Some distance bounds of branching processes and their
diffusion limits

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Abstract: We compute exact values respectively bounds of “distances” – in the sense of (transforms of) power divergences and relative entropy – between two discrete-time Galton-Watson branching processes with immigration GWI for which the offspring as well as the immigration is arbitrarily Poisson-distributed (leading to arbitrary type of criticality). Implications for asymptotic distinguishability behaviour in terms of contiguity and entire separation of the involved GWI are given, too. Furthermore, we determine the corresponding limit quantities for the context in which the two GWI converge to Feller-type branching diffusion processes, as the time-lags between observations tend to zero. Some applications to (static random environment like) Bayesian decision making and Neyman-Pearson testing are presented as well.

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1. Introduction

It is well known that “distances” in form of (relative-entropy covering) power divergences between finite measures are important for probability theory and statistics as well as their applications to various different research fields such as physics, information theory, econometrics, biology, speech and image recognition, transportation of (sorts of) “mass”, etc. For probability measures \( P_H, P_A \) on a measurable space \((\Omega, \mathcal{A})\) and parameter \( \lambda \in \mathbb{R} \) these power divergences – also known as Cressie-Read measures respectively generalized cross-entropy family – are defined as (see e.g. Liese and Vajda [47], [48])

\[
I_\lambda (P_A\|P_H) := \begin{cases} 
I (P_A\|P_H), & \text{if } \lambda = 1, \\
\frac{1}{\lambda(\lambda-1)} (H_\lambda (P_A\|P_H) - 1), & \text{if } \lambda \in \mathbb{R}\setminus\{0,1\}, \\
I (P_H\|P_A), & \text{if } \lambda = 0,
\end{cases}
\]

(1)

where

\[
I (P_A\|P_H) := \int_{\{p_H > 0\}} p_A \log \frac{p_A}{p_H} d\mu + \infty \cdot P_A(p_H = 0)
\]

(2)

is the relative entropy (Kullback-Leibler information divergence) and

\[
H_\lambda (P_A\|P_H) := \int_{\Omega} p_A^{1-\lambda} p_H^\lambda d\mu
\]

(3)

is the Hellinger integral of order \( \lambda \in \mathbb{R}\setminus\{0,1\} \); for this, we assume as usual without loss of generality that the probability measures \( P_H, P_A \) are dominated by some \( \sigma \)-finite measure \( \mu \), with densities

\[
p_A = \frac{dP_A}{d\mu} \quad \text{and} \quad p_H = \frac{dP_H}{d\mu}
\]

defined on \( \Omega \) (the zeros of \( p_H, p_A \) are handled in (2), (3) with the usual conventions). Apart from the relative entropy, other prominent examples of power divergences are the squared Hellinger distance \( \frac{1}{2} I_{1/2} (P_A\|P_H) \) and Pearson’s \( \chi^2 \)-divergence \( 2 I_2 (P_A\|P_H) \). Extensive studies about basic and advanced general facts on power divergences, Hellinger integrals and the related Renyi divergences of order \( \lambda \in \mathbb{R}\setminus\{0,1\} \)

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\[
R_\lambda (P_A\|P_H) := \frac{1}{\lambda(\lambda - 1)} \log H_\lambda (P_A\|P_H) , \quad \text{with } \log 0 = -\infty ,
\]
can be found e.g. in Liese and Vajda [47], [48], Jacod and Shiryaev [29]. For instance, the integrals in (2) and (3) do not depend on the choice of \( \mu \). As far as finiteness is concerned, for \( \lambda \in ]0,1[ \) one gets the rudimentary bounds
\[
0 \leq I_\lambda (P_A\|P_H) \leq \frac{1}{\lambda(\lambda - 1)} ,
\]
where the lower bound is achieved if and only if \( P_A = P_H \), and the upper bound is achieved if and only if \( P_A \perp P_H \) (singularity). For \( \lambda \notin ]0,1[ \), the power divergences \( I_\lambda (P_A\|P_H) \) and Hellinger integrals \( H_\lambda (P_A\|P_H) \) might be infinite, depending on the particular setup. For the sake of brevity, we only deal here with the case \( \lambda \in ]0,1[ \); the case \( \lambda \notin ]0,1[ \) will appear elsewhere.

Apart from the extensive literature on the relative-entropy cases \( \lambda (1 - \lambda) = 0 \), for \( \lambda (1 - \lambda) \neq 0 \) the evaluation of power divergences \( I_\lambda \) – respectively their straightforward transforms such as Hellinger integrals \( H_\lambda \) and Renyi divergences \( R_\lambda \) – have been investigated for various different contexts of (probability distributions associated with) stochastic processes, such as processes with independent increments (see e.g. Newman [57], Liese [44], Memin and Shiryaev [55], Jacod and Shiryaev [29], Liese and Vajda [47], Linkov and Shevlyakov [53], Poisson point processes (see e.g. Liese [45], Jacod and Shiryaev [29], Liese and Vajda [47]), diffusion processes respectively solutions of stochastic differential equations with continuous paths (see e.g. Kabanov et al. [33], Liese [46], Jacod and Shiryaev [29], Liese and Vajda [47], Vajda [69], Stummer [64], Stummer and Vajda [67]); further related literature can be found e.g. in references of the abovementioned papers and books.

Another important class of time-dynamic models is given by discrete-time branching processes, in particular Galton-Watson processes without immigration GW respectively with immigration GWI, which have numerous applications in biotechnology, population genetics, internet traffic research, clinical trials, asset price modelling and derivative pricing. (Transforms of) Power divergences have been used for supercritical Galton-Watson processes without immigration SUPGW for instance as follows: Feigin and Passy [15] study the problem to find an offspring distribution which is closest (in terms of relative entropy type distance) to the original offspring distribution and under which ultimate extinction is certain. Furthermore, Mordecki [56] gives an equivalent characterization for the stable convergence of the corresponding log-likelihood process to a mixed Gaussian limit, in terms of conditions on Hellinger integrals of the involved offspring laws. Moreover, Sriram and Vidyashankar [62] study the properties of offspring-distribution-parameters which minimize the squared Hellinger distance \( \frac{1}{2} I_{1/2} \) between the model offspring distribution and the corresponding non-parametric maximum likelihood estimator of Guttorp [19]. For the setup of GWI with Poisson offspring and nonstochastic immigration of constant value 1, Linkov and Lunyova [52] investigate the asymptotics of Hellinger integrals in order to deduce large deviation assertions in hypotheses testing problems.

In contrast to the abovementioned contexts, this paper pursues the following main goals:

(MG1) for any time horizon and any criticality scenario, to compute (non-rudimentary) lower and upper bounds – and sometimes even exact values – of the Hellinger integrals \( H_\lambda (P_A\|P_H) \) and power divergences \( I_\lambda (P_A\|P_H) \) \( (\lambda \in ]0,1[) \) of two Galton-Watson branching processes \( P_A, P_H \) with Poisson(\( \beta_A \)) respectively Poisson(\( \beta_H \)) distributed offspring as well as Poisson(\( \alpha_A \)) respectively Poisson(\( \alpha_H \)) distributed immigration. As a side effect, we also aim for corresponding asymptotic distinguishability results in terms of contiguity and entire separation.

(MG2) to compute the corresponding limit quantities for the context in which (a proper rescalation of) the two Galton-Watson processes with immigration converge to Feller-type branching diffusion processes, as the time-lags between the generation-size observations tend to zero.

(MG3) as an exemplary field of application, to indicate how to use the results of (MG1) for Bayesian decision making and Neyman-Pearson testing based on the sample path observations of the GWI-generation sizes, when the hypothesis law is given by \( P_H \) and the alternative law by \( P_A \); in a certain sense, this can also be interpreted in terms of a rudimentary static random environment.

Because of the involved Poisson distributions, these goals (which are potentially reasonable also for other types of offspring resp. immigration distributions) can be tackled with a high degree of tractability, which is
worked out in detail with the following structure: we first deal with the non-relative-entropy case $\lambda(1 - \lambda) \neq 0$. Section 2 contains the first basic result concerning Goal (MG1), which is then deepened in Section 3 in order to obtain – parameter constellation dependent – recursively computable exact values respectively recursively computable lower and upper bounds of $H_\lambda(P_A||P_N)$. Additionally, we construct related closed-form bounds in Section 4, which will also be used to achieve (the Hellinger-integral part of) Goal (MG2) in Section 5. The power divergences $I_\lambda(P_A||P_N)$ are treated in Section 6, complemented with the relative-entropy cases $\lambda(1 - \lambda) = 0$ of the Goals (MG1), (MG2). The subsequent Section 7 is concerned with Goal (MG3), whereas the Appendix contains main proofs and auxiliary lemmas.

2. Process setup and first basic result

Let $X_n$ denote the $n$th generation size of a discrete-time Galton-Watson process with immigration GWI. We use the recursive description

$$X_0 := \omega_0 \in \mathbb{N}; \quad X_n = \sum_{k=1}^{X_{n-1}} Y_{n-1,k} + \bar{Y}_n, \quad n \in \mathbb{N},$$

(5)

where $Y_{n-1,k}$ is the number of offspring of the $k$th object (e.g. organism, person) within the $(n-1)$th generation, and $\bar{Y}_n$ denotes the number of immigrating objects in the $n$th generation. Notice that we employ an arbitrary deterministic initial generation size $X_0$. We always assume that under the law $P_N$ (e.g. a hypothesis),

- the collection $Y := \{Y_{n-1,k}, n \in \mathbb{N}, k \in \mathbb{N}\}$ consists of independent and identically distributed (i.i.d.) random variables which are Poisson distributed with parameter $\beta_H > 0$,
- the collection $\bar{Y} := \{\bar{Y}_n, n \in \mathbb{N}\}$ consists of i.i.d. random variables which are Poisson distributed with parameter $\alpha_H \geq 0$ (where $\alpha_H = 0$ stands for the degenerate case of having no immigration),
- $Y$ and $\bar{Y}$ are independent.

In contrast, under the law $P_A$ (e.g. an alternative) the same is supposed to hold with parameters $\beta_A > 0$ (instead of $\beta_H > 0$) and $\alpha_A \geq 0$ (instead of $\alpha_H \geq 0$). Furthermore, let $(F_n)_{n \in \mathbb{N}}$ be the corresponding canonical filtration generated by $X := (X_n)_{n \in \mathbb{N}}$.

Basic and advanced facts on GWI (introduced by Heathcote [21]) can be found e.g. in the monographs of Athreya and Ney [2], Jagers [30], Asmussen and Hering [3], Haccou [20]; see also e.g. Heyde and Seneta [25], Basawa and Rao [4], Basawa and Scott [6], Sankaranarayanan [59], Wei and Winnicki [71], Winnicki [72], Guttrop [19] as well as Yanev [73] (and also the references therein all those) for adjacent fundamental statistical issues including the involved technical respectively conceptual challenges.

For the sake of brevity, wherever we introduce or discuss corresponding quantities simultaneously for both the hypothesis $\mathcal{H}$ and the alternative $\mathcal{A}$, we will use the subscript $\bullet$ as a synonym for either the symbol $\mathcal{H}$ or $\mathcal{A}$. For illustration, recall the well-known fact that the corresponding conditional probabilities $P_\bullet(X_n = \cdot | X_{n-1} = k)$ are again Poisson-distributed, with parameter $\beta_\bullet \cdot k + \alpha_\bullet$. In order to achieve a transparently representable structure of our results, we subsume the involved parameters as follows: let $P_{SP}$ be the set of all constellations $(\beta_A, \beta_H, \alpha_A, \alpha_H)$ of real-valued parameters $\beta_A > 0$, $\beta_H > 0$, $\alpha_A > 0$, $\alpha_H > 0$, such that $\beta_A \neq \beta_H$ or $\alpha_A \neq \alpha_H$ (or both). Furthermore, we write $P_{NI}$ for the set of all $(\beta_A, \beta_H, \alpha_A, \alpha_H)$ of real-valued parameters $\beta_A > 0$, $\beta_H > 0$, $\alpha_A = \alpha_H = 0$, such that $\beta_A \neq \beta_H$; this corresponds to the important special case of having no immigration. The resulting disjoint union will be denoted by $P = P_{SP} \cup P_{NI}$. A typical situation for applications in our mind is that one particular constellation $(\beta_A, \beta_H, \alpha_A, \alpha_H) \in P$ (e.g. obtained from theoretical or previous statistical investigations) is fixed, whereas – in contrast – the parameter $\lambda \in [0, 1]$ for the Hellinger integral or the power divergence might be chosen freely, e.g. depending on which “probability distance” one decides to choose for further analysis. At this point, let us emphasize that in general we will not make assumptions of the form $\beta_\bullet \geq 1$, i.e. upon the type of criticality.

To start with our investigations, we define the extinction time $\tau := \min\{l \in \mathbb{N} : X_m = 0 \text{ for all integers } m \geq l\}$ if this minimum exists, and $\tau := \infty$ else. Correspondingly, let $\mathcal{B} := \{\tau < \infty\}$ be the extinction set. It is well known that in the case $P_{NI}$ one gets $P_\bullet(\mathcal{B}) = 1$ if $0 < \beta_\bullet \leq 1$ and $P_\bullet(\mathcal{B}) \in [0, 1]$ if $\beta_\bullet > 1$. In
contrast, for $P_{SP}$ there always holds $P_{\bullet}(E) = 0$. Furthermore, for $P_{SP}$ the two laws $P_H$ and $P_A$ are equivalent, whereas for $P_{NI}$ the two restrictions $P_{H|B}$ and $P_{A|B}$ are equivalent (see e.g. Lemma 1.1.3 of Guttorp [19]); with a slight abuse of notation we shall henceforth omit $|_B$. Consistently, for fixed time $n \in \mathbb{N}_0$ we introduce $P_{A,n} := P_A|_{\mathcal{F}_n}$ and $P_{H,n} := P_H|_{\mathcal{F}_n}$ as well as the corresponding Radon-Nikodym-derivative

$$Z_n := \frac{dP_{A,n}}{dP_{H,n}}.$$  

Clearly, $Z_0 = 1$. By using the “rate functions” $f_\bullet(x) = \beta_\bullet x + \alpha_\bullet$ ($x \in [0, \infty]$), a version of (6) can be easily determined by calculating for each $\omega = (\omega_0, ..., \omega_n) \in \Omega := \mathbb{N}_0^n$

$$Z_n(\omega) = \prod_{k=1}^n Z_{n,k}(\omega) \quad \text{with} \quad Z_{n,k}(\omega) := \exp \left\{ - (f_A(\omega_{k-1}) - f_H(\omega_{k-1})) \right\} \frac{[f_A(\omega_{k-1}) - f_H(\omega_{k-1})]^{\lambda \omega_k}}{\omega_k!},$$

where for the last term we use the convention $(\frac{0}{0})^x = 1$ for all $x \in \mathbb{N}_0$. Furthermore, we define for each $\omega \in \Omega_n$

$$Z_{n,k}^{(\lambda)}(\omega) := \exp \left\{ - (\lambda f_A(\omega_{k-1}) + (1 - \lambda) f_H(\omega_{k-1})) \right\} \frac{[f_A(\omega_{k-1})]^\lambda (f_H(\omega_{k-1}))^{1-\lambda \omega_k}}{\omega_k!}$$

with the convention $(\frac{0}{0})^x = 1$ for the last term. Accordingly, with the choice $\mu = P_{H,n}$ one obtains from (3) the Hellinger integral $H_\lambda(P_{A,0}||P_{H,0}) = 1$, as well as for all $(\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P} \times [0, 1[$

$$H_\lambda(P_{A,1}||P_{H,1}) = \exp \left\{ (f_A(\omega_0))^\lambda (f_H(\omega_0))^{1-\lambda} - (\lambda f_A(\omega_0) + (1 - \lambda) f_H(\omega_0)) \right\}$$

and for all $n \in \mathbb{N} \setminus \{1\}$

$$H_\lambda(P_{A,n}||P_{H,n}) = E P_{H,n}[(Z_n)^\lambda] = \sum_{\omega_1=0}^{\infty} \cdots \sum_{\omega_n=0}^{\infty} \prod_{k=1}^n Z_{n,k}^{(\lambda)}(\omega)$$

$$= \sum_{\omega_1=0}^{\infty} \cdots \sum_{\omega_n=0}^{\infty} \prod_{k=1}^{n-1} Z_{n,k}^{(\lambda)}(\omega) \cdot e^{-(\lambda f_A(\omega_{n-1}) + (1 - \lambda) f_H(\omega_{n-1}))}$$

$$= \sum_{\omega_1=0}^{\infty} \cdots \sum_{\omega_n=0}^{\infty} \prod_{k=1}^{n-1} Z_{n,k}^{(\lambda)}(\omega) \cdot e^{f_A(\omega_{n-1})}(f_H(\omega_{n-1}))^{1-\lambda} - (\lambda f_A(\omega_{n-1}) + (1 - \lambda) f_H(\omega_{n-1})) \cdot \omega_n!$$

From (9), one can see that a crucial role for the exact calculation (respectively the derivation of bounds) of the Hellinger integral is played by the functions defined for $x \in [0, \infty[$

$$\phi_\lambda(x) := \varphi_\lambda(x) - f_\lambda(x), \quad \text{with} \quad (10)$$

$$\varphi_\lambda(x) := (f_A(x))^\lambda (f_H(x))^{1-\lambda}$$

and

$$f_\lambda(x) := \lambda f_A(x) + (1 - \lambda) f_H(x) = \alpha_\lambda \beta_\lambda + \alpha_\lambda \beta_\lambda x \quad \text{and} \quad (11)$$

where we have used the $\lambda$-weighted-averages $\beta_\lambda = \lambda \cdot \beta_A + (1 - \lambda) \cdot \beta_H$ and $\alpha_\lambda = \lambda \cdot \alpha_A + (1 - \lambda) \cdot \alpha_H$. According to Lemma A.1 in Appendix A.1, it follows for $\lambda \in [0, 1[ $ that $\phi_\lambda(x) \leq 0$ for all $x \in [0, \infty[$, and that $\phi(x) = 0$ iff $f_\lambda(x) = f_H(x)$. This is consistent with the corresponding generally valid upper bound

$$H_\lambda(P_{A,n}||P_{H,n}) \leq 1.$$  

As a first indication for our proposed method, let us start by illuminating the simplest case $\lambda \in [0, 1[$ and

$$\gamma := \alpha_H \beta_A - \alpha_A \beta_H = 0.$$  

This means that $(\beta_A, \beta_H, \alpha_A, \alpha_H) \in \mathcal{P}_{NI} \cup \mathcal{P}_{SP,1}$, where $\mathcal{P}_{SP,1}$ is the set of all (componentwise) strictly positive $(\beta_A, \beta_H, \alpha_A, \alpha_H)$ with $\beta_A \neq \beta_H$, $\alpha_A \neq \alpha_H$ and $\frac{\alpha_A}{\alpha_H} = \frac{\beta_A}{\beta_H} \neq 1$. In this situation, all the three functions (10) to (12) are linear. Indeed,

$$\varphi_\lambda(x) = p_\lambda^E + q_\lambda^E x$$

with $p_\lambda^E := \alpha_\lambda \alpha_H^{1-\lambda}$ and $q_\lambda^E := \beta_\lambda \beta_H^{1-\lambda}$ (where the index E stands for exact linearity). Clearly, $q_\lambda^E > 0$ on $\mathcal{P}_{NI} \cup \mathcal{P}_{SP,1}$, as well as $p_\lambda^E > 0$ on $\mathcal{P}_{SP,1}$ respectively $p_\lambda^E = 0$ on $\mathcal{P}_{NI}$. Furthermore,

$$\phi_\lambda(x) = r_\lambda^E + s_\lambda^E x$$

with $r_\lambda^E := p_\lambda^E - \alpha_\lambda = \alpha_\lambda \alpha_H^{1-\lambda} - (\lambda \alpha_A + (1 - \lambda) \alpha_H)$ and $s_\lambda^E := q_\lambda^E - \beta_\lambda = \alpha_\lambda \beta_H^{1-\lambda} - (\lambda \beta_A + (1 - \lambda) \beta_H)$. Due to Lemma A.1 one knows $s_\lambda^E < 0$ on $\mathcal{P}_{NI} \cup \mathcal{P}_{SP,1}$, as well as $r_\lambda^E < 0$ on $\mathcal{P}_{SP,1}$ respectively $r_\lambda^E = 0$ on $\mathcal{P}_{NI}$.
As it will be seen later on, such kind of linearity properties are useful for the recursive handling of the Hellinger integrals. However, only on the parameter set \( P_{NL} \cup P_{SP,1} \) the functions \( \varphi_\lambda \) and \( \phi_\lambda \) are linear. Hence, in the general case \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in P \times [0,1]\) we aim for linear lower and upper bounds

\[
\varphi_\lambda^L(x) := p_\lambda^L + q_\lambda^L x \leq \varphi_\lambda(x) \leq \varphi_\lambda^U(x) := p_\lambda^U + q_\lambda^U x ,
\]

(14)

\( x \in [0, \infty) \) (ultimately, \( x \in \mathbb{N}_0 \)), which lead to

\[
\phi_\lambda^L(x) := r_\lambda^L + s_\lambda^L x := (p_\lambda^L - \alpha_\lambda) + (q_\lambda^L - \beta_\lambda)x \leq \phi_\lambda(x) \leq \phi_\lambda^U(x) := r_\lambda^U + s_\lambda^U x := (p_\lambda^U - \alpha_\lambda) + (q_\lambda^U - \beta_\lambda)x ,
\]

(15)

\( x \in [0, \infty) \) (ultimately, \( x \in \mathbb{N}_0 \)). Of course, the involved slopes and intercepts should satisfy reasonable restrictions. For instance, because of the nonnegativity of \( \varphi_\lambda \) we require \( p_\lambda^U \geq p_\lambda^L \geq 0, q_\lambda^U \geq q_\lambda^L \geq 0 \) (leading to the nonnegativity of \( \varphi_\lambda^L, \varphi_\lambda^U \)). Furthermore, (9) and (13) suggest that \( p_\lambda^L \leq \alpha_\lambda, q_\lambda^L \leq \beta_\lambda \) which leads to the nonpositivity of \( \phi_\lambda^L \). Moreover, it is assumed that

\[
\text{at least one of the two inequalities } p_\lambda^U < \alpha_\lambda, q_\lambda^U < \beta_\lambda \text{ holds},
\]

(16)

and hence \( \phi_\lambda^U(x) < 0 \) for some (but not necessarily all) \( x \in [0, \infty] \). Notice that in (16) we do not demand the validity of both inequalities, which might lead to the effect that the constructed Hellinger integral upper bounds have to be cut off at 1 for some (but not all) observation horizons \( n \in \mathbb{N} \); see (21) below. For the formulation of our first assertions on Hellinger integrals, we make use of the following notation:

**Definition 2.1.** For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in P \times [0,1]\) and all \( p \in [0, \infty], q \in [0, \infty] \), let us define the sequences \( (a_n^{(p,q)})_{n \in \mathbb{N}_0} \) and \( (b_n^{(p,q)})_{n \in \mathbb{N}_0} \) recursively by

\[
a_0^{(p,q)} := 0 ; \quad a_n^{(p,q)} := e^{a_{n-1}^{(p,q)}} \cdot q - \beta_\lambda, \quad n \in \mathbb{N},
\]

(17)

\[
b_0^{(p,q)} := 0 ; \quad b_n^{(p,q)} := e^{b_{n-1}^{(p,q)}} \cdot p - \alpha_\lambda, \quad n \in \mathbb{N}.
\]

(18)

Notice the interrelation \( a_n^{(p,q)} = s_n^A \) and \( b_n^{(p,q)} = r_n^A \) for \( A \in \{E, L, U\} \). Clearly, for \( q \in [0, \infty], p \in [0, \infty] \), one has the linear interrelation

\[
b_n^{(p,q)} = \frac{p}{q} a_n^{(p,q)} + \frac{q}{q} \beta_\lambda - \alpha_\lambda, \quad n \in \mathbb{N}.
\]

(19)

Accordingly, we obtain fundamental Hellinger integral evaluations:

**Theorem 2.2.** (a) For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (P_{NL} \cup P_{SP,1}) \times [0,1] \), all initial population sizes \( \omega_0 \in \mathbb{N} \) and all observation horizons \( n \in \mathbb{N} \) one can recursively compute the exact value

\[
H_\lambda(P_{A,n}||P_{H,n}) = \exp \left\{ a_n^{(q_\lambda^U)} \omega_0 + \frac{\alpha_A}{\beta_A} \sum_{k=1}^n b_k^{(q_\lambda^U)} \right\} =: V_{\lambda,n},
\]

(20)

where \( \frac{\alpha_A}{\beta_A} \) can be equivalently replaced by \( \frac{\alpha_n}{\beta_n} \). Recall that \( q_\lambda^E := \beta_A^{-1} \lambda^{-1} \).

