $\|\phi\| = 1$, it is not evident for $\delta_0$ and $\delta_1$, since they are not even elements of the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Nevertheless one has deficiency indices $(2, 2)$ for $\delta_0$ and $\delta_1$ hence changing the value of $v_0$ and $v_1$ is a finite rank perturbation. \(\Box\)

A natural framework for the above question is a chain of Hilbert spaces $\mathcal{H}_s$. For an (unbounded) operator $H \geq 0$ on the Hilbert space $\mathcal{H}$ one can define for $s \geq 0$ a new Hilbert space $\mathcal{H}_s = D(|H|^{s/2})$ with the norm given by

$$\|\psi\|_s^2 = \langle \psi | (H + 1)^s \psi \rangle.$$  

As for $s < 0$, $\mathcal{H}_s(H)$ is the completion of $\mathcal{H}$ in the $\| \cdot \|_s$ norm. One can also construct it using the duality $\mathcal{H}_s = \mathcal{H}_{-s}$ in a natural way. For a $\phi \in \mathcal{H}_{-1}$ the functional $\psi \mapsto |\langle \psi, \phi \rangle|^2$ defines a quadratic form on $\mathcal{H}_1$, so by the standard form
perturbation theory, the operator $H_v = H + v\langle \phi \mid \cdot \rangle \phi$ is well defined and for its Borel transform $F_v$ and $z \in \mathbb{C} \setminus \mathbb{R}$ holds the relation $F_v(z) = \frac{F(z)}{1 + vz}$. Since both $\delta_0$ and $\delta_1$ are elements of $\mathcal{H}_{-1}$ for $H = -\frac{d^2}{dx^2}$ and $\mathcal{H} = L^2(\mathbb{R})$, the argumentation for rank one perturbations is still applicable for them, see [Sim2].
3 Kronig-Penney model

This chapter consists of the work done in the publication [DKS]. We study the Kronig-Penney model with random Dirac potentials on the lattice \( \mathbb{Z} \). It is known, that this model has critical energies at which the Lyapunov exponent vanishes and the density of states has a van Hove singularity. This leads to a non-trivial quantum diffusion even though the spectrum is known to be pure-point.

3.1 Main results

The Kronig-Penney model describes the motion of a particle in a one-dimensional system with singular potentials. We consider the case of a random \( \delta \)-potential on the points of the lattice \( \mathbb{Z} \). The Hamiltonian is given by

\[
H_\omega = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} v_n \delta_n, \tag{4}
\]

where \( \omega = (v_n)_{n \in \mathbb{Z}} \) are i.i.d. random variables. We assume that the distribution of the \( v_n \)'s has compact support and is nondegenerate, i.e. is not concentrated in a single point. The precise mathematical meaning of the \( \delta \)-potential in (4) is recalled in Section 3.2 below. If the \( v_n \)'s all have the same sign, the almost sure spectrum of \( H_\omega \) consists of an infinite number of bands, while the spectrum contains all positive reals if 0 is in the support of the distribution of the \( v_n \)'s [KM] (see Section 3.2 for details). Under rather general assumptions it is known that the spectrum of one-dimensional random Schrödinger operators is pure point and that the eigenfunctions are exponentially localized. We are not aware of any publication in which this is actually proved for model (4) with \( v_n \)'s distributed on an interval in \( \mathbb{R}^+ \), but the general methods [Sto] can be applied for this model (this is briefly sketched in the previous chapter, see, however, [HKK] where localization at the bottom of the spectrum is proved). For a model with a centered distribution of the random variables the pure point spectrum is proved in [DSS]. The spectrum of that operator however does not have a band structure and is given by \( \mathbb{R} \).

The main focus here is on the transport properties of the random Kronig-Penney Hamiltonian. We analyze the growth of the time-averaged \( q \)-th moment of the position operator \( X \) on \( L^2(\mathbb{R}) \):

\[
M_q(T) = \frac{2}{T} \int_0^\infty dt \ e^{-\frac{2t}{T}} e^{iH_\omega t} |X|^q e^{-iH_\omega t}, \quad q > 0. \tag{5}
\]

This operator is an integral operator and its integral kernel is denoted by \( \langle x|M_q(T)|y \rangle \). The diagonal entries \( \langle a|M_q(T)|a \rangle \) can be interpreted as the time-averaged moments of a wave packet initially localized in a Dirac state at \( a \in \mathbb{R} \) (which, of course, is not an element of Hilbert space). The following result shows that for sufficiently large \( q \) these moments grow with time \( T \), which is certainly not the typical behavior inside the localization regime.
Theorem 29. Suppose that $E(v) \neq 0$ and let $a \in \mathbb{R} \setminus \mathbb{Z}$. For every $\alpha > 0$ there is a positive constant $C_\alpha$ such that for $T > 1$
\[
E(a|M_q(T)|a) \geq C_\alpha T^{q(\frac{3}{2} - \frac{5}{3q}) - \alpha}, \quad q > \frac{5}{2}.
\] (6)

The heuristics leading to a non-trivial lower bound (6) on the wavepacket spreading is as follows. We will show that the localization length (given by the inverse of the Lyapunov exponent) diverges at certain critical energies $E_l$. At the same time, the density of states diverges at these energies so that there are many such states (see Theorem 30 below). The quantum motion in these states is more or less ballistic until it reaches the localization length. This decreasing fraction of delocalized states allows to prove the lower bound. Both these heuristics and the implementation of the idea in a proof are quite similar to the random polymer model [JSS]. There is, however, a crucial difference: the transfer matrices at the critical energies $E_l$ are all Jordan blocks in the random Kronig-Penney model, while they are random rotations at the critical energies of a random polymer model. This leads to a completely different behavior of the Lyapunov exponent and ultimately also for the moments in (6). Indeed, in the random polymer model the behavior is $\langle a|M_q(T)|a \rangle \sim T^{q \frac{3}{4}}$ with lower and upper bounds proved in [JSS] and [JS] respectively. Let us stress though that we do not claim (nor expect) that the lower bound (6) is optimal. Despite the fact that the lower bound is given for the expectation value of the integral kernel, it is valid for the random integral kernels on a set of configurations $\Omega(T, \alpha)^c$ explicitly defined below. The measure of $\Omega(T, \alpha)^c$ can be chosen for big $T$ close to one.

The proof and above heuristics are based on the vanishing of the Lyapunov exponent $\gamma^E$ at the critical energies $E_l = (\pi l)^2$, $l \in \mathbb{N}$, which have been known at least since the work of Ishii [Ish]. The Lyapunov exponent is defined in terms of the fundamental solution $T^E_\omega(x,y) \in \text{Sl}(2,\mathbb{R})$, $x,y \in \mathbb{R}$, of the first order linear equation in $\mathbb{R}^2$ associated to (4), by the formula
\[
\gamma^E = \lim_{N \to \infty} \frac{1}{N} \log(\|T^E_\omega(N,0) e\|),
\] (7)
where $e$ is an arbitrary unit vector in $\mathbb{R}^2$. The convergence is known to be almost sure [BL]. The next theorem summarizes our main results on the behavior of $\gamma^E$ as well as the integrated density of states $\mathcal{N}^E$ in the vicinity of the critical energies. The formal definition of $\mathcal{N}^E$ is recalled in Section 3.3.

Theorem 30. Let $E(v) \neq 0$. With the positive constants
\[
D_- = \frac{E(v^2 - E(v)^2)}{16 E(v) E_l}, \quad D_+ = \left(\frac{E(v)}{2E_l}\right)^\frac{1}{2},
\]
one has for $\varepsilon \geq 0$
\[
\gamma^{E_l - \varepsilon} = D_- \varepsilon + \mathcal{O}(\varepsilon^\frac{3}{2}), \quad \gamma^{E_l + \varepsilon} = D_+ \varepsilon^\frac{3}{2} + \mathcal{O}(\varepsilon^3),
\] (8)
and
\[
\mathcal{N}^{E_l - \varepsilon} = l - \frac{1}{\pi} D_+ \varepsilon^\frac{3}{2} + \mathcal{O}(\varepsilon).
\] (9)
The proof of these formulas is based on perturbation theory for products of random matrices around random Jordan blocks, and uses the techniques of [SS]. It is well-known that $-\gamma E + i\pi N^E$ are the boundary value of a Herglotz function $w^z$ which in terms of the Weyl-Titchmarch function $m^{E,\pm}_w$ defined below is given by $w^z = \mathbf{E}(m^{E,\pm}_w)$. It can be read off Theorem 30 that

$$
\mathbf{E}(m^{E,\pm}_{E_l+z}) = i\pi l - D_+ z^\frac{1}{2} + (D_- + iB) z + \mathcal{O}(z^\frac{3}{2}) ,
$$

where $B$ is some real constant. Then $\mathcal{N}^{E_l+\epsilon} = l + \frac{1}{\pi} B \epsilon + \mathcal{O}(\epsilon^\frac{3}{2})$. This constant $B$ vanishes if the support of the distribution of the $v_n$ is strictly positive because in this case $E_l$ is an upper band edge. We do not calculate $B$ below, but believe that it is in principle possible by the techniques presented below (but with considerable effort because the Fokker-Planck operators on $S^1$ used in [SS] become singular).

Formula (9) shows that the density of states conserves a one-dimensional van Hove singularity at $E_l$ (even though the model is random). This is due to the fact that the random potential cannot move any eigenvalue from below $E_l$ to above $E_l$ and vice versa [KN] (this follows from a basic Sturm-Liouville argument, see Section 3.3 for details). The existence of such singularities was already proved in [KN], but the expansion in (9) is new, as are the results on the Lyapunov exponent. For the above random Kronig-Penney model with positive potentials Gredeskul and Pastur [GP] have analyzed the density of states at the lower band edges (this is a Lifshitz tail type regime), see also [KN].

### 3.2 Basic analytical set-up

#### 3.2.1 Definition of the operator

Let us begin by considering $H_0 = -\partial^2$ as a symmetric operator on the Sobolev space $H^2_0(\mathbb{R} \setminus \mathbb{Z})$, namely $\psi \in L^2(\mathbb{R})$ with $\psi', \psi'' \in L^2(\mathbb{R})$ and such that $\psi(n) = \psi'(n) = 0$ for all $n \in \mathbb{Z}$. The deficiency spaces $\ker(H_0^\pm + i)$ are infinite dimensional and can be explicitly calculated. These deficiency spaces can be seen as direct sum of 2-dimensional subspaces attached to each point $n \in \mathbb{Z}$ (that is, the local deficiency indices are $(2,2)$ at each $n$). We are only interested in self-adjoint extensions of $H_0$ that are local in the sense that they only link the left and right limits at one point $n \in \mathbb{Z}$:

$$
\psi(n_\pm) = \lim_{\epsilon \downarrow 0} \psi(n \pm \epsilon), \quad \psi'(n_\pm) = \lim_{\epsilon \downarrow 0} \psi'(n \pm \epsilon) .
$$

The self-adjoint extension corresponding to the $\delta$-potentials in (4) is given by

$$
\begin{pmatrix}
\psi'(n_+) \\
\psi(n_+)
\end{pmatrix} =
\begin{pmatrix}
1 & v_n \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\psi'(n_-) \\
\psi(n_-)
\end{pmatrix}, \quad n \in \mathbb{Z} .
$$

This means that the domain $\mathcal{D}(H_\omega)$ is given by all functions in $H^2(\mathbb{R} \setminus \mathbb{Z})$ satisfying (10). This gives a precise meaning to the operator in (4). If one is interested in
δ'-interactions of the form \( \sum_n w_n \delta'_n \), then the matrix \( \begin{pmatrix} 1 & v_n \\ 0 & 1 \end{pmatrix} \) in (10) is replaced by \( \begin{pmatrix} 1 & 0 \\ w_n & 1 \end{pmatrix} \). The main features described in Theorems 29 and 30 are also valid in this case. For sake of concreteness, we stick to the model (4). Let us point out that it is also possible to consider mixed δ and δ' potentials, but then Theorems 29 and 30 do not hold (because the matrices describing the boundary conditions are not Jordan blocks in the same basis, which is an essential element in all arguments below).

### 3.2.2 Fundamental solutions

Now we are interested in formal solutions \( \psi \) of the Schrödinger equation \( H_\omega \psi = z \psi \) at a complex energy \( z \) (possibly not square integrable) which satisfy the boundary conditions (10). As this is a second order differential equation, it is as usual helpful to reformulate it as a system of first order equation for vector valued functions

\[
\Psi = \begin{pmatrix} \psi' \\ \psi \end{pmatrix} \in H^1(\mathbb{R}, \mathbb{C}^2), \quad \Psi(n+) = \begin{pmatrix} 1 & v_n \\ 0 & 1 \end{pmatrix} \Psi(n-) ,
\]

given by

\[
-\mathcal{J} \partial \Psi = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \Psi, \quad \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .
\]

Conversely, solving (12) with the boundary conditions (11) gives a solution \( \Psi \) the second component of which solves the Schrödinger equation \( H_\omega \psi = z \psi \). These solutions define the transfer matrices, also called fundamental solutions, by the equation

\[
\Psi(x) = T^{z}(x, y) \Psi(y) ,
\]

as well as the requirement to be right continuous in \( x \) and \( y \), namely \( T^{z}(x+, y+) = T^{z}(x, y) \). Combined with (11), this implies, in particular,

\[
T^{z}(n, y) = \begin{pmatrix} 1 & v_n \\ 0 & 1 \end{pmatrix} T^{z}(n-, y) , \quad y < n .
\]

Furthermore, we set for \( x < y \),

\[
T^{z}(x, y) = \left( T^{z}(y, x) \right)^{-1}.
\]

Then the concatenation identity holds for all \( x, y, t \in \mathbb{R} \):

\[
T^{z}(y, x) = T^{z}(y, t) T^{z}(t, x) ,
\]

Next let us calculate the transfer matrices on an interval in \( \mathbb{R} \setminus \mathbb{Z} \), for example \([y, x]\), by solving (12):

\[
T^{z}(x, y) = \exp \left( (x - y) \begin{pmatrix} 0 & -z \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} \cos(z^{\frac{1}{2}}(x - y)) & -z^{\frac{1}{2}} \sin(z^{\frac{1}{2}}(x - y)) \\ z^{-\frac{1}{2}} \sin(z^{\frac{1}{2}}(x - y)) & \cos(z^{\frac{1}{2}}(x - y)) \end{pmatrix} .
\]
Let us note that the second expression is an even function of \( z_1 \) so that the choice of square root is irrelevant. For sake of concreteness, we will always choose the principal value though. As \( T_\omega^z(x, y) \) is the exponential of a matrix with vanishing trace, it has unit determinant, namely \( T_\omega^z(x, y) \in \text{Sl}(2, \mathbb{C}) \). For real energies \( E \in \mathbb{R} \), one, moreover, has \( T_\omega^E(x, y) \in \text{Sl}(2, \mathbb{R}) \). It is useful to introduce the notations

\[
T_n^z = T_\omega^z(n, n - 1) .
\]

According to the above

\[
T_n^z = \begin{pmatrix} 1 & v_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(z_1^2) & -z_1^2 \sin(z_1^2) \\ z_1^2 \sin(z_1^2) & \cos(z_1^2) \end{pmatrix} . \tag{15}
\]

Therefore, at \( E_l = (\pi l)^2 \),

\[
T_n^{E_l} = (-1)^l \begin{pmatrix} 1 & v_n \\ 0 & 1 \end{pmatrix} . \tag{16}
\]

### 3.2.3 Basic spectral analysis

Here we restrict our attention first to the non-random case of the periodic Hamiltonian \( H_v \) defined by setting \( v_n = v > 0 \). As usual, the spectrum can be read off the transfer matrices \( T^E = T_n^E \) at real energy \( E = k^2 \in \mathbb{R} \), namely \( E \in \sigma(H_v) \) if and only if \( |\text{Tr}(T^E)| \leq 2 \). The band edges are therefore given by the solutions of the equation

\[
|\text{Tr}(T^E)| = \left| 2 \cos(k) + \frac{v}{k} \sin(k) \right| = 2 .
\]

Thus there exist positive constants \( c_l(v) \in \mathbb{R}_+ \) such that

\[
\sigma(H_v) = \bigcup_{l \in \mathbb{N}} [E_l - c_l(v), E_l] , \quad E_l = (\pi l)^2
\]

In particular, the right band edges are independent of \( v \). Thus they are also band edges of the random operator \( H_\omega \) if the distribution of the \( v_n \)'s has positive support. Moreover, at these energies the transfer matrices for any periodic approximant are given by

\[
T_\omega^{(\pi)^2}(n, m) = (-1)^{(n-m-1)} \begin{pmatrix} 1 & \sum_{j=m+1}^n v_j \\ 0 & 1 \end{pmatrix} , \tag{17}
\]

and therefore have a trace of modulus 2. A little bit more can be said (see [KM] and [GHK]).

**Proposition 31.** Consider the Kronig-Penney model with positive independent random potentials distributed identically according the probability distribution \( p \) with compact support in \( \mathbb{R}_+ \). Let \( v = \inf \text{ supp}(p) \). Then the almost-sure spectrum satisfies

\[
\sigma(H_\omega) = \sigma(H_v) .
\]
Proof. This follows from a standard Weyl sequence argument using operators with almost constant potential almost equal to $v$. 

The following variant is proved similarly.

**Proposition 32.** Consider the Kronig-Penney model with independent random potentials distributed identically according the probability distribution $p$ with compact support containing 0. Then the almost-sure spectrum satisfies

$$[0, \infty) \subset \sigma(H_\omega).$$

### 3.2.4 Estimates on the transfer matrices

Here the focus is on deriving an estimate on the transfer matrices at complex energies $z = E + \kappa, \kappa \in \mathbb{C}$, in terms of estimates at real energies $E > 0$. First of all, by analyticity one has

$$\mathcal{T}_n^z = \mathcal{T}^E_n + \kappa R_n,$$

for some matrix $R_n$ which is uniformly bounded in $n$. Using the concatenation relation (13) one deduces

$$\mathcal{T}^z(n, m) = \mathcal{T}^E(n, m) + \kappa \sum_{j=m+1}^{n} \mathcal{T}^z(n, j) R_j \mathcal{T}^E(j - 1, m).$$

Taking the norm in the above equation, estimating the r.h.s. and taking the supremum over $0 \leq m \leq n \leq N$ leads to the following lemma [Sim1].

**Lemma 33.** Set

$$c_1 = \sup_{0 \leq m \leq n \leq N} \|\mathcal{T}^E(n, m)\|, \quad c_2 = \sup_n \|R_n\|.$$

Then, as long as $|\kappa| c_1 c_2 N < 1$,

$$\sup_{0 \leq m \leq n \leq N} \|\mathcal{T}^{E+\kappa}(n, m)\| \leq \frac{c_1}{1 - |\kappa| c_1 c_2 N}.$$

### 3.2.5 Green functions and Weyl theory

At some instances below formulas connecting the transfer matrices to the Green functions will be used. Such a link is established by Weyl theory which is reviewed, e.g., in [KS1]. For this purpose, one cuts $H_\omega$ into two half-sided operators. This cut is usually done at the orgin 0, but as the model (4) has a singular potential there, it is more convenient to cut at some other point $a \in (0, 1)$. Then it is a basic fact that for every complex energy $z$ with $\Im m(z) > 0$ there are two unique $f^z_{\omega, \pm} \in L^2(\mathbb{R}_\pm)$ with $f^z_{\omega, \pm}(a) = 1$ and solving $H_\omega f^z_{\omega, \pm} = z f^z_{\omega, \pm}$ on $\mathbb{R}_\pm$. The $m$-functions are then defined by

$$m^z_{\omega, \pm} = \left(\partial_x f^z_{\omega, \pm}\right)(a).$$

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In terms of the transfer matrices, one has
\[ f_{\omega,\pm}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* T_{\omega}(x,a)^{\pm 1} \begin{pmatrix} \pm m_{\omega,\pm} \\ 1 \end{pmatrix}. \]

It is also known that the resolvent \((z - H_{\omega})^{-1}\) is an integral operator with the following jointly continuous kernel
\[ G_{\omega}(x, y) = \begin{cases} f_{\omega, -}(x)(m_{\omega, +}^2 + m_{\omega, -}^2)^{-1} f_{\omega, +}(y), & x \leq y, \\ f_{\omega, +}(x)(m_{\omega, +}^2 + m_{\omega, -}^2)^{-1} f_{\omega, -}(y), & y \leq x. \end{cases} \tag{18} \]

### 3.3 Lyapunov exponent and DOS at upper band edges

The purpose of this section is to prove Theorem 30. Let us begin with the Lyapunov exponent. It follows from (17) that
\[ \gamma_{E_l}^{E_l} = 0, \quad E_l = (l\pi)^2. \]

Then (8) describes how the Lyapunov exponent grows as one enters the spectrum. This will be achieved by a controlled perturbation theory using modified Prüfer variables. This technique also gives access to the density of states.

#### 3.3.1 Modified Prüfer variables

The basic fact motivating the use of modified Prüfer variables is that the Lyapunov exponent defined in (7) can be calculated by
\[ \gamma^E = \lim_{N \to \infty} E \frac{1}{N} \log(\|MT_N^E(N,0)M^{-1}e\|) \]
\[ = \lim_{N \to \infty} E \frac{1}{N} \log(\|(MT_N^E M^{-1}) \cdots (MT_1^E M^{-1})e\|), \tag{19} \]
where \(M\) is an arbitrary invertible \(2 \times 2\) matrix which may, moreover, depend on \(E\). This matrix \(M\) can be chosen later in such a manner that the building blocks \(MT_n^E M^{-1}\) are close to some adequately chosen normal form. The choice of \(M\) will be made in Section 3.3.3.

In order to telescope the matrix product in the Lyapunov exponent further below, let us next introduce a random dynamical system on the unit circle. The unit circle is identified with unit vectors in \(\mathbb{R}^2\) via
\[ e_\theta = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \theta \in [0, 2\pi). \]

Then there is a natural action of invertible real \(2 \times 2\) matrices on the unit circle given by
\[ e_{S_T(\theta)} = \frac{T e_\theta}{\|Te_\theta\|}. \tag{20} \]
In particular, the map $S_T$ is invertible and $S_T^{-1} = S_{T^{-1}}$. With the notation $u = \left( \begin{array}{c} 1 \\ 1 \end{array} \right)$, one has
\[ e^{2sS_T(\theta)} = \frac{\langle u | T | e^{\theta} \rangle}{\langle u | T | e^{\theta} \rangle} . \quad (21) \]
In connection with the calculation of the Lyapunov exponent, one now has to consider the random dynamical system (Markov process) on the unit circle generated by the random sequence $MT_{n}^{E}M^{-1}$ of transfer matrices:
\[ \theta_n = S_{\varepsilon,n}(\theta_{n-1}) , \]
where $\theta_0$ is some initial condition and
\[ S_{\varepsilon,n} = S_{MT_{\varepsilon}^{E}M^{-1}} . \quad (22) \]
The $\theta_n$ are called the $M$-modified Prüfer phases. If one sets $e_n = e^{\theta_n}$, they are explicitly given by
\[ e_n = \frac{MT_{n}^{E}M^{-1}e_{n-1}}{\|MT_{n}^{E}M^{-1}e_{n-1}\|} , \quad \|e_0\| = 1 . \quad (23) \]
Coming back to the Lyapunov exponent as given in (19), one now has
\[ \gamma^E = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} E \log \left( \|M^{-1}T_{n}^{E}Me_{n-1}\| \right) . \quad (24) \]
In particular, we now have the Lyapunov exponent given by a Birkhoff sum associated to the random dynamical system (23) (which again converges almost surely so that the expectation in 24 may be dropped).

### 3.3.2 The integrated density of states

The integrated density of states is defined by
\[ N^E = \lim_{N \to \infty} \frac{1}{N} \# \{ \text{eigenvalues of } H_{\omega,N} \leq E \} , \]
where $H_{\omega,N}$ is the restriction $H_{\omega}$ to $[0, N]$ with Dirichlet boundary conditions (the Dirac potentials at the boundaries 0 and $N$ vanish). It is known that the limit defining $N^E$ exists almost surely and is almost surely independent of $\omega$. At the critical energies $E_l$, one has
\[ N^E_l = 1 . \]
As pointed out in [KN] this follows from Sturm-Liouville oscillation theory because at energy $E_l$ and volume $N$ one always (for all values of the potential) has an eigenfunction with exactly $Nl$ zeros so that there are exactly $Nl$ eigenvalues below $E_l$. Roughly stated this means that there is no spreading of density of states through
each $E_l$ when the distribution of the impurities is changed. Furthermore, Sturm-Liouville oscillation theory allows to calculate the integrated density of states as a rotation number (this can be done at every energy). In the vicinity of the critical energies, it is, moreover, possible to use the rotation number $R^\varepsilon$ associated to the modified Prüfer phases defined by

$$R^\varepsilon = \frac{1}{\pi} \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \sum_{n=0}^{N-1} \left( S_{\varepsilon,n}(\theta_{n-1}) - \theta_{n-1} \right).$$

In fact, one then has

$$N_{E_l - \varepsilon} = l + R^\varepsilon.$$  

### 3.3.3 Choice of the basis change in the elliptic regime

Close to the critical energy $E_l = (\pi l)^2$, the transfer matrix given by (15) is close to a Jordan block. Let us begin by expanding $T_{E_l - \varepsilon}^n$ in $\varepsilon > 0$. For sake of simplicity, let us throughout assume that $l$ is even. As $(E_l - \varepsilon)^{\frac{1}{2}} = \pi l - \frac{\varepsilon}{2\pi l} + O(\varepsilon^2)$, one finds

$$T_{E_l - \varepsilon}^n = \begin{pmatrix} 1 & v_n \\ 0 & 1 \end{pmatrix} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \left( \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2E_l} & 0 \end{pmatrix} + O(\varepsilon^2) \right) \right].$$

To lowest order, namely for $\varepsilon = 0$, the dynamics (20) of the Jordan blocks have a unique, common, globally attractive fixed point $\theta = 0$ which, however, is not stable. Therefore the orbit is close to $\theta = 0$ most of the time. This feature is conserved after a small perturbation so that the invariant distribution of the $\theta_n$’s is to lowest order a Dirac peak. Following [DG, SS], it is a good idea to blow up the vicinity of the fixed point in order to detect the effect of disorder and calculate deviations of the invariant measure from the Dirac peak. The blow up will done by conjugating $T_{E_l - \varepsilon}^n$ with

$$M_1 = \begin{pmatrix} \varepsilon \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$  

Indeed,

$$M_1 T_{E_l - \varepsilon}^n M_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \frac{1}{2} \left( \begin{pmatrix} 0 & v_n \\ -\frac{1}{2E_l} & 0 \end{pmatrix} - \varepsilon \left( \begin{pmatrix} v_n & 0 \\ 0 & 0 \end{pmatrix} + O(\varepsilon^2) \right) \right),$$

which is close to the unit matrix. Next let us center the random variable $v_n$. Set $\bar{\nu} = \mathbb{E}(v)$ and $\tilde{v}_n = v_n - \mathbb{E}(v)$. Then one can split the $M_1$-modified transfer matrix into a deterministic and a random part:

$$M_1 T_{E_l - \varepsilon}^n M_1^{-1} = \begin{pmatrix} 1 - \varepsilon \frac{\bar{\nu}}{2E_l} & \varepsilon \frac{1}{2} \tilde{v} \\ -\varepsilon \frac{1}{2E_l} & 1 \end{pmatrix} + \tilde{v}_n \begin{pmatrix} -\varepsilon \frac{1}{2E_l} & \varepsilon \frac{1}{2} \\ 0 & 0 \end{pmatrix} + O(\varepsilon^2).$$

As long as $\varepsilon \tilde{v} \leq 4\pi l$, the deterministic summand is up to corrections of order $\varepsilon \frac{3}{2}$ conjugate to a rotation matrix $R_{-\eta \frac{1}{2}}$ by an angle $-\eta \varepsilon \frac{1}{2}$ where

$$\eta = \sqrt{\frac{\tilde{v}}{2E_l}}.$$
This is what we call an elliptic regime. An equivalent condition for the elliptic regime is that the absolute value of the trace is less than 2. In this situation it is good to conjugate (29) again with an adequate matrix $M_2$ in order to bring this rotation into the normal form given by $R_\beta = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}$. The good choice turns out to be

$$M_2 = \begin{pmatrix} \frac{1}{2} & -b \\ \frac{1}{2} & b \end{pmatrix}, \quad b = \left(\frac{\bar{v}E_l}{2}\right)^{\frac{1}{4}}.$$

Indeed, setting $M_\varepsilon = M_2 M_1$ one finds after some algebra

$$M_\varepsilon T_n^{E_l - \varepsilon} (M_\varepsilon)^{-1} = R_{-\eta \varepsilon^{\frac{1}{2}}} T_n^{E_l - \varepsilon} + \mathcal{O}(\varepsilon^{\frac{3}{2}}), \quad (30)$$

After this basis change $M_\varepsilon$ the dynamics consists to lowest order $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ of a deterministic rotation and a centered random perturbation which thus has a variance of order $\mathcal{O}(\varepsilon)$. On the one hand, perturbation theory of the Lyapunov exponent will be based on (30), but on the other it also allows to readily calculate the modified Prüfer dynamics (22) perturbatively by using (21):

$$S_{\varepsilon,n}(\theta) = \theta - \eta \varepsilon^{\frac{1}{2}} + (\sin(2\theta) - 1) \frac{\bar{v}_n}{2\sqrt{2\bar{v}E_l}} \varepsilon^{\frac{1}{2}} + \mathcal{O}(\varepsilon). \quad (31)$$

It is also possible to calculate the terms of $\mathcal{O}(\varepsilon)$, but this is not needed here.

3.3.4 Perturbative calculation of Birkhoff sums

For the calculation of the Lyapunov exponent and the IDOS by (24) and (25) respectively one needs to evaluate Birkhoff sums of the type

$$I_N(f) = \mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} f(\theta_n), \quad f \in C(S^1).$$

The following result is taken from [SS].

**Proposition 34.** For any $f \in C^1(S^1)$, one has

$$I_N(f) = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta) + \mathcal{O}(\varepsilon^{\frac{1}{2}}, (N\varepsilon^{\frac{1}{2}})^{-1}).$$

**Proof.** Because $I_N(f) = c + I_N(f - c)$ for $c = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta)$, one may assume that $\int_0^{2\pi} d\theta f(\theta) = 0$. Then $f$ has an antiderivative $F \in C^2(S^1)$. Using a Taylor expansion one obtains up to an error of order $\mathcal{O}(\varepsilon)$,

$$F(\theta_n) = F(S_{\varepsilon,n}(\theta_{n-1})) = F(\theta_{n-1}) + f(\theta_{n-1}) \varepsilon^{\frac{1}{2}} \left(\sin(2\theta_{n-1}) - 1\right) \frac{\bar{v}_n}{2\sqrt{2\bar{v}E_l}} - \eta.$$
As $\tilde{v}_n$ is centered and independent of $\theta_{n-1}$, taking the expectation and summing over $n$ shows:

$$I_N(F) = I_N(F) - \varepsilon^{1/2} \eta I_N(f) + O(\varepsilon, N^{-1}).$$

Dividing by $\varepsilon^{1/2} \eta$ finishes the proof. \hfill $\Box$

### 3.3.5 Calculation of the Lyapunov exponent

Here we conclude the controlled perturbation theory for the Lyapunov exponent using the formula (24). In view of (24), it is convenient to introduce the auxiliary random variables

$$\gamma_n = \log\left(\|M^\varepsilon T_n^{E_l-\varepsilon}(M^\varepsilon)^{-1} e^{\theta_{n-1}}\|\right).$$

Using (30), one finds

$$\gamma_n = -\frac{\tilde{v}_n}{2\sqrt{2} \bar{v} E_l} \cos(2\theta_{n-1}) \varepsilon^{1/2} + \frac{\tilde{v}_n}{2 \bar{v} E_l} \sin(2\theta_{n-1}) \varepsilon + \frac{\tilde{v}_n^2}{16 \bar{v} E_l} \left(1 - 2 \sin(2\theta_{n-1}) - \cos(4\theta_{n-1})\right) \varepsilon + O(\varepsilon^{3/2}). \quad (32)$$

Now $\tilde{v}_n$ is centered and independent of $\theta_{n-1}$ so that the expectation value of the terms in the first line vanishes. Furthermore, by Proposition 34

$$\lim_{N \to \infty} E \frac{1}{N} \sum_{n=1}^{N} \sin(2\theta_{n-1}) = O(\varepsilon^{3/2}),$$

and similarly for the Birkhoff sum of $\cos(4\theta_{n-1})$. Therefore, replacing (32) into (24) shows

$$\gamma_{E_l-\varepsilon} = \lim_{N \to \infty} E \frac{1}{N} \sum_{n=1}^{N} \gamma_n = E(\tilde{v}_n^2) \frac{1}{16 \bar{v} E_l} \varepsilon + O(\varepsilon^{3/2}),$$

which is already the formula for $\gamma_{E_l-\varepsilon}$ in Theorem 30.

### 3.3.6 Calculation of the IDOS

Next let us calculate the rotation number $R_{E_l-\varepsilon}$ defined in (25) by using (31). Again using that $\tilde{v}_n$ is centered, one finds $R_{E_l-\varepsilon} = -\frac{2}{\pi} \varepsilon^{1/2} + O(\varepsilon)$. By (26) this shows the formula for the IDOS in Theorem 30.

### 3.3.7 Choice of the basis change in the hyperbolic regime

In this section we consider the transfer matrices $T_n^{E_l+\varepsilon}$ with $\varepsilon > 0$. Of course, in (27) this only leads to a sign change in front of $\varepsilon$. Then the basis change $M_1$ gives instead of (29)

$$M_1 T_n^{E_l+\varepsilon} M_1^{-1} = \begin{pmatrix} 1 + \frac{\varepsilon}{2 E_l} & \frac{\varepsilon^{1/2} \bar{v}}{1} \\ \frac{\varepsilon^{1/2} \bar{v}}{2 E_l} & 1 \end{pmatrix} + \tilde{v}_n \begin{pmatrix} \varepsilon^{1/2} \bar{v} & 0 \\ \frac{\varepsilon^{1/2} \bar{v}}{2 E_l} & 0 \end{pmatrix} + O(\varepsilon^{3/2}). \quad (33)$$
The deterministic part now has trace larger than 2 and is thus conjugate to a hyperbolic matrix. Again using $M_2$, $M^\varepsilon$ and $\eta$ as above, one now finds up to an error of order $O(\varepsilon^{\frac{3}{2}})$

$$M^\varepsilon \mathcal{T}_n^{E_l+\varepsilon}(M^\varepsilon)^{-1} = \begin{pmatrix} 1 - \eta \varepsilon^{\frac{1}{2}} & 0 \\ \varepsilon^{\frac{1}{2}} & 1 \end{pmatrix} + \varepsilon^{\frac{1}{2}} \frac{v_n}{2\sqrt{2|v_l|^2}} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \varepsilon \frac{v_n}{4E_l} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$  

(34)

The phase dynamics in this representation is given by

$$S_{\varepsilon,n}(\theta) = \theta + \varepsilon^{\frac{1}{2}} \eta \sin(2\theta) + \varepsilon^{\frac{1}{2}} \frac{v_n}{2\sqrt{2|v_l|^2}} (\sin(2\theta) - 1) + O(\varepsilon).$$

It has $\theta = \frac{\pi}{2}$ as stable fixed point (and $\theta = 0$ as unstable one), up to errors of order $O(\varepsilon)$. Replacing this information in (34) allows to show (by proceeding as in the next chapter) that $\gamma_{E_l+\varepsilon} = \eta \varepsilon^{\frac{1}{2}} + O(\varepsilon^{\frac{3}{2}})$ which is the missing part of Theorem 30.

### 3.4 Transport bounds

The aim of this section is to prove Theorem 29.

#### 3.4.1 Set-up for the lower bound

By the results of Section 3.2.5 the resolvent is an integral operator with an integral kernel given by (18). From this follows that also the unitary groups of $H_\omega$ have an integral kernel and we will denote all these integral kernels with Dirac notation. Then

$$\langle a|M_q(T)|a \rangle = \frac{2}{T} \int_0^\infty dt \ e^{-\frac{2}{T}} \int dx \ |x|^q \langle a|e^{iH_\omega t}|x \rangle \langle x|e^{-iH_\omega t}|a \rangle.$$  

The point $a$ is used above in the construction of the resolvent, but can be chosen freely so that one obtains the lower bound for all $a \in \mathbb{R} \setminus \mathbb{Z}$.

Writing out the appearing matrix elements in spectral representation and using the identity

$$\int_0^\infty dt \ e^{-\frac{2}{T}} e^{iE' t} e^{-iE'' t} = \int \frac{dE}{2\pi} \frac{1}{E' - E - iT^{-1}} \frac{1}{E'' - E + iT^{-1}},$$

it follows that

$$\langle a|M_q(T)|a \rangle = \int dx \ |x|^q \frac{1}{T} \int \frac{dE}{\pi} \left| \langle x|(H_\omega - E - iT^{-1})^{-1}|a \rangle \right|^2.$$  

A lower bound is obtained by restricting the energy integral over an interval
\[ [E_l - \varepsilon_0, E_l] \text{ close to any of the critical energies:} \]
\[
\langle a | M_q(T) | a \rangle \geq \int dx |x|^q \int_{E_l - \varepsilon_0}^{E_l} \frac{dE}{\pi T} |\langle x | (H_\omega - E - i T^{-1})^{-1} | a \rangle|^2
\]
\[
= \int dx |x|^q \int_0^{\varepsilon_0} \frac{d\varepsilon}{\pi T} |\langle x | (H_\omega - E_l + \varepsilon - i T^{-1})^{-1} | a \rangle|^2
\]
\[
= \int dx |x|^q \int_0^{\varepsilon_0} \frac{d\varepsilon}{\pi T} |G^z_\omega(x, a)|^2,
\]
where in the last equality we set \( z = E_l - \varepsilon + iT^{-1} \). By (18) one has for \( x \geq a \)
\[
|G^z_\omega(x, a)|^2 = |G^z_\omega(a, a)|^2 \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| T^z_\omega(x, a) \begin{pmatrix} m^z_\omega^+ \\ 1 \end{pmatrix} \right|^2,
\]
and similarly for \( x \leq a \). Let us also note
\[
|G^z_\omega(a, a)|^2 = \frac{1}{|m^z_\omega^+ + m^z_\omega^-|^2}. \tag{35}
\]
Replacing thus shows, still with \( z = E_l - \varepsilon + iT^{-1} \), that \( \langle a | M_q(T) | a \rangle \) is bounded below by
\[
\int_0^\infty dx \int_0^{\varepsilon_0} \frac{d\varepsilon}{\pi T} |G^z_\omega(x, a)|^2 M^z_\omega(x, a),
\]
with
\[
M^z_\omega(x, a) = \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| T^z_\omega(x, a) \begin{pmatrix} m^z_\omega^+ \\ 1 \end{pmatrix} \right|^2 + \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| T^z_\omega(-x, a) \begin{pmatrix} m^z_\omega^- \\ 1 \end{pmatrix} \right|^2.
\]
Now it may happen that the appearing matrix elements vanish for some \( x \), but as the transfer matrices vary as given by (14) between the potentials, for two points \( x, x' \) within one interval of \( \mathbb{R} \setminus \mathbb{Z} \) satisfying \( x' \geq x + \frac{1}{2} \), one has
\[
\left| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| T^z_\omega(x, a) \begin{pmatrix} m^z_\omega^+ \\ 1 \end{pmatrix} \right|^2 + \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| T^z_\omega(x', a) \begin{pmatrix} m^z_\omega^+ \\ 1 \end{pmatrix} \right|^2
\]
\[
= \left| \begin{pmatrix} m^z_\omega^+ \\ 1 \end{pmatrix} \right| A T^z_\omega(x, a) \begin{pmatrix} m^z_\omega^+ \\ 1 \end{pmatrix} \right|^2
\]
\[
\geq C_1 \left| T^z_\omega(x, a) \begin{pmatrix} m^z_\omega^+ \\ 1 \end{pmatrix} \right|^2
\]
\[
\geq C_1 \frac{|m^z_\omega^-|^2 + 1}{\|T^z_\omega(x, a)\|^2},
\]
where \( C_1 \) is a uniform lower bound (for \( E = E_l - \varepsilon \in [E_l - \varepsilon_0, E_l] \) and \( T \geq 1 \) on the positive matrix
\[
A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + T^z_\omega(x', x)^* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} T^z_\omega(x', x),
\]
\[
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\]
and where in the last inequality it was also used that the transfer matrix is in \( \text{SL}(2, \mathbb{C}) \) so that its norm coincides with the norm of its inverse. A similar bound holds for \( x < 0 \). Of course, one can also replace \( x \) and \( x' \) by each other at the cost of another factor. Therefore the above implies

\[
\langle a | M_q(T) | a \rangle \geq \frac{C_2}{T} \int_0^N dx \int_0^{\varepsilon_0} dx' \langle G^z_\omega(a, a) \rangle \frac{|m_{\omega,+}|^2 + 1}{\left\| T^z_\omega(x, a) \right\|^2} + \frac{|m_{\omega,-}|^2 + 1}{\left\| T^z_\omega(-x, a) \right\|^2},
\]

(36)

where the upper boundary \( N \) on the space variable can be chosen at convenience later on. Actually, both \( N \) and \( \varepsilon_0 \) will be adequate functions of time \( T \). The bound (36) will be the starting point of the conclusion of the argument below.