(b) For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (P_{SP,1}\setminus P_{SP,1}) \times [0,1] \), all coefficients \( p_k^U \in [0, \infty], q_k^U \in [0, \infty], p_k^L \in [0, \min\{p_k^U, \alpha_\lambda\}], q_k^L \in [0, \min\{q_k^U, \beta_\lambda\}] \), such that (14) holds for all \( x \in \mathbb{N}_0 \) as well as (16), all initial population sizes \( \omega_0 \in \mathbb{N} \) and all observation horizons \( n \in \mathbb{N} \) one gets the recursive (i.e. recursively computable) bounds

\[
B_{\lambda,n}^L < H_\lambda(P_{A,n}||P_{H,n}) < B_{\lambda,n}^U ,
\]

where

\[
B_{\lambda,n}^L := \exp \left\{ a_n^{(q_\lambda^L)} \omega_0 + \sum_{k=1}^n b_k^{(p_k^L,q_k^L)} \right\} \quad \text{and} \quad B_{\lambda,n}^U := \min \left\{ \exp \left\{ a_n^{(p_k^U)} \omega_0 + \sum_{k=1}^n b_k^{(p_k^U,q_k^U)} \right\}, 1 \right\} .
\]

(21)

**Remark 2.3.** From the proof below one can see that both parts of Theorem 2.2 remain true for the cases \( \lambda \notin [0,1] \). For the (to our context) incompatible setup of GWI with Poisson offspring but nonstochastic immigration of constant value \( \lambda \), the exact values of the corresponding Hellinger integrals (i.e. an “analogue” of part (a)) was established in Linkov and Lunyova [52].

**Proof:**
We first prove the upper bound \( B_{\lambda,n}^U \). Let us fix \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda), p_k^U, q_k^U, \omega_0 \in \mathbb{N} \) as described in part (b).
From (8), (10), (11), (12) and (14) one gets immediately \( B_{\lambda,1}^L \), and with the help of (9) for all observation horizons \( n \in \mathbb{N} \setminus \{1\} \) (with the obvious shortcut for \( n = 2 \))
For part (b) in Theorem 2.2, we have assumed the existence of reasonable linear lower and upper bounds
3. Detailed analyses

\[ H_\lambda (P_{A,n}||P_{H,n}) = \sum_{\omega_1 = 0}^{\infty} \cdots \sum_{\omega_{n-1} = 0}^{\infty} \prod_{k=1}^{n-1} Z_{n,k}^{(\lambda)}(\omega) \cdot \exp \left\{ \varphi_\lambda(\omega_{n-1}) - f_\lambda(\omega_{n-1}) \right\} \]

\[ < \sum_{\omega_1 = 0}^{\infty} \cdots \sum_{\omega_{n-1} = 0}^{\infty} \prod_{k=1}^{n-1} Z_{n,k}^{(\lambda)}(\omega) \cdot \exp \left\{ (p_{\lambda}^U - \alpha_\lambda) + (q_{\lambda}^U - \beta_\lambda) \cdot \omega_{n-1} \right\} \]

\[ = \sum_{\omega_1 = 0}^{\infty} \cdots \sum_{\omega_{n-1} = 0}^{\infty} \prod_{k=1}^{n-1} Z_{n,k}^{(\lambda)}(\omega) \cdot \exp \left\{ b_1(p_{\lambda}^U, q_{\lambda}^U) + a_1(q_{\lambda}^U) \cdot \omega_{n-1} \right\} \]

\[ = \exp \left\{ b_1(p_{\lambda}^U, q_{\lambda}^U) \right\} \sum_{\omega_1 = 0}^{\infty} \cdots \sum_{\omega_{n-2} = 0}^{\infty} \prod_{k=1}^{n-2} Z_{n,k}^{(\lambda)}(\omega) \cdot \exp \left\{ a_1(q_{\lambda}^U) \cdot \varphi_\lambda(\omega_{n-2}) - f_\lambda(\omega_{n-2}) \right\} \]

\[ < \exp \left\{ b_1(p_{\lambda}^U, q_{\lambda}^U) \right\} \sum_{\omega_1 = 0}^{\infty} \cdots \sum_{\omega_{n-2} = 0}^{\infty} \prod_{k=1}^{n-2} Z_{n,k}^{(\lambda)}(\omega) \cdot \exp \left\{ b_2(p_{\lambda}^U, q_{\lambda}^U) + a_2(q_{\lambda}^U) \cdot \omega_{n-2} \right\} \]

\[ < \cdots < \exp \left\{ a_n(q_{\lambda}^U) \cdot \omega_0 + \sum_{k=1}^{n} b_k(p_{\lambda}^U, q_{\lambda}^U) \right\} . \]  (22)

Notice that for the strictness of the above inequalities we have used the fact that \( \phi_\lambda(x) < \phi_{\lambda}^U(x) \) for some (in fact, all but at most two) \( x \in \mathbb{N}_0 \) (cf. (p-xiv) below). Since for some admissible choices of \( p_{\lambda}^U, q_{\lambda}^U \) and some \( n \in \mathbb{N} \) the last term in (22) can become larger than 1, one needs to take into account the cutoff-point 1 arising from (13). Notice that without assumption (16), the last term in (22) would always be larger than 1 (and thus useless). The lower bound \( B_{\lambda,n}^L \) of part (b), as well as the exact value of part (a) follow from (9) in an analogous manner by employing \( p_{\lambda}^L, q_{\lambda}^L \) and \( p_{\lambda}^E, q_{\lambda}^E \) respectively. Furthermore, we use the fact that for \( (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_S \cup \mathcal{P}_S') \times [0,1] \) one gets from (19) the relation \( b_0(p_{\lambda}^E, q_{\lambda}^E) = \frac{\lambda}{q-\lambda} a_n(q_{\lambda}^E) \). For the sake of brevity, the corresponding straightforward details are omitted here. Although we take the minimum of the upper bound derived in (22) and 1, the inequality \( B_{\lambda,n}^L < B_{\lambda,n}^U \) is nevertheless valid: the reason is that for constituting a lower bound, the parameters \( p_{\lambda}^L, q_{\lambda}^L \) must fulfill either the conditions \([p_{\lambda}^L < 0 \text{ and } q_{\lambda}^L \leq 0]\) or \([p_{\lambda}^L \leq 0 \text{ and } q_{\lambda}^L < 0]\) (or both).

\[ \square \]

3. Detailed analyses

For part (b) in Theorem 2.2, we have assumed the existence of reasonable linear lower and upper bounds of \( \varphi_\lambda \) and \( \phi_\lambda \). In the following, we shall carry out a more detailed analysis addressing questions upon the non-uniqueness (and thus, flexibility) of the coefficients \( p_{\lambda}^U, q_{\lambda}^U, p_{\lambda}^L, q_{\lambda}^L \) in (14), their “optimal respectively reasonable choices”, as well as the corresponding behaviour of the Hellinger integrals \( H_\lambda (P_{A,n}||P_{H,n}) \) as the observation horizon \( n \) increases and finally converges to \( \infty \). Of course, the answers to these questions will depend on the (e.g. fixed) value of \( (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \) and the (e.g. selectable) value of \( \lambda \).

Before starting a closer inspection, notice by induction the general fact that for \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P} \times [0,1] \) and \( q \in [0,\infty[ \) the principal behaviour of the sequence \( \left( a_n^q \right)_{n \in \mathbb{N}} \) is strongly governed by its first element:

\[ (p-i) \ a_n^q \equiv 0, \]

\[ (p-ii) \ a_n^q \text{ is strictly negative and strictly decreasing,} \]

\[ (p-iii) \ a_n^q \text{ is strictly positive and strictly increasing,} \]

if \( a_1^q = q - \beta_\lambda = 0 \) (i.e. \( q = \beta_\lambda \)),

if \( a_1^q < 0 \),

if \( a_1^q > 0 \).
Due to the linear interrelation (19), the monotonicity carries over to the sequence \( (b^{(p,q)}_n)_{n \in \mathbb{N}_0} \) in the following way:

\[
\begin{align*}
(p-i) & \quad b^{(0,q)}_n \equiv -\alpha_\lambda < 0, \\
(p-ii) & \quad b^{(p,q)}_n \equiv p - \alpha_\lambda, \quad \text{if } q = \beta_\lambda, \\
(p-iii) & \quad \left( b^{(p,q)}_n \right)_{n \in \mathbb{N}} \text{ is strictly decreasing,} \quad \text{if } q < \beta_\lambda, \\
(p-iv) & \quad \left( b^{(p,q)}_n \right)_{n \in \mathbb{N}} \text{ is strictly increasing,} \quad \text{if } q > \beta_\lambda.
\end{align*}
\]

Notice that the sign of \( b^{(p,q)}_n \) might not be the same as the sign of \( a^{(q)}_n \) (see e.g. (p-i), (p-iv)). Finally, for the remaining case one trivially gets

\[
(p-viii) \quad a^{(q)}_n \equiv -\beta_\lambda, \quad b^{(p,0)}_n \equiv e^{-\beta_\lambda} \cdot p - \alpha_\lambda \quad (p \geq 0).
\]

Moreover, for \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P} \times ]0, 1[\) and \(q \in ]0, \infty[\) we shall sometimes use the function

\[
\xi^{(q)}_\lambda(x) := q \cdot e^x - \beta_\lambda, \quad x \in \mathbb{R},
\]

which has the following obvious properties:

\[
\begin{align*}
(p.ix) \quad \xi^{(q)}_\lambda \text{ is strictly increasing, strictly convex and smooth,} \\
(p.x) \quad \lim_{x \to -\infty} \xi^{(q)}_\lambda(x) = -\beta_\lambda < 0, \quad \lim_{x \to \infty} \xi^{(q)}_\lambda(x) = \infty.
\end{align*}
\]

With these auxiliary basic facts in hand, let us now start our detailed investigations of the time-behaviour \( n \mapsto H_\lambda(P_{A,n}||P_{H,n}) \) for the exactly treatable case (a) in Theorem 2.2.

### 3.1. Detailed analysis of the exact values

(aNI) **The non-immigration case** \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{NI} \times ]0, 1[\):

Recall that for this set-up we derived \( q^E_\lambda := \beta_A \beta_H^{1-\lambda} > 0 \) and \( p^E_\lambda := \alpha_A \lambda^{1-\lambda} = 0 \). According to Lemma A.1, one has \( q^E_\lambda < \beta_\lambda \) and thus, \( \left( a^{(q^E_\lambda)}_n \right)_{n \in \mathbb{N}} \) is strictly negative as well as strictly decreasing. Furthermore, because of (p-ix), (p-x) and \( a^{(q^E_\lambda)}_n < 0 \), the function \( \xi^{(q^E_\lambda)}_\lambda \) hits on \([-\infty, 0]\) the straight line \( id(x) := x \) once and only once. Consequently, \( \left( a^{(q^E_\lambda)}_n \right)_{n \in \mathbb{N}} \) converges to the unique solution \( x^{(q^E_\lambda)}_0 \in ]-\beta_\lambda, a^{(q^E_\lambda)}_1| \) of the equation

\[
\xi^{(q^E_\lambda)}_\lambda(x) = q^E_\lambda \cdot e^x - \beta_\lambda = x, \quad x < 0.
\]

Summing up, we have shown the following detailed behaviour of Hellinger integrals:

**Proposition 3.1.** For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{NI} \times ]0, 1[\) and all initial population sizes \( \omega_0 \in \mathbb{N} \) there holds

\[
\begin{align*}
(a) & \quad H_\lambda(P_{A,1}||P_{H,1}) = \exp \left\{ \left( \beta_A \beta_H^{1-\lambda} - \lambda \beta_A - (1 - \lambda)\beta_H \right) x_0 \right\} < 1, \\
(b) & \quad \text{the sequence } (H_\lambda(P_{A,n}||P_{H,n}))_{n \in \mathbb{N}} \text{ given by} \\
& \quad H_\lambda(P_{A,n}||P_{H,n}) = \exp \left\{ a^{q^E_\lambda}_n \omega_0 \right\} =: V_{\lambda,n} \\
& \quad \text{is strictly decreasing,} \\
(c) & \quad \lim_{n \to \infty} H_\lambda(P_{A,n}||P_{H,n}) = \exp \left\{ x^{(q^E_\lambda)}_0 \omega_0 \right\} \in ]0, 1[, \\
(d) & \quad \lim_{n \to \infty} \frac{1}{n} \log H_\lambda(P_{A,n}||P_{H,n}) = 0.
\end{align*}
\]
(aEF) The “equal-fraction-case” \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{S\mathcal{P},1} \times [0,1]$$: Again, one has \(q^\xi_A := \beta_A^\xi \beta_H^{1-\xi} \lambda > 0$$. Furthermore, \(p^\xi_A := \alpha_A^\xi \alpha_H^{1-\xi} \lambda > 0$$, which leads to the abovementioned relation \(p^\xi_A (q^\xi_A) = \frac{\alpha_A^\xi}{\beta_A^\xi} q^\xi_A$$. Hence, the results about the sequence \((q^\xi_{A_k})_{n \in \mathbb{N}}$$ coincide with those of the non-immigration case. This implies also that the sequence \(\left(\sum_{k=1}^n a_k(q^\xi_A)\right)_{n \in \mathbb{N}}$$ is strictly negative, strictly decreasing and converges to \(\infty\). Hence, we get

**Proposition 3.2.** For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{S\mathcal{P},1} \times [0,1]$$ and all initial population sizes \(\omega_0 \in \mathbb{N}$$ there holds

\[
\begin{align*}
(a) & \quad H_A(P_{A,1}||P_{H,1}) = \exp \left\{ \left( \beta_A^\lambda \beta_H^{1-\lambda} - \lambda \beta_A - (1-\lambda) \beta_H \right) \left( \omega_0 + \frac{\alpha_A}{\beta_A} \right) \right\} < 1, \\
(b) & \quad \text{the sequence} \ (H_A(P_{A,n}||P_{H,n}))_{n \in \mathbb{N}} \ \text{given by} \\
& \quad H_A(P_{A,n}||P_{H,n}) = \exp \left\{ a_n(q^\xi_A) \omega_0 + \frac{\alpha_A}{\beta_A} \sum_{k=1}^n a_k(q^\xi_A) \right\} =: V_{\lambda,n}
\end{align*}
\]

is strictly decreasing,

\[
(c) \quad \lim_{n \to \infty} H_A(P_{A,n}||P_{H,n}) = 0,
\]

\[
(d) \quad \lim_{n \to \infty} \frac{1}{n} \log H_A(P_{A,n}||P_{H,n}) = \frac{\alpha_A}{\beta_A} \lambda(q^\xi_A).
\]

**Remark 3.3.** For the (to our context) incompatible setup of GWI with Poisson offspring but nonstochastic immigration of constant value 1, an “analogue” of part (d) of Proposition 3.2 was established in Linkov and Lunyova [52].

### 3.2. Detailed analysis of the lower bounds

In this section we assume \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{S\mathcal{P}} \mathcal{P}_{S\mathcal{P},1} \times [0,1]$$ and thus \(\alpha_A > 0, \alpha_H > 0, \frac{\alpha_A}{\alpha_H} \neq \frac{\beta_A}{\beta_H}$$, \(\gamma \neq 0, f_A(x) > 0, f_H(x) > 0 \ (x \in [0,\infty[)$$ (x \in [0,\infty[). Concerning (15), let us derive a lower linear bound \(\phi^\xi_A(\cdot)$$ of \(\phi_A(\cdot)$$ which is in order. In order to achieve this, one can use the following straightforward properties of \(\phi_A(x), x \in [0,\infty[$$ (cf. (10)):

- (p-xi) \(\phi_A(0) = \alpha_A^\xi \alpha_H^{1-\xi} - \alpha_A \leq 0$$ (cf. Lemma A.1), with equality iff \(\alpha_A = \alpha_H$$ (together with \(\beta_A \neq \beta_H)$$.
- (p-xii) \(\phi^\xi_A(x) = \lambda \beta_A (f_A(x))^{\lambda-1} (f_H(x))^{-\lambda-1} (1-\lambda) \beta_H \left( f_A(x) \right)^{\lambda} \left( f_H(x) \right)^{-\lambda} - \beta_A > -\beta_A$$.
- (p-xiii) \(\lim_{x \to \infty} \phi^\xi_A(x) = \beta_A^\lambda \beta_H^{1-\lambda} - \beta_A \leq 0$$ (cf. Lemma A.1), with equality iff \(\beta_A = \beta_H$$ (together with \(\alpha_A \neq \alpha_H)$$.
- (p-xiv) \(\phi^\xi_A(\cdot) = -\lambda (1-\lambda) \left( f_A(x) \right)^{-\lambda-2} \left( f_H(x) \right)^{-\lambda-1} \beta_A < 0$$, i.e. the function \(\phi_A(\cdot)$$ is strictly concave; notice that \(\phi^\xi_A(0) = \lambda \beta_A (\alpha_A/\alpha_H)^{\lambda-1} + (1-\lambda) \beta_H (\alpha_A/\alpha_H)^{\lambda} - \beta_A$$ can be either negative (e.g. for \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (4, 2, 3, 1, 0.5)$$), or zero (e.g. for \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (4, 2, 4, 1, 0.5)$$), or positive (e.g. for \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (4, 2, 5, 1, 0.5)$$). Accordingly, the strict decreasingness and continuity of \(\phi^\xi_A(\cdot)$$ as well as (p-xiii) imply that \(\phi_A(\cdot)$$ can be either strictly decreasing, or can obtain its global maximum on \([0,\infty[$$, or only in the case \(\beta_A = \beta_H$$ can be strictly increasing.

- (p-xv) \(\lim_{x \to \infty} \phi_A(x) - (r^\xi_A + s^\xi_A x) = 0$$ for \(r^\xi_A := \lambda \alpha_A \left( \frac{\beta_A}{\beta_H} \right)^{\lambda-1} + (1-\lambda) \alpha_H \left( \frac{\beta_A}{\beta_H} \right)^{\lambda} - 1$$ and \(s^\xi_A := \beta_A^\lambda \beta_H^{1-\lambda} - \beta_A \leq 0$$; notice that \(s^\xi_A = 0$$ iff \(\beta_A = \beta_H$$ (together with \(\alpha_A \neq \alpha_H$$). Furthermore, \(\phi_A(0) < r^\xi_A$$ (cf. Lemma A.1). If \(\alpha_A = \alpha_H$$ (and thus \(\beta_A \neq \beta_H$$) then the intercept \(r^\xi_A$$ is strictly positive, whereas for the case \(\alpha_A \neq \alpha_H$$ the intercept \(r^\xi_A$$ can take any sign (take e.g. \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (3.7, 0.9, 2.0, 1.0, 0.5)$$ for \(r_A > 0$$, \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (3.6, 0.9, 2.0, 1.0, 0.5)$$ for \(r_A = 0$$, \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (3.5, 0.9, 2.0, 1.0, 0.5)$$ for \(r_A < 0$$).

From (p-xi) to (p-xv) it is easy to see that for all current parameter constellations the particular choices

\[
p^\xi_A := \alpha_A^\xi \alpha_H^{1-\xi} > 0, \quad q^\xi_A := \beta_A^\lambda \beta_H^{1-\lambda} > 0
\]

– which correspond to the choices

\[
r^\xi_A := \alpha_A^\xi \alpha_H^{1-\xi} - \alpha_A \leq 0, \quad s^\xi_A := \beta_A^\lambda \beta_H^{1-\lambda} - \beta_A \leq 0
\]
in (15) (and at least one of the two last inequalities is strict) – lead to the tightest lower bound $B_{\lambda,n}^L$ for $H_\lambda(P_{A,n}||P_{H,n})$ in (21). This situation coincides partially with those in Section 3.1. Formally, $p_\lambda^L = p_\lambda^E$ and $q_\lambda^L = q_\lambda^E$, but because of $\gamma \neq 0$ the relation $b_\lambda(p_\lambda^L,q_\lambda^L) = a_\lambda^A(q_\lambda^L)$ is in general not valid anymore and has to be replaced by the relation (cf. (19))

$$b_\lambda(p_\lambda^L,q_\lambda^L) = \left( \frac{\alpha_\lambda}{\beta_\lambda} \right)^\lambda \left( \frac{\alpha_H}{\beta_H} \right)^{1-\lambda} a_\lambda^A(q_\lambda^L) + \left( \frac{\alpha_\lambda}{\beta_\lambda} \right)^{1-\lambda} \frac{\alpha_H}{\beta_H} \beta_\lambda - \alpha_\lambda, \ n \in \mathbb{N}. \quad (25)$$

Hence, for a better distinguishability and easier reference we stick to the $L$–notation here. Nevertheless, the behaviour of the sequence $\left( a_\lambda^A(q_\lambda^L) \right)_{n \in \mathbb{N}}$ coincides exactly with that of the sequence $\left( a_\lambda^E(q_\lambda^L) \right)_{n \in \mathbb{N}}$ in the Subsections 3.1(aNI), (aEF). In particular $\left( a_\lambda^E(q_\lambda^L) \right)_{n \in \mathbb{N}}$ is strictly negative, strictly decreasing and converges to the unique solution $x_0(q_\lambda^L) \in (-\infty, a_\lambda^E(q_\lambda^L)]$ of the equation

$$\xi_\lambda(x) = q_\lambda^E \cdot e^x - \beta_\lambda = x, \ x < 0. \quad (26)$$

Consequently, because of (25) and $b_\lambda(p_\lambda^L,q_\lambda^L) = \alpha_\lambda^A \omega_0(1-\lambda) - \alpha_\lambda \leq 0$ (cf. (18)), the sequence $\left( b_\lambda(p_\lambda^L,q_\lambda^L) \right)_{n \in \mathbb{N}} \setminus \{1\}$ is strictly negative and strictly decreasing. As in Subsection 3.1(aEF), we obtain

**Proposition 3.4.** For all $(\beta_\lambda, \beta_H, \alpha_\lambda, \alpha_H, \lambda) \in (\mathcal{P}_{SP}\setminus\mathcal{P}_{SP,1}) \setminus [0,1]$ and all initial population sizes $\omega_0 \in \mathbb{N}$ there holds

\begin{itemize}
    \item[(a)] $B_{\lambda,1}^L := \exp \left\{ \left( \beta_\lambda \frac{1-\lambda}{\beta_H} - \lambda_\beta - (1-\lambda)\beta_\lambda \right) \omega_0 + \left( \alpha_\lambda \frac{1-\lambda}{\beta_H} - \lambda_\alpha - (1-\lambda)\alpha_\lambda \right) \right\} < 1$,
    \item[(b)] the sequence $\left( B_{\lambda,n}^L \right)_{n \in \mathbb{N}}$ of lower bounds for $H_\lambda(P_{A,n}||P_{H,n})_{n \in \mathbb{N}}$ given by $B_{\lambda,n}^L := \exp \left\{ a_\lambda^E(q_\lambda^L) \omega_0 + \sum_{k=1}^{n} b_\lambda(p_\lambda^L,q_\lambda^L) \right\}$ is strictly decreasing,
    \item[(c)] $\lim_{n \to \infty} B_{\lambda,n}^L = 0$ ,
    \item[(d)] $\lim_{n \to \infty} \frac{1}{n} \log B_{\lambda,n}^L = - \frac{p_\lambda^L}{q_\lambda^L} \left( x_0(q_\lambda^L) + \beta_\lambda \right) - \alpha_\lambda$ .
\end{itemize}

### 3.3. Detailed analysis of the upper bounds

As above, we again assume $(\beta_\lambda, \beta_H, \alpha_\lambda, \alpha_H, \lambda) \in (\mathcal{P}_{SP}\setminus\mathcal{P}_{SP,1}) \setminus [0,1]$ throughout this section. In contrast to the treatment of the lower bounds in Section 3.2, the finetuning of the upper bounds is more involved. Because of the strict concavity of the function $\phi_\lambda(\cdot)$ (cf. (p-xiv)), there is in general no overall best linear upper bound of $\phi_\lambda(\cdot)$ within the framework (15). Different reasonable goals might lead to different reasonable choices of $p_\lambda^U, q_\lambda^U$ (and thus of $s_\lambda^U, t_\lambda^U$) which might imply different behaviour of the corresponding sequence $\left( B_{\lambda,n}^U \right)_{n \in \mathbb{N}}$ of upper bounds in (21). This can be conjectured from the following immediate monotonicity properties:

\begin{itemize}
    \item[(p-xvi)] $0 \leq q_1 < q_2 \implies a_\lambda^{(q_1)} < a_\lambda^{(q_2)}$ for all $n \in \mathbb{N}$.
    \item[(p-xvii)] Trivially, $b_\lambda^{(0,q_1)} = b_\lambda^{(0,q_2)} \equiv -\alpha_\lambda$. In contrast, let $p \in \mathbb{N}$ be fixed; then, $0 \leq q_1 < q_2 \implies b_\lambda^{(p,q_1)} < b_\lambda^{(p,q_2)}$ for all $n \in \mathbb{N}$.
    \item[(p-xviii)] Let $q \in \mathbb{N}$ be fixed. Then, $0 \leq p_1 < p_2 \implies b_\lambda^{(p_1,q)} < b_\lambda^{(p_2,q)}$ for all $n \in \mathbb{N}$.
    \item[(p-xix)] $0 \leq p_1 < p_2, 0 \leq q_1 < q_2 \implies b_\lambda^{(p_1,q_1)} < b_\lambda^{(p_2,q_2)}$ for all $n \in \mathbb{N}$.
    \item[(p-x)] For the case $0 \leq p_1 < p_2, 0 \leq q_2 < q_1$ there is in general no dominance assertion for $b_\lambda^{(p_1,q_1)}, b_\lambda^{(p_2,q_2)}$ which holds for all $n \in \mathbb{N}$; take e.g. $(\beta_\lambda, \beta_H, \alpha_\lambda, \alpha_H, \lambda) = (1, 0.6, 3.3, 0.5), p_1 = 3.4641, q_1 = 0.7785$ for which $\phi_\lambda(\cdot)$ corresponds to the secant line through the points $\phi_\lambda(0)$ and $\phi_\lambda(1)$, as well as $p_2 = 3.4857, q_2 = 0.7746$ for which $\phi_\lambda(\cdot)$ corresponds to the asymptote of $\phi_\lambda(\cdot)$, and inspect the first six values of of the corresponding $b_\lambda$–sequence.
\end{itemize}

The properties (p-xvi) to (p-x) have corresponding effects on the behaviour $b_\lambda(p_\lambda^U,q_\lambda^U) \mapsto B_{\lambda,n}^U = \min \left\{ \exp \left\{ a_\lambda^E \omega_0 + \sum_{k=1}^{n} b_\lambda^{(p_\lambda^U,q_\lambda^U)} \right\}, 1 \right\}$ (cf. (21)) of the upper bounds. For instance,
for any fixed admissible intercept $p^U_{\lambda}$ one would provide the smallest admissible $q^U_{\lambda}$ in order to achieve the smallest possible upper bound; due to (p-xiv) this implies that on the ultimately relevant subdomain $\mathbb{N}_0$, the linear function $\phi^U_{\lambda}(\cdot)$ should hit $\phi_{\lambda}(\cdot)$ in at least one but at most two points (tangent or secant line). Furthermore, we require for the rest of the section that $p^U_{\lambda} > 0$ and $q^U_{\lambda} > 0$, because otherwise $r^U_{\lambda} < \phi_{\lambda}(0)$ and $s^U_{\lambda} < \omega (\text{cf. (p-xv)})$ which contradicts to the nature of linear upper bounds of $\phi_{\lambda}$.

The (only partially restricted) choice of parameters $p^U_{\lambda}$, $q^U_{\lambda}$ for the upper bounds $B^U_{\lambda,n}$ can be made according to different, partially incompatible, "optimality-" respectively "goodness-" criteria, such as:

- (Ga) very good tightness for $n \geq N$ for some fixed large $N \in \mathbb{N}$, or
- (Gb) for a fixed initial population size $\omega_0 \in \mathbb{N}$ there holds $B^U_{\lambda,n} < 1$ for all $n \in \mathbb{N}$, or
- (Gc) there holds $B^U_{\lambda,n} < 1$ for all $n \in \mathbb{N}$ and all $\omega_0 \in \mathbb{N}$ (strict improvement of the general upper bound (13)).

For the sake of brevity, we investigate only goal (Gc) (with the exception of Subsection 3.3(a7) and Theorem 6.3) which can be achieved if (and "nearly but not fully" iff) (16) holds; this can be seen from

$$ B^U_{\alpha,1} = \min \left\{ \exp \left\{ a^U_{1}(q^U_{\lambda}) \omega_0 + b^U_{1}(q^U_{\lambda}) \right\}, 1 \right\} = \min \left\{ \exp \left\{ (q - \beta_{\lambda}) \omega_0 + (p - \alpha_{\lambda}) \right\}, 1 \right\} $$

and the properties (p-i) to (p-vii). Furthermore, (p-xiv) and (p-xv) imply that the slope $s_\lambda := q_{\lambda}^U - \beta_{\lambda}$ in (15) should be greater or equal to the limit slope $s_\lambda$ which leads to the restriction $q_{\lambda}^U \geq \beta_{\lambda} - \beta_0$. Moreover, since $s_\lambda \leq 0$, the intercept $r_{\lambda} := p^U_{\lambda} - \alpha_{\lambda}$ in (15) should be greater or equal to $\phi_{\lambda}(0)$ and thus, $p^U_{\lambda} \geq \alpha_0$.

By comparing the above established lower and upper parameter-bounds, from Lemma A.1 it follows that the case $q^U_{\lambda} < \beta_{\lambda}$ automatically implies $\beta_A \neq \beta_H$ whereas the case $p^U_{\lambda} < \alpha_{\lambda}$ leads to $\alpha_A \neq \alpha_H$. In consistence with (p-xiv), various different parameter constellations can lead to different Hellinger-integral-upper-bound details, which we investigate in the following.

(a1) The case $\mathcal{P}_{SP,2}$ of all (componentwise) strictly positive $(\beta_{\lambda}, \beta_H, \alpha_A, \alpha_H)$ with $\beta_H \neq \beta_A$, $\alpha_A = \alpha_H$

We have $\phi_{\lambda}(0) = 0$ (cf. (p-xi)), $\phi_{\lambda}(0) = 0$ (cf. (p-xii)). Thus, the only admissible intercept choice is $r^U_{\lambda} = 0 = p^U_{\lambda} - \alpha_{\lambda} = b^U_{1}(q^U_{\lambda})$ (i.e. $p^U_{\lambda} = \alpha_{\lambda} = \alpha_* > 0$), and the minimal admissible slope which implies (15) for $x \in \mathbb{N}$ is given by $s^U_{\lambda} = \frac{\phi_{\lambda}(1) - \phi_{\lambda}(0)}{1} = q^U_{\lambda} - \beta_{\lambda} = a^U_{1}(q^U_{\lambda}) < 0$ (i.e. $q^U_{\lambda} = (\alpha_* + \beta_A) \lambda (\alpha_* + \beta_H)^{1-\lambda} - \alpha_* > 0$).

Analogously to Subsection 3.1(aNI), one can derive that $(a^U_{n}(q^U_{\lambda}))_{n \in \mathbb{N}}$ is strictly negative, strictly decreasing, and converges to the unique solution $x_0(q^U_{\lambda}) \in [0, \infty], a^U_{1}(q^U_{\lambda})$ of the equation

$$ \xi^U_{\lambda}(x^U_{\lambda}(x)) = q^U_{\lambda} \cdot e^x - \beta_{\lambda} = x, \ x < 0 \ . $$

Moreover, in the same manner as in Section 3.2, the sequence $(b^U_{n}(q^U_{\lambda}))_{n \in \mathbb{N} \setminus \{1\}}$ is strictly negative and strictly decreasing. This leads to

**Proposition 3.5.** For all $(\beta_{\lambda}, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{SP,2} \times [0,1]$ and all initial population sizes $\omega_0 \in \mathbb{N}$ there holds

(a) $B^U_{\alpha,1} := \exp \left\{ \left( q^U_{\lambda} - \beta_{\lambda} \right) \omega_0 + \left( p^U_{\lambda} - \alpha_{\lambda} \right) \right\} < 1$,  
(b) $\left( B^U_{\lambda,n} \right)_{n \in \mathbb{N}}$ of upper bounds for $(H_{\lambda}(P_{\lambda,n}||P_{H,n}))_{n \in \mathbb{N}}$ given 

\[ \lim_{n \to \infty} B^U_{\alpha,1} = 0 = \lim_{n \to \infty} H_{\lambda}(P_{\lambda,n}||P_{H,n}), \]

(d) $\lim_{n \to \infty} \frac{1}{n} \log B^U_{\alpha,1} = \frac{p^U_{\lambda}}{q^U_{\lambda}} \left( x^U_{\lambda} + \beta_{\lambda} \right) - \alpha_{\lambda}$ .

In contrast to $\mathcal{P}_{SP,2}$, the constellation $\mathcal{P}_{SP,3}$ of all (componentwise) strictly positive $(\beta_{\lambda}, \beta_H, \alpha_A, \alpha_H)$ with $\alpha_A = \alpha_H$, $\beta_A = \beta_H$ and $\frac{\alpha_A}{\alpha_H} \neq \frac{\beta_A}{\beta_H}$ is divided into three main parts as follows: because of Lemma A.1 one
(a2) The case \( P_{\text{SP},3a}^{\lambda,\leq 0} \) of all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in P_{\text{SP},3ab} \times [0, 1] \) for which \( \lambda \beta_A (\alpha_A/\alpha_H)^{\lambda - 1} + (1 - \lambda) \beta_H (\alpha_A/\alpha_H)^{\lambda} - \beta_H \leq 0 \) holds.

From (p-xi) and (p-xii), one gets \( \phi_0(0) < 0 \) and \( \phi'_0(0) \leq 0 \). For the latter, both the strict negativity as well as the vanishing can appear in the current parameter setup, take e.g. \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (1.8, 0.9, 2.8, 0.7, 0.5)\) for \( \phi'_0(0) = 0 \) and \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (1.8, 0.9, 2.7, 0.7, 0.5)\) for \( \phi'_0(0) < 0 \). In the current setup, \( \phi_0 \) is a strictly negative, strictly decreasing, and – due to (p-xiv) – strictly concave function (and thus, the assumption \( \alpha_Y - \alpha_X - \gamma < 0 \) is superfluous here). In contrast to Subsection (a1), one has the flexibility to choose the intercept \( p^{\text{U}}_X \) from the nonempty interval \([\alpha^X, \alpha^X - \gamma, \alpha^X]\) and the slope \( q^{\text{U}}_X \) from the nonempty interval \([\beta^X, \beta^X + \gamma, \beta^X]\), subject to the constraints that \((p^{\text{U}}_X, q^{\text{U}}_X) \neq (\alpha^X, \beta^X)\) and \( \phi_0(x) = (p^{\text{U}}_X - \alpha^X) + (q^{\text{U}}_X - \beta^X) x \) (cf. (15)). Of course, one way to obtain a reasonable choice of intercept and slope is the search for the optimum

\[
(p^{\text{U}}_X, q^{\text{U}}_X) := \text{argmin}_{(p,q)} \left\{ \exp \left( a_n^{(q)} \omega_0 + \sum_{k=1}^{n} b_k^{(p,q)} \right) \right\}
\]

(28)

subject to the abovementioned constraints. However, the corresponding result generally depends on the choice of the initial population size \( \omega_0 \) and the observation horizon \( n \). Hence, there is in general no overall optimal choice of \( p^{\text{U}}_X, q^{\text{U}}_X \) (without the incorporation of further goal-dependent constraints such as \( \lim_{n \to \infty} H_\lambda(P_{A,n} \mid P_{H,n}) = 0 \)) in case of \( \lim_{n \to \infty} H_\lambda(P_{A,n} \mid P_{H,n}) = 0 \). By the way, due to the recursive nature of the sequences in (28) and the nontriviality of the constraints, this optimization problem seems to be not straightforward to solve, in general.

Inspired from Subsection (a1), a more pragmatic but yet reasonable choice is the following: take any intercept \( p^{\text{U}}_X \in [\alpha^X, \alpha^X - \gamma, \alpha^X] \) such that \((p^{\text{U}}_X, q^{\text{U}}_X) = (\alpha^X, \beta^X)\) (i.e. \( \alpha^X = (\alpha_A + \beta_A)^{\lambda} (\alpha_H + \beta_H)^{1 - \lambda} - p^{\text{U}}_X + \alpha_H \geq (\alpha_A + 2\beta_A)^{\lambda} (\alpha_H + 2\beta_H^{1 - \lambda})\) and \( q^{\text{U}}_X := \phi_0(1) - (p^{\text{U}}_X - \alpha^X) + \beta^X = (\alpha_A + \beta_A)^{\lambda} (\alpha_H + \beta_H)^{1 - \lambda} - p^{\text{U}}_X \)), which corresponds to a linear function \( \phi_0^X \) which is

- (a) nonpositive on \( \mathbb{N}_0 \) and strictly negative on \( \mathbb{N} \),
- (b) larger than or equal to \( \phi_0 \) on \( \mathbb{N}_0 \), strictly larger than \( \phi_0 \) on \( \mathbb{N} \setminus \{1, 2\} \), and equal to \( \phi_0 \) at the point \( x = 1 \) ("discrete tangent or secant line through \( x = 1 \)).

One can easily see that (due to the restriction (14)) not all \( p^{\text{U}}_X \in [\alpha^X, \alpha^X - \gamma, \alpha^X] \) might qualify for the current purpose. For the particular choice \( p^{\text{U}}_X = \alpha^X \alpha_H^{-\lambda} \) and \( q^{\text{U}}_X = (\alpha_A + \beta_A)^{\lambda} (\alpha_H + \beta_H)^{1 - \lambda} - \alpha_H \alpha^X^{-1 - \lambda} \) one obtains \( r^{\text{U}}_X = p^{\text{U}}_X - \alpha^X = b_1(q^{\text{U}}_X, \phi_0^X) < 0 \) (cf. Lemma A.1) and \( s^{\text{U}}_X = q^{\text{U}}_X - \beta^X = \phi_0(1) - (p^{\text{U}}_X - \alpha^X) + \beta^X = a_1(q^{\text{U}}_X) < 0 \) (secant line through \( \phi_0(0) \) and \( \phi_0(1) \)). Hence, analogously to Subsection (a1) one can derive that \( \left( a_n^{(q)} \right)_{n \in \mathbb{N}} \) is strictly negative, strictly decreasing, and converges to the unique solution \( x_0^{(q)} \in ] - \infty, a_1(q^{\text{U}}_X) \) of equation (27). Moreover, the sequence \( \left( b_n^{(p_X, q^{\text{U}}_X)} \right)_{n \in \mathbb{N} \setminus \{1\}} \) is strictly negative and strictly decreasing. Thus, all the assertions (a), (b), (c), (d) of Proposition 3.5 hold for all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in P_{\text{SP},3a}^{\lambda,\leq 0}\) and all initial population sizes \( \omega_0 \in \mathbb{N} \).

(a3) The case \( P_{\text{SP},3b}^{\lambda,>0} \) of all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in P_{\text{SP},3ab} \times [0, 1] \) for which \( \lambda \beta_A (\alpha_A/\alpha_H)^{\lambda - 1} + (1 - \lambda) \beta_H (\alpha_A/\alpha_H)^{\lambda} - \beta_H > 0 \) holds.

In this situation (which appears e.g. for \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (1.8, 0.9, 2.9, 0.7, 0.5)\)) one gets from (p-xi) and (p-xii) the two inequalities \( \phi_0(0) < 0 \) and \( \phi'_0(0) > 0 \). Furthermore, in accordance with the arguments in the forefront of Subsection (a2), \( \phi_0 \) is a strictly negative, strictly concave, hump-shaped (cf. (p-xiii)) function. One can proceed similarly to (a2). Indeed, let \( x_{\text{max}} := \text{argmax}_{x \in [0, \infty]} \phi_0(x) \) which is the unique solution of
\[ \lambda \beta_A \left[ \left( \frac{f_A(x)}{f_R(x)} \right)^{\lambda-1} - 1 \right] + (1 - \lambda) \beta_H \left[ \left( \frac{f_A(x)}{f_R(x)} \right)^\lambda - 1 \right] = 0, \quad x \in ]0, \infty[, \quad (29) \]

(cf. (p-xii), (p-xiv)); notice that \( x^* \) formally satisfies the equation (29) but does not qualify because of the current restriction \( x^* < 0 \).

Let us first inspect the case \( \phi_\lambda([x_{max}]) > \phi_\lambda([x_{max}]+1) \), where \([x]\) denotes the integer part of \(x\). Consider the subcase \( \phi_\lambda([x_{max}]) + [x_{max}] (\phi_\lambda([x_{max}]) - \phi_\lambda([x_{max}]+1)) \leq 0 \), which means that the secant line through \( \phi_\lambda([x_{max}]) \) and \( \phi_\lambda([x_{max}]+1) \) possesses a non-positive intercept. If this situation is preferable to choose as intercept any \( p^U_\lambda - \alpha_\lambda = b^U_{q^U_\lambda} = r^U_\lambda \in \phi_\lambda([x_{max}]) \) and as corresponding slope \( q^U_\lambda - \alpha_\lambda = a^U(q^U_\lambda) = s^U_\lambda = \frac{\phi_\lambda([x_{max}]) - r^U_\lambda}{(x_{max}) - 0} \leq 0 \) (notice that the corresponding line \( \phi^U_\lambda \) is on \([x_{max}], \infty[, \leq\) strictly larger than the secant line through \( \phi_\lambda([x_{max}]) \) and \( \phi_\lambda([x_{max}]+1) \).

In the other subcase \( \phi_\lambda([x_{max}]) + [x_{max}] (\phi_\lambda([x_{max}]) - \phi_\lambda([x_{max}]+1)) > 0 \), one can choose any intercept \( p^U_\lambda - \alpha_\lambda = b^U_{q^U_\lambda} = r^U_\lambda \in \phi_\lambda([x_{max}]), 0 \) and as corresponding slope \( q^U_\lambda - \alpha_\lambda = a^U(q^U_\lambda) = s^U_\lambda = \frac{\phi_\lambda([x_{max}]) - r^U_\lambda}{(x_{max}) - 0} \leq 0 \) (notice that the corresponding line \( \phi^U_\lambda \) is on \([x_{max}], \infty[, \leq\) strictly larger than the secant line through \( \phi_\lambda([x_{max}]) \) and \( \phi_\lambda([x_{max}]+1) \)).

If \( \phi_\lambda([x_{max}]) \leq \phi_\lambda([x_{max}]+1) \), one can proceed as above by substituting the crucial pair of points \(([x_{max}], [x_{max}]+1)\) with \(([x_{max}]+1, [x_{max}]+2)\) and examining the analogous two subcases.

With the accordingly derived \( p^U_\lambda, q^U_\lambda \) one gets in all four (sub)cases exactly the same kind of behaviour of the sequences \( \left( a^U_n(q^U_\lambda) \right)_{n \in \mathbb{N}}, \left( b^U_n(q^U_\lambda) \right)_{n \in \mathbb{N}} \) as in Subsection (2a). Hence, all the assertions (a), (b), (c), (d) of Proposition 3.5 hold for all \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{\lambda, 0}^{\lambda_3} \) and all initial population sizes \( \omega_0 \in \mathbb{N} \).

(a4) The case \( \mathcal{P}_{\lambda_3}^{\lambda_3} \) of all (componentwise) strictly positive \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H) \) with \( \alpha_A \neq \alpha_H \), \( \beta_\lambda \neq \beta_H, \frac{\alpha_A}{\beta_H} \neq \frac{\beta_A}{\beta_H} \) and \( \frac{\alpha_A}{\beta_H} - \frac{\beta_A}{\beta_H} \in ]0, \infty[ \cap \mathbb{N} \).

The only difference to Subsection (a3) is that the maximum value of \( \phi_\lambda(\cdot) \) now achieves 0, at the positive non-integer point \( x_{max} = x^* = \frac{\omega_0 - \beta_A}{\beta_A - \beta_H} \in ]0, \infty[ \cap \mathbb{N} \) (take e.g. \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H, \lambda) = (1.8, 0.9, 1.2, 3.0, 0.5) \) as an example). Due to (p-xi), (p-xii) and (p-xiv) one gets automatically \( \lambda \beta_A (\alpha_A/\alpha_H)^{\lambda-1} + (1 - \lambda) \beta_H (\alpha_A/\alpha_H)^\lambda - \beta_\lambda > 0 \) for all \( \lambda \in ]0, 1[ \). This situation can be treated exactly as in (a3). Consequently, all the assertions (a), (b), (c), (d) of Proposition 3.5 hold for all \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{\lambda_3}^{\lambda_3} \times ]0, 1[ \) and all initial population sizes \( \omega_0 \in \mathbb{N} \).

(a5) The case \( \mathcal{P}_{\lambda_3}^{\lambda_3} \) of all (componentwise) strictly positive \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H) \) with \( \alpha_A \neq \alpha_H \), \( \beta_\lambda \neq \beta_H, \frac{\alpha_A}{\beta_H} \neq \frac{\beta_A}{\beta_H} \) and \( \frac{\alpha_A}{\beta_H} - \frac{\beta_A}{\beta_H} \in \mathbb{N} \).