**Remark** Let us show how one may go over to the \( M^\varepsilon \)-modified transfer matrices at this point. First of all, \( M^\varepsilon = M_2 M_1 \) satisfies \( \| M^\varepsilon \| \leq C_3 \) and \( \|(M^\varepsilon)^{-1}\| \leq C_3\varepsilon^{-\frac{1}{2}} \) because \( \| M_1 \| = 1 \), \( \| M_1^{-1} \| = \varepsilon^{-\frac{1}{2}} \) and \( C_3 = \| M_2 \| = \| M_2^{-1} \| \). Thus \( E(a | M_q(T) | a) \) is bounded below by

\[
\frac{C_4}{T} \int_0^N dx \int_0^{\varepsilon_0} dx' \langle G^z_\omega(a, a) \rangle |\tilde{M}_{\omega}^z(x, a)|^2.
\]

(37)

with

\[
\tilde{M}_{\omega}^z(x, a) = \left( \frac{|m_{\omega,+}|^2 + 1}{\left\| M^\varepsilon T^z_\omega(x, a) (M^\varepsilon)^{-1} \right\|^2} + \frac{|m_{\omega,-}|^2 + 1}{\left\| M^\varepsilon T^z_\omega(-x, a) (M^\varepsilon)^{-1} \right\|^2} \right)^{\frac{1}{2}}.
\]

Replacing (35) for \( |G^z_\omega(a, a)|^2 \) one can now attempt to use the deterministic lower bound

\[
\frac{|m_{\omega,+}|^2 + 1}{|m_{\omega,+} + m_{\omega,-}|^2} + \frac{|m_{\omega,-}|^2 + 1}{|m_{\omega,+} + m_{\omega,-}|^2} \geq \frac{1}{2} \frac{|m_{\omega,+}|^2 + |m_{\omega,-}|^2 + 2}{|m_{\omega,+} + m_{\omega,-}|^2} \geq \frac{1}{2},
\]

(38)

so that either the first or the second summand in (37) is bounded below by \( \frac{1}{4} \). One is then tempted to combine this with Jensen’s inequality

\[
E \left( \frac{1}{\left\| M^\varepsilon T^z_\omega(x, a) (M^\varepsilon)^{-1} \right\|^2} \right) \geq \exp \left( -E \left( \log(\left\| M^\varepsilon T^z_\omega(x, a) (M^\varepsilon)^{-1} \right\|^2) \right) \right).
\]

Now one just calculates the expectation in the exponent exactly as the Lyapunov exponent was calculated. However, there is a flaw in this argument: one cannot decorrelate the two bounds because there is no control for which \( \omega \) the lower bound by \( \frac{1}{4} \) holds. Therefore, one needs to show that \( \| T^z_\omega(x, a) \| \lesssim \varepsilon^{\gamma^*} \) grows as most as indicated by the Lyapunov exponent \( \gamma^* \) at least with high probability. This is shown in the next section. An alternative would be to show that the two terms on the l.h.s. of (38) are of equal size with high probability. Another way out is to consider the model on the half-line with adequate boundary conditions (not Dirichlet). Then there is only one term and arguing with Jensen as above works.
3.4.2 Large deviations

Here it will be shown that the $M^\varepsilon$-modified transfer matrices appearing in (37) remain small on adequate scales with high probability. For this purpose, let us work first at real energies, neglect the difference from $x$ and $a$ to the integers, and also insert an arbitrary initial unit vector $e_0 \in \mathbb{R}^2$. Working with modified Prüfer variables as in Section 3.3 in order to telescope the matrix product and then using (32) shows

$$
\log(\|M^\varepsilon T^E_{\omega} - \varepsilon(n,m)(M^\varepsilon)^{-1}e_0\|) = \sum_{j=m+1}^{n} \gamma_j = \varepsilon \frac{1}{2} \sum_{j=m+1}^{n} X_j + O((n-m)\varepsilon),
$$

(39)

where

$$
X_j = -\frac{\tilde{v}_{j}}{2\sqrt{2\pi E^2}} \cos(2\theta_{j-1}).
$$

These $X_j$ are martingales, namely centered random variables depending on $v_i$, $i \leq j$, but not on the future $v_i$, $i > j$. They also depend on $\theta_0$ and $\varepsilon$. Let us begin with a standard upper bound on the large deviations of sums of the $X_j$.

**Lemma 35.** Let $\alpha > 0$. Set $Z^\varepsilon(n,m) = \sum_{j=m+1}^{n} X_j$ and

$$
\Omega^\varepsilon_N(\alpha, e_0) = \left\{ \omega \in \Omega \mid \sup_{-N \leq m \leq n \leq N} |Z^\varepsilon(n,m)| \geq N^{1+\alpha} \right\}.
$$

Then there is a constant $C_5 > 0$ such that

$$
P(\Omega^\varepsilon_N(\alpha, e_0)) \leq C_5 N^2 e^{-N^\alpha}.
$$

**Proof.** As the distribution of the potentials $v_j$ is compactly supported, one has $-c \leq X_j \leq c$ for some $c > 0$. Therefore the convexity of the exponential implies

$$
P_{E_j}(e^{\beta X_j}) \leq \cosh(\beta c) \leq e^{\frac{1}{2}(c\beta)^2},
$$

where $E_j$ is the expectation over $v_j$ only. For $\lambda > 0$ and $\beta > 0$, one thus has

$$
P\left( \left\{ Z^\varepsilon(n,m) \geq \lambda \right\} \right) = P\left( \left\{ e^{\beta Z^\varepsilon(n,m)} \geq e^{\beta \lambda} \right\} \right) \leq e^{-\lambda \beta} E_j(e^{\beta Z^\varepsilon(n,m)})
$$

$$
= e^{-\lambda \beta} E_{n-1} E_j(e^{\beta X_j} e^{\beta Z^\varepsilon(n-1,m)})
$$

$$
\leq e^{-\lambda \beta} e^{\frac{1}{2}(c\beta)^2} E_{n-1}(e^{\beta Z^\varepsilon(n-1,m)})
$$

$$
\leq e^{-\lambda \beta} e^{\frac{1}{2}(n-m)(c\beta)^2},
$$

where the last inequality is obtained iteratively. A similar bound holds for $\lambda < 0$ and $\beta < 0$. With $\lambda = N^{1+\alpha}$ and $\beta = N^{-\frac{1}{2}}$ it follows, as $n \leq N$,

$$
P\left( \left\{ |Z^\varepsilon(n,m)| \geq N^{1+\alpha} \right\} \right) \leq 2 e^{\frac{1}{2}(c\beta)^2} e^{-N^\alpha}.
$$

Summing over $m$ and $n$ completes the proof. \qed

Combining the lemma with (39) leads to the following.
Corollary 36. For $\omega \notin \Omega_N(\alpha, \varepsilon_0)$ one has for all $-N \leq m \leq n \leq N$ and some constant $C_6$

$$\log (\|M^\varepsilon T_\omega^{E_l - \varepsilon}(n, m)(M^\varepsilon)^{-1}e_0\|) \leq N^{1/2} \varepsilon^{1/2} + C_6 N \varepsilon .$$

Corollary 37. There is a set $\Omega_N^\varepsilon(\alpha)$ satisfying

$$\mathbb{P}(\Omega_N^\varepsilon(\alpha)) \leq 2 C_5 N^2 e^{-N^\alpha} ,$$

such that for $\omega \notin \Omega_N^\varepsilon(\alpha)$, $C_6$ as above and a constant $C_7$ one has for all $-N \leq m \leq n \leq N$

$$\|T_\omega^{E_l - \varepsilon}(n, m)\| \leq C_7 \varepsilon^{-1/2} \exp \left( N^{1/2} \varepsilon^{1/2} + C_6 N \varepsilon \right) .$$

Proof. In order to control the norms of the transfer matrices it is sufficient to control their action on 2 initial vectors because for any $2 \times 2$ matrix $A$,

$$\|A\| = \sup_{\theta \in [0, \pi]} \|Ae^{\theta}\| \leq \sqrt{2} \max_{\theta = 0, \pi/2} \|Ae^{\theta}\| .$$

Hence it is sufficient to prove probabilistic bounds on (39) for two initial conditions, namely one can set $\Omega_{N, \varepsilon}^\varepsilon(\alpha) = \Omega_{N, \varepsilon}^\varepsilon(\alpha, \varepsilon_0) \cup \Omega_{N, \varepsilon}^\varepsilon(\alpha, \varepsilon_{\pi/2})$ which hence satisfies (40) and it follows from Corollary 36 that for $\omega \notin \Omega_N^\varepsilon(\alpha)$ and $-N \leq m \leq n \leq N$ one has

$$\log (\|M^\varepsilon T_\omega^{E_l - \varepsilon}(n, m)(M^\varepsilon)^{-1}\|) \leq \sqrt{2} \left( N^{1/2} \varepsilon^{1/2} + C_6 N \varepsilon \right) .$$

Now the result follows from the bounds $\|M^\varepsilon\| \leq C_3$ and $\|(M^\varepsilon)^{-1}\| \leq C_3 \varepsilon^{-1/2}$ derived before equation 37 above. \qed

Finally, we are going to combine Corollary 37 with Lemma 33. The constant $c_2$ in that lemma is equal to the r.h.s. of (41) as long as $\omega \notin \Omega_N^\varepsilon(\alpha)$. Therefore, as long as

$$|\kappa| \leq \frac{1}{2 c_2 c_3 N} = \frac{\varepsilon^{1/2}}{N} \frac{1}{2 c_2 C_11} \exp \left( -N^{1/2} \varepsilon^{1/2} - C_{10} N \varepsilon \right) ,$$

one has for $\omega \notin \Omega_N^\varepsilon(\alpha)$ the bound

$$\|T_\omega^{E_l - \varepsilon + \kappa}(n, m)\| \leq 2 C_{11} \varepsilon^{-1/2} \exp \left( N^{1/2} \varepsilon^{1/2} + C_{10} N \varepsilon \right) .$$

This means that the bound (41) transposes, up to a factor 2, to a complex neighborhood of energies the size of which is given by (43).

The aim is now to use the bound (44) in (36) and for that purpose the free parameters $\varepsilon_0$ and $N$ have to be coupled to the time $T$. In order for (44) to be of any use for all $\varepsilon < \varepsilon_0$, we first of all choose

$$\varepsilon_0 = N^{-1-2\alpha} .$$

Then the exponential factors in both (43) and (44) are of order 1 and (43) becomes

$$|\kappa| \leq C_8 N^{-3/2-\alpha} .$$
for some constant $C_8$. Now the size of the balls of radius $\varepsilon\Delta = N^{-\frac{3}{2} - \alpha}$ (in complex energy) is much smaller than the interval of size $\varepsilon_0 = N^{-1-2\alpha}$ that we want to cover. However, it can be covered by $N^{\frac{1}{2}}$ intervals of size $\varepsilon\Delta$. Thus setting

$$\Omega_N(\alpha) = \bigcup_{j=1,\ldots,N^{\frac{1}{2}}} \Omega_N^{E_j - j\varepsilon\Delta}(\alpha),$$

one still has a set with sufficiently small probability on the complement of which (44) holds uniformly in an adequate set of complex energies. More precisely, we have proved the following.

**Proposition 38.** Let $\alpha > 0$. There exists a set $\Omega_N(\alpha)$ satisfying

$$\mathbb{P}(\Omega_N(\alpha)) \leq 2C_5 N^{\frac{3}{2}} e^{-N^{\alpha}},$$

such that there are constants $C_8, C_9$ so that for all $\varepsilon \leq N^{-1-2\alpha}$ and $\delta \leq C_8 N^{-\frac{3}{2} - \alpha}$ one has for $\omega \notin \Omega_N(\alpha)$

$$\sup_{-N \leq m \leq n \leq N} \| T_{\omega}^{E_j} - \varepsilon + \delta(n, m) \| \leq C_9 \varepsilon^{-\frac{1}{2}}.$$ 

We do not expect this result to be optimal, but rather that the same bound holds for a larger set of complex energies given by $\delta \leq C_8 N^{-1-\alpha}$, but could not obtain such a better estimate.

### 3.4.3 Conclusion of the proof of the lower bound

In (36), we now choose $N$ and $\varepsilon_0$ as

$$N = (C_8 T)^{\frac{3}{2} + 2\alpha}, \quad \varepsilon_0 = N^{-1-2\alpha} = (C_8 T)^{-\frac{\frac{2}{3} + 4\alpha}{3+2\alpha}}.$$ 

Due to Proposition 38 one then has

$$\mathbf{E}\left< a | M_q(T) | a \right> \geq \frac{C_2}{T} \int_0^N dx x^q \mathbf{E} \chi_{\Omega_N(\alpha)^c}(\omega) \int_0^{\varepsilon_0} d\varepsilon \left| G_\omega^*(a, a) \right|^2 \tilde{m}_\omega^\varepsilon \frac{\varepsilon}{(C_9)^2},$$

where $\tilde{m}_\omega^\varepsilon = (|m_{\omega,+}^\varepsilon|^2 + |m_{\omega,-}^\varepsilon|^2 + 2)$ and $\chi_{\Omega_N(\alpha)^c}$ is the indicator function onto the complement of $\Omega_N(\alpha)$. Now one can use the deterministic bound (38) to conclude that

$$\left< a | M_q(T) | a \right> \geq \frac{2C_2}{T} \int_0^N dx x^q \int_0^{\varepsilon_0} d\varepsilon \frac{\varepsilon}{(C_9)^2} (1 - \mathbb{P}(\Omega_N(\alpha))).$$

Thus there are constants $C_{10}$ and $C_{11}$ such that

$$\left< a | M_q(T) | a \right> \geq C_{10} \frac{\varepsilon_0^2 N^{q+1}}{T} = C_{11} T^{-1-\frac{\frac{2}{3} + 4\alpha}{3+2\alpha} + (q+1)\frac{2}{3+2\alpha}}.$$ 

If $\alpha$ is redefined, this proves Theorem 29.
4 Gaussian fluctuations of products of random matrices close to the identity

This chapter consists of the work done in the publication [DraS]. Products of random matrices exhibit Gaussian fluctuations around almost surely convergent Lyapunov exponents. If the distribution of the random matrices is supported by a small neighborhood of the identity, one speaks about an anomaly and such anomalies can have either an elliptic, hyperbolic or parabolic nature. For all cases, the variance of the Gaussian fluctuations is calculated perturbatively at anomalies of random $2 \times 2$ matrices. This requires a detailed analysis of the associated random dynamical system on the unit circle and its invariant measure.

4.1 Main results

The analysis of random finite difference equations such as harmonic chains or the one-dimensional Anderson model naturally leads to study products of random $2 \times 2$ matrices. Asymptotically these products converge to a Gaussian process which is characterized by the Lyapunov exponent and its variance [BL]. Often one is interested in a perturbative regime of small randomness. For example, the zero frequency limit of a harmonic chain is of this type, as well as the analysis of band center and band edges of the Anderson model [Ish, KW, DG]. Also the behavior near the so-called critical energies of a random Kronig-Penney model is of this perturbative nature [DKS]. It is the object of this work to develop a rigorously controlled perturbation theory for both the Lyapunov exponent and its variance within a generic model covering all the situations alluded to above, and potentially others.

More precisely, let us consider a family $(T_{\lambda,n})_{n \geq 1}$ of random matrices in $\text{Sl}(2, \mathbb{R})$ which is of the form

$$T_{\lambda,n} = \exp(\lambda P_n + \lambda^2 Q_{\lambda,n}), \quad (47)$$

where $\lambda \geq 0$ is a small coupling parameter and $P_n, Q_{\lambda,n} \in \text{sl}(2, \mathbb{R})$. The matrix $Q_{\lambda,n}$ is supposed to be analytic in $\lambda$, and $P_n$ and $Q_{\lambda,n}$ are independent and identically distributed on a bounded set. The distribution of $T_{\lambda,n}$ is thus supported in a neighborhood of size $\lambda$ around the identity. Averaging over this measure is denoted by $E$. Associated to any random sequence of matrices are products $\prod_{n=1}^{N} T_{\lambda,n} = T_{\lambda,N} \cdots T_{\lambda,1}$ and their asymptotics satisfy a 0-1 law characterized by the Lyapunov exponent $\gamma_{\lambda}$ [BL]. It can be calculated by

$$\gamma_{\lambda} = \lim_{N \to \infty} \frac{1}{N} E \left[ \log(\| \prod_{n=1}^{N} T_{\lambda,n} e \|) \right], \quad (48)$$

with some initial condition $e \in \mathbb{R}^2, \|e\| = 1$. Furthermore, a central limit theorem
[Tut, BL] states that the expression
\[
\frac{1}{\sqrt{N}} \left( \log(\| \prod_{n=1}^{N} T_{\lambda,n} e \|) - N \gamma_{\lambda} \right)
\] (49)
converges to a centered Gaussian law with a variance denoted by \( \sigma_{\lambda} \). Both the variance and the Lyapunov exponent are independent of the initial condition \( e \). Clearly the distribution of \( \mathcal{P}_n \) should have a crucial influence on the values of \( \gamma_{\lambda} \) and \( \sigma_{\lambda} \). The following theorem summarizes the main results of the paper.

**Theorem 39.** Let the i.i.d. random matrices \( T_{\lambda,n} \) be of the form (47) with the distribution supported on a bounded set. Furthermore, it is supposed that in each of the cases below a certain linear combination of the entries of \( \mathcal{P}_n \) specified below has a strictly positive variance.

(i) If the averaged perturbation is elliptic in the sense that \( \det(\mathbb{E}[\mathcal{P}_n]) > 0 \), then there is a constant \( C_e \), which is calculated explicitly from the variances of the entries of \( \mathcal{P}_n \) in Section 4.3, such that
\[
\gamma_{\lambda} = C_e \lambda^2 + \mathcal{O}(\lambda^3), \quad \sigma_{\lambda} = C_e \lambda^2 + \mathcal{O}(\lambda^3).
\]

(ii) If the averaged perturbation is hyperbolic in the sense that \( \det(\mathbb{E}[\mathcal{P}_n]) < 0 \), then with \( C_h = \sqrt{-\det(\mathbb{E}[\mathcal{P}_n])} \),
\[
\gamma_{\lambda} = C_h \lambda + \mathcal{O}(\lambda^{\frac{3}{2}}), \quad \sigma_{\lambda} = \mathcal{O}(\lambda^{\frac{3}{2}}).
\]

(iii) If the perturbation is centered \( \mathbb{E}[\mathcal{P}_n] = 0 \), then there exists constants \( C_s \) and \( C'_s \) such that
\[
\gamma_{\lambda} = C_s \lambda^2 + \mathcal{O}(\lambda^3), \quad \sigma_{\lambda} = C'_s \lambda^2 + \mathcal{O}(\lambda^3).
\]

There have been numerous works in both the physics and the mathematics literature on products of random matrices of the form (47). One is an influential work of Kappus and Wegner [KW] on so-called band center anomalies for the Lyapunov exponent at the band center of the one-dimensional Anderson model. The term *anomaly* reflects that the standard perturbation theory breaks down and that this leads to an enhancement of the Lyapunov exponent. These anomalies and also the band edge were then further studied by Derrida and Gardner [DG]. In the sequel, many rigorous works analyzed the Lyapunov exponent of the Anderson model in these particular citations [BK, CK, SVW, SB, SS]. In [SS] any model of the form (47) was called an anomaly and such anomalies were further distinguished to be of first order elliptic if \( \det(\mathbb{E}[\mathcal{P}_n]) > 0 \), of first order hyperbolic if \( \det(\mathbb{E}[\mathcal{P}_n]) < 0 \), and of second order if \( \mathbb{E}[\mathcal{P}_n] = 0 \). This terminology will be used below and corresponds to the three cases in the theorem. Let us point out that the parabolic first order case...
4 Gaussian fluctuations of products of random matrices close to the identity

\[
\det \left( E \left[ \mathcal{P}_n \right] \right) = 0 \text{ and } E[\mathcal{P}_n] \neq 0 \text{ is not covered by the theorem, but can be considered as non-generic.}
\]

The main novelty of Theorem 39 are the formulas for the variances. In fact, the Lyapunov exponents in (i) and (iii) were calculated in [SS] and [SB] respectively, and the case of (ii) was sketched in [SS, DKS], but there are actually considerable technical difficulties to make the analysis rigorous, see Section 4.4. On the other hand, no other work on variances at anomalies is known to us. The paper [SSS] developed a perturbation theory for the variance in situations where the lowest order matrix \( \mathcal{T}_{0,n} \) is a non-trivial rotation rather than the identity. This corresponds to energies away from the band center and the band edges in the Anderson model. To control the variances in Theorem 39 it is necessary to go far beyond the analysis of [SSS].

In the first order elliptic case of Theorem 39(i), the result implies the equality
\[
\sigma_\lambda = \gamma_\lambda + O(\lambda^3).
\]
Hence the asymptotic distribution of the product of random matrices is described by a single parameter, up to errors of higher order. In the framework of random Schrödinger operators this is referred to as single parameter scaling. It was already shown to hold away from anomalies in [SSS], but only to lowest order in perturbation theory. This is now extended to the first order elliptic regime. For an Anderson model, this covers energies inside of the band, but close to a band edge (see Section 4.6). On the other hand, energies outside of the band, but close to a band edge, correspond to a first order hyperbolic anomaly as in Theorem 39(ii). In this situation, the Lyapunov exponent is given by its deterministic value, up to fluctuations which are of much smaller order as expected (because the system is a random perturbation around a hyperbolic one, albeit a very weakly hyperbolic one). In particular, there is no single parameter scaling in this regime. We are not able to determine whether \( \sigma_\lambda \) is of even lower order than \( O(\lambda^{3/2}) \). In the centered case of Theorem 39(iii), the Lyapunov exponent and variance are of the same order of magnitude, and the formulas in Section 4.5 show that an equality \( C_s = C'_s \) is to be considered a coincidence so that single parameter scaling does not hold in this case.

In the final Section 4.6 of the paper, it is shown how Theorem 39 can be applied to the Anderson model as well as two other models, namely a harmonic chain and the Kronig-Penny model. Both have already been considered in the work of Ishii [Ish], but there has been continuous and also recent interest in them (e.g. [AH] and [DKS] respectively). The remainder of the article is mainly devoted to proofs. Apart from perturbative formulas and recursive arguments, the main technical tool is the analysis of singular differential operators.

4.2 From random products to a random dynamical system

This section contains some preparatory material, most of which is already contained in [BL, SS, SSS], but is needed to fix notations and make this work self-contained.
4.2.1 Dynamics on the unit circle

For any \( P = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \text{sl}(2, \mathbb{R}) \) one has \( P^2 = -\det(P) \mathbf{1}_2 \) which implies

\[
\exp(\lambda P) = \cosh(\lambda d) \mathbf{1}_2 + \frac{\sinh(\lambda d)}{d} P, \quad d = \sqrt{-\det(P)} \in \mathbb{C}.
\]

A dynamics \( \theta \in \mathbb{S}^1_x \mapsto S_P(\theta) \in \mathbb{S}^1_x \) on \( \mathbb{S}^1_x = [0, \pi] \) is induced by

\[
e_{S_{\lambda P}(\theta)} = \frac{\exp(\lambda P) e_\theta}{\|\exp(\lambda P) e_\theta\|}, \quad e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.
\]

This dynamics can readily be calculated in terms of harmonics in \( \theta \), namely with \( v = \binom{1}{i} \)

\[
e^{2i S_{\lambda P}(\theta)} = \frac{v^* \exp(\lambda P) e_\theta}{\|v^* \exp(\lambda P) e_\theta\|} = \frac{\cosh(\lambda d) e^\theta + \frac{\sinh(\lambda d)}{d} v^* P e_\theta}{\cosh(\lambda d) e^{-\theta} + \frac{\sinh(\lambda d)}{d} v^* P e_\theta},
\]

so that in terms of the entries of \( P \)

\[
e^{2i(S_{\lambda P}(\theta)-\theta)} = \frac{\cosh(\lambda d) + \frac{\sinh(\lambda d)}{d} \left( ae^{-2\theta} + \frac{i}{2} (c-b) + \frac{i}{2} (c+b) e^{-2\theta} \right)}{\cosh(\lambda d) + \frac{\sinh(\lambda d)}{d} \left( ae^{2\theta} - \frac{i}{2} (c-b) - \frac{i}{2} (c+b) e^{2\theta} \right)}.
\]

Expanding in \( \lambda \) therefore leads to

\[
S_{\lambda P}(\theta) = \theta + \lambda p(\theta) + \frac{1}{2} \lambda^2 p(\theta) \partial_\theta p(\theta) + \mathcal{O}(\lambda^3),
\]

where, still for \( P = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \),

\[
p(\theta) = -a \sin(2\theta) + \frac{i}{2} (c-b) + \frac{i}{2} (c+b) \cos(2\theta) = -a \sin(2\theta) - b \sin^2(\theta) + c \cos^2(\theta).
\]

(50)

All the above applies directly to \( T_{\lambda,n} \). However, in order to deal with the various types of anomalies, it will be advantageous to make an appropriate basis change \( M \in \text{Gl}(2, \mathbb{R}) \) from \( T_{\lambda,n} \) to

\[
T_{\lambda,n} = M T_{\lambda,n} M^{-1} = \exp(\lambda P_n + \lambda^2 Q_{\lambda,n}),
\]

with \( P_n = M P_n M^{-1} \) and \( Q_{\lambda,n} = M Q_{\lambda,n} M^{-1} \) respectively. The adequate choice of \( M \) will be made for each type of anomaly further below. Let us denote the dynamics associated with \( T_{\lambda,n} \) by \( S_{\lambda,n} = S_{\lambda P_n + \lambda Q_{\lambda,n}} \). According to the above,

\[
S_{\lambda,n}(\theta) = \theta + \lambda p_n(\theta) + \lambda^2 q_n(\theta) + \frac{1}{2} \lambda^2 p_n(\theta) \partial_\theta p_n(\theta) + \mathcal{O}(\lambda^3),
\]

(51)

where \( p_n \) and \( q_n \) are defined by (50) from the entries of \( P_n \) and \( Q_{0,n} \) respectively. Now given any sequence \( T_{\lambda,n} \) or equivalently \( P_n, Q_{\lambda,n} \) for \( n \geq 1 \) and any initial condition \( \theta_0 \in \mathbb{S}^1_x \), one obtains a sequence

\[
\theta_{n+1} = S_{\lambda,n}(\theta_n).
\]

If \( P_n \) and \( Q_{0,n} \) are random, this provides a random dynamical system on the circle \( \mathbb{S}^1_x \).

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4.2.2 Invariant measure

There is an invariant measure $\nu_\lambda$ on $S^1_{\pi}$, corresponding to the action of the family $(T_{\lambda,n})_{n \in \mathbb{N}}$, which is given by

$$
\int_0^\pi f(\theta) \nu_\lambda(d\theta) = E \int_0^\pi f(S_{\lambda,n}(\theta)) \nu_\lambda(d\theta), \quad f \in C(S^1_{\pi}).
$$

From the Furstenberg theorem [BL] follows that the invariant measure is unique and Hölder continuous, provided the Lyapunov exponent of the associated system is positive. This actually holds in all cases considered here. Of interest will be the following ergodic sums

$$I_{\lambda,N}(f) = \frac{1}{N} E \sum_{n=0}^{N-1} f(\theta_n),$$

and especially in their limit

$$I_{\lambda}(f) = \lim_{N \to \infty} I_{\lambda,N}(f).$$

This limit can also be expressed in the terms of the invariant measure $\nu_\lambda$ for which several notations are used:

$$I_{\lambda}(f) = \int_0^\pi f(\theta) \nu_\lambda(d\theta) = \nu_\lambda(f) = E_{\nu_\lambda}[f].$$

As in [SSS] the calculation of the variance $\sigma_\lambda$ is based on a control of the correlation sum

$$J_{\lambda}(f) = E_1 \left[ \sum_{n=1}^{\infty} \left( f(\theta_n) - \nu_\lambda(f) \right) \right].$$

Here $J_{\lambda}(f)$ depends on the initial condition $\theta_0$ and $E_n$ denotes the expectation over all $T_{\lambda,m}$ with $m \geq n$. Hence $E_1 = E$. It is shown in [BL] that positivity of the Lyapunov exponent implies that $J_{\lambda}(f)$ is finite for Hölder continuous functions. Let us point out that $f \mapsto J_{\lambda}(f)$ is linear and that constant functions are in the kernel of this map. A quantitative bound is given in Proposition 40 below.

4.2.3 The Lyapunov exponent and the variance

Let us introduce a family $(g_n)_{n \geq 1}$ of random variables by

$$g_n = \frac{1}{2} \log(\|T_{\lambda,n}e_{\theta_{n-1}}\|^2).$$

The Lyapunov exponent can then be defined as

$$\gamma_\lambda = \lim_{N \to \infty} \frac{1}{N} E \sum_{n=1}^{N} g_n = I_{\lambda}(g), \quad (52)$$
where \( g : S^1 \to \mathbb{R} \) is given by \( g(\theta_{n-1}) = E_n[g_n] \), while the variance is

\[
\sigma_\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n,k=1}^{N} E_{\nu_\lambda} \left[ (g_n - \gamma_\lambda) (g_k - \gamma_\lambda) \right] \\
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} E_{\nu_\lambda} \left[ g_n^2 - \gamma_\lambda^2 + 2 \sum_{m=1}^{N-n} \left( g_n g_{n+m} - \gamma_\lambda^2 \right) \right].
\]

The positivity of the Lyapunov exponent implies again [BL] that the sum over \( m \) is convergent even if \( N \to \infty \). Furthermore, \( E_n(g_{n+m}) \) converges exponentially fast to \( \gamma_\lambda \) as \( m \to \infty \) [BL, SSS] and the summands in the sum over \( n \) converge in expectation so that (with integration w.r.t. \( \nu_\lambda \) over \( \theta_0 \))

\[
\sigma_\lambda = E_{\nu_\lambda} \left[ g_n^2 - \gamma_\lambda^2 + 2 \sum_{m=2}^{\infty} (g_n g_m - \gamma_\lambda^2) \right] \\
= E_{\nu_\lambda} \left[ g_n^2 - \gamma_\lambda^2 + 2 g_1 E_2 \left( g_m - \gamma_\lambda \right) \right] \\
= E_{\nu_\lambda} \left[ g_1^2(\theta_0) - \gamma_\lambda^2 + 2 g_1(\theta_0)J_\lambda(g)(\theta_1) \right] \\
= E_{\nu_\lambda} \left[ g^2 - \gamma_\lambda^2 + 2 E(g_1 J_\lambda(g) \circ S_{\lambda,1}) \right].
\]

### 4.2.4 Basic perturbative formulas and estimates

Replacing the expansion

\[
\|T_{\lambda,n} e_\theta\|^2 = 1 + 2 \lambda e_\theta^* P_n e_\theta + \lambda^2 \left( 2 e_\theta^* Q_n e_\theta + e_\theta^* P_n^* P_n e_\theta - \det(P_n) \right) + O(\lambda^3),
\]

into the definition of \( g_n \), leads to

\[
g_n = \lambda e_{\theta_{n-1}}^* P_n e_{\theta_{n-1}} + \lambda^2 \left( e_{\theta_{n-1}}^* Q_n e_{\theta_{n-1}} + \frac{1}{2} e_{\theta_{n-1}}^* P_n^* P_n e_{\theta_{n-1}} - \frac{1}{2} \det(P_n) - \left( e_{\theta_{n-1}}^* P_n e_{\theta_{n-1}} \right)^2 \right),
\]

up to terms of order \( O(\lambda^3) \). Using

\[
e_\theta^* A e_\theta = \frac{\text{Tr}(A)}{2} + \frac{a - d}{2} \cos(2\theta) + \frac{b + c}{2} \sin(2\theta), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

one can rewrite \( g_n \) in terms of the entries of

\[
P_n = \begin{pmatrix} p_{1,n} & p_{2,n} \\ p_{3,n} & -p_{1,n} \end{pmatrix} \quad \text{and} \quad Q_n = \begin{pmatrix} q_{1,n} & q_{2,n} \\ q_{3,n} & -q_{1,n} \end{pmatrix}
\]

and obtain

\[
g_n = O(\lambda^3) + \lambda \left( p_{1,n} \cos(2\theta_{n-1}) + \frac{1}{2} (p_{2,n} + p_{3,n}) \sin(2\theta_{n-1}) \right) \\
+ \lambda^2 \left( \frac{1}{8} [4p_{1,n}^2 + (p_{2,n} + p_{3,n})^2] + [q_1 + \frac{1}{4} (p_{3,n}^2 - p_{2,n}^2)] \cos(2\theta_{n-1}) \right) \\
+ \frac{1}{2} [q_{2,n} + q_{3,n} + p_{1,n} (p_{2,n} - p_{3,n})] \sin(2\theta_{n-1}) \\
- \frac{1}{8} [4p_{1,n}^2 - (p_{2,n} + p_{3,n})^2] \cos(4\theta_{n-1}) - \frac{1}{2} [p_{1,n} (p_{2,n} + p_{3,n})] \sin(4\theta_{n-1}) \right). \]

\[
\]
Using these formulas, the following quantitative \textit{a priori} estimate on $J_{\lambda}(f)$ is proved in [SSS].

**Proposition 40.** There is a constant $C$ such that for any Hölder continuous function $f$ with Hölder norm $\|f\|_{\alpha}$,

$$J_{\lambda}(f) \leq \frac{C \|f\|_{\alpha}}{\lambda^{\gamma_{\lambda}}}.$$ 

### 4.3 Elliptic first order anomaly

This section considers the elliptic first order anomaly and proves item (i) of Theorem 39. As $\det(E[P_n]) > 0$, there is a basis transformation by $M$ such that the matrix $E[P_n]$ is the generator of a rotation:

$$P_n = \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix} + \tilde{P}_n = \begin{pmatrix} \tilde{p}_{1,n} & \tilde{p}_{2,n} - \eta \\ \tilde{p}_{3,n} + \eta & -\tilde{p}_{1,n} \end{pmatrix},$$

where $\eta = \sqrt{\det E[P_n]}$ and $\tilde{P}_n$ is centered. Thus to lowest order the dynamics is a deterministic rotation perturbed by random (centered) fluctuations:

$$S_{\lambda,n}(\theta) = \theta + \lambda \eta \theta + \lambda \left( \frac{1}{2}(\tilde{p}_{2,n} - \tilde{p}_{3,n}) + \frac{1}{2}(\tilde{p}_{2,n} + \tilde{p}_{3,n}) \cos(2\theta) - \tilde{p}_{1,n} \sin(2\theta) \right) + \mathcal{O}(\lambda^2).$$

**Proposition 41.** [SS] Up to corrections, the invariant measure at an elliptic first order anomaly is given for $f \in C^2(S^1_n)$ by the Lebesgue measure:

$$I_{\lambda}(f) = \frac{1}{\pi} \int_{0}^{\pi} f(\theta) d\theta + \mathcal{O}(\lambda).$$

**Proof.** Expanding $f \in C^2(S^1_n)$ in Taylor series and taking the expectation provides

$$E_{n+1}[f(S_{\lambda,n}(\theta))] = f(\theta_n) + \lambda \eta f'(\theta_n) + \mathcal{O}(\lambda^2).$$

By summing this up one obtains

$$I_{\lambda,N}(f) = I_{\lambda,N}(f) + \lambda \eta I_{\lambda,N}(f') + \mathcal{O}(\lambda^2, N^{-1}),$$

and since $f - \frac{1}{\pi} \int_{0}^{\pi} f(\theta) d\theta$ has an antiderivative, the statement follows. \hfill \Box

In particular, $I_{\lambda}(\cos(2\cdot)) = I_{\lambda}(\sin(2\cdot)) = \mathcal{O}(\lambda)$. This allows to calculate the Lyapunov exponent. Using (55) one gets from (54)

$$g_n = \lambda \left( \tilde{p}_{1,n} \cos(2\theta_{n-1}) + \frac{1}{2}(\tilde{p}_{2,n} + \tilde{p}_{3,n}) \sin(2\theta_{n-1}) \right) + \lambda^2 \left( \frac{1}{8}[4\tilde{p}_{1,n}^2 + (\tilde{p}_{2,n} + \tilde{p}_{3,n})^2] \\
+ [q_1 + \frac{1}{4}(\tilde{p}_{3,n}^2 - \tilde{p}_{2,n}^2) - \frac{1}{2}\eta(\tilde{p}_{2,n} + \tilde{p}_{3,n})] \cos(2\theta_{n-1}) \\
+ \frac{1}{2}[q_{2,n} + q_{3,n} + \tilde{p}_{1,n}(\tilde{p}_{2,n} - \tilde{p}_{3,n} - 2\eta)] \sin(2\theta_{n-1}) \\
- \frac{1}{8}[4\tilde{p}_{1,n}^2 - (\tilde{p}_{2,n} + \tilde{p}_{3,n})^2] \cos(4\theta_{n-1}) - \frac{1}{2}[\tilde{p}_{1,n}(\tilde{p}_{2,n} + \tilde{p}_{3,n})] \sin(4\theta_{n-1}) \right) + \mathcal{O}(\lambda^3),$$

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so that the Lyapunov exponent is
\[
\gamma_\lambda = \frac{1}{8} \lambda^2 \left( 4 \text{Var}(\hat{p}_1) + \text{Var}(\hat{p}_2 + \hat{p}_3) \right) + \mathcal{O}(\lambda^3). \tag{57}
\]

Here we omitted the indexation with \(n\), since the \(P_n\)'s are distributed identically. This formula also already appeared in [SS]. The calculation of the variance uses the following lemma.

**Lemma 42.** Let \(f \in C^{3+\alpha}(\mathbb{S}_n^1)\) with \(\alpha > 0\). Choose \(c \in \mathbb{R}\) such that the function \(f - c\) has an antiderivative \(F \in C^{3+\alpha}(\mathbb{S}_n^1)\). Then, independent of the choice of \(F\),
\[
J_\lambda(f) = -\frac{1}{\lambda \eta} \left( F(\theta_0) - \nu_\lambda(F) \right) + \mathcal{O}(1).
\]

**Proof.** Let us begin by noting that Proposition 40 combined with \(\gamma_\lambda = \mathcal{O}(\lambda^2)\) as given by (57) shows that \(J_\lambda(f) = \mathcal{O}(\lambda^{-3})\) because \(f\) is a Hölder continuous function. Furthermore, one has \(J_\lambda(f - c) = J_\lambda(f)\) so that it is possible to suppose \(c = 0\) and \(\int_0^\pi f(\theta) \, d\theta = 0\) from now on. Next let us recall the Taylor formula
\[
F(\theta + \epsilon) = F(\theta) + \epsilon F'(\theta) + \frac{1}{2} \epsilon^2 F''(\theta + \bar{\epsilon})
\]
where \(0 \leq \bar{\epsilon}/\epsilon \leq 1\). Replacing \(\theta = \theta_n\) and \(\epsilon = \theta_{n+1} - \theta_n = S_n,\lambda_n(\theta_n) - \theta_n\) and taking the average of (56) leads to
\[
\mathbb{E}_n F(\theta_{n+1}) = \mathbb{E}_n F(S_n,\lambda_n(\theta_n)) = F(\theta_n) + \lambda \eta F'(\theta_n) + \lambda^2 g(\theta_n)
\]
where \(g\) is some Hölder continuous function because \(f \in C^{1+\alpha}(\mathbb{S}_n^1)\). Replacing this into \(J_\lambda\) and separating the first summand, one has
\[
J_\lambda(F) = \left( F(\theta_0) - \mathbb{E}_n \nu_\lambda(F) \right) + J_\lambda(F) + \lambda \eta J_\lambda(F') + \lambda^2 J_\lambda(g).
\]
Therefore, for \(f = F'\),
\[
J_\lambda(f) = -\frac{1}{\eta \lambda} \left( F(\theta_0) - \nu_\lambda(F) + \lambda^2 J_\lambda(g) \right).
\]

As \(J_\lambda(g) = \mathcal{O}(\lambda^{-3})\) by the a priori estimate, this implies \(J_\lambda(f) = \mathcal{O}(\lambda^{-2})\). Up to now only \(f \in C^{1+\alpha}(\mathbb{S}_n^1)\) was used. As \(f \in C^{2+\alpha}(\mathbb{S}_n^1)\), one has \(g \in C^{1+\alpha}(\mathbb{S}_n^1)\) and hence also \(J_\lambda(g) = \mathcal{O}(\lambda^{-2})\). This implies \(J_\lambda(f) = \mathcal{O}(\lambda^{-2})\) for \(f \in C^{2+\alpha}(\mathbb{S}_n^1)\). One further iteration is needed to calculate the value of \(J_\lambda(f)\). As \(f \in C^{3+\alpha}(\mathbb{S}_n^1)\), one has \(g \in C^{2+\alpha}(\mathbb{S}_n^1)\) so that \(J_\lambda(g) = \mathcal{O}(\lambda^{-1})\). Replacing this in (58) concludes the proof. \(\square\)

Lemma 42 implies \(J_\lambda(\cos(j \cdot \cdot)) = -\frac{1}{\eta \lambda} \sin(\theta_0) + \mathcal{O}(1)\) and \(J_\lambda(\sin(j \cdot \cdot)) = \frac{1}{\eta \lambda} \cos(\theta_0) + \mathcal{O}(1)\). For the calculation of the variance the index \(n\) is again suppressed:
\[
J_\lambda(g) = \frac{\lambda}{2 \eta} \left( -\mathbb{E}[q_1 + \frac{1}{4}(\hat{p}_2^2 - \hat{p}_3^2)] \sin(2\theta_0) + \frac{1}{2} \mathbb{E}[q_2 + q_3 + \hat{p}_1(\hat{p}_2 - \hat{p}_3)] \cos(2\theta_0) 
+ \frac{1}{16} \mathbb{E}[4\hat{p}_1^2 - (\hat{p}_2 + \hat{p}_3)^2] \sin(4\theta_0) - \frac{1}{4} \mathbb{E}[\hat{p}_1(\hat{p}_2 + \hat{p}_3)] \cos(4\theta_0) \right) + \mathcal{O}(\lambda^2)
\]
Replacing into (53) shows that \(\sigma_\lambda = \gamma_\lambda + \mathcal{O}(\lambda^3)\) with \(\gamma_\lambda\) given by (57).

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4 Gaussian fluctuations of products of random matrices close to the identity
4 Gaussian fluctuations of products of random matrices close to the identity

4.4 Hyperbolic first order anomaly

At a hyperbolic first order anomaly it is possible to choose a basis change $M$ such that

$$P_n = \begin{pmatrix} p_{n,1} & p_{n,2} \\ p_{n,3} & -p_{n,1} \end{pmatrix}, \quad E(p_{n,2}) = E(p_{n,3}) = 0, \quad \eta = E(p_{n,1}) > 0.$$  

(59)

Then the dynamics is to lowest order (higher orders are irrelevant and thus suppressed in the following)

$$S_{\lambda,n}(\theta) = \theta + \lambda \left( -p_{n,1} \sin(2\theta) - p_{n,2} \sin^2(\theta) + p_{n,3} \cos^2(\theta) \right) + \mathcal{O}(\lambda^2).$$  

(60)

Hence the averaged dynamics is simply $E S_{\lambda,n}(\theta) = \theta - \lambda \overline{E}(p_{n,1}) \sin(2\theta) + \mathcal{O}(\lambda^2)$ which has $\theta = 0$ as a stable and $\theta = \frac{\pi}{2}$ as an unstable fixed point, and no other fixed points. The averaged dynamics is hyperbolic and any initial point $\theta_0$ ultimately reaches the stable fixed point 0, except if $\theta_0 = \frac{\pi}{2}$ is the unstable fixed point. The first aim is to show that this behavior is stable in the sense that the invariant measure $\nu_\lambda$ converges to a Dirac peak on the stable fixed point as $\lambda \to 0$. This is actually a delicate endeavor as will become apparent shortly. Once $\nu_\lambda$ is analyzed, Lyapunov exponent and variance can be deduced.