The only difference to Subsection (a4) is that the maximum value of \( \phi_\lambda(\cdot) \) now achieves 0 at the integer point \( x_{max} = x^* = \frac{\omega_0 - \beta_A}{\beta_A - \beta_H} \in \mathbb{N} \) (take e.g. \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H, \lambda) = (1.8, 0.9, 1.2, 3.0, 0.5) \) as an example). Under the restriction that \( \exp \left\{ \sum_{k=1}^n b^U_{q^U_\lambda} \right\} \leq 1 \) for all \( n \in \mathbb{N} \) and all \( \omega_0 \in \mathbb{N} \), our method leads to the choices \( r^U_\lambda = 0 \) as well as \( s^U_\lambda = 0 \). Consequently, \( B^U_{\lambda, n} = 1 \), which coincides with the general upper bound (13), but violates the abovementioned desired goal (Gc).

However, by using a conceptually different method we can nevertheless prove the convergence

\[
\lim_{n \to \infty} H_\lambda(P_{A,n}||P_{H,n}) = 0
\] (30)

(which will be used for the study of entire separation below). This will be done in Appendix A.1.

As a next step, let us investigate the last possible parameter constellation:

...
(a6) The case \( \mathcal{P}_{SP,4} \) of all (componentwise) strictly positive \((\beta_A, \beta_H, \alpha_A, \alpha_H)\) with \(\alpha_A \neq \alpha_H\), \(\beta_A = \beta_H\)

This is the only case where \(\phi_A(\cdot)\) is strictly negative and strictly increasing, with \(\lim_{x \to -\infty} \phi_A(x) = \lim_{x \to -\infty} \phi_A'(x) = 0\), leading to the choices \(r^U_\lambda = 0\) as well as \(s^U_\lambda = 0\) under the restriction that \(\exp\left( a_n^U \lambda \omega_0 + \sum_{k=1}^n b_k^U \lambda q_k^U \right) \leq 1\) for all \(n \in \mathbb{N}\) and all \(\omega_0 \in \mathbb{N}\). Consequently, \(B^U_{\lambda,n} = 1\), which is consistent with the general upper bound (13), but violates the above-mentioned desired Goal (Gc). Unfortunately, the proof method of (30) can’t be carried over to the current setup (see Appendix A.1).

(a7) Alternative bounds for \( \mathcal{P}_{SP,2} \cup \mathcal{P}_{SP,3ab} \cup \mathcal{P}_{SP,3c} \cup \mathcal{P}_{SP,3d} \)

Within this last subsection, let us exceptionally ignore the Goal (Gc). Correspondingly, for the derivation of an upper bound \(B^U_{\lambda,n}\), one can use the asymptote of \(\varphi_\lambda\) given in (p-xv) to end up with \(\bar{p}^U_\lambda := \bar{r}_\lambda + \alpha_\lambda = \lambda \alpha_A \left( \frac{\beta_A}{\beta_H} \right)^{\lambda-1} + (1 - \lambda) \alpha_H \left( \frac{\beta_A}{\beta_H} \right)^{\lambda} \) as well as \(\bar{q}^U_\lambda = \bar{s}_\lambda + \beta_\lambda = \beta^A \lambda^{-\beta_A} \beta_H^{\lambda-1}\). Clearly, \(\bar{p}^U_\lambda > p^U_\lambda = \alpha_\lambda^{\lambda-1}\) by Lemma A.1 and \(\bar{q}^U_\lambda = q_\lambda^U\). Furthermore, \(\bar{q}^U_\lambda < \beta_\lambda\) and thus (16) holds, since we have excluded \(\mathcal{P}_{SP,4}\). However – depending on the choice of \((\beta_A, \beta_H, \alpha_A, \alpha_H)\) – the intercept \(\bar{r}_\lambda = \bar{p}^U_\lambda - \alpha_\lambda\) may become strictly positive, and hence

\[
\bar{B}^U_{\lambda,n} := \exp \left( a_n^U \lambda \omega_0 + b_1^U \lambda q_1^U \right) = \exp \left( \bar{p}^U_\lambda - \beta_\lambda \right) \cdot \omega_0 + \bar{p}^U_\lambda - \alpha_\lambda
\]

may become larger than 1. However, according to properties (p-ii) and (p-vi) the sequence

\[
n \mapsto \bar{B}^U_{\lambda,n} := \exp \left( a_n^U \lambda \omega_0 + \sum_{k=1}^n b_k^U \lambda q_k^U \right) = \exp \left( \bar{p}^U_\lambda - \beta_\lambda \right) \cdot \omega_0 + \frac{\bar{p}^U_\lambda}{q_\lambda^U} \sum_{k=1}^n a_k^U \lambda q_k^U
\]

may become smaller than 1. Let us therefore define for all \(n \in \mathbb{N}\) and all \(\lambda \in [0,1] \text{ by } \mathcal{P}_{SP,3d} \times [0,1] \}

\[
\bar{B}^U_{\lambda,n} := \min \left\{ \bar{B}^U_{\lambda,n}, 1 \right\}
\]

which can be used as an upper bound for the case \(\mathcal{P}_{SP,3d} \times [0,1] \}

For the other cases \((\mathcal{P}_{SP,2} \times [0,1] \cup \mathcal{P}_{SP,3a} \cup \mathcal{P}_{SP,3b} \cup (\mathcal{P}_{SP,3c} \times [0,1]) \) all the assertions (a), (b), (c) of Proposition 3.5 remain valid for replacing \(B^U_{\lambda,n}\) by the improved upper bound

\[
B^U_{\lambda,n,impr} := \min \left\{ B^U_{\lambda,n}, \bar{B}^U_{\lambda,n} \right\} < 1
\]

In fact, for all these parameter classes there are concrete examples such that the upper bound \(B^U_{\lambda,n,impr}\) really improves the upper bound \(B^U_{\lambda,n}\) for all \(n \in \mathbb{N}\) (i.e. \(\bar{B}^U_{\lambda,n} < B^U_{\lambda,n}\)). For \(\mathcal{P}_{SP,2} \times [0,1] \) take e.g. \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (0.8, 0.6, 2, 0.5)\) and \(\omega_0 = 10\), with \(p^U_A = 2.021\), \(q^U_A = 0.693\), instead of the proposed choice \(p^U_A = 2\) and \(q^U_A = 0.698\). For \(\mathcal{P}_{SP,3a}^{\lambda \leq 0}\) take e.g. \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (0.8, 0.6, 2, 1.9, 0.5)\) and \(\omega_0 = 10\), with \(p^U_A = 1.963\), \(q^U_A = 0.693\), instead of the proposed choice \(p^U_A = 1.949\) and \(q^U_A = 0.696\). For \(\mathcal{P}_{SP,3b}^{\lambda > 0}\) take e.g. \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (0.8, 0.6, 2, 1.1, 0.5)\) and \(\omega_0 = 10\), with \(p^U_A = 1.501\), \(q^U_A = 0.693\), instead of the (amongst others proposed) choice \(p^U_A = 1.483\) and \(q^U_A = 0.699\). For \(\mathcal{P}_{SP,3c} \times [0,1] \) take e.g. \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) = (1.15, 2, 1.8, 0.5)\) and \(\omega_0 = 10\), with \(p^U_A = 1.960\), \(q^U_A = 1.225\), instead of the (amongst others proposed) choice \(p^U_A = 1.897\) and \(q^U_A = 1.249\).

### 3.4. Asymptotic distinguishability

For each \(n \in \mathbb{N}_0\), let \((\Omega_n, \mathcal{F}_n)\) be a measurable space equipped with two probability measures \(\hat{\mathcal{P}}_n, \mathcal{P}_n\). The following two general types of asymptotic distinguishability are well known (see e.g. LeCam [42], Liese and Vajda [47], Jacod and Shiryaev [29], Linkov [51], and the references therein):

(CEA) the sequence \((\hat{\mathcal{P}}_n)_{n \in \mathbb{N}_0}\) is contiguous to the sequence \((\mathcal{P}_n)_{n \in \mathbb{N}_0}\) – in symbols, \((\hat{\mathcal{P}}_n) \subset (\mathcal{P}_n)\) – if for all sequences \(A_n \in \mathcal{F}_n\) with \(\lim_{n \to \infty} \mathcal{P}_n(A_n) = 0\) there holds \(\lim_{n \to \infty} \hat{\mathcal{P}}_n(A_n) = 0\).
(CEb) the sequences \((P_n)_{n \in \mathbb{N}_0}\) and \((\overline{P}_n)_{n \in \mathbb{N}_0}\) are entirely separated (completely asymptotically separable) – in symbols, \((P_n) \Delta (\overline{P}_n)\) – if there exist a sequence \(n_m \uparrow \infty\) as \(m \uparrow \infty\) and for each \(m \in \mathbb{N}_0\) an \(A_{n_m} \in \mathcal{F}_{n_m}\) such that \(\lim_{m \to \infty} \overline{P}_{n_m}(A_{n_m}) = 1\) and \(\lim_{m \to \infty} P_{n_m}(A_{n_m}) = 0\).

The corresponding negations will be denoted by \(\overline{\Delta}\) and \(\overline{\overline{\Delta}}\). As demonstrated in the abovementioned references for a general context,

(CEb) holds iff \(\lim_{n \to \infty} H_\lambda \left( (\overline{P}_n) || P_n \right) = 0\) for some (or equivalently, all) \(\lambda \in ]0,1[\); furthermore,

(CEa) holds iff \(\lim_{n \to \infty} H_\lambda \left( (\overline{P}_n) || P_n \right) = 1\).

Combining these results with the respective part (c) of Propositions 3.1, 3.2 and 3.5 as well as the connected investigations of Subsections 3.3(a2) to (a5), we obtain the following

**Corollary 3.6.** (a) For all \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in (P_{SP,1} \cup P_{SP,2} \cup P_{SP,3a} \cup P_{SP,3c} \cup P_{SP,3d})\) and all initial population sizes \(\omega_0 \in \mathbb{N}\), the corresponding sequences \((P_{A,n})_{n \in \mathbb{N}_0}\) and \((P_{H,n})_{n \in \mathbb{N}_0}\) are entirely separated.

(b) For all \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in P_{NI}\) with \(\beta_A \leq 1\) and all initial population sizes \(\omega_0 \in \mathbb{N}\), the sequence \((P_{A,n})_{n \in \mathbb{N}_0}\) is contiguous to \((P_{H,n})_{n \in \mathbb{N}_0}\).

(c) For all \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in P_{NI}\) with \(\beta_A > 1\) and all initial population sizes \(\omega_0 \in \mathbb{N}\), the sequence \((P_{A,n})_{n \in \mathbb{N}_0}\) is neither contiguous to nor entirely separated to \((P_{H,n})_{n \in \mathbb{N}_0}\).

**Remarks 3.7.** (i) Assertion (c) of Corollary 3.6 contrasts the case of Gaussian processes with independent increments where one gets either either separation or mutual contiguity (see e.g. Liese and Vajda [47]).

(ii) By putting Corollary 3.6(b) and (c) together, we obtain for different “criticality pairs” in the non-immigration case \(P_{NI}\) the following asymptotic distinguishability types:

\[ (P_{A,n}) \overline{\Delta} (P_{H,n}) \text{ if } \beta_A \leq 1, \beta_H \leq 1; \quad (P_{A,n}) \overline{\overline{\Delta}} (P_{H,n}) \text{ if } \beta_A \leq 1, \beta_H > 1; \]
\[ (P_{A,n}) \overline{\overline{\overline{\Delta}}} (P_{H,n}) \text{ if } \beta_A > 1, \beta_H \leq 1; \quad (P_{A,n}) \overline{\overline{\overline{\overline{\Delta}}}} (P_{H,n}) \text{ if } \beta_A > 1, \beta_H > 1; \]

in particular, for \(P_{NI}\) the sequences \((P_{A,n})_{n \in \mathbb{N}_0}\) and \((P_{H,n})_{n \in \mathbb{N}_0}\) are not completely asymptotically inseparable (indistinguishable).

(iii) In the light of the abovementioned (CEa) resp. (CEb) characterizations by means of Hellinger integral limits, the finite-time-horizon results on Hellinger integrals given in Theorem 2.2, Section 3 and also in the following Section 4 can loosely be interpreted as “finite-sample (rather than asymptotic) distinguishability” assertions.

4. Closed-form bounds

Depending on the parameter constellation, we have given bounds respectively exact values for the Hellinger integrals, which can be obtained with the help of recursions (17) (together with (19) respectively (p-viii)) which are “stepwise fully evaluable” but generally seem not to admit a closed-form representation in the observation horizons \(n\); consequently, the exact time-behaviour of (the bounds of) the Hellinger integrals can generally not be seen explicitly. To avoid this intransparency (at the expense of losing some precision) one can approximate (17) by a recursion that allows for a closed-form representation. Accordingly, we shall employ (context-adapted) linear inhomogeneous difference equations

\[
\tilde{a}_0 := 0; \quad \tilde{a}_n := \tilde{\xi}(\tilde{a}_{n-1}) + \rho_{n-1}, \quad n \in \mathbb{N}, \quad \text{with (31)}
\]

\[
\tilde{\xi}(x) := c + d \cdot x, \quad x \in ]-\infty,0[, \quad (32)
\]

\[
\rho_{n-1} := K_1 \cdot x^{n-1} + K_2 \cdot _{\nu}^{\nu^{n-1}}, \quad n \in \mathbb{N}, \quad (33)
\]

for some constants \(c \in ]-\infty,0[, \ d \in ]0,1[, \ K_1, K_2, x, \nu \in \mathbb{R} \) with \(0 \leq \nu < x < d\). As usual, one gets the closed-form representation

\[
\tilde{a}_n = \tilde{a}_n^{\text{hom}} + \tilde{c}_n \quad \text{with } \tilde{a}_n^{\text{hom}} = c \cdot \frac{1 - d^n}{1-d} \quad \text{and } \tilde{c}_n = K_1 \cdot \frac{d^n - x^n}{d-x} + K_2 \cdot \frac{d^n - \nu^n}{d-\nu}, \quad (34)
\]

which immediately leads for all \(n \in \mathbb{N}\) to

\[
\sum_{k=1}^{n} \tilde{a}_k = \left( \frac{K_1}{d-x} + \frac{K_2}{d-\nu} - \frac{c}{1-d} \right) \cdot \frac{d \cdot (1 - d^n)}{1-d} - \frac{K_1 \cdot x \cdot (1-x^n)}{(d-x)(1-x)} - \frac{K_2 \cdot \nu \cdot (1-\nu^n)}{(d-\nu)(1-\nu)} + \frac{c \cdot n}{1-d}. \quad (35)
\]
Notice that for the special case $K_2 = -K_1 > 0$ one has from (33) for all integers $n \geq 2$ the relation $\rho_{n-1} < 0$ and thus $\bar{a}_n - \bar{a}_{n-1} > 0$, leading to

$$\bar{c}_n < 0 \quad \text{and} \quad \sum_{k=1}^{n} \bar{c}_k < 0.$$  

(36)

In the following, we appropriately apply (31)-(35) to the different parameter contexts of Section 3.

### 4.1. Closed-form lower bounds

Let $(\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P} \times [0, 1]$. We have seen in the Sections 3.1 and 3.2 that the determination of the exact values and the lower bounds had (more or less) identical structure: choose $q_0^* := q_0 = q_1^* = \beta_A \beta_H^{1-\lambda} > 0$, compute the sequence $(a_n(q_0^*))_{n \in \mathbb{N}_0}$ by the nonlinear recursion (cf. (17), (23))

$$a_0(q_0^*) := 0 \quad ; \quad a_n(q_0^*) := \xi_\lambda(q_n^*)(a_{n-1}^*)^(-1), \ n \in \mathbb{N},$$  

(37)

choose $p^* := p_f^* = \alpha_A \beta_H^{1-\lambda} \geq 0$, compute (cf. (19))

$$b_n^*(p_f^*, q_0^*) = \left( \frac{\alpha_A}{\beta_A} \right)^\lambda \left( \frac{\alpha_H}{\beta_H} \right)^{1-\lambda} a_n(q_0^*)^\lambda + \left( \frac{\alpha_A}{\beta_A} \right)^\lambda \left( \frac{\alpha_H}{\beta_H} \right)^{1-\lambda} \beta_\lambda - \alpha_\lambda, \ n \in \mathbb{N},$$

and finally end up with (cf. (21), (20)) exp \{$a_n(q_0^*) \omega_0 + \sum_{k=1}^{n} b_k^*(p_f^*, q_0^*) \}$ which is either interpreted as bound $B_{\lambda,n}^*$ in the parameter case $(\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_\mathcal{S} \cap \mathcal{P}_\mathcal{SP}_1) \times [0, 1]$ or as exact value $V_{\lambda,n}$ in the parameter case $(\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_\mathcal{S} \cap \mathcal{P}_\mathcal{SP}_1) \times [0, 1]$ (where we achieved some further simplifications above). Since $(a_n(q_0^*))_{n \in \mathbb{N}}$ is strictly negative, strictly decreasing and converges to the unique solution $x_0(q_0^*) \in ]-\infty, a_1(q_0^*)]$ (i.e. we replace the nonlinear function $\xi_\lambda(q_0^*)$ by $q_0^* \cdot e^x - \beta_\lambda$ by the tangent line of $\xi_\lambda(q_0^*)$ at $x = x_0(q_0^*)$ defined by

$$\xi_\lambda(q_0^*)^\prime(x) := x_0(q_0^*) \left( 1 - q_0^* \cdot e^{x_0(q_0^*)} \right) + q_0^* \cdot e^{x_0(q_0^*)} \cdot x, \quad x \in [x_0(q_0^*), 0],$$

(40)

and reduce the error we face by adding the “correction-term”

$$\varphi_{n-1}^*(q_0^*) := \frac{1}{2} \cdot \left( q_0^* \cdot e^{x_0(q_0^*)} \right)^2 < 0.$$  

(41)

In other words, by means of the two functions on the domain $[0, \infty[$

$$q \mapsto d(q)^T := q \cdot e^{x_0(q)}, \quad q \mapsto \Gamma(q) := \frac{1}{2} \cdot e^{x_0(q)} \cdot \left( x_0(q) \right)^2 = \frac{d(q)^T}{2} \cdot \left( x_0(q) \right)^2$$

we use (31), (32), (33) with constants $d := d(q)^T \in ]0, 1[, c := x_0(q_0^*) \cdot \left( 1 - d(q)^T \right) \in ]-\infty, 0[, \ K_1 := \Gamma(q_0^*) > 0, \ \varphi := \left( d(q)^T \right)^2 \in \mathbb{R} \setminus \{d, 1\}, \ K_2 := 0, \ \nu := 0$. Let us first present some fundamental properties which will be proved in Appendix A.2:

**Lemma 4.1.** For all $(\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P} \times [0, 1]$ there holds:

(a) $\varphi_n(q_0^*) < a_n(q_0^*)$, for all $n \in \mathbb{N}$.

(b) The sequence $(\varphi_n(q_0^*))_{n \in \mathbb{N}}$ is strictly decreasing.

(c) $\lim_{n \to \infty} a_n(q_0^*) = \lim_{n \to \infty} a_n(q_0^*) = x_0(q_0^*)$. 


Applying Theorem 2.2, Lemma 4.1 as well as the formulae (19), (34) and (35), one gets

**Theorem 4.2.** For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P} \times [0, 1]\) and all initial population sizes \(\omega_0 \in \mathbb{N}\) the following assertions hold:

(a) for all observation horizons \(n \in \mathbb{N}\) the Hellinger integral can be bounded from below by the closed-form bounds \(H_\lambda(P_{A,n} \| P_{H,n}) > C_{\lambda,n}^L\) given by

\[
C_{\lambda,n}^L := \exp \left\{ 2 - \frac{\beta_\lambda^*}{q_\lambda^*} \frac{d(q_\lambda^*, T)^n}{1 - d(q_\lambda^*, T)^n} \right\} \left( 1 - \left( \frac{d(q_\lambda^*, T)^n}{1 + d(q_\lambda^*, T)^n} \right) \right) \cdot n
\]

where for all \(n \in \mathbb{N}\)

\[
\Delta_n(q_\lambda^*) := \Gamma(q_\lambda^*), \quad \left( \frac{d(q_\lambda^*, T)^n}{1 - d(q_\lambda^*, T)^n} \right) > 0 \quad \text{and}
\]

\[
\beta_\lambda^* := \frac{\beta_\lambda}{q_\lambda^*} \cdot \Gamma(q_\lambda^*) \cdot \left( \frac{1 - (d(q_\lambda^*, T)^n)}{1 - d(q_\lambda^*, T)^n} \right) \left( 1 - \frac{d(q_\lambda^*, T)^n}{1 + d(q_\lambda^*, T)^n} \right) > 0.
\]

(b) the sequence \((C_{\lambda,n}^L)_{n \in \mathbb{N}}\) is strictly decreasing.

(c) for all observation horizons \(n \in \mathbb{N}\)

\[
C_{\lambda,n}^L < \begin{cases} B_{\lambda,n}^L, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{SP} \setminus \mathcal{P}_{SP, 1}) \times [0, 1], \\ V_{\lambda,n}, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{NI} \cup \mathcal{P}_{SP, 1}) \times [0, 1]. \end{cases}
\]

(d) \(\lim_{n \to \infty} C_{\lambda,n}^L = \begin{cases} 0, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{NI} \cup \mathcal{P}_{SP, 1}) \times [0, 1], \\ \exp \left\{ \omega_0 (q_\lambda^*) \right\} > 0, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{NI} \times [0, 1], \end{cases}\)

which coincides with \(\lim_{n \to \infty} H_\lambda(P_{A,n} \| P_{H,n})\) for all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P} \setminus \mathcal{P}_{SP, 1}) \times [0, 1]\).

(e) \(\lim_{n \to \infty} \frac{1}{n} \log H_\lambda(P_{A,n} \| P_{H,n})\) for all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{NI} \cup \mathcal{P}_{SP, 1}) \times [0, 1]\) respectively with \(\lim_{n \to \infty} \frac{1}{n} \log B_{\lambda,n}^L\) for all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{SP, 2} \cup \mathcal{P}_{SP, 3a} \cup \mathcal{P}_{SP, 3b}) \times [0, 1].\)

**Remark 4.3.** Notice that the formula (43) simplifies in the parameter case \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{SP, 1} \times [0, 1]\), since then it holds \(p_\lambda^* / q_\lambda^* = \frac{\alpha_A}{\beta_A} = \frac{\alpha_H}{\beta_H}\) and therewith \((p_\lambda^* / q_\lambda^*) \cdot \beta_A - \alpha_A = 0; \) for the case \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{NI} \times [0, 1],\) one can even use the stronger relation \(p_\lambda^* = 0 = \alpha_A.\)

In order to get an “explicit” lower bound which does not rely on the implicitly given fixed point \(x_0^*(q_\lambda^*)\), one can replace the latter by a close explicit lower approximate \(\Delta_0(q_\lambda^*) < x_0^*(q_\lambda^*)\) and proceed completely analogously, leading to a smaller lower bound (say) \(C_{\lambda,n}^L < C_{\lambda,n}^L\) in assertion (a) of Theorem 4.2; in the corresponding assertions (b), (c) and (d) one then has to replace \(C_{\lambda,n}^L\) by \(C_{\lambda,n}^L\) and \(x_0^*(q_\lambda^*)\) by \(\Delta_0(q_\lambda^*)\). For instance, one could choose

\[
\Delta_0(q_\lambda^*) := e^{-h(q_\lambda^*)} \cdot \left( 1 - q_\lambda^* \right) - \sqrt{\left( 1 - q_\lambda^* \right)^2 - 2 \cdot q_\lambda^* \cdot e^{h(q_\lambda^*)} \cdot (q_\lambda^* - \beta_A) \right),
\]

where

\[
h(q_\lambda^*) := \begin{cases} \max \left\{ -\beta_A : \frac{q_\lambda^* - \beta_A}{1 - q_\lambda^*} \right\}, & \text{if } q_\lambda^* < 1, \\ -\beta_A, & \text{if } q_\lambda^* \geq 1; \end{cases}
\]

this will be used as an auxiliary tool for the diffusion-limit-concerning proof of Lemma A.3(c) in the appendix. If \(q_\lambda^* < 1\), the term \(\frac{q_\lambda^* - \beta_A}{1 - q_\lambda^*}\) represents the existing negative intersection of the tangent of \(q_\lambda^*\) at \(x = 0\) and the bisectrix. Clearly, \(h(q_\lambda^*) < x_0^*(q_\lambda^*)\). By (46), \(\Delta_0(q_\lambda^*)\) is the unique negative solution of \(\Delta_0(q_\lambda^*)(x) = x\) with the quadratic function...
\[ Q_{\lambda}^{(q)}(x) := \frac{q^*}{2} e^{b(q^*)} \cdot x^2 + q^*_\lambda \cdot x + q^*_\lambda - \beta_\lambda. \]

Notice that \( Q_{\lambda}^{(q^*)}(0) = \xi_{\lambda}^{(q^*)}(0), \) \( \frac{dQ_{\lambda}^{(q^*)}}{dx}(0) = \frac{d\xi_{\lambda}^{(q^*)}}{dx}(0), \) \( \frac{d^2Q_{\lambda}^{(q^*)}}{dx^2}(x) < \frac{d^2\xi_{\lambda}^{(q^*)}}{dx^2}(x) \) for all \( x \in [x_0^{(q^*)}, 0], \) and thus \( Q_{\lambda}^{(q^*)}(x) < \xi_{\lambda}^{(q^*)}(x) \) for all \( x \in [x_0^{(q^*)}, 0], \) which leads to the desired \( x_0^{(q^*)} < x_0^{(q^*)}. \)

### 4.2. Closed-form upper bounds

In order to achieve closed-form upper bounds, we principally proceed as in the previous Section 4.1. However, the situation is now more diverse since we have to start from Section 3.3 which carries much more “nonuniqueness” respectively variety than the corresponding Sections 3.1 and 3.2 which we used as a starting point for the investigations in Section 4.1.