4.4.1 Random dynamics on $S^1$ with two fixed points

Let us consider for a function $F \in C^2(S^1)$

$$F(S_{\lambda,n}(\theta)) = F(\theta) + \lambda p_n(\theta) F'(\theta) + \mathcal{O}(\lambda^2\|F\|_{C^2}).$$

where $p_n(\theta) = -p_{n,1} \sin(2\theta) - p_{n,2} \sin^2(\theta) + p_{n,3} \cos^2(\theta)$. Thus the associated ergodic sums satisfy

$$I_{\lambda,N}(F) = \frac{1}{N} \sum_{n=0}^{N-1} E[F(\theta_n)] = I_{\lambda,N}(F) + \lambda I_{\lambda,N}(E[p_n] F') + \mathcal{O}(\lambda^2\|F\|_{C^2}, N^{-1}),$$

so that for every function $f = \mathcal{M} F$ in the range of the first order differential operator $\mathcal{M} : C^1(S^1) \to C(S^1)$, defined by

$$(\mathcal{M} F)(\theta) = \eta \sin(2\theta) F'(\theta),$$

(61)

one has $I_{\lambda,N}(f) = \mathcal{O}(\lambda\|F\|_{C^2}, N^{-1})$ and thus, upon integration w.r.t. the invariant probability measure $\nu_\lambda$, one has $\nu_\lambda(f) = \mathcal{O}(\lambda\|F\|_{C^2})$. Every function in $\operatorname{Ran}(\mathcal{M})$ has zeroes of order at least 1 in both fixed points, but $\operatorname{Ran}(\mathcal{M})$ contains sufficiently many functions to conclude that the weight of $\nu_\lambda$ is of order $\mathcal{O}(\lambda^{\frac{1}{2}})$ outside of $\lambda^{\frac{1}{2}}$-neighborhoods of the fixed points. Indeed, let us choose $F(\theta) = -\cos(2\theta)/2\eta$ so that $f(\theta) = (\mathcal{M} F)(\theta) = \sin^2(2\theta)$. Thus $I_{\lambda}(\sin^2(2\cdot)) = \mathcal{O}(\lambda)$ so that, using the indicator function $\chi_A$ on $A = \{ \theta \in S^1 | \sin^2(2\theta) > \lambda^{\frac{1}{2}} \}$ which is bounded by $\chi_A(\theta) \leq \lambda^{-\frac{1}{2}} \sin^2(2\theta)$, one deduces $\nu_\lambda(A) = \mathcal{O}(\lambda^{\frac{1}{2}})$. 
Remark It is possible to implement this line of argument for many other functions. For example, taking $F(\theta) = \sin(2\theta)$ one obtains $I_\lambda(\sin(4\cdot)) = \mathcal{O}(\lambda)$. Analogously, with $F(\theta) = \sin((2 + 4k)\theta)$ for $k \in \mathbb{N}$, one obtains inductively $I_\lambda(\sin(4k\cdot)) = \mathcal{O}(\lambda)$. Similarly, one obtains $I_\lambda(\cos(4k\cdot)) = 1 + \mathcal{O}(\lambda)$ for $k \in \mathbb{N}$. Furthermore, using directly the dynamics (60)

$$
\frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} \theta_n = \frac{1}{N} \sum_{n=0}^{N-2} \mathbb{E} \theta_n - \lambda \frac{1}{N} \sum_{n=0}^{N-2} \mathbb{E} \sin(2\theta_n) + \mathcal{O}(\lambda^2, N^{-1}),
$$

so that $I_\lambda(\sin(2\cdot)) = \mathcal{O}(\lambda)$ and, by the above procedure, $I_\lambda(\sin(2k\cdot)) = \mathcal{O}(\lambda)$ for all $k \in \mathbb{N}$. \(\diamondsuit\)

However, the above strategy does not allow to say what the distribution of $\nu_\lambda$-weight of balls centered at the two fixed points 0 and $\frac{\pi}{2}$ actually is. For example, one cannot evaluate $I_\lambda(\cos(2\cdot))$ by this method (its value is then equal to $I_\lambda(\cos((2 + 4k)\cdot))$ for $k \in \mathbb{N}$) because the function $\theta \mapsto \cos(2\theta)$ takes different values at the two fixed points. To go further it is necessary to zoom the dynamics into the unstable neighborhood of this point. This will be done by a change of variables induced by a special Möbius transformation, in a manner explained in [SS]:

$$
\left(\begin{array}{cc}
\lambda^{-\frac{1}{2}} & 0 \\
0 & 1
\end{array}\right) \exp\left(\lambda\begin{pmatrix} p_1 & p_2 \\ p_3 & -p_1 \end{pmatrix}\right) \left(\begin{array}{cc}
\lambda^{\frac{1}{2}} & 0 \\
0 & 1
\end{array}\right) = \exp\left(\lambda^{\frac{1}{2}}\begin{pmatrix} 0 & p_2 \\ 0 & -p_1 \end{pmatrix}\right) + \lambda\begin{pmatrix} p_1 & 0 \\ 0 & -p_1 \end{pmatrix} + \mathcal{O}(\lambda^3),
$$

where the abbreviation $p_{n,j} = p_j$ was used. The dynamics induced by the r.h.s. will be denoted by $\hat{S}_{\lambda,n} : S^1_\pi \to S^1_\pi$ and the associated random dynamics by $(\hat{\theta}_n)_{n \geq 0}$. All objects in this rescaled representation will carry a hat in the sequel. One finds from (51)

$$
\hat{S}_{\lambda,n}(\hat{\theta}) = \hat{\theta} - \lambda^{\frac{1}{2}} p_{n,2} \sin^2(\hat{\theta}) + \lambda\left(\frac{1}{2} p_{n,2}^2 \sin^2(\hat{\theta}) - p_{n,1}\right) \sin(2\hat{\theta}) + \mathcal{O}(\lambda^3).
$$

Now the lowest order term (in $\lambda^{\frac{1}{2}}$) is centered, and it is precisely its square of its order that has a non-vanishing mean. Hence one is a situation where the techniques of [SS], in particular Proposition 3(vii), can be applied. However, we present a self-contained argument here in order to show that almost all $\nu_\lambda$-weight lies in a neighborhood of the stable fixed point 0.

**Proposition 43.** Suppose $\mathbb{E}[p_2^2] > 0$. Let $\chi_B$ be the indicator function on $B = \{\theta \in S^1_\pi | |\theta| > \lambda^{\frac{1}{2}}\}$. Then

$$
\nu_\lambda(B) = \mathcal{O}(\lambda^{\frac{3}{2}}).
$$

**Proof:** Underlying the above basis change to the rescaled dynamics is the diffeomorphism $t_\lambda : S^1_\pi \to S^1_\pi$ defined by

$$
t_\lambda(\hat{\theta}) = \tan^{-1}(\lambda^{-\frac{1}{2}} \tan(\hat{\theta})),
$$

It zooms into the unstable fixed point $\frac{\pi}{2}$ and keeps 0 conserved. Then $\hat{S}_{\lambda,n} = t_\lambda^{-1} \circ S_{\lambda,n} \circ t_\lambda$ and one is led to study the random rescaled dynamics $(\hat{\theta}_n)_{n \geq 0}$ given
by $\hat{\theta}_n = t_{\lambda}^{-1}(\theta_n)$. The aim is to control the Birkhoff sums $\hat{I}_N(\hat{f}_{\lambda})$ of the rescaled dynamics for the functions $\hat{f}_{\lambda} = f \circ t_{\lambda}$, which are readily connected to the original ones via

$$I_{\lambda,N}(f) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} f(\theta_n) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} f(t_{\lambda}(\hat{\theta}_n)) = \hat{I}_{\lambda,N}(\hat{f}_{\lambda}).$$

With the aim of controlling such Birkhoff sums let us again expand for a given function $\hat{F} \in C^2(S^1_{\pi})$ using (63) and $\mathbb{E}(p_2) = 0$:

$$\mathbb{E} \hat{F}(\hat{S}_{\lambda,n}(\hat{\theta})) = \hat{F}(\hat{\theta}) + \lambda \frac{\mathbb{E}(p_2^2)}{2} \left( \sin^4(\hat{\theta}) \hat{F}''(\hat{\theta}) + (\sin^2(\hat{\theta}) - 2k) \sin(2\hat{\theta}) \hat{F}'(\hat{\theta}) \right) + O(\lambda^3 \|\hat{F}\|_{C^2}),$$

where $k = \mathbb{E}(p_1)/\mathbb{E}(p_2^2) > 0$. Defining a first order differential operator $\hat{\mathcal{L}} : C^1(S^1_{\pi}) \to C(S^1_{\pi})$ by

$$\left( \hat{\mathcal{L}} \hat{g} \right)(\hat{\theta}) = \sin^4(\hat{\theta}) \hat{g}'(\hat{\theta}) + (\sin^2(\hat{\theta}) - 2k) \sin(2\hat{\theta}) \hat{g}(\hat{\theta}),$$

one concludes that the Birkhoff sums satisfy

$$\hat{I}_{\lambda,N}(\hat{F}) = \hat{I}_{\lambda,N}(\hat{F}') + \lambda \frac{\mathbb{E}(p_2^2)}{2} \hat{I}_{\lambda,N}(\hat{\mathcal{L}} \hat{F}') + O(\lambda^3 \|\hat{F}\|_{C^2}, N^{-1}).$$

Therefore the function $\hat{\rho} = \hat{\mathcal{L}} \hat{F}'$ satisfies

$$\hat{I}_{\lambda,N}(\hat{\rho}) = O\left( \lambda^3 \|\hat{F}\|_{C^2}, N^{-1} \right).$$

Let us choose in (65)

$$\hat{\rho}(\hat{\theta}) = \exp\left( -k \lambda^2 \cot^2(\hat{\theta}) \right),$$

then solve $\hat{\rho} = \hat{\mathcal{L}} \partial_{\hat{\theta}} \hat{F}$ for $\hat{F}$ and show that this solution satisfies $\|\hat{F}\|_{C^2} \leq C$ uniformly in $\lambda$ (note that $\hat{F}$ depends on $\lambda$ via $\hat{\rho}$). Hence one can first solve the first order equation $\hat{\mathcal{L}} \hat{g} = \hat{f}$ for $\hat{g}$. There is actually a two-parameter family of solutions $\hat{g}$ in the neighborhood of the singularity 0 due to Proposition 3(vii) in [SS] (because the order of singularities and the signs of the coefficients are precisely as required there). The first parameter is fixed by requiring $\hat{g}$ to be continuous on all $S^1_{\pi}$ (periodicity) and the second one by adding an adequate multiple of the (smooth non-negative) ground state $\hat{g}_0$ satisfying $\hat{\mathcal{L}} \hat{g}_0 = 0$, explicitly given by

$$\hat{g}_0(\hat{\theta}) = \frac{\exp\left( -k \sin^{-2}(\hat{\theta}) \right)}{\sin^2(\hat{\theta})}.$$

This assures that $\hat{g}$ has vanishing integral and therefore has an antiderivative $\hat{F}$. This explicit solution $\hat{g} = \hat{F}'$ of $\hat{\mathcal{L}} \hat{g} = \hat{\rho}$ is (as in [SS])

$$\hat{F}'(\hat{\theta}) = \frac{\exp\left( -k \sin^{-2}(\hat{\theta}) \right)}{\sin^2(\hat{\theta})} \left( C_1 + \int_{C_2}^{\hat{\theta}} \hat{\rho}(\hat{s}) \frac{\exp\left( k \sin^{-2}(\hat{s}) \right)}{\sin^2(\hat{s})} d\hat{s} \right).$$
The function $\hat{F}'$ and its derivative $\hat{F}''$ are obviously regular away on an interval $I \subset S_1^+$ not containing $\hat{\theta} = 0$ with uniform bounds on $I$ that are uniform in $\lambda$ (because $\hat{\rho} \leq 1$ uniformly in $\lambda$). More delicate is the behavior at $\hat{\theta} = 0$ because the integrand become singular. However, by de l’Hôpital’s rule:

$$
\lim_{\hat{\theta} \to 0} \hat{F}'(\hat{\theta}) = \lim_{\hat{\theta} \to 0} \frac{C_1 + \int_{C_2} \hat{\rho}(\hat{s}) \sin^{-2}(\hat{s}) \exp\left(k \sin^{-2}(\hat{s})\right) d\hat{s}}{\exp\left(k \sin^{-2}(\hat{\theta})\right) \sin^2(\hat{\theta})}
= \lim_{\hat{\theta} \to 0} \frac{\hat{\rho}(\hat{\theta}) \sin^{-2}(\hat{\theta})}{(1 - \cos(2\hat{\theta}) - 2k) \cot(\hat{\theta})},
$$

which vanishes due to the special form of $\hat{\rho}$. Next calculating $\hat{F}'''$ from the differential equation $\hat{L}\hat{F}' = \hat{f}$,

$$
\lim_{\hat{\theta} \to 0} \hat{F}'''(\hat{\theta}) = \lim_{\hat{\theta} \to 0} \frac{\hat{\rho}(\hat{\theta}) - \left(\sin^2(\hat{\theta}) - 2k\right) \sin(2\hat{\theta}) \hat{F}'(\hat{\theta})}{\sin^4(\hat{\theta})}.
$$

Since both denominator and numerator tend to 0 for $\hat{\theta} \to 0$, de l’Hôpital’s rule can be applied. However, using it directly is not helpful and it is better to first divide both by $(\sin^2(\hat{\theta}) - 2k) \sin(2\hat{\theta}) \hat{F}'(\hat{\theta})$ and define $\hat{\rho}(\hat{\theta}) = \frac{\hat{\rho}(\hat{\theta})}{(\sin^2(\hat{\theta}) - 2k) \sin(2\hat{\theta})}$ and use that $(\frac{1}{\sin^2(\theta)})' = \frac{1}{(\sin^2(\theta))'} = \frac{1}{\cos^2(\theta)}$. Thus one obtains

$$
\lim_{\hat{\theta} \to 0} \hat{F}'''(\hat{\theta}) = \lim_{\hat{\theta} \to 0} \frac{\hat{\rho}'(\hat{\theta})}{1 + \hat{\rho}'(\hat{\theta})} = 0.
$$

In concludes that the $C^1$-norm of $\hat{F}'$ is uniformly bounded in $\lambda$. Let us note that [SS, Proposition 3] presents an iterative procedure showing that also further derivatives of $\hat{F}$ existent and are bounded.

Now let us consider the transformed $\rho(\theta) = \hat{f} \circ t_{1,\lambda}^{-1}(\theta) = \exp\left(-k \lambda^{-\frac{1}{2}} \cot(\theta)\right)$. Then $I_{\lambda, N}(\rho) = \hat{I}_N(\hat{\rho}) = \mathcal{O}(\lambda^{\frac{1}{2}} \|\hat{F}'\|_{C^2})$. Now set $A' = \{\theta \in S_1^+ | ||\theta - \frac{\pi}{2}|| \leq \lambda^{\frac{1}{2}}\}$. Then there is a constant $C$ such that $\chi_{A'} \leq C \rho$ and therefore $\nu_\lambda(A') \leq C' \lambda^{\frac{3}{2}}$. On the other hand, it was already shown above that the set $A = \{\theta \in S_1^+ | \sin^2(2\theta) > \lambda^{\frac{3}{2}}\}$ satisfies $\nu_\lambda(A) = \mathcal{O}(\lambda^{\frac{3}{2}})$. Combining these estimates concludes the proof.

**Corollary 44.** Suppose $\mathbb{E}[p_2^2] > 0$. For every function $f \in C^1(S_1^+)$

$$
\nu_\lambda(f) = f(0) + \mathcal{O}(\|f\|_{C^1} \lambda^{\frac{3}{2}}),
$$

while for $f \in C^2(S_1^+)$

$$
\nu_\lambda(f) = f(0) + \mathcal{O}(\|f\|_{C^2} \lambda^{\frac{3}{2}}).
$$

In particular, every sequence $(\nu_{\lambda_k})_{k \geq 1}$ of invariant measures with $\lambda_k \to 0$ converges weakly to a Dirac peak at the stable fixed point $0$. 

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Proof: Let $B = \{ \theta \in S^1_n \mid |\theta| > \lambda^\frac{1}{2} \}$ and $B^c = S^1_n \setminus B$. Then by Proposition 43
\[
\nu_\lambda(f) = O(\|f\|_\infty \lambda^{\frac{1}{2}}) + \int_B f(\theta) \nu_\lambda(d\theta).
\]
Expanding $f(\theta)$ in a Taylor series of order 1 in $\lambda$ implies the first claim. For the second, let us replace $f$ by $\hat{f}(\theta) = f(\theta) - \frac{1}{2} f'(0) \sin(4\theta)$ so that $\hat{f}'(0) = 0$. As $\nu_\lambda(\sin(4\cdot)) = O(\lambda)$ by the argument preceding (62), one has $\nu_\lambda(\hat{f}) = \nu_\lambda(f) + O(\lambda)$. Now a Taylor series of $\hat{f}$ to second order in $\lambda$ leads to the second claim. \qed

4.4.2 Evaluation of the Lyapunov exponent and its variance

Corollary 44 can be applied to the function $f(\theta) = \cos(2\theta)$ and implies $I_\lambda(\cos(2\cdot)) = 1 + O(\lambda^{\frac{1}{2}})$. Furthermore, it was already shown using (62) that $I_\lambda(\sin(2\cdot)) = O(\lambda)$. This allows us to calculate the Lyapunov exponent based on the expansion (54) up to order $\lambda:
\[
\gamma_\lambda = \lambda \eta + O(\lambda^{\frac{3}{2}}).
\]
This shows the first formula in Theorem 39(ii) under the condition $E[p_2^2] > 0$ on the variance of the entries of $P_n$. The second claim in Theorem 39(ii) is an estimate on the variance $\sigma_\lambda$. This will be based on (53). As it is already known that $g_1$, $g$ and $\gamma_\lambda$ are of order $O(\lambda)$, it is thus sufficient to show that $J_\lambda(g) = O(\lambda^{\frac{1}{2}})$, which results from the following

Proposition 45. Assuming $\theta_0 = 0$. Then for any $f \in C^{3+\alpha}(S)$ with $\alpha > 0$
\[
J_\lambda(f) = O(\lambda^{-\frac{1}{2}}).
\]

Proof. Again let us use the rescaled dynamics defined in (63). Let $\hat{\lambda}$ be defined by (64) and set $\hat{f}_\lambda = f \circ t_\lambda$ as above. Furthermore, let $\hat{\nu}_\lambda$ be the invariant measure of the rescaled dynamics. Then $\nu_\lambda(f) = \hat{\nu}_\lambda(\hat{f})$ and there is a connection between the correlation sums of the two processes:
\[
J_\lambda(f) = \mathbb{E}_1 \left[ \sum_{n=1}^{\infty} \left( f(\theta_n) - \nu_\lambda(f) \right) \right] = \mathbb{E}_1 \left[ \sum_{n=1}^{\infty} \left( \hat{f}(\theta_n) - \hat{\nu}_\lambda(\hat{f}_\lambda) \right) \right] = \hat{J}_\lambda(\hat{f}_\lambda).
\]
For $\hat{f} \in C^{1+\alpha}(S^1_n)$ with $\hat{f} = \hat{\lambda} \hat{F}'$, a Taylor expansion with some Hölder continuous $\hat{g}$ gives
\[
\hat{J}_\lambda(\hat{F}') = \hat{F}'(\theta_0) - \hat{\nu}_\lambda(\hat{F}) + \hat{J}_\lambda(\hat{F}) - \lambda \hat{J}_\lambda(\hat{\lambda} \hat{F}') + \lambda^\frac{3}{2} \hat{J}_\lambda(\hat{g}),
\]
so that
\[
\hat{J}_\lambda(\hat{\lambda} \hat{F}') = \frac{1}{\lambda} \left( \hat{F}'(\theta_0) - \hat{\nu}_\lambda(\hat{F}) \right) + \lambda^\frac{3}{2} \hat{J}_\lambda(\hat{g}).
\]
Because $\gamma_\lambda = O(\lambda)$, the $a\ priori$ estimate gives $\hat{J}_\lambda(\hat{g}) = O(\lambda^{-\frac{1}{2}})$ and one hence obtains $\hat{J}_\lambda(\hat{\lambda} \hat{F}') = O(\lambda^{-\frac{1}{2}})$. As mentioned before, every Hölder continuous function

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with zero at 0 lies in the range of \( \hat{\mathcal{L}} \). Furthermore, \( \hat{J}_\lambda \) is invariant under the translation by a constant. Thus, \( \hat{J}_\lambda(\hat{g}) = \hat{J}_\lambda(\hat{g} - \hat{g}(0)) \). By (65) the invariant measure \( \hat{\nu}_\lambda \) is concentrated at 0, so the difference \( \hat{F}(\theta_0) - \hat{\nu}_\lambda(\hat{F}) \) is of order \( \lambda^{\frac{1}{2}} \) because \( \theta_0 = 0 \). Recursively it follows that \( \hat{J}_\lambda(\hat{f}) = O(\lambda^{-\frac{3}{2}}) \) for every function \( \hat{f} \in C^{3+\alpha}(S^1_n) \). Hence for a function \( f_\lambda = \hat{f} \circ \tau^{-1}_\lambda \) one has

\[
J_\lambda(f_\lambda) = \hat{J}_\lambda(\hat{f}) = O(\lambda^{-\frac{3}{2}}).
\]

Next for a given smooth function \( f \) let us define

\[
\tilde{f}(\theta) = f(\theta) - f(0) - (f(\frac{\pi}{2}) - f(0)) \sin(t\lambda^{-1}(\theta)) - c_\lambda \sin(2t\lambda^{-1}(\theta)),
\]

with the constant defined by

\[
c_\lambda = \frac{1}{\pi} \int_{S^1_n} \frac{f(\theta) - f(0) - (f(\frac{\pi}{2}) - f(0)) \sin(t\lambda^{-1}(\theta))}{\sin(2\theta)} d\theta < \infty.
\]

This constant is uniformly bounded in \( \lambda \) as can be seen by analyzing the contributions around the singularities. For example, for small \( a > 0 \) one has \( \int_{-a}^{a} \frac{f(\theta) - f(0)}{\sin(2\theta)} d\theta \leq C \) and using that also

\[
\int_{-a}^{a} \frac{\sin(t\lambda^{-1}(\theta))}{\sin(2\theta)} d\theta = \int_{-a}^{a} \left( \frac{\lambda \tan^2(\theta)}{1 + \lambda \tan^2(\theta)} \right)^{\frac{1}{2}} \frac{1}{\sin(2\theta)} d\theta \leq \int_{-a}^{a} \left( \frac{\tan^2(\theta)}{1 + \tan^2(\theta)} \right)^{\frac{1}{2}} \frac{1}{\sin(2\theta)} d\theta,
\]

is bounded, one sees that also the second summand has a uniformly bounded contribution. Furthermore \( \tilde{f}(0) = \tilde{f}(\frac{\pi}{2}) = 0 \) and

\[
\int_{S^1_n} \frac{\sin(2t\lambda^{-1}(\theta))}{\sin(2\theta)} d\theta = \int_{S^1_n} \lambda^{\frac{1}{2}} \sin(\lambda\theta^2) + \cos(\theta) d\theta = \int_{S^1_n} (t\lambda^{-1})'(\theta) d\theta = \pi,
\]

so that also \( \int_{S^1_n} \frac{\tilde{f}(\theta)}{\sin(2\theta)} d\theta = 0 \). By (61) one concludes that \( \tilde{f} \) lies in the range of \( \mathcal{M} \).

By (67) one now concludes \( J_\lambda(\hat{f}) = J_\lambda(f) + O(\lambda^{-\frac{3}{2}}) \) and it thus only remains to analyze the correlation sum of \( \tilde{f} = \mathcal{M}F \). Expanding \( F \in C^{2+\alpha}(S^1_n) \) in Taylor series and using the operator \( \mathcal{M} \) defined in (61), one gets

\[
J_\lambda(F) = F(\theta_0) - \nu_\lambda(F) + J_\lambda(F) - \lambda \eta J_\lambda(\mathcal{M}F) + \lambda^2 J_\lambda(g),
\]

for some Hölder continuous function \( g \), since \( F \in C^{2+\alpha}(S) \). By Proposition 40 and \( \gamma_\lambda = O(\lambda) \), \( g \) satisfies the a priori bound \( J_\lambda(g) = O(\lambda^{-2}) \). Hence

\[
J_\lambda(\mathcal{M}F) = \frac{1}{\lambda \eta} \left( F(\theta_0) - \nu_\lambda(F) + O(1) \right),
\]

so that \( J_\lambda(\mathcal{M}F) = O(\lambda^{-1}) \). By the above applied to \( g \) one has \( J_\lambda(g) = J_\lambda(\tilde{g}) + O(\lambda^{-\frac{3}{2}}) = O(\lambda^{-1}) \). Thus actually \( J_\lambda(\mathcal{M}F) = O(\lambda^{-\frac{3}{2}}) \). This finishes the proof. \( \square \)
4.5 Second order anomalies

At a second order anomaly, the term of order $O(\lambda)$ in (51) is centered. It is then not necessary to carry out a basis change $M$, but we nevertheless set $P_n = P_n$ and $Q_n(\lambda) = Q_n(\lambda)$. Let us begin by recalling some results from [SB, SS] which are needed below. First of all, one is naturally led to study the second order differential operator on $C^2(S^1)$ defined by

$$\mathcal{L} = \frac{1}{2} E_n[p_n^2] \partial_\theta^2 + E_n[q_n + \frac{1}{2} p_n p'_n] \partial_\theta,$$

because for $F \in C^2(S^1)$ one then has

$$E_n F(S_n, \lambda(\theta)) = F(\theta) + \lambda^2 (\mathcal{L} F)(\theta) + O(\lambda^3).$$

From this one can again control Birkhoff sums in the range of $\mathcal{L}$. For sake of simplicity (weaker hypothesis are possible), it will be assumed that $E_n[p_n^2] > 0$. This assures that $\mathcal{L}$ is elliptic. Let also introduce the formal adjoint $\mathcal{L}^* = \partial_\theta^2 \frac{1}{2} E_n[p_n^2] - \partial_\theta E_n[q_n + \frac{1}{2} p_n p'_n]$. Then [SS, Theorem 3] implies the following where scalar product and orthogonal complement are taken in $L^2(S^1)$.

**Theorem 46.** Under the above ellipticity hypothesis, one has $\dim(\text{Ran}(\mathcal{L})) = 1$ and there is a unique smooth and non-negative function $\rho \in \text{Ker}(\mathcal{L}^*)$ with unit integral. For every function $f \in C^1(S^1)$ one has

$$I_{\lambda,N}(f) = \langle \rho | f \rangle + O(\lambda, (\lambda N)^{-1}),$$

where $\langle \rho | f \rangle = \int_{S^1} f(\theta) \rho(\theta) \, d\theta$.

Using Theorem 46 and (52) as well as (54), it is straightforward to write out the expansion for the Lyapunov exponent given in Theorem 39(iii) with rigorous control on the error terms. The formula for $C_s$ was already given in [SB] and invokes $\rho$ which, in general, cannot be calculated explicitly. For the calculation of the variance, one needs again to control $J_\lambda(f)$.

**Lemma 47.** For a function $f \in C^{5+\alpha}(S^1)$ with $\alpha > 0$ the correlation sum $J_\lambda(f)$ is given by

$$J_\lambda(f) = -\frac{1}{\lambda^2}(F(\theta_0) - \nu_\lambda(F)) + O(\frac{1}{\lambda}),$$

with a periodic function $F = \mathcal{L}^{-1}(f - \langle f | \rho \rangle)$ defined uniquely up to a constant.

**Proof.** Let us first recall [SS] that the range $\text{Ran}(\mathcal{L})$ includes all smooth functions that are orthogonal to $\rho$ in the $L^2$-sense. Hence $F$ is well-defined. Furthermore, $J_\lambda$ is invariant under shifts by a constant. Therefore $J_\lambda(f) = J_\lambda(\mathcal{L} F)$. As in the proof of Lemma 42 a Taylor expansion of $F \in C^{5+\alpha}(S^1)$ shows

$$J_\lambda(F) = F(\theta_0) - \nu_\lambda(F) + J_\lambda(F) + \lambda^2 J_\lambda(\mathcal{L} F) + \lambda^3 J_\lambda(g),$$

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where $g$ is a Hölder-continuous residual function for which an priori bound $J_\lambda(g) = O(\lambda^{-3})$ holds. From this one deduces $J_\lambda(LF) = O(\lambda^{-2})$ and hence $J_\lambda(f) = O(\lambda^{-2})$. As all this equally well applies to $g$, namely $J_\lambda(g) = O(\lambda^{-2})$, the equation actually already implies the claim. □

The calculation of $\sigma_\lambda$ will again be based on (53). By (54) one has $g_n = \lambda h_n + \lambda^2 f_n + O(\lambda^3)$ with trigonometric polynomials $h_n$ and $f_n$. By hypothesis, one has $E[h_n] = 0$. Hence the function $g = E_n[g_n]$ has an expansion given by $g = \lambda^2 f + O(\lambda^3)$ with $f = E[f_n]$. Let $F = \mathcal{L}^{-1}(f - \langle \rho | f \rangle)$. Cancelling out the $\lambda$-factors, one then concludes from Lemma 47

$$J_\lambda(g) = \nu_\lambda(F) - F(\theta_0) + O(\lambda).$$

As $\gamma_\lambda = O(\lambda^2)$, one thus has

$$\sigma_\lambda = E_{\nu_\lambda \mathcal{E}_1} \left[ \lambda^2 h_1^2 + (\lambda h_1 + \lambda^2 f_1) J_\lambda(g) \right] + O(\lambda^3).$$

Replacing $J_\lambda(g)$ one sees that due to $E[h_1] = 0$ the variance satisfies $\sigma_\lambda = O(\lambda^2)$. In principle the above formula gives all the lowest order contributions, provided the contribution of order $O(\lambda)$ to $J_\lambda(g)$ can be calculated. This provides $C'_s$ and it appears to be a coincidence for it to be equal to $C_s$.

### 4.6 Examples

#### 4.6.1 Harmonic chain

The first model to be studied is the harmonic chain with a stationary equation of motion

$$-m_n \omega^2 u_n = u_{n+1} - 2u_n + u_{n-1}.$$

For $n$ from 1 to $N$, where $u_n$ is a displacement of an $n$-th atom from its equilibrium position and $m_n$ it's i.i.d. randomly distributed positive mass. Furthermore the distribution function should have compact support bounded away from zero. The displacements $u_n$ are assumed to depend on time monochromatically with frequency $\omega$ an the strength of the harmonic strings between the nearest neighboring atoms is taken to be unity. The boundary condition for a finite chain is given by $u_0 = u_{N+1} = 0$ and for the semi-infinite chain is just $u_0 = 0$. Then the transfer matrix is given by

$$T_{n,\omega} = \begin{pmatrix} 2 - \omega^2 m_n & -1 \\ 1 & 0 \end{pmatrix}.$$

This is a $\text{SL}(2, \mathbb{R})$ matrix, which is conjugate to a rotation, but lies in the vicinity of the Jordan block. So first it should be transformed to reveal this structure. By the basis change with the matrix $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ one obtains

$$M_1 T_{n,\omega} M_1^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - m_n \omega^2 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$
Now one blows up the vicinity of the stable point by conjugation with the matrix
\[ M_2 = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \]
\[ M_2 M_1 T_{n,\omega} (M_2 M_1)^{-1} = 1 + \omega \begin{pmatrix} 0 & 1 \\ -m_n & 0 \end{pmatrix} - \omega^2 \begin{pmatrix} m_n & 0 \\ 0 & 0 \end{pmatrix}. \]

The last basis change is done with the matrix
\[ M_3 = \begin{pmatrix} -\sqrt{E[m_n]} & 0 \\ 0 & 1 \end{pmatrix} \]
\[ T_{n,\omega} = M_3 M_2 M_1 T_{n,\omega} (M_3 M_2 M_1)^{-1} = 1 + \omega \begin{pmatrix} 0 & -\sqrt{E[m_n]} \\ m_n & 0 \end{pmatrix} - \omega^2 \begin{pmatrix} m_n & 0 \\ 0 & 0 \end{pmatrix}. \]

This is a first order elliptic order model, thus the Lyapunov exponent \( \gamma_\omega \) is given by
\[ \gamma_\omega = \frac{1}{8} \omega^2 \frac{E[v_n^2] - E[m_n]^2}{E[m_n]} + O(\omega^3) = \sigma_\omega + O(\omega^3). \]

as well as the fluctuation coefficient \( \sigma_\omega \). The Lyapunov exponent is known at least since the paper of [Ish].

### 4.6.2 Anderson model at band edges

Next model to be studied is the tight binding electron model with the stationary Schrödinger equation with an energy \( E \)
\[ E \psi_n = \psi_{n+1} + \psi_{n-1} + \lambda v_n \psi_n. \] (68)

For \( n \) from 1 to \( N \), where \( \psi_n \) is the probability amplitude and \( v_n \) is the randomly i.i.d. distributed site energy of an electron on site \( n \). The distribution function is assumed to be bounded and centered on a compact interval. The transfer energy between the nearest neighboring sites is taken to be unity. As boundary condition let us choose \( \psi_0 = 0 \). The transfer matrix is given by
\[ T_{n,\omega} = \begin{pmatrix} E - \lambda v_n & -1 \\ 1 & 0 \end{pmatrix}. \]

For \( \lambda = 0 \), one has an anomaly for \( |E| = 2 \). Let us focus on one of the cases and choose \( E = 2 + w \lambda \) for some \( w \in \mathbb{R} \). The behaviour of this system depends strongly on the sign of \( w \). If \( w < 0 \) one has an elliptic first order anomaly. From the technical point of view, this problem can be dealt with the same procedures as the previous one substituting \( \omega^2 \) by \( \lambda \) and \( E[m_n] \) by \(-w\). In this case the Lyapunov exponent \( \gamma_\lambda \) and the perturbation coefficient \( \sigma_\lambda \) are given by
\[ \gamma_\lambda = \frac{1}{8} \lambda \frac{E[v_n^2] - w^2}{w} + O(\lambda^\frac{3}{2}) = \sigma_\lambda + O(\lambda^\frac{3}{2}). \] (69)
The case \( w > 0 \) is hyperbolic first order anomaly. The basis changes are done by the matrices \( M_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \), \( M_2 = \begin{pmatrix} \lambda \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \) and \( M_3 = \begin{pmatrix} \frac{w}{2} & 1 \\ -\frac{w}{2} & 1 \end{pmatrix} \) respectively.

In this case the Lyapunov exponent \( \gamma_\lambda \) is given by

\[
\gamma_\lambda = \lambda \frac{1}{2} w \frac{1}{2} + \mathcal{O}(\lambda^{\frac{3}{4}}).
\]

and the perturbation coefficient \( \sigma_\lambda \) is not exceeding the order \( \mathcal{O}(\lambda^{\frac{3}{4}}) \). A numerical simulation shows, that it could be even smaller.

### 4.6.3 Kronig-Penney model

This model was already studied by the authors in [DKS]. The Kronig-Penny model describes the motion of a particle in a one-dimensional system with singular potentials. The case of weighted \( \delta \)-potentials with random weights \( v_n \) on the points of the lattice \( \mathbb{Z} \) was considered. The distribution of the \( v_n \)'s is assumed to have compact support and is nondegenerate, i.e. is not concentrated in a single point. Its expectation value is denoted by \( \bar{v} \) and the centered random weights are denoted by \( \tilde{v}_n = v_n - \bar{v} \). The so-called critical energies of the system are \( E_l = (\pi l)^2, l \in \mathbb{N} \).

The transfer matrix for energies \( E_l - \varepsilon \) on the left of the spectral boundary of the unperturbed system after a basis change is

\[
T_n^{E_l - \varepsilon} = R_{-\eta \varepsilon^{\frac{1}{2}}} \begin{pmatrix} 1 & \varepsilon \frac{1}{2} \tilde{v}_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \varepsilon \frac{1}{2} \tilde{v}_n \\ 1 \end{pmatrix} - \varepsilon \frac{\bar{v}}{4E_l} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \varepsilon \frac{\tilde{v}_n}{2E_l} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathcal{O}(\varepsilon^{\frac{3}{2}}).
\]

Where \( R_{-\eta \varepsilon^{\frac{1}{2}}} \) is a rotation with an angle \( \eta = \sqrt{\frac{\bar{v}}{2E_l}} \). This is again an elliptic case, where \( \varepsilon^{\frac{1}{2}} \) plays the role of \( \lambda \). Then the Lyapunov exponent \( \gamma^{E_l - \varepsilon} \) and the perturbation coefficient \( \sigma^{E_l - \varepsilon} \) are up to the order \( \varepsilon^{\frac{3}{2}} \) given by

\[
\gamma^{E_l - \varepsilon} = \frac{E_l v_n^2 - \bar{v}^2}{16 \bar{v} E_l} \varepsilon + \mathcal{O}(\varepsilon^{\frac{3}{2}}) = \sigma^{E_l - \varepsilon} + \mathcal{O}(\varepsilon^{\frac{3}{2}}).
\]

The transfer matrix for energies \( E_l + \varepsilon \) on the right of the spectral boundary of the unperturbed system after a basis change is

\[
T_n^{E_l + \varepsilon} = \begin{pmatrix} 1 - \eta \varepsilon^{\frac{1}{2}} & 0 \\ 1 + \eta \varepsilon^{\frac{1}{2}} & 0 \end{pmatrix} + \varepsilon^{\frac{1}{2}} \frac{\tilde{v}_n}{2\sqrt{2\bar{v} E_l}} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \varepsilon \frac{v_n}{4E_l} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \mathcal{O}(\varepsilon^{\frac{3}{2}}).
\]

This is a hyperbolic case. The Lyapunov exponent \( \gamma^{E_l + \varepsilon} \) is given by

\[
\gamma^{E_l + \varepsilon} = \frac{\bar{v}}{2E_l} \varepsilon^{\frac{1}{2}} + \mathcal{O}(\varepsilon^{\frac{3}{2}}).
\]

and the perturbation coefficient \( \sigma^{E_l + \varepsilon} \) is of higher order \( \mathcal{O}(\varepsilon^{\frac{3}{2}}) \). The Lyapunov exponents were already obtained by the authors in the previous work [DKS], but the proofs and the error estimates there are given too briefly.
5 Localization and Chern numbers for weakly disordered BdG operators

This chapter consists of the work done in the publication [DDS]. After a short discussion of various random Bogoliubov-de Gennes (BdG) model operators and the associated physics, the Aizenman-Molchanov method is applied to prove Anderson localization in the weak disorder regime for the spectrum in the central gap. This allows to construct random BdG operators which have localized states in an interval centered at zero energy. Furthermore, techniques for the calculation of Chern numbers are reviewed and applied to two non-trivial BdG operators, the $p+ip$ wave and $d+id$ wave superconductors.

5.1 Introductory comments

BdG Hamiltonians describing the electron gas in a superconductor are of the block form

$$ H_{\mu} = \frac{1}{2} \begin{pmatrix} h - \mu & \Delta \\ -\Delta & -(\bar{h} - \mu) \end{pmatrix} \tag{75} $$

where the operator $h = h^*$ acting on a one-particle complex Hilbert space $\mathcal{H}$ with complex structure $\mathbb{C}$ describes a single electron, $\mu \in \mathbb{R}$ is the chemical potential, and $\Delta$, also an operator on $\mathcal{H}$, is called the pairing potential or pair creation potential. In (75) and below, the complex conjugate of an operator $A$ on $\mathcal{H}$ is defined by $\overline{A} = \mathbb{C}A\mathbb{C}$. The pairing potential satisfies the so-called BdG equation

$$ \Delta^* = -\overline{\Delta}, \tag{76} $$

assuring the self-adjointness of $H_{\mu}$. Throughout this work both $h$ and $\Delta$ are bounded operators. Hence $H_{\mu}$ is a bounded self-adjoint operator on the particle-hole Hilbert space $\mathcal{H}_{ph} = \mathcal{H} \otimes \mathbb{C}^2_{ph}$. The factor $\mathbb{C}^2_{ph}$ is called the particle-hole fiber. In the associated grading, the BdG Hamiltonian has the particle hole symmetry (PHS)

$$ K^* H_{\mu} K = -H_{\mu}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{77} $$

The BdG Hamiltonian (75) is obtained from the BCS model by means of a self-consistent mean-field approximation [dG]. In the associated second quantized operator on Fock space (quadratic in the creation and annihilation operators), the off-diagonal entries $\Delta$ and $-\overline{\Delta}$ lead to annihilation and creation of Cooper pairs respectively. Various standard tight-binding models for $\Delta$ are described in Section 5.2 below.

There are various reasons to consider the operator entries of $H_{\mu}$ to be random [AZ]. First of all, in a so-called dirty superconductor one can have a random potential just as in any alloy or semiconductor. Moreover, it is reasonable to model the mean field by a random process (even though random in time may seem more adequate).
Hence all entries of (75) can be random operators. For mesoscopic systems it is even reasonable to assume these entries to be random matrices [AZ]. However, in the models considered this paper a spatial structure is conserved by supposing that both $h$ and $\Delta$ only contain finite range hopping operators on $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^r$ where $r$ is the number of internal degrees of freedom and the complex structure is induced by complex conjugation.

BdG Hamiltonians having only the symmetry (77) are said to be in Class D of the Altland-Zirnbauer (AZ) classification. If furthermore a time-reversal symmetry is imposed, one obtains the Classes AIII and DIII depending on whether spin is even or odd. Particularly interesting are also models with a SU(2) spin rotation invariance. Then [AZ, DS] the Hamiltonian (with odd spin) can be written as a direct sum of spinless Hamiltonians $H_{\mu}^{\text{red}}$ satisfying

$$I^* H_{\mu}^{\text{red}} I = -H_{\mu}^{\text{red}}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (78)$$

This is also a PHS, but an odd one because $I^2 = -1$, while the PHS (77) is said to be even because $K^2 = 1$. Operators with an odd PHS are said to be in the AZ Class C. It is also possible to have a spin rotation symmetry only around one axis (namely, a U(1)-symmetry), which can then be combined with a time reversal symmetry (TRS) and this leads to operators lying in other AZ Classes [SRFL]. Let us point out one immediate consequence of the PHS (either even or odd):

**Proposition 48.** If $H = H^*$ has a PHS, then the spectrum satisfies $\sigma(H) = -\sigma(H)$. Therefore the energy 0 is a reflection point of the spectrum and hence special. It is furthermore shown in Section 5.3 below that the integrated density of states for covariant BdG Hamiltonians is symmetric around 0 and that it is generic that 0 either lies in a gap or in a pseudo gap.

Since the late 1990’s there has been a lot of interest in topological properties of BdG Hamiltonians which can be read of the Fermi projection $P_{\mu} = \chi(H_{\mu} \leq 0)$ on particle-hole Hilbert space $\mathcal{H}_{\text{ph}}$, namely the spectral projection on all negative energy states of $H_{\mu}$. Here the focus is on the two-dimensional case $d = 2$. Then the Fermi projection can have non-trivial Chern numbers. For disordered systems, these invariants are defined as in [BES] and enjoy stability properties, see Section 5.5. Non-triviality of the Chern number makes the system into a so-called topological insulator [SRFL]. This leads to a number of interesting physical phenomena. For Class D, one has a quantized Wiedemann-Franz law [Vis, SF] and Majorana zero energy states at half-flux tubes [RG], while for Class C one is in the regime of the spin quantum Hall effect [SMF, RG]. A rigorous analysis of these effects will be provided in [DS]. For periodic models, the Chern numbers can be calculated using the transfer matrices or the Bloch functions. Two techniques to carry out these calculations are discussed and applied in Section 5.6.