Notice first that for the subcases \( \mathcal{P}_{SP,3ab} \times [0,1] \) and \( \mathcal{P}_{SP,3c} \times [0,1] \) (cf. Subsections 3.3(a2),(a3),(a4)) one can achieve a closed-form upper bound without further investigations: if one chooses \( q^*_\lambda = \beta_\lambda \) (and thus, the slope \( s_{\lambda}^U = 0 \)), then by properties (p-i), (p-v) one has \( a_{\lambda}^{(q)} = 0 \) (i.e. recursion (17) is trivial), \( b_{\lambda}^{(p,q)} = p - \alpha_\lambda < 0 \) and hence
\[ B_{\lambda,n}^U = \exp \left\{ a_{\lambda}^{(q^*)} \cdot \omega_0 + \sum_{k=1}^n b_{\lambda}^{(p,q)} \cdot \ast \right\} = \exp \left\{ n \cdot (p_{\lambda}^U - \alpha_\lambda) \right\} \quad n \to \infty. \quad (47) \]
However, there might exist (and for \( \mathcal{P}_{SP,\leq 0} \) definitely exists) choices \( (p_{\lambda}^U, q_{\lambda}^U) \) which lead to (fully or eventually partially) tighter upper bounds \( B_{\lambda,n}^U \) but for which the non-linear recursion (17) is nontrivial. Such potential cases, for which in particular \( 0 < q_{\lambda}^U < \beta_\lambda \) and \( 0 < p_{\lambda}^U \leq \alpha_\lambda \) holds, will be treated in the following; since the parameter constellation \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{SP,3d} \cup \mathcal{P}_{SP,4} \times [0,1] \) does not meet this requirement, let us fix \( (\beta_\lambda, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{NI} \cup \mathcal{P}_{SP,1} \cup \mathcal{P}_{SP,2} \cup \mathcal{P}_{SP,3ab} \cup \mathcal{P}_{SP,3c} \times [0,1] \) where we also include the two setups \( \mathcal{P}_{NI} \cup \mathcal{P}_{SP,1} \) for which we want to replace the recursive, non-closed-form exact values by closed-form upper bounds. For this situation, we determined recursive upper bounds respectively exact values in a (more or less) identical structure which is also very close to the one given by (37) to (38): choose \( q_{\lambda}^G \) for \( G = U \) respectively \( G = E \) subject to the corresponding parameter case which leads to \( a_{\lambda}^{(q^G)} = s_{\lambda}^G = q_{\lambda}^G - \beta_\lambda < 0, \) compute the (rest of the) sequence \( \left( a_n^{(q^G)} \right)_{n \in \mathbb{N}_0} \) by the nonlinear recursion (cf. (17), (23))
\[ a_{n}^{(q^G)} := \xi_{\lambda}^{(q^G)}(a_{n-1}^{(q^G)}), \quad n \in \mathbb{N}, \quad (48) \]
choose \( p_{\lambda}^G \) subject to the corresponding parameter case and evaluate
\[ b_{n}^{(p^G,q^G)} := \frac{p_{\lambda}^G}{q_{\lambda}^G} \cdot a_n^{(q^G)} + \frac{p_{\lambda}^G}{q_{\lambda}^G} \cdot \beta_\lambda - \alpha_\lambda, \quad n \in \mathbb{N}, \quad (49) \]
which leads to the desired bound \( B_{\lambda,n}^G = \exp \left\{ a_{\lambda}^{(q^G)} \cdot \omega_0 + \sum_{k=1}^n b_{\lambda}^{(p^G,q^G)} \cdot \ast \right\} \) (cf. part (b) of Proposition 3.5). According to (p-ii), the fundamentally important sequence \( \left( a_n^{(q^G)} \right)_{n \in \mathbb{N}} \) is strictly negative, strictly decreasing and converges to the unique solution \( x_0^{(q^G)} \in ]-\infty, a_{1}^{(q^G)}[ \) of the equation
\[ \xi_{\lambda}^{(q^G)}(x) = q_{\lambda}^G \cdot e^x - \beta_\lambda = x, \quad x < 0. \quad (50) \]
For an upper bound of the sequence \( a_{\lambda}^{(q^G)} \) we introduce the recursion
\[ a_{0}^{(q^G)} := 0 ; \quad a_{n}^{(q^G)} := \xi_{\lambda}^{(q^G)}(a_{n-1}^{(q^G)}) + p_{\lambda}^{(q^G)}, \quad n \in \mathbb{N}, \quad (49) \]
i.e. we replace the nonlinear function \( \xi_{\lambda}^{(q^G)}(x) = q_{\lambda}^G \cdot e^x - \beta_\lambda \) by the secant line of \( \xi_{\lambda}^{(q^G)} \) across its arguments \( x_0^{(q^G)} \) and 0, defined by
\[ \xi_{\lambda}^{(q^G),S}(x) := q_{\lambda}^G - \beta_\lambda + \frac{x_0^{(q^G)} - (q_{\lambda}^G - \beta_\lambda)}{x_0^{(q^G)}} \cdot x, \quad x \in [x_0^{(q^G)}, 0], \quad (50) \]
and reduce the error we face by adding the “correction-term”

\[ \theta^{(q^G)}_{n^{-1}} := -\frac{1}{2} \cdot \left( x_0^{(q^G)} \right)^2 \cdot \left( q^G_{\lambda} - e^{-\nu(q^G)} \right) \cdot \left( 1 - \frac{x_0^{(q^G)} - (q^G - \beta_\lambda)}{x_0^{(q^G)}} \right)^{n-1} < 0. \]  

(51)

In other words, by means of (42) and the function on the domain \([0, \infty[\]

\[ q \mapsto d(q),S := \frac{x_0(q) - (q - \beta_\lambda)}{x_0^{(q^G)}} \]  

(52)

we use (31), (32), (33) with the constants \(d := d(q^G) \cdot S \in [d(q^G),1], \ c := q^G - \beta_\lambda \in [0, \infty[ \), \( K_1 := -\nu(q^G) < 0, \ \alpha := d(q^G) \cdot T \in (0, d], \ K_2 := -K_1, \ \nu := d(q^G) \cdot T \cdot d(q^G) \cdot S \in [0, \alpha[ \). The following fundamental properties will be proved in the appendix:

**Lemma 4.4.** For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P} \setminus \{ \mathcal{P}_{SP,3d} \cup \mathcal{P}_{SP,4} \} \times [0, 1[\ it holds

(a) \( \frac{\theta^{(q^G)}}{\alpha_n(q^G)} \geq \frac{\theta^{(0)}}{\alpha_n^{(0)}} \) for all \( n \in \mathbb{N} \), with equality iff \( n = 1 \).

(b) The sequence \( \frac{\theta^{(q^G)}}{\alpha_n(q^G)} \) is strictly decreasing.

(c) \( \lim_{n \to \infty} \frac{\theta^{(q^G)}}{\alpha_n(q^G)} = \lim_{n \to \infty} \frac{\theta^{(0)}}{\alpha_n^{(0)}} = x_0^{(q^G)} \).

Applying Theorem 2.2, Lemma 4.4 as well as the formulae (19), (34) and (35), one obtains

**Theorem 4.5.** For all \((\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{NI} \cup \mathcal{P}_{SP,1} \cup \mathcal{P}_{SP,2} \cup \mathcal{P}_{SP,3ab} \cup \mathcal{P}_{SP,3c}) \times [0, 1[\) and \( \text{all initial population sizes } \omega_0 \in \mathbb{N} \) the following assertions hold:

(a) for all \( n \in \mathbb{N} \) the Hellinger integral can be bounded from above by the closed-form bounds \( H_X(P_{A,n} || P_{H,n}) < C_{\lambda,n} \) given by

\[ C_{\lambda,n}^{G} := \exp \left\{ x_0^{(q^G)} \left[ \omega_0 - \frac{p_G^{G}}{q^G_{\alpha}} \cdot \frac{d(q^G, S)}{d(q^G, T)} \right] \left( 1 - \left( \frac{d(q^G, S)}{d(q^G, T)} \right)^n \right) + \left( \frac{p_G^{G}}{q^G_{\alpha}} \left( \beta_\lambda + x_0^{(q^G)} \right) - \alpha_\lambda \right) \cdot n \right\} \]

\[ - \left( \alpha_n^{(q^G)} \cdot \omega_0 - \theta^{(q^G)}_{n^{-1}} \right), \quad \text{where all } n \in \mathbb{N} \]

\[ \frac{\theta^{(q^G)}}{\alpha_n(q^G)} := \Gamma^{(q^G)} \left[ \left( d(q^G, S) - d(q^G, T) \right)^n \cdot \left( d(q^G, S) - d(q^G, T) \right) \right] > 0 \]  

and

\[ \frac{\theta^{(q^G)}}{\alpha_n(q^G)} := \frac{\theta^{(q^G)}}{\alpha_n(q^G)} \cdot \frac{p_G^{G} \cdot d(q^G, T)}{q^G} \left[ 1 - \left( \frac{d(q^G, S) - d(q^G, T)}{d(q^G, S) - d(q^G, T)} \right)^n \right] + \left( \frac{p_G^{G}}{q^G_{\alpha}} \left( \beta_\lambda + x_0^{(q^G)} \right) - \alpha_\lambda \right) \cdot n \]

\[ > 0. \]  

(53)

(54)

The parameters \( 0 < q^G_\lambda < \beta_\lambda, 0 < p_G^{G} \leq \alpha_\lambda \) can be chosen subject to the restrictions explained in the parameter-adequate Subsections 3.3(a1),(a2),(a3),(a4) for \( G = U \) respectively Subsections 3.1(aN),(aEF) for \( G = E \).

(b) the sequence \( \left( C_{\lambda,n}^{G} \right)_{n \in \mathbb{N}} \) is strictly decreasing.

(c) for all observation horizons \( n \in \mathbb{N} \)

\[ C_{\lambda,n}^{G} \geq \begin{cases} B_{A,n}^U, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{SP,2} \cup \mathcal{P}_{SP,3ab} \cup \mathcal{P}_{SP,3c}) \times [0, 1[, \\
V_{A,n}, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{NI} \cup \mathcal{P}_{SP,1}) \times [0, 1[, \\
\end{cases} \]

with equality iff \( n = 1 \).

(d) \( \lim_{n \to \infty} C_{\lambda,n}^{G} = \begin{cases} 0, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{SP} \setminus (\mathcal{P}_{SP,3d} \cup \mathcal{P}_{SP,4}) \times [0, 1[, \\
\exp \left\{ \omega_0 \cdot x_0^{(q^G)} \right\} > 0, & \text{if } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{NI} \times [0, 1[, \\
\end{cases} \]

\[ = \lim_{n \to \infty} H_X(P_{A,n} || P_{H,n}). \]

(e) \( \lim_{n \to \infty} \frac{1}{n} \log C_{\lambda,n}^{G} = \frac{p_G^{G}}{q^G_{\alpha}} \left( x_0^{(q^G)} + \beta_\lambda \right) - \alpha_\lambda \) which coincides with

\[ \lim_{n \to \infty} \frac{1}{n} \log H_X(P_{A,n} || P_{H,n}) \text{ for all } (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{NI} \cup \mathcal{P}_{SP,1}) \times [0, 1[ \) respectfully with \( \lim_{n \to \infty} \frac{1}{n} \log B_{A,n}^U \) for all \( (\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in (\mathcal{P}_{SP,2} \cup \mathcal{P}_{SP,3ab} \cup \mathcal{P}_{SP,3c}) \times [0, 1[, \)

\[ . \]
Notice that the strict positivity in (54) and (55) can be easily seen from (36).

**Remark 4.6.** The formula (53) simplifies in the parameter case $(\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{SP,1} \times [0,1]$ which results in \(\frac{F}{q_\lambda} \left( \beta_\lambda + x_0(q_\lambda^G) \right) - \alpha_\lambda = \frac{F}{4} \cdot x_0(q_\lambda^G).\) For the case $(\beta_A, \beta_H, \alpha_A, \alpha_H, \lambda) \in \mathcal{P}_{N} \times [0,1]$, one can even use the stronger relation $p_\lambda^E = 0 = \alpha_\lambda$.

In order to get an “explicit” upper bound which does not rely on the implicitly given fixed point $x_0(q_\lambda^G)$ ($G \in \{U, E\}$), one can replace the latter by a close explicit upper approximate $\bar{x}_0(q_\lambda^G)$ and proceed completely analogously, leading to a larger upper bound (say) $\overline{C}_{\lambda,n}^G > C_{\lambda,n}^G$ in assertion (a) of Theorem 4.5; in the corresponding assertions (b), (c) and (d) one then has to replace $C_{\lambda,n}^G$ by $\overline{C}_{\lambda,n}^G$ and $x_0(q_\lambda^E)$ by $\bar{x}_0(q_\lambda^E)$. One possibility along these lines is the choice

$$\bar{x}_0(q_\lambda^G) := \frac{1}{q_\lambda^G} \left( 1 - q_\lambda^G \right) - \sqrt{(1 - q_\lambda^G)^2 - 2 \cdot q_\lambda^G \cdot (q_\lambda^G - \beta_\lambda)}, \quad (56)$$

which is exactly the unique negative solution to the quadratic equation

$$\overline{C}_{\lambda}(q_\lambda^G)(x) := \frac{q_\lambda^G}{2} \cdot x^2 + q_\lambda^G \cdot x + q_\lambda^G - \beta_\lambda = x.$$

By inspection of the first two derivatives, one gets $\overline{C}_{\lambda}(q_\lambda^G)(x) > \xi_\lambda(q_\lambda^G)(x)$ for all $x \in [-\infty, 0]$, and thus $\bar{x}_0(q_\lambda^G) > x_0(q_\lambda^G)$. Notice that the additional fundamental requirement $\overline{x}_0(q_\lambda^G) < q_\lambda^G - \beta_\lambda$ holds for parameter constellations for which $x_0(q_\lambda^G) > -1$ for any $x_0(q_\lambda^G) \leq \bar{x}_0(q_\lambda^G)$, since then one has $\frac{dG_\lambda(q_\lambda^G)}{dx}(x) > 0$ for all $x \in [x_0(q_\lambda^G), 0]$. Such a situation will be used as an auxiliary tool for the proof of Lemma A.3(c) in the appendix.

**5. Hellinger integral bounds in the diffusion limit**

One can show that a properly rescaled Galton-Watson process with immigration (GWI) converges weakly to a diffusion process $\tilde{X} := \{\tilde{X}_t, t \in [0, \infty]\}$ which is the unique, strong, nonnegative – and in case of $\frac{\eta}{\kappa} \geq \frac{1}{2}$ strictly positive – solution of the stochastic differential equation (SDE) of the form

$$d\tilde{X}_t = \left( \eta - \kappa \tilde{X}_t \right) dt + \sigma \sqrt{\tilde{X}_t} dW_t, \quad t \in [0, \infty], \quad \tilde{X}_0 \in [0, \infty] \text{ given,} \quad (57)$$

where $\eta \in [0, \infty]$, $\kappa \in [0, \infty]$, $\sigma \in [0, \infty]$ are constants and $W_t$ denotes a standard Brownian motion with respect to the underlying probability measure $P$; see e.g. Feller [16], Jirina [31], Lamperti [36], [37], Lindvall [49], [50], Grimvall [18], Jagers [30], Borovkov [7], Ethier and Kurtz [13], Durrett [12] for the non-immigration case corresponding to $\eta = 0$, $\kappa \geq 0$, Kawazu and Watanabe [34], Wei and Winnicki [70], Winnicki [72] for the immigration case corresponding to $\eta \neq 0$, $\kappa = 0$, as well as Sriram [61] for the general case $\eta \in [0, \infty]$, $\kappa \in \mathbb{R}$, Feller-type branching processes of the form (57), which are special cases of continuous state branching processes with immigration (see e.g. Kawazu und Watanabe [34], Li [43], as well as Dawson and Li [10] for imbeddings to affine processes) play for instance an important role in the modelling of the term structure of interest rates, cf. the seminal Cox-Ingersoll-Ross CIR model [9] and the vast follow-up literature thereof. Furthermore, (57) is also prominently used as (a special case of) Cox and Ross’s [8] constant elasticity of variance CEV asset price process, as (part of) Heston’s [22] stochastic asset-volatility framework, as a model of neuron activity (see e.g. Lansky and Lanska [39], Giorno et al. [17], Lanska et al. [41], Lansky et al. [40], Ditløven and Lansky [11], Höpfner [27], Lansky and Ditløven [38]), as a time-dynamic description of the nitrous oxide emission rate from the soil surface (see e.g. Pedersen [58]), as well as a model for the individual hazard rate in a survival analysis context (see e.g. Aalen and Gjessing [1]).

Along these lines of branching-type diffusion limits, it makes sense to consider the solutions of two SDEs (57) with different fixed parameter sets $(\eta, \kappa, \sigma)$ and $(\eta, \kappa, \eta, \sigma)$, determine for each of them a corresponding approximating GWI, investigate the Hellinger integral between the laws of these two GWI, and finally calculate the Hellinger integral (bounds) limit as the GWI approach their SDE solutions. Notice that for technicality reasons (which will be explained below), the constants $\eta$ and $\sigma$ ought to be independent of $\mathcal{A}, \mathcal{H}$ in our current context.
In order to make the abovementioned limit procedure rigorous, it is reasonable to work with appropriate approximations such that in each convergence step \( m \) one faces the setup \((\mathcal{P}_N \cup \mathcal{P}_{SP,1}) \times [0,1]\) (i.e. the non-immigration or the equal-fraction case), where the corresponding Hellinger integral can be calculated exactly in a recursive way (cf. Section 3.1). Let us explain the details in the following.

Consider a sequence of GWI \((X^{(m)})_{m \in \mathbb{N}}\) with probability laws \(P^{(m)}\) on a measurable space \((\Omega, \mathcal{A})\), where as above the subscript \( \bullet \) stands for either the hypothesis \( \mathcal{H} \) or the alternative \( \mathcal{A} \). Analogously to (5), we use for each fixed step \( m \in \mathbb{N} \) the representation

\[
X^{(m)}_n := \sum_{k=1}^{X^{(m)}_{n-1}} Y^{(m)}_{n-1,k} + \tilde{Y}^{(m)}_n, \quad n \in \mathbb{N}, \quad X^{(m)}_0 \in \mathbb{N}\text{ given,}
\]

(58)

where under the law \(P^{(m)}\)

- the collection \(Y^{(m)} := \{Y^{(m)}_{n-1,k}, n \in \mathbb{N}, k \in \mathbb{N}\}\) consists of i.i.d. random variables which are Poisson distributed with parameter \(\beta^{(m)} > 0\),
- the collection \(\tilde{Y}^{(m)} := \{\tilde{Y}^{(m)}_n, n \in \mathbb{N}\}\) consists of i.i.d. random variables which are Poisson distributed with parameter \(\alpha^{(m)} \geq 0\),
- \(Y^{(m)}\) and \(\tilde{Y}^{(m)}\) are independent.

From arbitrary drift-parameters \(\eta \in [0, \infty[, \kappa \in [0, \infty[, \sigma > 0\), and diffusion-term-parameter \(\sigma > 0\), we construct the offspring-distribution-parameter and the immigration-distribution parameter of the sequence \((X^{(m)}_n)_{n \in \mathbb{N}}\) by

\[
\beta^{(m)} := 1 - \frac{\kappa}{\sigma^2}, \quad \alpha^{(m)} := \beta^{(m)} \cdot \frac{\eta}{\sigma^2}.
\]

(59)

Here and henceforth, we always assume that the approximation step \( m \) is large enough to ensure that \(\beta^{(m)} \in [0,1]\) and at least one of \(\beta^{(m)}_{\mathcal{A}}, \beta^{(m)}_{\mathcal{H}}\) is strictly less than 1; this will be abbreviated by \( m \in \mathbb{N}\). Let us point out that – as mentioned above – our choice entails the best-to-handle setup (59) for the required absolute continuity in (6) both models at stake have to “live” on the same time-scale \( \tau^{(m)} := \lfloor \sigma^2 mt \rfloor \). For this setup, one obtains the following convergence result:

**Theorem 5.1.** Let \(\eta \in [0, \infty[, \kappa \in [0, \infty[, \sigma \in [0, \infty[, \alpha \in \mathbb{R}\) and \(\tilde{X}^{(m)}\) be as defined in (58) to (60). Furthermore, let us suppose that \(\lim_{m \to \infty} \frac{1}{m} X^{(m)}_0 = \tilde{X}_0 > 0\) and denote by \(D([0, \infty[, [0, \infty])\) the space of right-continuous functions \(f : [0, \infty[ \to [0, \infty[\) with left limits. Then the sequence of processes \((\tilde{X}^{(m)}_t)_{m \in \mathbb{N}}\) converges in distribution in \(D([0, \infty[, [0, \infty])\) to a diffusion process \(\tilde{X}\) which is the unique strong, nonnegative – and in case of \(\frac{\eta}{\sigma^2} \geq \frac{1}{2}\) strictly positive – solution of the SDE

\[
d\tilde{X}_t = (\eta - \kappa \tilde{X}_t) dt + \sigma \sqrt{\tilde{X}_t} dW^*_t, \quad t \in [0, \infty[, \quad \tilde{X}_0 \in [0, \infty[\text{ given,}
\]

(61)

where \(W^*_t\) denotes a standard Brownian motion with respect to the limit probability measure \(P^*_\bullet\).