Anderson localization is of importance for topological insulators just as it is for the quantum Hall effect. It provides localized states near zero energy which are
responsible for the stability and measurability of effects related to the topological invariants. For this purpose, it is interesting to have a mathematical proof of Anderson localization in the relevant regimes of adequate models, and to show that the topological invariants are indeed stable. The present paper provides such proofs for BdG models in the weakly disordered regime and shows that localized states near zero energy can be produced by an adequate choice of the parameters, see the discussion after Theorem 49 in Section 5.4 resumed in Figure 4. Let us point out that the localization proof transposes directly to yet other classes of models of interest, the chiral unitary class (AZ Class AIII which also has spectral symmetry as in the one in Proposition 48) as well as operators with odd time reversal symmetry (AZ Class AII), but no detailed discussion is provided for these cases. For the chiral unitary class this is an important input for the existence of non-commutative higher winding numbers [PS].

On a technical level, this is achieved basically by combining known results. The Aizenman-Molchanov method [AM] in its weak disorder version [Aiz] provides a framework that can be followed closely, with adequate modifications related to the fact that one has to deal with matrix valued potentials and hopping amplitudes. This has recently be tackled by Elgart, Shamis and Sodin [ESS], but these authors focused on the strong disorder regime and the models do not seem to cover quite what is needed in connection with the questions addressed above. Furthermore, our arguments seem (to us) a bit more streamlined, and are closer to the original analysis in [Aiz]. The stability of the Chern numbers under disordered perturbation is then obtained just as in [RS]. The multiscale analysis [FS] is another method, historically the first one, to prove localization. This approach has been followed by Kirsch, Müller, Metzger and Gebert in [KMM, GM] for certain models of BdG type in Class CI.

5.2 BdG Hamiltonians in tight-binding representation

This section merely presents the basic tight-binding models used to model and numerically analyze dirty superconductors with particular focus on the form of the pairing potentials. These terms are of relevance both for high-temperature superconductors (e.g. [WSS, Sca]) as well as for topological insulators [SRFL]. The electron Hilbert space $\mathcal{H}$ in (75) is chosen to be $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^r$. The $r \in \mathbb{N}$ internal degrees of freedom are used to describe a spin as well as possibly a sublattice degree of freedom, or larger unit cells (e.g. [ASV] contains a very detailed description of the sublattice degree for the honeycomb lattice as well as various spin-orbit interactions). Let us focus on the two-dimensional case $d = 2$, and, just for sake of concreteness, let the one-electron Hamiltonian be given by

$$ h = S_1 + S_1^* + S_2 + S_2^* + \lambda \sum_{l \in \mathbb{Z}^2} \pi_l^* V_l \pi_l. \quad (79) $$

Here $S_1$ and $S_2$ are the shift operators on $\ell^2(\mathbb{Z}^2)$, $\pi_l^* : \mathbb{C}^r \to \mathcal{H}$ is the partial isometry onto the spin and internal degrees of freedom over site $l \in \mathbb{Z}^d$ and the $V_l = V_{l^*}$ are
After discrete Fourier transform is often chosen to be translation invariant and the numbers characterizing under the change. Here $E_n$ are odd under a 90 degree rotation. Hence the eigenvalues are of such translation invariant pairing potentials are presented. They do not cover all cases studied in the literature [WSS, Sca], but hopefully the most important ones. Here is the list, expressed in terms of the shift operators and the spin operators $s^1, s^2, s^3$ represented on $\mathbb{C}^{2s+1}$, which is part of the fiber $\mathbb{C}^r$. Several of the pairing potentials are graphically represented in Figure 1. Thinking of atomic orbitals, this also explains the nomenclature.

\[
\begin{align*}
\Delta_s &= \delta_s \iota s^2, \\
\Delta_{s^*} &= \delta_{s^*} (S_1 + S_1^* + S_2 + S_2^*) \iota s^2, \\
\Delta_{p_x} &= \delta_{p_x} (S_1 - S_1^*) s^1, \\
\Delta_{p_y} &= \delta_{p_y} (S_1 - S_1^*) s^1, \\
\Delta_{p_{\pm}} &= \delta_p (S_1 - S_1^* \pm \iota(S_2 - S_2^*)) , \\
\Delta_p &= \delta_p s^1, \\
\Delta'_p &= \delta_p (S_1 - S_1^* \pm \iota(S_2 - S_2^*) s^3), \\
\Delta_{d_{xy}} &= \delta_{d_{xy}} (S_1 - S_1^*)(S_2 - S_2^*) \iota s^2, \\
\Delta_{d_{x^2-y^2}} &= \delta_{d_{x^2-y^2}} (S_1 + S_1^* - S_2 - S_2^*) \iota s^2, \\
\Delta_{d_{\pm}} &= \delta_{d_{\pm}} \pm \iota \Delta_{d_{xy}}, \\
\end{align*}
\]

All the constants $\delta$ are real so that one readily checks that (76) holds in all cases. Here $p$-wave pairing potentials correspond to hopping terms which are anti-symmetric under the change $S_j \leftrightarrow S_j^*$, while $s$-wave and $d$-wave pairing potentials are symmetric under this change. Furthermore, the $s$-wave is rotation symmetric and the $d$-wave odd under a 90 degree rotation $(S_1, S_2, S_1^*, S_2^*) \leftrightarrow (S_2, S_1^*, S_2^*, S_1)$. Next follow two very concrete examples that show that the pairing potential can open a central gap (namely, a gap of $H_\mu$ around zero energy). In Section 5.5 it is discussed under which circumstances these models lead to non-trivial topology of the Bloch bundles.

**Example 1** This example is about a spinless $p \pm \iota p$ model which is hence in Class D and is relevant for the thermal quantum Hall effect (see [Vis, SF, DS] for details). The Hilbert space is simply $l^2(\mathbb{Z}^2)$, namely $r = 1$. The one-electron Hamiltonian $h$ is (79) with $\lambda = 0$ and the pairing potential given by (83). Thus the BdG Hamiltonian is

\[
H_\mu = \frac{1}{2} \left( S_1 + S_1^* + S_2 + S_2^* - \mu \delta_p (S_1 - S_1^* \pm \iota(S_2 - S_2^*)) \right).
\]

After discrete Fourier transform

\[
H_{\mu}(k) = \begin{pmatrix}
\cos(k_1) + \cos(k_2) - \frac{\mu}{2} & \delta_p (\iota \sin(k_1) \mp \sin(k_2)) \\
\delta_p (-\iota \sin(k_1) \mp \sin(k_2)) & -\cos(k_1) - \cos(k_2) + \frac{\mu}{2}
\end{pmatrix}.
\]

Hence the eigenvalues are

\[
E_\eta(k) = \eta \left( (\cos(k_1) + \cos(k_2) - \frac{\mu}{2})^2 + \delta_p^2 (\sin^2(k_1) + \sin^2(k_2)) \right)^{\frac{1}{2}}, \quad \eta \in \{-1, 1\}.
\]

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This shows that a central gap of size \( g(\delta_p, \mu) = 2 \min_{k \in \mathbb{T}^2} E_+(k) \) opens if \(|\mu| \neq 0, 4\) and \( \delta_p \neq 0 \). If \( \mu = 0 \), the gap is closed for all \( \delta_p \). Furthermore, the gap satisfies the upper bound \( g(\delta_p, \mu) \leq 2E_+(0, \pi) = |\mu| \). For \( \mu \) sufficiently small compared to \( \delta_p \), one even has \( g(\delta_p, \mu) = |\mu| \) so that the gap closes linearly in \(|\mu|\).

Example 2

A \( d \pm i d \) model is of interest in connection with the spin quantum Hall effect (see [SMF, RG, DS]). Here the spin is \( s = \frac{1}{2} \) so that \( r = 2 \). Again \( h \) is the discrete Laplacian (tensorized with \( 1 \) on the spin degree of freedom) and \( \Delta \) is given by (88). As this contains an \( r s^2 \) in the spin component, the interaction as well as \( h \) are SU(2) invariant. For sake of simplicity let us assume \( \delta_{d,2,2} = \delta_{d,x} = \delta_d \). The BdG Hamiltonian \( H_\mu \) is a \( 4 \times 4 \) matrix, but it can be written as a direct sum \( H_\mu = H_\mu^+ \oplus H_\mu^- \) of \( 2 \times 2 \) operator matrices \( H_\mu^\pm \) satisfying the odd PHS \( I^* H_\mu^\pm I = -H_\mu^\pm \), so that \( H_\mu^\pm \) are in Class C. Their Fourier transforms \( H_\mu^\pm(k) \) are given by

\[
\begin{pmatrix}
\cos(k_1) + \cos(k_2) - \frac{\mu}{2} & \delta_d (\cos(k_1) - \cos(k_2) \mp i \sin(k_1) \sin(k_2)) \\
\delta_d (\cos(k_1) - \cos(k_2) \pm i \sin(k_1) \sin(k_2)) & -\cos(k_1) - \cos(k_2) + \frac{\mu}{2}
\end{pmatrix}
\]

Such a direct sum decomposition is always possible in presence of an SU(2)-invariance [AZ, DS]. The two Bloch bands of \( H_\mu^\pm \), indexed by \( \eta \in \{-1, 1\} \), are

\[
E_\eta(k) = \eta \left( (\cos(k_1) + \cos(k_2) - \frac{\mu}{2})^2 + \delta_d^2 (\cos(k_1) \cos(k_2) - 1)^2 \right)^{\frac{1}{2}}.
\]  

Again a central gap opens for \( \delta_d \neq 0 \) and \(|\mu| \neq 0, 4\) and its size satisfies \( g(\delta_d, \mu) \leq 2E_+(0, 0) \) and \( g(\delta_d, \mu) \leq 2E_+(\pi, \pi) \), so that \( g(\delta_d, \mu) \leq |4 - |\mu||. \)
Figure 2: Schematic representation of the DOS $\rho^{(2)}$ and corresponding $\rho$ in the two generic situations for periodic operators in dimension $d = 2$: with pseudo gap and true gap.

5.3 Density of states of covariant BdG Hamiltonians

This short section discusses an extension of Proposition 48, namely a symmetry of the integrated density of states (IDS) of a covariant family of BdG operators (with even or odd PHS). Let $S_j$, $j = 1, \ldots, d$, denote the shifts on $H = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^r$, naturally extended to the particle-hole Hilbert space $H_{ph} = H \otimes \mathbb{C}^2_{ph}$. A strongly continuous family $A = (A_\omega)_{\omega \in \Omega}$ of bounded operators on $H_{ph}$ is called covariant if

$$S_j A(\omega) S_j^{-1} = A(T_j \omega), \quad j = 1, \ldots, d. \quad (90)$$

Here $\Omega$ is a compact space (of disorder or crystalline configurations) which is furnished with an action $T = (T_1, \ldots, T_d)$ of the translation group $\mathbb{Z}^d$. Furthermore, there is given an invariant and ergodic probability measure on $\Omega$. Let us now consider a family $H = (H(\omega))_{\omega \in \Omega}$ of BdG Hamiltonians satisfying the covariance relation (90). We are interested in the cases, where $H$ has a well-defined IDS $\mathcal{N}$. By general ergodic principles, this is given for a large variety of models, where the potential only depends on the distance between the sites, not on the sites itself, see [CFKS] and [ESS]. Usually, one chooses the normalization condition $\mathcal{N}(-\infty) = 0$, but here we rather choose to impose $\mathcal{N}(0) = 0$ in view of Proposition 48. With this normalization one has for $E \geq 0$

$$\mathcal{N}(E) = \mathbf{E} \ Tr \langle 0 | \chi_{[0,E]}(H(\omega)) | 0 \rangle, \quad \mathcal{N}(-E) = -\mathbf{E} \ Tr \langle 0 | \chi_{[-E,0]}(H(\omega)) | 0 \rangle,$$

where $\text{Tr}$ denotes the trace over $\mathbb{C}^r \otimes \mathbb{C}^2_{ph}$ and $\mathbf{E}$ the average over the invariant measure on $\Omega$, and the notation $\langle n | A | m \rangle = \pi_n A \pi_m^* \in \text{Mat}(r \times r; \mathbb{C})$ is used in order to stress the similarity with the scalar case $r = 1$. Now the BdG symmetry implies

$$\mathcal{N}(E) = -\mathcal{N}(-E). \quad (91)$$
Furthermore, the IDS $\mathcal{N}$ can be nicely expressed in terms of the IDS $\mathcal{N}^{(2)}$ of the positive operator $H^2$ given by

$$\mathcal{N}^{(2)}(E) = \mathcal{E} \text{ Tr} \langle 0 | \chi_{[0,E]}(H(\omega)^2)|0 \rangle.$$  

Using the symmetry (91) one finds for $E \geq 0$

$$\mathcal{N}(E) = \frac{1}{2} \sqrt{\mathcal{N}^{(2)}(E^2)}.$$  

If $\mathcal{N}$ and $\mathcal{N}^{(2)}$ are absolutely continuous with density of states (DOS) $\rho(E)$ and $\rho^{(2)}(E)$ respectively, then one deduces

$$\rho(E) = |E| \rho^{(2)}(E^2).$$

This shows that generically a periodic BdG operator in dimension $d=2$ either has a gap in the DOS or a so-called linear pseudo-gap, namely the DOS $\rho$ vanishes linearly at 0, up to lower order corrections. These two generic cases are illustrated in Figure 2, while non-generic cases are given in Figure 3.

### 5.4 Localization for BdG Hamiltonians

The one-particle BdG operators considered in this section act on the particle-hole Hilbert space $H_{ph} = H \otimes \mathbb{C}^2_{ph}$ with $H = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^r$. It is of the form

$$H_{\mu,\lambda} = H_{\mu,0} + \lambda V, \quad V = \sum_{l \in \mathbb{Z}^d} \sum_{|j| \leq R} v_{j,l} \pi^*_l \pi_{l+j} W_j$$

where $\lambda > 0$ is a coupling constant, $\pi_l : H_{ph} \rightarrow \mathbb{C}^r \otimes \mathbb{C}^2_{ph}$ is the partial isometry onto the spin and particle-hole space over site $l \in \mathbb{Z}^d$, and for each $j$ the matrices $W_j$ act on $\mathbb{C}^2 = \mathbb{C}^r \otimes \mathbb{C}^2_{ph}$, and the $v_{j,l}$ are real random numbers. It is assumed that $W_{-j} = W_j^*$ and $v_{j,l} = v_{-j,l+j}$ which assures that $V$ is self-adjoint. The operator $H_{\mu,\lambda}$ is supposed to satisfy the even or odd PHS and possibly TRS, depending on which symmetry class is to be described. Furthermore, $H_{\mu,0}$ is supposed to be a translation invariant (or at least periodic) BdG operator with chemical potential $\mu$ and interest will be mainly in the situation where $H_{\mu,0}$ already contains a central spectral gap, e.g. opened by a constant pairing potential. The framework of (93) allows, in particular, to cover all the models discussed in Section 5.2, but also allows for random or periodic pairing potentials and spin orbit interactions. Some examples of matrices $W_j$ are

$$W_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W_{(1,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad W_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

Then $W_{(0,0)}$ allows to model a random potential contained in $h$ as in (79), but also allows to vary the chemical potential via $H_{\mu',\lambda} = H_{\mu,\lambda} + (\mu' - \mu) \mathbf{1}_{\ell^2(\mathbb{Z}^d)} \otimes W_{(0,0)}$. On the other hand, $W_{(1,0)}$ and $W_{(0,1)}$ can be used to describe a random pairing potential.
Figure 3: Schematic representation of the DOS $\rho^{(2)}$ and corresponding $\rho$ in various non-generic cases for periodic operators in dimension $d = 2$. Note that the first picture would be generic in dimension $d = 4$ (linear vanishing of the DOS at the bottom of the spectrum of $H^2$), while the third one is generic in dimension $d = 1$ (square root singularity at the bottom of the spectrum of $H^2$).
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for spinless \( p \pm ip \) waves. Other random pairing potentials form the list of Section 5.2 can also be described by operators of the form (93). There is a number of technical hypothesis imposed on all these objects:

**Hypothesis:** The \( v_{j,l} = v_{-j,l+j} \) are independent random variables for all \( l \in \mathbb{Z}^d \) and \( |j| \leq R \). For each fixed \( j \) the distributions \( \nu_j \) of the \( v_{j,l} \) are identically \( (in \ l) \) and uniformly \( \alpha \)-Hölder continuous with \( 1 - \nu_j([−a,a]) \) decaying in a faster than any polynomial.

Let us introduce the \( 2r \times 2r \) Green matrices at energy \( z \in \mathbb{C} \) by

\[
G_z^{\mu,\lambda}(n,m) = \pi_n(z-H_{\mu,\lambda})^{-1}\pi_m^*.
\]

**Theorem 49.** Suppose that the random BdG Hamiltonian satisfies the above hypothesis. Let \( I \subset \mathbb{R} \) be a compact spectral interval with \( D = \text{dist}(I,\sigma(H_{\mu,0})) > 0 \) and let \( s < 1 \) be sufficiently small. Then there exist constants \( C_1, C_2 \) and \( C_3 \) such that for \( \lambda \leq C_1 D^{1+s} \) the Hilbert-Schmidt norm of the Green matrix satisfies

\[
\mathbf{E}\left(\|G_z^{\mu,\lambda}(n,m)\|_2^s\right) \leq C_2 e^{-s C_3 D |n-m|},
\]

for all \( z = E + i\epsilon \) with \( E \in I \) and independently of \( \epsilon \neq 0 \).

Before starting with the proof let us investigate under which circumstance the statement of the theorem is not void, namely when there are energies in the spectrum of \( H_{\mu,\lambda} \) satisfying the hypothesis needed to prove the exponential decay (outside of the spectrum it already holds due to a simple Combes-Thomas estimate stated in Proposition 51 below). This point was already discussed by Aizenman [Aiz], but here it is, moreover, of particular importance to produce models with localized states at zero energy. Suppose that one only adds a random potential (that is, only the term with \( W_{(0,0)} \) above). If the support of \( \nu = \nu_{(0,0)} \) is \( [-r,r] \), then the almost sure spectrum of \( H_{\mu,\lambda} \) grows from the band edges at least as \( \lambda r \) until the central gap closes. More precisely, by a standard probabilistic argument the spectrum of the BdG operator with the above random potential is equal to the spectrum of the deterministic BdG model with chemical potential shifted from \( \mu > 0 \) to \( \mu - \lambda r \). Assuming that the gap is closed for the periodic model with vanishing chemical potential (as in the two examples), the gap of the random model is closed for \( \lambda \geq \frac{r}{\mu} \). Furthermore, it closes linearly in the two examples considered above (and thus also Fig. 4), but this is not important for the following. On the other hand, the condition \( \lambda \leq C_1 D^{1+s} \) assures a localization regime by Theorem 49 and this condition is independent of \( r \) (as long as the moments of the distribution of the random potential remain uniformly bounded). Hence choosing \( r \) sufficiently large guarantees the existence of an interval centered at zero energy containing only localized states. Let us mention that this does not address the important question about the fate of the states near a pseudo gap of a periodic model when a random perturbation is added (raising the density of states in the pseudo gap of the periodic model). Indeed it was supposed here that the gap for the model without disorder is opened by some mechanism such as the pairing potential.
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Another comment is that the Hypothesis does not contain any minimal coupling condition. In fact, this would be necessary in a strong coupling regime, but for weak disorder results one only shows that the decay of the free Green functions (given by Combes-Thomas) is conserved under adequate perturbations. Before proceeding to the proof, let us note a consequence of (94) that is important for the definition of the Chern numbers in the next section. The following corollary can be deduced by a technique put forward in [AG], see also Theorem 5.1 in [PS] for a detailed argument. Physically interesting is only the case $E = 0$ and for this case it pends on Theorem 49 only if there is spectrum at 0.

Corollary 50. Let $I \subset \mathbb{R}$ be a compact energy interval on which (94) holds. Then for any $E \in I$ which is not a boundary point of $I$ and any $\alpha > 0$, the Fermi projection $P_\mu(E) = \chi(H_\mu \leq E)$ satisfies

$$E \| \langle n | P_\mu(E) | m \rangle \| < C_4 |n - m|^{-\alpha},$$

where $C_4$ is a constant depending on $\alpha$.

The proof of Theorem 49 uses the resolvent identity

$$G^z_{\mu,\lambda}(n, m) = G^z_{\mu,0}(n, m) + \lambda \sum_{l \in \mathbb{Z}^d} \sum_{|j| \leq R} v_{j,l} G^z_{\mu,0}(n, l + j) W_j G^z_{\mu,\lambda}(j, m).$$

These matrices will be estimated using the Hilbert-Schmidt norm $\| \cdot \|_2$ for matrices acting on the fibers $\mathbb{C}^{2r}$:

$$\| G^z_{\mu,\lambda}(n, m) \|_2 \leq \| G^z_{\mu,0}(n, m) \|_2 + \lambda \sum_{l \in \mathbb{Z}^d} \sum_{|j| \leq R} |v_{j,l}| \| G^z_{\mu,0}(n, l+j) \|_2 \| W_j \|_2 \| G^z_{\mu,\lambda}(l, m) \|_2.$$
Now let us take the expectation value over the randomness, as well as the
∈ η
Let us drop the indices
s< estimate in terms of the matrices
E
operator on ℓ
for adequate energies
z
| Suppose that
Now a uniform upper bound on the
∥∥∥∥ one has
Since
Now recall the bound
∥∥∥∥ which is obtained by a so-called Combes-Thomas estimate (which will be
used only for H_{μ,0} below).

Proposition 51. Suppose that H_{μ,λ} has finite range R, namely π^*_n H_{μ,λ} π_m = 0 for |n − m| > R. Then there are constants C_5, C_6 and C_7 such that
\|
G_{μ,λ}^z (n, m)\|_2 \leq \frac{C_5}{D(z)} \exp \left( -C_6 \arcsinh (C_7 D(z)) |n − m| \right),
with D(z) = \text{dist}(z, \sigma(H_{μ,λ})).

Proof. Let us drop the indices μ, λ and choose one direction j ∈ \{1, ..., d\}. For η ∈ ℝ, set H(η) = e^{η X_j} H e^{-η X_j} where X_j is the jth component of the position operator on ℓ^2(ℤ^d), naturally extended to H_{μ,α}. Then one has the following norm estimate in terms of the matrices H_{n,m} = π^*_n H π_m
\|
H(η) − H\| = \sup_{\|\phi\|=1} \sum_{n,m} \|H_{n,m}(1 − e^{(n_j−m_j)η})φ_m\|^2
\leq \sup_{\|\phi\|=1} \sum_{n,m} \left( \sum_{m} \|H_{n,m}\|^2 |1 − e^{(n_j−m_j)η}|^2 \right) \|φ_m\|^2
\leq \sup_m \left( \sum_{n} \|H_{n,m}\|^2 |1 − e^{(n_j−m_j)η}|^2 \right).

Now a uniform upper bound on the H_{n,m} and |1 − e^x| ≤ 2 \sinh(|x|) implies
\|
H(η) − H\| \leq C_8 \sinh(R|η|).

Now recall the bound \| (1 + B)^{-1} \| ≤ 1 + \| B (1 + B)^{-1} \| holding for any operator B for any operator with invertible 1 + B. Using this for B = (H(η) − H)(H − z)^{-1}, one has
\|
(H(η) − z)^{-1}\| \leq \left( \| (H − z)^{-1}\|^{-1} − \|H(η) − H\| \right)^{-1}.

Since \| (H − z)^{-1}\| ≤ D(z)^{-1}, the choice R|η| = \arcsinh (D(z)/2C_7) therefore leads to the bound \| (H(η) − z)^{-1}\| ≤ 2/D(z). The desired estimate now follows from the
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identity \( \langle n | (H - z)^{-1} | m \rangle = e^{\eta(m_j - n_j)} \langle n | (H(\eta) - z)^{-1} | m \rangle \) by choosing the adequate sign for \( \eta \).

In order to use the Combes-Thomas estimate as a starting point for a perturbative analysis, one has to prove the following decorrelation estimate

\[
E(|v_{j,l}|^s \| G_{\mu,\lambda}^z(l, m) \|_2^s) \leq C_{\alpha,s} E(|v_{j,l}|^s) E(\| G_{\mu,\lambda}^z(l, m) \|_2^s).
\]

Its proof is deferred to the end of this paragraph. Once this is proved, the following subharmonicity argument applied to \( \tau(n) = \tau_0(n, m) \) and \( \tau_0(n) = \tau_0^*(n, m) \) for every fixed \( m \) concludes the proof.

**Lemma 52.** Suppose that \( \tau_0 : \mathbb{Z}^d \to \mathbb{R}_{\geq 0} \) satisfies

\[ \tau_0(n) \leq c_1 e^{\tau_2|n|}. \]

Then, if another function \( \tau : \mathbb{Z}^d \to \mathbb{R}_{\geq 0} \) satisfies the subharmonicity estimate

\[ \tau(n) \leq \tau_0(n) + c_3 \sum_{l \in \mathbb{Z}^d} \tau_0(n - l) \tau(l) \]

with \( c_3 \leq c_4 c_2/c_1 \) and a constant \( c_4 \) sufficiently small only depending on the dimension \( d \), this function satisfies

\[ \tau(n) \leq c_5 e^{\tau_2|n|}, \]

with \( c_5 \) of order 1.

**Proof.** Let us introduce an operator \( T : \ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\mathbb{Z}^d) \) by

\[ (T \tau)(n) = c_3 \sum_{l \in \mathbb{Z}^d} \tau_0(n - l) \tau(l). \]

Then

\[ \|T\|_{\ell^\infty \to \ell^\infty} = \sum_{n \in \mathbb{Z}^d} c_3 \tau_0(n) \leq c_1 c_3 \sum_{n \in \mathbb{Z}^d} e^{\tau_2|n|} < 1, \]

where the last equality holds for \( c_3 c_1/c_2 \) sufficiently small. Iterative application of the subharmonicity inequality then shows

\[ \tau(n) \leq \sum_{i=0}^{\infty} (T^i \tau_0)(n). \]

Let us set \( b(n) = e^{\tau_2|n|} \) and consider this as a multiplication operator on as well as an element of \( \ell^\infty(\mathbb{Z}^d) \). Telescoping then shows

\[ \tau(n) \leq b(n) \sum_{i=0}^{\infty} \left( (b^{-1} T b)^i \right)^{\tau_0} \leq b(n) \sum_{i=0}^{\infty} \|b^{-1} T b\|^i_{\ell^\infty \to \ell^\infty} \frac{\tau_0}{b} \|b\|^i_{\ell^\infty \to \ell^\infty}. \]

Now \( \frac{\tau_0}{b} \) is bounded by hypothesis, and

\[ \|b^{-1} T b\|^i_{\ell^\infty \to \ell^\infty} \leq \sup_n \sum_l e^{\tau_2|n|} c_3 e^{\tau_2|n-l|} e^{-\tau_2|l|} \leq \frac{c_3 c_6}{c_2}. \]
Now supposing again that $c_3$ is sufficiently small such that $c_3c_6 < c_2$, the result follows by summing the geometric series in the above estimate. \hfill \Box

Let us briefly show how this allows to conclude the proof of Theorem 49. Due to Proposition 51, one can choose $c_1 = (C_5/D)^s$ and $c_2 = sC_6C_7DC_9$ with $C_9$ depending on the size of $I$. Furthermore, $c_3 = \lambda^sC_{\alpha,s}C_{10}$ with $C_{10}$ bounding the moments of the random variable and the norms of $W_j$. Then the smallness assumption in Lemma 52 reads $\lambda^sC_{\alpha,s}C_{10} \leq c_4sC_6C_7DC_9(D/C_5)^s$, which is the small coupling assumption in Theorem 49.

For the proof of the decorrelation estimate (96) the dependence of $G_{\mu,\lambda}^z(l, m)$ on $v_{j,l}$ has to be determined in an explicit manner. This can be done via a standard perturbative formula which is written in a manner that can be immediately applied to the present model if one sets $v = v_{j,l}$ and $W = W_j$.

**Lemma 53.** Let us consider the following splitting of the Hamiltonian

$$H_{\mu,\lambda} = \tilde{H}_{\mu,\lambda} + \lambda v(W + W^*) ,$$

with an operator $\tilde{H}_{\mu,\lambda}$ which does not depend on the real parameter $v$. Let $\pi_v : \mathcal{H} \to \text{Ran}(W + W^*)$ be the partial isometry onto $\text{Ran}(W + W^*)$. Then for any $z \in \mathbb{C} \setminus \mathbb{R}$

$$(z - H_{\mu,\lambda})^{-1} = (z - \tilde{H}_{\mu,\lambda})^{-1} - (z - \tilde{H}_{\mu,\lambda})^{-1} \pi_v^* T_v \pi_v (z - \tilde{H}_{\mu,\lambda})^{-1} ,$$

where the finite-dimensional $T$-matrix is given by

$$T_v = \left((\lambda v \pi_v(W + W^*)\pi_v^*)^{-1} - \pi_v(z - \tilde{H}_{\mu,\lambda})^{-1}\pi_v^* \right)^{-1} .$$

The standard algebraic proof of Lemma 53 can be found e.g. in [BS, Lemma 8]. Let us point out that $W + W^* = \pi_v^* \pi_v(W + W^*)\pi_v^* \pi_v$ and that $\pi_v(W + W^*)\pi_v^*$ is invertible by construction. This inverse is again a self-adjoint operator. As the operator $\pi_v(z - \tilde{H}_{\mu,\lambda})^{-1}\pi_v^*$ has positive imaginary part, the inverse defining $T_v$ actually exists.

Next let us suppose $W$ and $W^*$ have finite range. Then $T_v$ is a finite dimensional matrix which can be calculated using Lagrange formula. This shows that $T_v$ is a rational function of $v$. Applying this to the situation sketched above, it follows that $G_{\mu,\lambda}^z(l, m)$ is a rational function of $v_{j,l}$. Therefore the decorrelation estimate (96) follows from the following lemma.

**Proposition 54.** Let $f$ and $g$ be rational functions given by a fraction of polynomials with degree at most $N$. If $\nu$ is a uniformly $\alpha$-Hölder continuous measure with $\nu([-R, R]^c)$ decaying faster than any polynomial, and $s < \alpha/2N$, then uniformly in $f$ and $g$ for some constant $C_{\alpha,s}$ depending only on $s$ and $\alpha$

$$E_{\nu}(|fg|^s) \leq C_{\alpha,s} E_{\nu}(|f|^s) E_{\nu}(|g|^s) ,$$

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Proof. First of all, it is possible to approximate \( \nu \) with a measure with support contained in \([-R, R]\) because \(|f^a, |f|, |g|\) grow at most polynomially and have integral singularities so that all factors can be arbitrarily well approximated for large \( R \). Thus from now on the support of \( \nu \) lies in \([-R, R]\). Let us start from the Cauchy-Schwarz inequality
\[
E_\nu(|f^a|) \leq E_\nu(|f|^{2s})^{1/2} E_\nu(|g|^{2s})^{1/2}.
\]
Hence it is sufficient to show that
\[
\frac{E_\nu(|f|^{2s})^{1/2}}{E_\nu(|f|)} \leq (C_{\nu,s})^{1/2},
\]
for all rational functions \( f \). Let \( a = (a_1, \ldots, a_{N'}) \) and \( b = (b_1, \ldots, b_{N''}) \) be the zeros of the numerator and denominator respectively with \( N', N'' \leq N \), namely
\[
f(x) = \frac{\prod_{a=1}^{N'} (x - a_n)}{\prod_{n=1}^{N''} (x - b_n)}.\]
Let us introduce the function
\[
F(a, b) = \frac{E_\nu(|f|^{2s})^{1/2}}{E_\nu(|f|)}.\]
It will be shown that \( F \) is continuous in \((a, b)\) and is bounded by a uniform constant outside of the ball \(|(a, b)|_\infty \leq 2R\). The continuity of \((a, b) \mapsto E_\nu(|f|)\) and \((a, b) \mapsto E_\nu(|f|^{2s})\) follows from the integrability of the singularities (resulting from the Hölder continuity of \( \nu \) with sufficiently large \( \alpha \)) and the fact that the integral of a continuous family of integrable functions is again continuous. Furthermore, the function \((a, b) \mapsto E_\nu(|f|)\) is bounded from below by 0 (uniformly on every compact). Both of these facts require some standard analytical verification using the Hölder continuity of \( \nu \). Combined they show the continuity of \( F \). It thus follows that \( F \) is bounded on every compact set, in particular on \(|(a, b)|_\infty \leq 2R\).

Now suppose that \(|(a, b)|_\infty > 2R\). Let \( a_1, \ldots, a_{M'} \) and \( b_1, \ldots, b_{M''} \) be the zeros with modulus smaller than or equal to \( 2R \) (after having renumbered). Let us set
\[
g(x) = \frac{\prod_{a=1}^{M'} (x - a_n)}{\prod_{n=1}^{M''} (x - b_n)}, \quad G(a, b) = \frac{E_\nu(|g|^{2s})^{1/2}}{E_\nu(|g|)}.
\]
Note that \( G \) only depends on the first \( M' \) and \( M'' \) of the \( a \)'s and \( b \)'s. By the same argument as for \( F \), \( G \) is uniformly bounded on the set defined by \(|a_n| \leq 2R\) for \( n' = 1, \ldots, M' \) and \(|b_n| \leq 2R\) for \( n = 1, \ldots, M'' \). For \( n' > M' \) and \( n > M'' \) one has \(|a_n| \geq 2R\) and \(|b_n| \geq 2R\) and therefore, for every \( x \in [-R, R] \), the following bounds hold
\[
|a_n' - x| \leq |a_n' - x| + R, \quad |b_n - x| \leq |b_n - x| + R.
\]
Consequently
\[
F(a, b) \leq \left( \prod_{n' = M'+1}^{N'} |a_{n'}| + R \prod_{n = M'+1}^{N''} |b_n| + R \right)^s G(a, b) \leq 3^{(N' + N'')} G(a, b).
\]

As there are a finite number of possibilities to choose \(G\) and the bound is independent of \(R\), the claim follows. \(\Box\)

### 5.5 Chern numbers and their stability

First let us recall [BES] the definition of the Chern number of a covariant family \(P = (P(\omega))_{\omega \in \Omega}\) of projections on \(\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^r\):

\[
\text{Ch}(P) = 2\pi i \mathbf{E} \text{Tr} \langle 0 | [P[X_2, P], [X_1, P]] | 0 \rangle,
\]

where \(X_1\) and \(X_2\) are the two components of the position operators on \(\ell^2(\mathbb{Z}^2)\), \(\mathbf{E}\) denotes the disorder average, and the projection is required to satisfy the so-called Sobolev condition

\[
\sum_{j=1,2} \mathbf{E} \text{Tr} \langle 0 | |X_j, P| |^2 | 0 \rangle < \infty.
\]

This condition assures that (97) is well-defined. For the (particle-hole space) Fermi projection \(P_{\mu, \lambda} = \chi(H_{\mu, \lambda} \leq 0)\) the condition (98) holds if the central gap remains open (by the Helffer-Sjöstrand formula combined with a Combes-Thomas estimate) or, due to Corollary 50, if \(E = 0\) lies in the Aizenman-Molchanov localization regime of \(H_{\mu, \lambda}\). On the other hand, the condition (98) also assures that \(\text{Ch}(P)\) is an integer given by the index of a Fredholm operator [BES]. Hence one expects \(\text{Ch}(P)\) to have some homotopy invariance properties. In particular, one may expect \(\text{Ch}(P_{\mu, \lambda})\) to be independent of the disorder coupling constant \(\lambda\) and the chemical potential \(\mu\) under adequate conditions. For quantum Hall systems, this was checked in [RS] and we claim here that the argument directly carries over to the BdG case to prove the following.

**Theorem 55.** Let \(\mathcal{R} \subset \{ (\mu, \lambda) \in \mathbb{R}^2 \}\) be a connected set such that for every \(\lambda\) the energy \(E = 0\) lies in an open interval for which the bound (94) holds for the BdG Hamiltonian \(H_{\mu, \lambda}\) of the form (93). Then \((\mu, \lambda) \in \mathcal{R} \mapsto \text{Ch}(P_{\mu, \lambda}) \in \mathbb{Z}\) is constant.

Due to this stability result, it is particularly important to calculate the Chern number without disorder, that is, for a periodic system with a central gap. This is the object of the next section.

### 5.6 Computation of Chern numbers

Two methods for the calculation are briefly presented in this section and applied to the two examples of Section 5.2. The first one from [ASV] applies to periodic
BdG Hamiltonians $H_{\mu}$ containing only nearest neighbor and next nearest neighbor hopping terms but arbitrarily (large) fibers. It uses merely the transfer matrices combined with basic numerics. The second, more conventional method applies whenever it is possible to write the Hamiltonian as linear combination of Clifford algebra generators [DL]. In order to deal with the examples one merely has to use Pauli matrices. As to the first method, one begins with a partial discrete Fourier transform, say in the $k_1$-direction. Rewriting the fibers of the Hamiltonian as

$$H_{\mu}(k_1) = S_2^* a(k_1) + b(k_1) + a(k_1)^* S_2,$$

defines $2r \times 2r$ matrices $a(k_1)$ and $b(k_1)$. Whenever $a(k_1)$ is invertible (which is almost surely the case), one next sets

$$T(k_1) = \begin{pmatrix} -b(k_1)a(k_1)^{-1} & -a(k_1)^* \\ a(k_1)^{-1} & 0 \end{pmatrix}. $$

This is a $4r \times 4r$ matrix which conserves $I$ given in (78), namely $T(k_1)^* I T(k_1) = I$. The generalized eigenspaces of $T(k_1)$ associated with all eigenvalues of modulus strictly less than 1 (namely the contracting ones) constitute an $2r$-dimensional $I$-Lagrangian plane in $\mathbb{C}^{4r}$. Let a basis of this space form the column vectors of a $4r \times 2r$ matrix $\Phi(k_1)$ and then define a $2r \times 2r$ matrix $U(k_1)$ by

$$U(k_1) = \frac{1}{1 - i 1} \Phi(k_1) \left( \left( \frac{1}{1 - i 1} \Phi(k_1) \right)^{-1} \right).$$

(99)

It turns out that $U(k_1)$ is unitary and that the following holds.

**Theorem 56.** [ASV] Let $0$ lie in a gap of $H_{\mu}$ and set $P_{\mu} = \chi(H_{\mu} \leq 0)$. Then

$$\text{Ch}(P_{\mu}) = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi i} \text{Tr} \left( U(k_1)^* \partial_{k_1} U(k_1) \right).$$

Let us apply this theorem to calculate the Chern number of the $p+ip$ model discussed in Example 1 in Section 5.2. Then $r = 1$ so the unitary of size $2 \times 2$ that is then calculated numerically from the contracting eigenvector of the $4 \times 4$ matrix $T(k_1)$ as a function of $k_1 \in [-\pi, \pi]$. The phase of its eigenvalues is plotted for three sets of parameters in Figure 5.

The second method is illustrated by calculating the Chern number of the Fermi projection of the $d + id$-wave Hamiltonians $H_{\mu}^{\pm}$ given in Example 2 of Section 5.2. First rewrite the Hamiltonian as a linear combination of the Pauli matrices

$$H_{\mu}^{\pm}(k) = \begin{pmatrix} p_3(k) & p_1(k) \mp ip_2(k) \\ p_1(k) \pm ip_2(k) & -p_3(k) \end{pmatrix},$$

where $p_1(k) = \delta_d(\cos(k_1) - \cos(k_2))$, $p_2(k) = \delta_d \sin(k_1) \sin(k_2)$, $p_3(k) = \cos(k_1) + \cos(k_2) - \frac{\mu}{2}$. 

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Figure 5: Plot of the phase $k_1 \in [-\pi, \pi] \mapsto U(k_1)$ for the clean $p+ip$ wave superconductor described in Example 1, with $(\delta_p, \mu) = (0.3, -0.5)$, $(\delta_p, \mu) = (0.3, -0.01)$ and $(\delta_p, \mu) = (0.3, 0.01)$. The Chern numbers of $\text{Ch}(P_\mu)$ are $-1$, $-1$ and $1$ respectively.

Note that at $\mu = 0$ the central gap closes and becomes a pseudo gap, so that there is a transition at this value.

The lower Bloch band is $E_-(k) = -(p_1(k)^2 + p_2(k)^2 + p_3(k)^2)^{\frac{1}{2}}$ and a central gap opens for $\delta_p \neq 0$ and $|\mu| \neq 0, 4$, as already pointed out above. The normalized eigenfunction for the lower eigenvalue, spanning the range of the Fermi projection, can be written in two ways

$$
\psi(k) = C_-(k) \left( p_1(k) \mp ip_2(k) \right), \quad \phi(k) = C_+(k) \left( \frac{E_-(k) + p_3(k)}{p_1(k) \pm ip_2(k)} \right),
$$

where $C_{\pm}(k) = (2E_-(-k)(E_-(k) \pm p_3(k)))^{-\frac{1}{2}}$. However, both of these vector functions may vanish for certain values of $k$ (at which then the normalization constants are singular). In fact, $p_1(k) = p_2(k) = 0$ holds at $k_I = (0,0)$ and $k_{II} = (\pi, \pi)$. For $\mu > 4$, one has $E_-(k) + p_3(k) < 0$ so that $\phi$ defines a global section, implying that the bundle is trivializable and has vanishing Chern number. Similarly, for $\mu < -4$, $\psi$ is a global section so that again the Chern number vanishes. Now let us come to the case $|\mu| < 4$ where both $\phi$ and $\psi$ have zeros in $k_I$ and $k_{II}$ respectively. Let us introduce the transition function $\theta$ defined on $\mathbb{T}^2/\{k_I, k_{II}\}$ by

$$
\phi(k) = e^{i\theta(k)} \psi(k), \quad \theta(k) = \arctan \left( \frac{\pm p_2(k)}{p_1(k)} \right).
$$

Now set $P_\mu^- = \chi(H_\mu^- < 0)$. By [BT, Sect. 20] and using the closed path $\Gamma$ given by $t \in [0, 2\pi] \mapsto (\epsilon \cos(t), \epsilon \sin(t))$ with small $\epsilon > 0$,

$$
\text{Ch}(P_\mu^\pm) = \frac{1}{2\pi} \oint_{\Gamma} \mathrm{d}\theta = \frac{\epsilon}{2\pi} \int_0^{2\pi} \mathrm{d}t \left( -\Theta_1'(t) \sin(t) + \Theta_2'(t) \cos(t) \right),
$$

where $\Theta_j'(t) = (\partial_k, \theta)(\epsilon \cos(t), \epsilon \sin(t))$. A straightforward computation provides

$$
\Theta_1'(t) = \pm \frac{2\sin(t)}{\epsilon} + O(\epsilon), \quad \Theta_2'(t) = \mp \frac{2\cos(t)}{\epsilon} + O(\epsilon).
$$

Replacing and taking the limit $\epsilon \to 0$ shows

$$
\text{Ch}(P_\mu^\pm) = \mp 2, \quad \text{if} \quad |\mu| < 4.
$$
References


References


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