Notice that the condition \(\frac{\eta}{\sigma^2} \geq \frac{1}{2}\) can be interpreted in our approximation setup (59) as \(\alpha^{(m)} \geq \beta^{(m)}/2\), which quantifies the intuitively reasonable indication that if the probability \(P^*_\bullet[\tilde{Y}^{(m)}_n = 0] = e^{-\alpha^{(m)}}\) of having no immigration is small enough relative to the probability \(P^*_\bullet[\tilde{Y}^{(m)}_{n-1,k} = 0] = e^{-\beta^{(m)}}\) of having no offspring \((m \in \mathbb{N})\), then the limiting diffusion \(\tilde{X}\) never hits zero almost surely.

The corresponding proof of Theorem 5.1 – which is outlined in Appendix A.3 – is an adaption of the proof of Theorem 9.1.3 in Ethier and Kurtz [13] which deals with drift-parameters \(\eta = 0, \kappa = 0\) in the SDE...
For each approximation step \( m \) and each observation horizon \( \tau_i^{(m)} \), the corresponding Hellinger integrals

\[
H_\lambda \left( P_{A_i[\sigma^2 mt]} \left| \right| P_{R_i[\sigma^2 mt]} \right)
\]

obey the results of

(ap1) the Propositions 3.1 (for \( \eta = 0 \)) and 3.2 (for \( \eta \in ]0, \infty[ \)), as far as recursively computable exact values are concerned,

(ap2) Theorem 4.2 and Theorem 4.5, as far as closed-form bounds are concerned; recall that the current setup is of type \( (P_{NI} \cup P_{SP,1}) \times ]0, 1[ \), and thus we can use the simplifications proposed in the Remarks 4.3 and 4.6.

In order to obtain the desired limits of the Hellinger integrals \( H_\lambda \left( P_{A_i[\sigma^2 mt]} \left| \right| P_{R_i[\sigma^2 mt]} \right) \) respectively of their closed-form bounds as \( m \to \infty \), one faces the following problems; in accordance with Section 3.1, for each fixed \( m \) in (ap1) one has to choose the parameters \( P_\lambda^{(m)} := \left( 0 \alpha_m^{(m)} \right)^{\lambda} \left( 0 \alpha_m^{(m)} \right)^{1-\lambda} \), \( q_{\lambda}^{(m)} := \left( \beta_m^{(m)} \right)^{\lambda} \left( \beta_m^{(m)} \right)^{1-\lambda} \), which in particular determine the fundamental sequence \( \alpha_n^{(m)} \in \mathbb{N} := \left( \left( 0 \alpha_n^{(m)} \right) \right)_{n \in \mathbb{N}} \) (cf. (17)). This enters in the appropriate versions of part (b) in Propositions 3.1 and 3.2 respectively in form of \( a_{\left( q_{\lambda}^{(m)} \right)} \), and the correspondingly arising convergences (as \( m \to \infty \)) seem to be not (straightforwardly) tractable due to the recursive nature of (17). In contrast, for the closed-form bounds of Section 4 the desired convergences are tractable, which will be worked out in the following. To begin with, let us explicitly formulate the results of the application of Theorem 4.2 (where Remark 4.3 applies) and Theorem 4.5 (where Remark 4.6 applies) to the current setup. For this, we use the following SDE-parameter constellations (which are consistent with (59) in combination with our requirement to work here only on \( (P_{NI} \cup P_{SP,1}) \times ]0, 1[ \)):

let \( P_{NI} \) be the set of all \( (\kappa_A, \kappa_H, \eta) \) for which \( \eta = 0, \kappa_A \in ]0, \infty[ , \kappa_H \in ]0, \infty[ \) with \( \kappa_A \neq \kappa_H \); furthermore, denote by \( P_{SP,1} \) the set of all \( (\kappa_A, \kappa_H, \eta) \) for which \( \eta \in ]0, \infty[ , \kappa_A \in ]0, \infty[ , \kappa_H \in ]0, \infty[ \) with \( \kappa_A \neq \kappa_H \). On \( P_{NI} \cup P_{SP,1} \) there hold for \( m \in \mathbb{N} \) the useful restrictions \( q_{\lambda}^{(m)} \in ]0, 1[ \) and \( \beta_m^{(m)} \in ]0, 1[ \). For the sake of brevity, let us henceforth use the abbreviations

\[
x_0^{(m)} := x_0^{(q_{\lambda}^{(m)})}, \Gamma^{(m)} := \Gamma^{(q_{\lambda}^{(m)})} = \frac{q_{\lambda}^{(m)}}{2} \cdot e_0^{(m)} \cdot \left( q_{\lambda}^{(m)} \right)^2, \quad d^{(m), \text{S}} := d^{(q_{\lambda}^{(m)})}, \quad d^{(m), \text{T}} := d^{(q_{\lambda}^{(m)})}, \quad d^{(m), \text{S}} := \frac{x_0^{(m)} - q_{\lambda}^{(m)} - \beta_m^{(m)}}{x_0^{(m)}} \]

and

\[d^{(m), \text{T}} := \frac{x_0^{(m)} - q_{\lambda}^{(m)} - \beta_m^{(m)}}{x_0^{(m)}}.
\]

By the above considerations, the Theorems 4.2 and 4.5 (together with their remarks) adapt to the current setup as follows:

**Corollary 5.2.** For all \( (\kappa_A, \kappa_H, \eta, \lambda) \in (P_{NI} \cup P_{SP,1}) \times ]0, 1[ \), all \( t \in ]0, \infty[ \), all approximation steps \( m \in \mathbb{N} \) and all initial population sizes \( X_0^{(m)} \in \mathbb{N} \) the Hellinger integral can be bounded by

\[
\exp \left\{ x_0^{(m)} \cdot \left[ X_0^{(m)} - \frac{\eta}{\sigma^2} d^{(m), \text{T}} \left| \right| \sigma^2 mt \right] \right\} \left( 1 - \left( d^{(m), \text{T}} \right)^{\left| \sigma^2 mt \right|} \right)
\]

\[
+ x_0^{(m)} \frac{\eta}{\sigma^2} \left| \sigma^2 mt \right| + \frac{\sigma^2 mt}{\sigma^2 mt} \left( X_0^{(m)} + \beta_m^{(m)} \right)
\]

\[
\leq H_\lambda \left( P_{A_i[\sigma^2 mt]} \left| \right| P_{R_i[\sigma^2 mt]} \right)
\]

\[
\leq \exp \left\{ x_0^{(m)} \cdot \left[ X_0^{(m)} - \frac{\eta}{\sigma^2} d^{(m), \text{S}} \left| \right| \sigma^2 mt \right] \right\} \left( 1 - \left( d^{(m), \text{S}} \right)^{\left| \sigma^2 mt \right|} \right)
\]

\[
+ x_0^{(m)} \frac{\eta}{\sigma^2} \left| \sigma^2 mt \right| - \frac{\sigma^2 mt}{\sigma^2 mt} \left( X_0^{(m)} - \beta_m^{(m)} \right),
\]

where we define analogously to (44), (45), (54) and (55).
\[ \zeta_n^{(m)} := \Gamma^{(m)} \cdot \frac{(d^{(m)}.T)^{n-1} (1 - (d^{(m)}.T)^n)}{1 - d^{(m)}.T} > 0, \]
\[ \eta_n^{(m)} := \frac{\eta}{\sigma^2} \cdot \Gamma^{(m)} \cdot \frac{(1 - (d^{(m)}.T)^n)}{(1 - d^{(m)}.T)^2} \cdot \left( 1 - d^{(m)}.T \cdot \frac{1 + (d^{(m)}.T)^n}{1 + d^{(m)}.T} \right) \geq 0, \]
\[ \xi_n^{(m)} := \Gamma^{(m)} \cdot \frac{d^{(m)}.S^{n-1} (1 - (d^{(m)}.T)^n)}{d^{(m)}.S - d^{(m)}.T} > 0, \]
\[ \psi_n^{(m)} := \frac{\eta}{\sigma^2} \cdot \Gamma^{(m)} \cdot \frac{d^{(m)}.T}{1 - d^{(m)}.T} \cdot \left( \frac{1 - (d^{(m)}.S d^{(m)}.T)^{n-1} (1 - (d^{(m)}.T)^n)}{d^{(m)}.S - d^{(m)}.T} \right) \geq 0. \]

Notice that the bounds (63) simplify significantly in the case \((\kappa_A, \kappa_H, \eta, \lambda) \in \bar{\mathcal{P}}_{N1} \times ]0,1[\) for which \(\eta = 0\) holds.

Let us finally present the corresponding desired limit assertions as the approximation step \(m\) tends to infinity, by making use of the quantities
\[ \kappa_\lambda := \lambda \kappa_A + (1 - \lambda) \kappa_H > 0 \quad \text{as well as} \quad \Lambda_\lambda := \sqrt{\lambda \kappa^2_A + (1 - \lambda) \kappa^2_H} > \kappa_\lambda : \quad (64) \]

**Theorem 5.3.** Let the initial SDE-value \(\bar{X}_0 \in [0, \infty]\) be arbitrary but fixed, and suppose that \(\lim_{m \to \infty} \frac{1}{m} X_0^{(m)} = \bar{X}_0\). Then, for all \(t \in [0, \infty[\) and all \((\kappa_A, \kappa_H, \eta, \lambda) \in (\bar{\mathcal{P}}_{N1} \cup \bar{\mathcal{P}}_{SP,1}) \times ]0,1[\) the Hellinger integral limit can be bounded by
\[ D_{\lambda,t}^U := \exp \left\{ - \frac{\Delta_1 - \kappa_\lambda}{\sigma^2} \left[ \bar{X}_0 - \frac{\eta}{\lambda_\lambda} \right] (1 - e^{-\lambda_\lambda \cdot t}) - \frac{\eta}{\sigma^2} (\Lambda_\lambda - \kappa_\lambda) \cdot t \right\} \]
\[ + L_{\lambda}^{(1)}(t) \cdot \bar{X}_0 + \frac{\eta}{\sigma^2} \cdot L_{\lambda}^{(2)}(t) \]
\[ \leq \lim_{m \to \infty} H_{\lambda} \left( \frac{P_{\lambda}^{(m)}}{P_{\lambda}(\sigma^2 m^2 t)} \right) \left\| P_{\lambda}(\sigma^2 m^2 t) \right\| \]
\[ \leq \exp \left\{ - \frac{\Delta_1 - \kappa_\lambda}{\sigma^2} \left[ \bar{X}_0 - \frac{\eta}{\lambda_\lambda} \right] (1 - e^{-\frac{1}{2}((\Lambda_\lambda + \kappa_\lambda) \cdot t)}) - \frac{\eta}{\sigma^2} (\Lambda_\lambda - \kappa_\lambda) \cdot t \right\} \]
\[ - U_{\lambda}^{(1)}(t) \cdot \bar{X}_0 - \frac{\eta}{\sigma^2} \cdot U_{\lambda}^{(2)}(t) \]
\[ =: D_{\lambda,t}^U, \tag{66} \]

where for all \(t \geq 0\)
\[ L_{\lambda}^{(1)}(t) := \frac{(\Lambda_\lambda - \kappa_\lambda)^2}{2 \sigma^2} \cdot e^{-\Lambda_\lambda \cdot t} \cdot (1 - e^{-\Lambda_\lambda \cdot t}) > 0, \tag{67} \]
\[ L_{\lambda}^{(2)}(t) := \frac{1}{4} \cdot \frac{(\Lambda_\lambda - \kappa_\lambda)^2}{\Lambda_\lambda} \cdot (1 - e^{-\Lambda_\lambda \cdot t})^2 > 0, \tag{68} \]
\[ U_{\lambda}^{(1)}(t) := \frac{(\Lambda_\lambda - \kappa_\lambda)^2}{\sigma^2} \cdot \left[ \frac{e^{-\frac{1}{2}((\Lambda_\lambda + \kappa_\lambda) \cdot t)} - e^{-\Lambda_\lambda \cdot t}}{2 \cdot \Lambda_\lambda} - \frac{e^{-\frac{1}{2}((\Lambda_\lambda + \kappa_\lambda) \cdot t)} - (1 - e^{-\Lambda_\lambda \cdot t})}{2 \cdot \Lambda_\lambda} \right] \geq 0, \tag{69} \]
\[ U_{\lambda}^{(2)}(t) := \frac{(\Lambda_\lambda - \kappa_\lambda)^2}{\Lambda_\lambda} \cdot \left[ \frac{1 - e^{-\frac{1}{2}((\Lambda_\lambda + \kappa_\lambda) \cdot t)} - e^{-\Lambda_\lambda \cdot t} - e^{-\frac{1}{2}((\Lambda_\lambda + \kappa_\lambda) \cdot t)}}{\Lambda_\lambda - \kappa_\lambda} \right] \geq 0. \tag{70} \]

Notice that the components \(L_{\lambda}^{(i)}(t)\) and \(U_{\lambda}^{(i)}(t)\) \((i = 1,2)\) do not depend on the parameter \(\eta\), and that the bounds (66) and (66) simplify significantly in the case \((\kappa_A, \kappa_H, \eta, \lambda) \in \bar{\mathcal{P}}_{N1} \times ]0,1[\), for which \(\eta = 0\) holds.

6. Power divergences and relative entropy

All the results of the previous sections carry correspondingly over from the Hellinger integrals \(H_{\lambda}(\cdot||\cdot)\) \((\lambda \in [0,1[)\) to the power divergences \(I_{\lambda}(\cdot||\cdot)\) by virtue of the relation (cf. (1))
In particular, this leads to bounds on $I_\lambda(P_A||P_H)$ which are tighter than the general rudimentary bound (4) connected with (13). Furthermore, it is well known that in general the relative entropy defined by (2)

$$I(P_A||P_H) = \lim_{\lambda \to 1} I_\lambda(P_A||P_H),$$

see e.g. Liese and Vajda [47]. Accordingly, for our context of GWI we can use (71) in combination with the recursive exact values respectively recursive lower bounds of Theorem 2.2 and Section 3.2 to obtain the following closed-form exact values respectively closed-form upper bounds of the relative entropy $I(P_{A,n}||P_{H,n})$:

**Theorem 6.1.** (a) For all $(\beta_A,\beta_H,\alpha_A,\alpha_H) \in (PNI \cup P_{SP,1})$, all initial population sizes $\omega_0 \in \mathbb{N}$ and all observation horizons $n \in \mathbb{N}$

$$I(P_{A,n}||P_{H,n}) = \left\{ \begin{array}{ll}
\beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) + \beta_H \\
+ \alpha_A \left[ \beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) - \beta_A \left( 1 - \beta_A \right) \right] \cdot \left( 1 - (\beta_A)^n \right) \\
+ \alpha_A \left[ \beta_A \log \left( \frac{\lambda}{\beta_A} \right) - \beta_A \cdot \beta_H \right] + \alpha_H \cdot n, & \text{if } \beta_A \neq 1,
\end{array} \right. \quad (72)$$

(b) For all $(\beta_A,\beta_H,\alpha_A,\alpha_H) \in P_{SP}\setminus P_{SP,1}$, all initial population sizes $\omega_0 \in \mathbb{N}$ and all observation horizons $n \in \mathbb{N}$ it holds $I(P_{A,n}||P_{H,n}) \leq E^U_n$, where

$$E^U_n := \left\{ \begin{array}{ll}
\beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) + \beta_H \\
+ \alpha_A \left[ \beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) - \beta_A \left( 1 - \beta_A \right) \right] \cdot \left( 1 - (\beta_A)^n \right) \\
+ \alpha_A \left[ \beta_A \log \left( \frac{\lambda}{\beta_A} \right) - \beta_A \cdot \beta_H \right] + \alpha_H \cdot n, & \text{if } \beta_A \neq 1,
\end{array} \right. \quad (73)$$

Remark 6.2. The $n$-behaviour (of the bounds of) the relative entropy $I(P_{A,n}||P_{H,n})$ in Theorem 6.1 is influenced by the following facts:

- $\beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) + \beta_H \geq 0$ with equality iff $\beta_A = \beta_H$.
- In the case $\beta_A \neq 1$ of (73), there holds $\alpha_A \left[ \beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) - \beta_A \left( 1 - \beta_A \right) \right] \cdot \left( 1 - (\beta_A)^n \right) \geq 0$ with equality iff $\alpha_A = \alpha_H$ and $\beta_A = \beta_H$.

In contrast, in order to derive (semi-)closed-form lower bounds of the relative entropy $I(P_{A,n}||P_{H,n})$ we use (71) in combination with the recursive upper bounds of Theorem 2.2(b) and appropriately adapted detailed analyses along the lines of Section 3.3. This amounts to

**Theorem 6.3.** For all $(\beta_A,\beta_H,\alpha_A,\alpha_H) \in P_{SP}\setminus P_{SP,1}$, all initial population sizes $\omega_0 \in \mathbb{N}$ and all observation horizons $n \in \mathbb{N}$

$$I(P_{A,n}||P_{H,n}) \geq E^L_n := \sup_{k \in \mathbb{N}_0, \ y \in [0,\infty]} \left\{ E^L_{y,n}, E^L_{k,n}, E^L_{n,\text{hor}} \right\} \in [0,\infty], \quad (74)$$

where for all $y \in [0,\infty]$ we define the possibly negatively valued – finite bound component

$$E^L_{y,n} := \left\{ \begin{array}{ll}
\beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) + \beta_H \\
+ \alpha_A \left[ \beta_A \left( \log \left( \frac{\lambda}{\beta_A} \right) - 1 \right) - \beta_A \left( 1 - \beta_A \right) \right] \cdot \left( 1 - (\beta_A)^n \right) \\
+ \alpha_A \left[ \beta_A \log \left( \frac{\lambda}{\beta_A} \right) - \beta_A \cdot \beta_H \right] + \alpha_H \cdot n, & \text{if } \beta_A \neq 1,
\end{array} \right. \quad (75)$$

and for all $k \in \mathbb{N}_0$ the possibly negatively valued – finite bound component
Of course, as a less tight but less involved explicit lower bound of the relative entropy $z$ with $n$, initial population $\omega$ and – possibly negatively valued – finite bound component $n$ observation horizons $P$

It seems that the optimization problem in (74) admits in general only an implicitly representable solution.

Furthermore, on $\mathcal{P}_{SP,4}$ we set $E_{n,hor}^L := 0$ for all $n \in \mathbb{N}$ whereas on $\mathcal{P}_{SP \setminus (\mathcal{P}_{SP,1} \cup \mathcal{P}_{SP,4})}$ we define

$$E_{n,hor}^L := \left\{ \begin{array}{ll} (\alpha_A + \beta_A z^*) \cdot \left[ \log \left( \frac{\alpha_A + \beta_A z^*}{\alpha_F + \beta_F z^*} \right) - 1 \right] + \alpha_H + \beta_H z^* \cdot n, & n \in \mathbb{N}, \end{array} \right.$$ (77)

with $z^* := \arg\max_{x \in \mathbb{N}} \left\{ (\alpha_A + \beta_A x) \left[ -\log \left( \frac{\alpha_A + \beta_A x}{\alpha_F + \beta_F x} \right) + 1 \right] - (\alpha_H + \beta_H x) \right\}$. In the subcases $\mathcal{P}_{SP,2} \cup \mathcal{P}_{SP,3a} \cup \mathcal{P}_{SP,3b} \cup \mathcal{P}_{SP,4}$ one gets even $E_{n,hor}^L > 0$ for all $\omega \in \mathbb{N}$ and all $n \in \mathbb{N}$. In the subcase $\mathcal{P}_{SP,3d}$, one obtains for each fixed $n \in \mathbb{N}$ and each fixed $\omega \in \mathbb{N}$ the strict positivity $E_{n,hor}^L > 0$ if $(\frac{d}{dy} E_{y,n,hor}^L)(y^*) \neq 0$, where $y^* := \frac{\alpha_A - \beta_A}{\alpha_H - \beta_H} \in \mathbb{N}$ and hence

$$\left( \frac{\partial}{\partial y} E_{y,n,hor}^L \right)(y^*) = \begin{cases} \frac{(\beta_A - \beta_H)^n}{\alpha_A \beta_H - \alpha_F \beta_A} \cdot \left[ \omega_n - \frac{\alpha_A}{1 - \beta_A} \right] \cdot \left[ \frac{\beta_A - \beta_H}{\beta_A} \cdot \left( 1 + \frac{\alpha_A (\beta_A - \beta_H)}{(\alpha_F - \beta_A)(\alpha_H - \beta_A)} \right) \right] \cdot n, & \text{if } \beta_A \neq 1, \\
-\frac{(1 - \beta_H)^n}{\alpha_A \beta_H - \alpha_F \beta_A} \cdot \left[ \frac{\alpha_A \beta_H}{\beta_A} \cdot n^2 + (\omega + \frac{\alpha_A}{1 - \beta_A}) \cdot n \right] - (1 - \beta_H)^2 \cdot n, & \text{if } \beta_A = 1. \end{cases}$$ (78)

Remark 6.4. Consider the exemplary parameter setup $(\beta_A, \beta_H, \alpha_A, \alpha_H) = (\frac{1}{3}, \frac{2}{3}, 2, 1) \in \mathcal{P}_{SP,3d}$. For initial population $\omega = 3$ it holds $(\frac{d}{dy} E_{y,n,hor}^L)(y^*) = 0$ for all $n \in \mathbb{N}$, whereas for $\omega \neq 3$ one obtains $(\frac{d}{dy} E_{y,n,hor}^L)(y^*) \neq 0$ for all $n \in \mathbb{N}$.

It seems that the optimization problem in (74) admits in general only an implicitly representable solution. Of course, as a less tight but less involved explicit lower bound of the relative entropy $I(P_A, n | P_H, n)$ one can use any term of the form $\max \left\{ E_{y,n,sec}^L, E_{y,n,hor}^L \right\}$ (y \in [0, \infty[, k \in \mathbb{N})), as well as the following

Corollary 6.5. (a) For all $(\beta_A, \beta_H, \alpha_A, \alpha_H) \in \mathcal{P}_{SP \setminus \mathcal{P}_{SP,1}}$, all initial population sizes $\omega \in \mathbb{N}$ and all observation horizons $n \in \mathbb{N}$

$$I(P_A, n | P_H, n) \geq E_n^L \geq E_n := \max \left\{ E_{\infty,n,sec}^L, E_{0,n,sec}^L, E_{n,hor}^L \right\} \in [0, \infty[, \right.$$ with $E_n^L$ defined by (77), with – possibly negatively valued – finite bound component $E_{\infty,n,sec}^L := \lim_{n \to \infty} E_{y,n,sec}^L$, where

$$E_{\infty,n}^L := \begin{cases} \frac{\beta_A (\log (\frac{\alpha_A}{1 - \beta_A}) - 1) + \beta_H}{\beta_A (1 - \beta_A)} \cdot \left[ \omega_n - \frac{\alpha_A}{1 - \beta_A} \right] \cdot (1 - (\beta_A)^n), & \text{if } \beta_A \neq 1, \\
+ \frac{\alpha_A \beta_A (\log (\frac{\alpha_A}{1 - \beta_A}) - 1) + \beta_H}{\beta_A (1 - \beta_A)} + \alpha_A \left( 1 - \frac{\beta_A}{\beta_A} \right) + \alpha_H \left( 1 - \frac{\beta_A}{\beta_A} \right) \cdot n, & \text{if } \beta_A = 1, \end{cases}$$

and – possibly negatively valued – finite bound component

$$E_{\infty,n,sec}^L := \lim_{n \to \infty} E_{y,n,sec}^L,$$
Definition, the Bayes decision rule making BDM and Neyman-Pearson testing NPT. In BDM, we decide between an action associated with immigration on the other hand. In order to indicate the concrete applicability of our combining in-

As already mentioned in the introduction, there are numerous applications of both ingredients – power diver-

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within the framework of Section 5, one gets for all initial SDE-values \( \tilde{X}_0 \in [0, \infty[, \) all observation horizons \( t \in [0, \infty[, \) and all parameter constellations \( (\kappa_A, \kappa_H, \eta) \in (\mathcal{P}_N) \cup (\mathcal{P}_{SP,1}) \)

\[
\lim_{m \to \infty} I \left( P_{A}^{(m)} \right| \sigma^2 \right) = \lim_{m \to \infty} \lim_{n \to \infty} I_A \left( P_{A}^{(m)} \right| \sigma^2 \right)
\]

\[
= \left\{ \begin{array}{ll}
\frac{(\kappa_A - \gamma_{\sigma^2})^2}{2 \kappa_A} \cdot \left[ (\tilde{X}_0 - \gamma_{\sigma^2}) \cdot (1 - e^{-\kappa_A t}) + \eta \cdot t \right], & \text{if } \kappa_A > 0, \\
\frac{\gamma_{\sigma^2}}{2}, & \text{if } \kappa_A = 0,
\end{array} \right.
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} I_A \left( P_{A}^{(m)} \right| \sigma^2 \right).
\]

7. Applications

As already mentioned in the introduction, there are numerous applications of both ingredients – power diver-

dependences resp. Hellinger integrals resp. relative entropy on the one hand and Galton-Watson branching processes with immigration on the other hand. In order to indicate the concrete applicability of our combining in-

investigations, for the sake of brevity we confine ourselves to some issues in the context of Bayesian decision making BDM and Neyman-Pearson testing NPT. In BDM, we decide here between an action \( d_H \) “associated with” the (say) hypothesis law \( P_H \) and an action \( d_A \) “associated with” the (say) alternative law \( P_A \), based on the sample path observation \( X_n := \{ X_l \mid l \in \{0, 1, \ldots, n\} \} \) of the GWI-generation-sizes up to observation horizon \( n \in \mathbb{N} \). Following the lines of Stummer and Vajda [67] (adapted to our branching process context), for BDM let us consider as admissible decision rules \( \delta_n : \Omega_n \to \{ d_H, d_A \} \) the ones generated by all path sets \( G_n \in \Omega_n \) through

\[
\delta_n(X_n) := \delta_G_n(X_n) := \begin{cases}
d_A, & \text{if } X_n \in G_n, \\
d_H, & \text{if } X_n \notin G_n,
\end{cases}
\]

as well as loss functions of the form

\[
\left( L(d_H, H) \quad L(d_H, A) \right) = \left( L(d_A, H) \quad L(d_A, A) \right)
\]

with pregiven constants \( L_A > 0, L_H > 0 \) (e.g. arising as bounds from quantities in worst-case scenarios); notice that in (80), \( d_H \) is assumed to be a zero-loss action under \( H \) and \( d_A \) a zero-loss action under \( A \). Per definition, the Bayes decision rule \( \delta_{G_n,\min} \) minimizes – over \( G_n \) – the mean decision loss

\[
\mathcal{L}(\delta_{G_n}) := p_H^{\text{prior}} \cdot \mathcal{L}_H \cdot \mathcal{P}_H \left( \delta_{G_n}(X_n) = d_H \right) + p_A^{\text{prior}} \cdot \mathcal{L}_A \cdot \mathcal{P}_A \left( \delta_{G_n}(X_n) = d_A \right) 
\]

\[
= p_H^{\text{prior}} \cdot \mathcal{L}_H \cdot \mathcal{P}_H(n) + p_A^{\text{prior}} \cdot \mathcal{L}_A \cdot \mathcal{P}_A(n, \Omega - G_n)
\]

for given prior probabilities \( p_H^{\text{prior}} = \Pr(H) \in [0,1] \) for \( H \) and \( p_A^{\text{prior}} := \Pr(A) = 1 - p_H^{\text{prior}} \) for \( A \). As a side remark let us mention that, in a certain sense, the involved model (parameter) uncertainty expressed by the “superordinate” Bernoulli-type law \( \Pr = Bin(1, p_H^{\text{prior}}) \) can also be reinterpreted as a rudimentary
static random environment caused e.g. by a random Bernoulli-type external static force. By straightforward
calculations, one gets with (6) the minimizing path set \( G_{n, \min} = \left\{ Z_n \geq \frac{p_{n}^{\text{prior}} L_n}{p_{n}^{\text{prior}} A_n} \right\} \) leading to the minimal
mean decision loss, i.e. the Bayes risk,
\[ \mathcal{R}_n := \min_{G_n} \mathcal{L}(\delta_{G_n}) = \mathcal{L}(\delta_{G_{n, \min}}) = \int_{\Omega_n} \min \left\{ p_{n}^{\text{prior}} L_n, p_{n}^{\text{prior}} A_n Z_n \right\} dP_{\mathcal{H}, n}. \] (82)

Notice that – by straightforward standard arguments – the alternative decision procedure
take action \( d_A \) (resp. \( d_L \)) if \( L_{\mathcal{H}} \cdot p_{n}^{\text{post}}(\mathcal{X}_n) / \min \left\{ p_{n}^{\text{prior}} L_n, p_{n}^{\text{prior}} A_n \right\} = \min \left\{ p_{n}^{\text{prior}} L_n, p_{n}^{\text{prior}} A_n \right\} \) with posterior probabilities \( p_{n}^{\text{post}}(\mathcal{X}_n) := \frac{p_{n}^{\text{prior}} L_n / p_{n}^{\text{prior}} A_n}{\min \left\{ p_{n}^{\text{prior}} L_n, p_{n}^{\text{prior}} A_n \right\}} \) for all \( n \in \mathbb{N} \)
with \( p_{n}^{\text{prior}} \in [0,1], \lambda \in [0,1] \) and \( n \in \mathbb{N} \) the upper bound
\[ \mathcal{R}_n \leq \Lambda_A^{\frac{1}{\lambda} - \lambda} H_{\lambda}(P_{A,n}||P_{\mathcal{H}, n}), \quad \text{with } \Lambda_{\mathcal{H}} := p_{n}^{\text{prior}} L_n, \quad \Lambda_A := (1 - p_{n}^{\text{prior}}) L_A, \] (83)
as well as the lower bound
\[ (\mathcal{R}_n)_{\min\{\lambda, 1 - \lambda\}} \cdot (\Lambda_A + \Lambda_{\mathcal{H}} - \mathcal{R}_n)_{\max\{\lambda, 1 - \lambda\}} \geq \Lambda_A^{\frac{1}{\lambda} - \lambda} H_{\lambda}(P_{A,n}||P_{\mathcal{H}, n}), \]
which implies in particular the “direct” lower bound
\[ \mathcal{R}_n \geq \frac{\Lambda_A^{\frac{1}{\lambda} - \lambda}}{(\Lambda_A + \Lambda_{\mathcal{H}})_{\max\{\lambda, 1 - \lambda\}}} \cdot (H_{\lambda}(P_{A,n}||P_{\mathcal{H}, n}))_{\max\{\lambda, 1 - \lambda\}}. \] (84)

By using (83) (respectively (84)) together with the exact values and the upper (respectively lower) bounds of the Hellinger integrals \( H_{\lambda}(P_{A,n}||P_{\mathcal{H}, n}) \) derived in the preceding sections, we end up with upper (respectively lower) bounds of the Bayes risk \( \mathcal{R}_n \). For different types of – mainly parameter estimation (squared-error type loss function) concerning – Bayesian analyses based on GW(I) generation size observations, see e.g. Jagers [30], Heyde [23], Heyde and Johnstone [24], Johnson et al. [32], Basawa and Rao [4], Basawa and Scott [6], Scott [60], Guttorp [19], Yanev and Tsokos [74], Mendoza and Gutierrez-Pena [54], and the references therein.

Alternatively to the BDM applications above, let us now briefly deal with the corresponding NPT framework
with randomized tests \( T_n : \Omega_n \rightarrow [0,1] \) of the hypothesis \( P_{\mathcal{H}} \) against the alternative \( P_A \), based on the
GWI-generation-size sample path observations \( \mathcal{X}_n := \{ X_l : l \in \{ 0, 1, \ldots, n \} \} \). In contrast to (81), (82) a Neyman-Pearson test minimizes – over \( T_n \) – the type II error probability \( \int_{\Omega_n} (1 - T_n) \cdot dP_{A,n} \) in the class of the tests for which the type I error probability \( \int_{\Omega_n} T_n \cdot dP_{A,n} \) is at most \( \varsigma \in [0,1] \). The corresponding minimal type II error probability
\[ \mathcal{E}_\varsigma(P_{A,n}||P_{\mathcal{H}, n}) := \inf_{T_n : \int_{\Omega_n} T_n \cdot dP_{A,n} \leq \varsigma} \int_{\Omega_n} (1 - T_n) \cdot dP_{A,n} \]
can for all \( \varsigma \in [0,1], \lambda \in [0,1], n \in \mathbb{N} \) be bounded from above by
\[ \mathcal{E}_\varsigma(P_{A,n}||P_{\mathcal{H}, n}) \leq \min \left\{ (1 - \lambda) \cdot \left( \frac{\lambda}{\varsigma} \right)^{\lambda(1 - \lambda)} \cdot (H_{1 - \lambda}(P_{A,n}||P_{\mathcal{H}, n}))^{1/(1 - \lambda)} \right\}, \] (85)
which is an adaption of a general result of Kraft [35], see also Liese and Vajda [47] as well as Stummer and Vajda [67]. Hence, by combining (85) with the exact values respectively upper bounds of the Hellinger integrals \( H_{1 - \lambda}(P_{A,n}||P_{\mathcal{H}, n}) \) from the preceding sections, we obtain for our context of GWI with Poisson offspring and Poisson immigration (including the non-immigration case) some upper bounds of \( \mathcal{E}_\varsigma(P_{A,n}||P_{\mathcal{H}, n}) \), which can also be immediately rewritten as lower bounds for the power \( 1 - \mathcal{E}_\varsigma(P_{A,n}||P_{\mathcal{H}, n}) \) of a most powerful test at level \( \varsigma \). In contrast to such finite-time-horizon results, for the (to our context) incompatible setup of GWI with Poisson offspring but nonstochastic immigration of constant value 1, the asymptotic rates of decrease as \( n \rightarrow \infty \) of the unconstrained type II error probabilities as well as the type I error probabilities were studied in Linkov and Lunyova [52] by a different approach employing also Hellinger integrals. Some other types of GW(I) concerning Neyman-Pearson testing investigations different to ours can be found e.g. in Basawa and Scott [5], Feigin [14], Sweeting [68], Basawa and Scott [6], and the references therein.

For the sake of brevity, a further more detailed discussion of GWI statistical issues along the lines of this
section as well as power-divergences-connected goodness-of-fit investigations will appear in a forthcoming paper.
Appendix A: Proofs and auxiliary lemmas

A.1. Tool and proof for Section 3

Lemma A.1. For all real numbers $x, y, z > 0$ and all $\lambda \in [0, 1]$ one has

$$x^\lambda y^{1-\lambda} - (\lambda x z^{\lambda-1} + (1-\lambda) y z^\lambda) \leq 0$$

with equality iff $\frac{x}{y} = z$.

**Proof of Lemma A.1** For fixed $\tilde{x} := x z^{\lambda-1} > 0$, $\tilde{y} := y z^\lambda > 0$ with $\tilde{x} \neq \tilde{y}$ we inspect the function $g$ on $[0, 1]$ defined by $g(\lambda) := \frac{\tilde{x} z^{\lambda-1} - (\lambda \tilde{x} + (1-\lambda) \tilde{y})}{\tilde{y}}$ which satisfies $g(0) = g(1) = 0$, $g'(0) = \tilde{y} \log(\tilde{x}/\tilde{y}) - (\tilde{x} - \tilde{y}) < \tilde{y}((\tilde{x}/\tilde{y}) - 1) - (\tilde{x} - \tilde{y}) = 0$ and which is strictly convex. Thus, the assertion follows immediately by taking into account the obvious case $\tilde{x} = \tilde{y}$.

**Proof of Formula (30):** For the parameter constellation in Subsection 3.3(a5) we employ as upper bound for $\phi_\lambda(x)$, $x \in \mathbb{N}_0$ the function

$$\bar{\phi}_\lambda(x) := \begin{cases} \phi_\lambda(0), & \text{if } x = 0, \\ 0, & \text{if } x > 0. \end{cases}$$

Notice that this method is rather crude, and gives in the other cases treated in the Subsections 3.3(a1) to (a4) worse bounds than those derived there. For the calculation of the Hellinger integral, we first set $\epsilon := 1 - e^{\phi_\lambda(0)} \in [0, 1]$. Hence, we obtain for all $n \in \mathbb{N}\setminus\{1\}$

$$\begin{align*}
\sum_{\omega_{n-1} = 0}^\infty \frac{[\varphi_\lambda(\omega_{n-2})]^{\omega_{n-1}}}{\omega_{n-1}!} \cdot \exp\{\bar{\phi}_\lambda(\omega_{n-1})\} &\leq \sum_{\omega_{n-1} = 0}^\infty \frac{[\varphi_\lambda(\omega_{n-2})]^{\omega_{n-1}}}{\omega_{n-1}!} \cdot \exp\{\bar{\phi}_\lambda(\omega_{n-1})\} \\
&= \exp\{\varphi_\lambda(\omega_{n-2})\} - \epsilon = \exp\{\varphi_\lambda(\omega_{n-2})\} \cdot [1 - \epsilon \cdot \exp\{-\varphi_\lambda(\omega_{n-2})\}] \\
&\leq \exp\{\varphi_\lambda(\omega_{n-2}) - \epsilon \cdot e^{-\varphi_\lambda(\omega_{n-2})}\}.
\end{align*}$$

In the current setup of Subsection 3.3(a5) we have $\beta_A \neq \beta_H$, which means that $\lim_{x \to -\infty} \phi_\lambda(x) = -\infty$ (cf. (p-xiii)). But this together with the nonnegativity of $\varphi_\lambda$ implies $\sup_{x \in \mathbb{N}_0} \exp\{\phi_\lambda(x) - \epsilon \cdot e^{-\varphi_\lambda(x)}\} =: \delta < 1$. Incorporating these considerations as well as the formulae (7) to (12), we get for $n = 1$ the relation $H_\lambda (P_{A, n}||P_{H, n}) = \exp\{\phi_\lambda(\omega_{n-1})\} < 1$ and for all $n \in \mathbb{N}\setminus\{1\}$ as a continuation of formula (9) (with the obvious shortcut for $n = 2$)

$$H_\lambda (P_{A, n}||P_{H, n}) = \sum_{\omega_1 = 0}^\infty \cdots \sum_{\omega_n = 0}^\infty \prod_{k=1}^n Z_{n,k}(\omega)$$

$$= \sum_{\omega_1 = 0}^\infty \cdots \sum_{\omega_{n-1} = 0}^\infty \prod_{k=1}^{n-1} Z_{n,k}(\omega) \cdot \exp\{f_A(\omega_{n-1})\} \cdot \exp\{f_H(\omega_{n-1})\}$$

$$\leq \delta \cdot \sum_{\omega_1 = 0}^\infty \cdots \sum_{\omega_{n-2} = 0}^\infty \prod_{k=1}^{n-2} Z_{n,k}(\omega) \leq \cdots \leq \delta^{[n/2]}.$$
A.2. Proofs of Section 4

Proof of Lemma 4.1 Recall the fundamental nonlinear recursion of \((a_n^{(q,k)})_{n \in \mathbb{N}_0}\) (cf. (37), (38)), the corresponding “substitute” inhomogeneous linear recursion of \((\tilde{a}_n^{(q,k)})_{n \in \mathbb{N}_0}\) (cf. (39), (40), (41)) and its homogenous linear relative \((\tilde{a}_n^{(q,k),\text{hom}})_{n \in \mathbb{N}_0}\) (cf. (31), (32)) which by (34) and (40) takes the form

\[
\tilde{a}_0^{(q,k),\text{hom}} := 0, \quad \tilde{a}_n^{(q,k),\text{hom}} := \xi^{(q,k)}_\lambda T(\tilde{a}_{n-1}^{(q,k),\text{hom}}) = x_0^{(q,k)} \left(1 - (d^{(q,k)}).T\right)^n, \quad n \in \mathbb{N}, \tag{87}
\]

with \(d^{(q,k)}.T = q_\lambda^* \cdot e^{x_0^{(q,k)}} \in [0,1] \). By construction, one has

\[
\tilde{a}_n^{(q,k),\text{hom}} < a_n^{(q,k)} \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{as well as} \quad \lim_{n \to \infty} \tilde{a}_n^{(q,k),\text{hom}} = \lim_{n \to \infty} a_n^{(q,k)} = x_0^{(q,k)}. \tag{88}
\]

As an auxiliary step, let us compare \(x \mapsto \xi^{(q,k)}_\lambda (x) = q_\lambda^* \cdot e^x - \beta_\lambda\) with the quadratic function

\[
\Upsilon^{(q,k)}_\lambda (x) := \frac{q_\lambda^*}{2} e^{2x} + q_\lambda^* e^x \left(1 - x^{(q,k)}_0\right) - x^{(q,k)}_0 \left(1 - q_\lambda^* e^x + \frac{q_\lambda^*}{2} e^{2x}\right)\tag{89}
\]

Clearly, we have the relations \(\Upsilon^{(q,k)}_\lambda (x_0^{(q,k)}) = x_0^{(q,k)} = \xi^{(q,k)}_\lambda (x_0^{(q,k)})\), \(\frac{\partial \Upsilon^{(q,k)}_\lambda}{\partial x} (x_0^{(q,k)}) = q_\lambda^* e^{x_0^{(q,k)}} = \frac{\partial \xi^{(q,k)}_\lambda}{\partial x} (x_0^{(q,k)})\), and \(\frac{\partial^2 \Upsilon^{(q,k)}_\lambda}{\partial x^2} (x) < \frac{\partial^2 \xi^{(q,k)}_\lambda}{\partial x^2} (x) \) for all \(x \in [x_0^{(q,k)}, 0]\). Hence, \(\Upsilon^{(q,k)}_\lambda (\cdot)\) is on \([x_0^{(q,k)}, 0]\) a strict lower functional bound of \(\xi^{(q,k)}_\lambda (\cdot)\). We are now ready to prove part (a) by induction. For \(n = 1\), we easily see that \(\tilde{a}_1^{(q,k)} < a_1^{(q,k)}\) iff \(\left(\frac{x_0^{(q,k)}}{2} - \frac{x_0^{(q,k)}}{2} - 1\right) < 0\), and the latter is obviously true. To continue, let us assume that \(\tilde{a}_n^{(q,k)} \leq a_n^{(q,k)}\) holds. From this, (41), (87) and (88) we obtain

\[
0 < a_n^{(q,k)} = \frac{q_\lambda^*}{2} e^{2x_0^{(q,k)}} \left(x_0^{(q,k)} \cdot x_0^{(q,k)}\right)^n = \frac{q_\lambda^*}{2} e^{2x_0^{(q,k)}} \left(\tilde{a}_n^{(q,k),\text{hom}} - x_0^{(q,k)}\right)^2
\]

\[
\frac{q_\lambda^*}{2} e^{2x_0^{(q,k)}} \left(a_n^{(q,k)} - x_0^{(q,k)}\right)^2 = \Upsilon^{(q,k)}_\lambda \left(a_n^{(q,k)} - d^{(q,k)}).T \cdot a_n^{(q,k)} - x_0^{(q,k)} \cdot (1 - d^{(q,k)}).T\right)
\]

\[
\xi^{(q,k)}_\lambda \left(a_n^{(q,k)} - d^{(q,k)}.T \cdot a_n^{(q,k)} - x_0^{(q,k)} \cdot (1 - d^{(q,k)}).T\right) < a_{n+1}^{(q,k)} - d^{(q,k)}.T \cdot \tilde{a}_n^{(q,k),\text{hom}} - x_0^{(q,k)} \cdot (1 - d^{(q,k)}).T\).
\]

Thus, \(\tilde{a}_n^{(q,k)} \leq a_n^{(q,k)}\) holds. In order to show (b), we make use of the straightforward representation

\[
\tilde{a}_n^{(q,k)} = \sum_{k=0}^{n-1} \left(d^{(q,k)}.T\right)^{n-k} \left(\tilde{a}_k^{(q,k)} + x_0^{(q,k)} \cdot (1 - d^{(q,k)}.T)\right)
\]

which implies that the sequence \(\left(\tilde{a}_n^{(q,k),\text{hom}}\right)_{n \in \mathbb{N}}\) is strictly decreasing since for all \(k \in \mathbb{N}_0\) there holds by (41)

\[
\tilde{a}_k^{(q,k)} + x_0^{(q,k)} \cdot (1 - d^{(q,k)}.T) \leq \Upsilon^{(q,k)}_\lambda (0) = \xi^{(q,k)}_\lambda (0) = q_\lambda^* - \beta_\lambda < 0.
\]

The final assertion follows immediately from (88) and the closed-form representation (34) with the choices \(K_1, K_2, \kappa, \nu, c\) given just right after (42).

Proof of Lemma 4.4 For \(\mathbb{P}_{SP,3d} \cup \mathbb{P}_{SP,4}\) we deal with the fundamental nonlinear recursion of \((a_n^{(q)})_{n \in \mathbb{N}_0}\), \(G \in \{E, U\}\) (cf. (48), (27)), the corresponding “substitute” inhomogeneous linear recursion of \((\tilde{a}_n^{(q)})_{n \in \mathbb{N}_0}\) (cf. (49), (50), (51)) and its homogenous linear counterpart \((\tilde{a}_n^{(q),\text{hom}})_{n \in \mathbb{N}_0}\) (cf. (31), (32)) which by (34) and (50) takes the form

\[
\tilde{a}_0^{(q),\text{hom}} := 0, \quad \tilde{a}_n^{(q),\text{hom}} := \xi^{(q)}_\lambda S(\tilde{a}_{n-1}^{(q),\text{hom}}) = x_0^{(q)} \left(1 - (d^{(q)}).S\right)^n, \quad n \in \mathbb{N}, \tag{89}
\]
with \(d(q_G^G)^{S} = 1 - \frac{\partial q_G^G}{\partial x_{0}(G)} \in ]d(q_G^G)^{T}, 1[\). By construction, we obtain
\[
\mu_1(q_G^G)^{hom} = \alpha_1(q_G^G)^{hom} = \frac{a_n(q_G^G)^{hom}}{\alpha_n(q_G^G)^{hom}} > \alpha_n(q_G^G)^{hom} \quad \text{for all } n \in \mathbb{N} \setminus \{1\}, \text{ and } \lim_{n \to \infty} \pi_n(q_G^G)^{hom} = \lim_{n \to \infty} a_n(q_G^G)^{hom} = x_0(q_G^G)^{hom}.
\]  
(90)

In analogy to the Proof of Lemma 4.1, we use the quadratic function
\[
\mathbf{T}_\lambda^{G}(G(x)^{q}) := \frac{q_G^G}{2} e^{x_0(G(x)^{q})} \cdot x^2 + \left(1 - \frac{q_G^G}{2} e^{x_0(G(x)^{q})} \cdot x_0(G(x)^{q}) - \frac{q_G^G - \beta}{x_0(G(x)^{q})} \right) \cdot x + q_G^G - \beta
\]
which satisfies \(\mathbf{T}_\lambda^{G}(G(x)^{q}) = x_0(G(x)^{q}) = \xi_\lambda^G(x_0(G(x)^{q}))\), \(\mathbf{T}_\lambda^{G}(0) = q_G^G - \beta = \xi_\lambda^G(0)\), and \(\frac{\partial^2 \mathbf{T}_\lambda^{G}(G(x)^{q})}{\partial x^2} < \frac{\partial^2 q_G^G}{\partial x^2}(x) \) for all \(x \in ]x_0(G(x)^{q}), 0]\). Hence, \(\mathbf{T}_\lambda^{G}(\cdot)\) is on \(]x_0(G(x)^{q}), 0]\) a strict upper functional bound of \(\xi_\lambda^G(\cdot)\). To start with the proof of part (a), let us first observe for \(n = 1\) the obvious relation \(\mu_1(q_G^G)^{hom} = q_G^G - \beta = \alpha_1(q_G^G)^{hom} = 0\).

Furthermore, let us assume that \(\mu_n(q_G^G)^{hom} > \alpha_n(q_G^G)^{hom} \quad (n \in \mathbb{N})\) holds. From this, (51), (89), (90) and the appropriately adapted version of \(\mu_n(q_G^G)^{hom}\) we obtain the desired inequality \(\mu_n(q_G^G)^{hom} > \alpha_n(q_G^G)^{hom}\) by estimating
\[
0 > \mu_n(q_G^G)^{hom} = \frac{(x_0(G(x)^{q})^2}{2} \cdot \left(\frac{G(x)^{q}}{x_0(G(x)^{q})} \right)^{n+1} \frac{\alpha_n(q_G^G)^{hom}}{\alpha_n(q_G^G)^{hom}} = \frac{q_G^G}{2} e^{x_0(G(x)^{q})} \cdot \left(\frac{\alpha_n(q_G^G)^{hom}}{x_0(G(x)^{q})} \right) \cdot \mu_n(q_G^G)^{hom}
\]
which implies that the sequence \(\left(\frac{\pi_n(q_G^G)^{hom}}{\pi_n(q_G^G)^{hom}}\right)_{n \in \mathbb{N}}\) is strictly decreasing since for all \(k \in \mathbb{N}_0\) one has \(\mu_k(q_G^G)^{hom} = (q_G^G - \beta)_k < 0\). Finally, part (c) follows immediately from (90) and the closed-form representation (34) with the choices \(K_1, K_2, x, \nu, c\) given just right after (52).

\[\square\]

### A.3. Proofs of Section 5

**Proof of Theorem 5.1** As already mentioned above, one can adapt the proof of Theorem 9.1.3 in Ethier-Kurtz [13] who deal with drift-parameters \(\eta = 0, \kappa = 0\), and the different setup of \(\sigma - \text{independent time-scale}\) and a sequence of critical Galton-Watson processes without immigration with general offspring distribution. For the sake of brevity, we basically outline here only the main differences to their proof; for similar limit investigations involving offspring/immigration distributions and parametrizations which are incompatible to ours, see e.g. Sriram [61].

As a first step, let us define the generator
\[
A_\bullet f(x) := \left(\eta - \kappa \bullet x\right) f'(x) + \frac{\sigma^2}{2} \cdot x \cdot f''(x), \quad f \in C_c^\infty([0, \infty)),
\]
which corresponds to the diffusion process \(\tilde{X}\) governed by (61). In connection with (58), we study
\[
T_\bullet^{(m)} f(x) := \mathbb{E} \left[ f\left(\frac{1}{m} \sum_{k=1}^{m x} Y_{0,k}^{(m)} + \tilde{Y}_{0}^{(m)}\right)\right], \quad x \in E^{(m)} := \left\{\frac{1}{m} N_0, \quad f \in C_c^\infty([0, \infty)), \right\}
\]
where the \(Y_{0,k}^{(m)}, \tilde{Y}_{0}^{(m)}\) are independent and (Poisson-\(\beta^{(m)}\)) respectively Poisson-\(\alpha^{(m)}\) distributed as the members of the collection \(Y^{(m)}\) respectively \(\tilde{Y}^{(m)}\). By the Theorems 8.2.1 and 1.6.5 as well as Corollary 4.8.9 of [13] it is sufficient to show
\[
\lim_{m \to \infty} \sup_{x \in E^{(m)}} \sigma^2 m \left(T_\bullet^{(m)} f(x) - f(x)\right) = A_\bullet f(x) = 0, \quad f \in C_c^\infty([0, \infty)).
\]  
(91)

But (91) follows mainly from the next
Lemma A.2. Let
\[ S_{n}(m) := \frac{1}{\sqrt{n}} \left( \sum_{k=1}^{n} (\gamma_{0,k}^{(m)} - \beta_{*}^{(m)}) + \frac{1}{\sqrt{n}} - \alpha_{*}^{(m)} \right), \quad n \in \mathbb{N}, \ m \in \mathbb{N}, \]
with the usual convention \( S_{0}^{(m)} := 0 \). Then for all \( m \in \mathbb{N} \), \( x \in E^{(m)} \) and all \( f \in C_{c}^{\infty}([0, \infty)) \)
\[ \epsilon_{(m)}(x) := EP_{*} \left[ \int_{0}^{1} \left( S_{mx}^{(m)} \right)^{2} x(1-v) \left( f'' \left( \beta_{*}^{(m)} \cdot x + \alpha_{*}^{(m)} \right) + \sqrt{\frac{x}{m}} S_{mx}^{(m)} \right) - f''(x) \right] dv \]
\[ = \frac{1}{\sigma^{2}} \left[ \epsilon^{2} \cdot \left( T_{*}(f(x) - f(x)) - A_{*} f(x) \right) + R^{(m)} \right], \quad \text{where} \quad \lim_{m \to \infty} R^{(m)} = 0. \] (92)

PROOF OF LEMMA A.2 Let us fix \( f \in C_{c}^{\infty}([0, \infty)) \). From the involved Poissonian expectations it is easy to see that
\[ \lim_{m \to \infty} \sigma^{2} m \left( T_{*}(f(0) - f(0)) - A_{*} f(0) \right) = 0, \]
and thus (92) holds for \( x = 0 \). Accordingly, we next consider the case \( x \in E^{(m)} \setminus \{0\} \), with fixed \( m \in \mathbb{N} \). From
\[ EP_{*} \left[ \left( S_{mx}^{(m)} \right)^{2} \right] = \beta_{*}^{(m)} + \frac{\alpha_{*}^{(m)}}{m} \]
we obtain
\[ EP_{*} \left[ \left( S_{mx}^{(m)} \right)^{2} x f''(x) \right] \int_{0}^{1} (1-v) dv = \frac{1}{2} \left( \beta_{*}^{(m)} \cdot x + \alpha_{*}^{(m)} \right) f''(x) =: a_{mx} \frac{f''(x)}{2} =: \frac{a_{mx}}{2}. \] (93)
Furthermore, with \( b_{mx} := b := a + \sqrt{x/m} S_{mx}^{(m)} \)
\[ \int_{0}^{1} f'' \left( \beta_{*}^{(m)} \cdot x + \alpha_{*}^{(m)} \right) + \sqrt{x/m} S_{mx}^{(m)} \right) dv = \frac{1}{\sqrt{m}} \cdot \frac{1}{S_{mx}^{(m)}} \int_{a}^{b} f''(y) dy = \sqrt{m} \cdot \frac{f''(b) - f''(a)}{S_{mx}^{(m)}} \] (94)
as well as
\[ \int_{0}^{1} y f'' \left( \beta_{*}^{(m)} \cdot x + \alpha_{*}^{(m)} \right) \sqrt{x/m} S_{mx}^{(m)} \right) \right) dy = \sqrt{m} \cdot \frac{1}{S_{mx}^{(m)}} \int_{a}^{b} y f''(y) dy \]
\[ = \sqrt{m} \cdot \frac{f''(b)}{S_{mx}^{(m)}} + \frac{m}{S_{mx}^{(m)}} \right) \left( \frac{f(a) - f(b)}{2} \right). \] (95)

With our choice \( \beta_{*}^{(m)} = 1 - \frac{\alpha_{*}^{(m)}}{m} \) and \( \alpha_{*}^{(m)} = \beta_{*}^{(m)} \cdot \eta
- \kappa_{*} \cdot \frac{x}{m} \), a Taylor expansion of \( f \) at \( x \) gives
\[ f(a) = f(x) + \frac{1}{\sigma^{2} x} \cdot f'(x) \left( \beta_{*}^{(m)} \cdot \eta - \kappa_{*} \cdot \frac{x}{m} \right) + o \left( \frac{x}{m} \right), \] (96)
where for the case \( \eta = \kappa = 0 \) we use the convention \( o \left( \frac{1}{m} \right) \equiv 0 \). Combining (93) to (96) and the centering
\[ EP_{*} \left[ S_{mx}^{(m)} \right] = 0, \]
the left hand side of equation (92) becomes
\[ EP_{*} \left[ \int_{0}^{1} \left( S_{mx}^{(m)} \right)^{2} x(1-v) \left( f'' \left( \beta_{*}^{(m)} \cdot x + \alpha_{*}^{(m)} \right) \right) \right] dv \]
\[ = EP_{*} \left[ \sqrt{m x} \cdot S_{mx}^{(m)} \right] \left( f''(b) - f''(a) \right) \right) - EP_{*} \left[ \sqrt{m x} \cdot S_{mx}^{(m)} \cdot f''(b) + m \cdot f(a) - f(b) \right] \]
\[ - \frac{1}{2} \left( \beta_{*}^{(m)} \cdot x + \alpha_{*}^{(m)} \right) \cdot f''(x) \]
\[ = m \cdot \left( EP_{*} \left[ f(b) - f(a) \right] \right) - \frac{1}{2} \left( \beta_{*}^{(m)} \cdot x + \alpha_{*}^{(m)} \right) \cdot f''(x) \]
\[ = m \cdot \left( EP_{*} \left[ f \left( \frac{1}{m} \sum_{k=1}^{m} \gamma_{0,k}^{(m)} \right) \right] \right) - f(x) \right) - \frac{1}{\sigma^{2} A_{*} f(x) \right) \]
\[ + \frac{1}{\sigma^{2}} \left[ (\eta - \kappa_{*} \cdot \frac{x}{m} - \beta_{*}^{(m)} \cdot \eta + \kappa_{*} \cdot \frac{x}{m} \right) \cdot f'(x) + \frac{x}{2} \left( 1 - \beta_{*}^{(m)} - \alpha_{*}^{(m)} \right) \cdot f''(x) - m \cdot o \left( \frac{1}{m} \right) \]
which immediately leads to the right hand side of (92).

To proceed with the proof of Theorem 5.1, we obtain for \( m \geq 2\kappa_\ast /\sigma^2 \) the inequality \( \beta^{(m)}_\ast \geq 1/2 \) and accordingly for all \( v \in ]0,1[ \), \( x \in E^{(m)} \)

\[
\beta^{(m)}_\ast x + \frac{\alpha^{(m)}_\ast}{m} + v \sqrt{\frac{x}{m} \cdot \sigma^{(m)}_{mx}} = (1 - v) \cdot x \cdot \beta^{(m)}_\ast + \left( 1 - v \cdot \frac{\alpha^{(m)}_\ast}{m} + v \left( \sum_{k=1}^{m} \tilde{Y}_{0,k}^{(m)} + \tilde{Y}_0 \right) \right) \geq x \cdot \frac{1 - v}{2}.
\]

Suppose that the support of \( f \) is contained in the interval \( [0,c] \). Correspondingly, for \( v \leq 1 - 2c/x \) the integrand in \( \epsilon^{(m)}(x) \) is zero and hence with (97) we can estimate

\[
\left| \int_0^1 \left( S^{(m)}_{mx} \right)^2 x(1-v) \left( f'' \left( \beta^{(m)}_\ast x + \frac{\alpha^{(m)}_\ast}{m} + v \sqrt{\frac{x}{m} \cdot \sigma^{(m)}_{mx}} \right) - f''(x) \right) dv \right| 
\leq \int_0^{1/(1 - 2c/x)} \left( S^{(m)}_{mx} \right)^2 x(1-v) \cdot 2 \left\| f'' \right\|_\infty \; dv \leq x \cdot \left( S^{(m)}_{mx} \right)^2 \left( 1 \wedge \frac{2c}{x} \right)^2 \left\| f'' \right\|_\infty.
\]

From this, one can deduce \( \lim_{m \to \infty} \sup_{x \in E^{(m)}} \epsilon^{(m)}(x) = 0 \) and thus (91) – in the same manner as at the end of the proof of Theorem 9.1.3 in [13] (by means of the dominated convergence theorem).

The following lemma is the main tool for the proof of Theorem 5.3 below.

**Lemma A.3.** Let \((\kappa_\Delta, \kappa_H, \eta, \lambda) \in (\overline{P}_N \cup \overline{P}_{SP,1}) \times ]0,1[\). By using the quantities \( \kappa_\lambda := \lambda \kappa_\Delta + (1 - \lambda) \kappa_H > 0 \) and \( \Lambda_\lambda := \sqrt{\lambda \kappa_\Delta + (1 - \lambda) \kappa_H} > \kappa_\lambda \) from (64), one gets for all \( t > 0 \)

(a) \( \lim_{m \to \infty} m \cdot (1 - q^{(m)}_\lambda) \geq \frac{\kappa_\lambda}{\sigma^2} > 0 \).

(b) \( \lim_{m \to \infty} m^2 \cdot a^{(m)}_1 = \frac{\lambda(1 - \lambda)(\kappa_\Delta - \kappa_H)^2}{2\sigma^4} = -\frac{\Lambda_\lambda^2 - \kappa_\lambda^2}{2\sigma^4} < 0 ; \lim_{m \to \infty} m \cdot (1 - \beta^{(m)}_\lambda) = \frac{\kappa_\lambda}{\sigma^2} > 0. \)

(c) \( \lim_{m \to \infty} m \cdot x^{(m)}_0 = -\frac{\Lambda_\lambda - \kappa_\lambda}{\sigma^2} < 0 ; \lim_{m \to \infty} m^2 \cdot \Gamma^{(m)} = \frac{(\Lambda_\lambda - \kappa_\lambda)^2}{2\sigma^4} > 0. \)

(d) \( \lim_{m \to \infty} m \cdot (1 - d^{(m)}_S) = \frac{\Lambda_\lambda + \kappa_\lambda}{2\sigma^2} > 0. \)

(e) \( \lim_{m \to \infty} m \cdot (1 - d^{(m)} \cdot T) = \frac{\Lambda_\lambda}{\sigma^2} > 0; \lim_{m \to \infty} m^2 \cdot x^{(m)}_0 \cdot (1 - d^{(m)} \cdot T) = -\frac{\Lambda_\lambda \cdot (\Lambda_\lambda - \kappa_\lambda)}{\sigma^4} < 0. \)

(f) \( \lim_{m \to \infty} m \cdot (1 - d^{(m)}_S \cdot d^{(m)} \cdot T) = \frac{3\Lambda_\lambda + \kappa_\lambda}{2\sigma^2} > 0. \)

(g) \( \lim_{m \to \infty} \left( d^{(m)} \cdot S \right)^{\sigma^2 \cdot m} = \exp \left\{ -\frac{\Lambda_\lambda + \kappa_\lambda}{2} \cdot t \right\} < 1. \)

(h) \( \lim_{m \to \infty} \left( d^{(m)} \cdot T \right)^{\sigma^2 \cdot m} = \exp \left\{ -\Lambda_\lambda \cdot t \right\} < 1. \)

(i) \( \lim_{m \to \infty} \left( d^{(m)} \cdot S \cdot d^{(m)} \cdot T \right)^{\sigma^2 \cdot m} = \exp \left\{ -\frac{3\Lambda_\lambda + \kappa_\lambda}{2} \cdot t \right\} < 1. \)

(j) \( \lim_{m \to \infty} m \cdot \zeta^{(m)}_{[\sigma^2 \cdot m]} = \frac{(\Lambda_\lambda - \kappa_\lambda)^2}{2\sigma^2} \cdot e^{-\Lambda_\lambda \cdot t} \cdot (1 - e^{-\Lambda_\lambda \cdot t}) > 0. \)

(k) \( \lim_{m \to \infty} d^{(m)}_{[\sigma^2 \cdot m]} = \frac{\eta}{4\sigma^2} \cdot \left( \frac{\Lambda_\lambda - \kappa_\lambda}{\Lambda_\lambda} \right)^2 \cdot (1 - e^{-\Lambda_\lambda \cdot t})^2 \geq 0. \)

(l) \( \lim_{m \to \infty} m \cdot \zeta^{(m)}_{[\sigma^2 \cdot m]} = \frac{(\Lambda_\lambda - \kappa_\lambda)^2}{\sigma^2} \cdot \left[ \frac{e^{-\frac{1}{2}(\Lambda_\lambda + \kappa_\lambda) \cdot t} - e^{-\Lambda_\lambda \cdot t}}{\Lambda_\lambda - \kappa_\lambda} - \frac{e^{-\frac{1}{2}(\Lambda_\lambda + \kappa_\lambda) \cdot t} (1 - e^{-\Lambda_\lambda \cdot t})}{2 \cdot \Lambda_\lambda} \right] \geq 0. \)

(m) \( \lim_{m \to \infty} d^{(m)}_{[\sigma^2 \cdot m]} = \frac{\eta}{\sigma^2} \cdot \left( \frac{\Lambda_\lambda - \kappa_\lambda}{\Lambda_\lambda} \right)^2 \cdot \left[ 1 - e^{-\frac{1}{2}(3\Lambda_\lambda + \kappa_\lambda) \cdot t} - e^{-\Lambda_\lambda \cdot t} \cdot e^{-\frac{1}{2}(\Lambda_\lambda + \kappa_\lambda) \cdot t} \right] \geq 0. \)

**Proof of Lemma A.3** For each of the assertions (a) to (m), we will make use of l’Hospital’s rule. To begin with, we obtain for arbitrary \( \mu, \nu \in \mathbb{R} \)
\[
\lim_{m \to \infty} m \cdot \left[ 1 - \left( \beta^{(m)}_\mathcal{A} \right)^{\nu} \left( \beta^{(m)}_\mathcal{H} \right)^{\nu} \right] = \lim_{m \to \infty} m^2 \cdot \left[ \mu \cdot \left( \beta^{(m)}_\mathcal{A} \right)^{\mu-1} \left( \beta^{(m)}_\mathcal{H} \right)^{\nu} \frac{K_\mathcal{A}}{\sigma^2 m^2} + \nu \cdot \left( \beta^{(m)}_\mathcal{A} \right)^{\mu} \left( \beta^{(m)}_\mathcal{H} \right)^{-1} \frac{K_\mathcal{H}}{\sigma^2 m^2} \right] = \mu \frac{K_\mathcal{A}}{\sigma^2} + \nu \frac{K_\mathcal{H}}{\sigma^2}.
\]

From this, (a) follows immediately and (b) can be deduced by
\[
\lim_{m \to \infty} m^2 \cdot a_1^{(m)} = \lim_{m \to \infty} \frac{m}{2\sigma^2} \cdot \left[ \lambda \cdot \kappa_\mathcal{A} \left( 1 - \left( \beta^{(m)}_\mathcal{A} \right)^{\lambda} \left( \beta^{(m)}_\mathcal{H} \right)^{-\lambda} \right) + (1 - \lambda) \cdot \kappa_\mathcal{H} \left( 1 - \left( \beta^{(m)}_\mathcal{A} \right)^{\lambda} \left( \beta^{(m)}_\mathcal{H} \right)^{-\lambda} \right) \right] = -\frac{\lambda(1 - \lambda)(\kappa_\mathcal{A} - \kappa_\mathcal{H})^2}{2\sigma^4}.
\]

For the proof of the first part of (c), we rely on the inequalities \( x^{(m)}_0 \leq x^{(m)} \leq x^{(m)}_0 \) (m \( \in \mathbb{N} \)), where \( x^{(m)}_0 \) and \( x^{(m)} \) are the obvious notational adaptations of (46) and (56), respectively. By using (a) and (b), one can calculate
\[
\lim_{m \to \infty} m \cdot \mathcal{I}^{(m)}_0 = \lim_{m \to \infty} (q^{(m)}_\lambda)^{-1} \cdot \left[ m \cdot (1 - q^{(m)}_\lambda) - \sqrt{(m \cdot (1 - q^{(m)}_\lambda))^2 - 2 \cdot q^{(m)}_\lambda \cdot m^2 \cdot a_1^{(m)}} \right] = -\frac{\Lambda_\lambda - \kappa_\lambda}{\sigma^2}.
\]

From (46), (a), (b) and \( \lim_{m \to \infty} \beta^{(m)}_\lambda = 1 \) we obtain the limit
\[
\lim_{m \to \infty} h \left( q^{(m)}_\lambda \right) = \lim_{m \to \infty} \max \left\{ -\beta^{(m)}_\lambda ; \frac{d_1^{(m)}}{1 - q^{(m)}_\lambda} \right\} = \lim_{m \to \infty} \frac{1}{m} \cdot \frac{m^2 \cdot a_1^{(m)}}{m \cdot (1 - q^{(m)}_\lambda)} = 0,
\]
which implies
\[
\lim_{m \to \infty} m \cdot \mathcal{I}^{(m)}_0 = \lim_{m \to \infty} \frac{e^{-h(q^{(m)}_\lambda)}}{q^{(m)}_\lambda} \cdot \left[ m \cdot (1 - q^{(m)}_\lambda) - \sqrt{(m \cdot (1 - q^{(m)}_\lambda))^2 - 2e^{h(q^{(m)}_\lambda)} \cdot d_1^{(m)} \cdot m^2 \cdot a_1^{(m)}} \right] = -\frac{\Lambda_\lambda - \kappa_\lambda}{\sigma^2}
\]
and thus the first part of (c). The second part is an immediate consequence thereof. Assertion (d) follows from (b) and (c) by
\[
\lim_{m \to \infty} m \cdot (1 - d^{(m)}_\mathcal{S}) = \lim_{m \to \infty} \frac{m^2 \cdot a_1^{(m)}}{m \cdot x^{(m)}_0} = \frac{\Lambda_\lambda + \kappa_\lambda}{2\sigma^2}.
\]

For the first part of (e), we use the general limit \( \lim_{x \to 0} \frac{e^{\frac{1}{x}} - 1}{x} = 1 \), to get with (a) and (c)
\[
\lim_{m \to \infty} m \cdot (1 - d^{(m)}_\mathcal{T}) = \lim_{m \to \infty} \left( m \cdot (1 - q^{(m)}_\lambda) - q^{(m)}_\lambda \cdot m \cdot x^{(m)}_0 \frac{e^{\frac{\mathcal{S}^{(m)}}{x^{(m)}_0}} - 1}{x^{(m)}_0} \right) = \frac{\Lambda_\lambda}{\sigma^2}.
\]

From this and (c), the second part of (e) is obvious. The limit (f) can be obtained from (d) and (e). The assertions (g) respectively (h) respectively (i) follow from (d) respectively (e) respectively (f) by using the general relation \( \lim_{m \to \infty} \left( 1 + \frac{d^{(m)}}{m} \right)^m = e^{\lim_{m \to \infty} x^{(m)}_e} \). The last four limits (j) to (m) are straightforward implications of (a) to (i).

**Proof of Theorem 5.3** It suffices to compute the limits of the bounds given in Corollary 5.2 as \( m \) tends to infinity. This is done by applying Lemma A.3 which provides corresponding limits of various involved quantities. Accordingly, for all \( t > 0 \) the lower bound (66) can be obtained from (62) by
\[
\lim_{m \to \infty} \exp \left\{ x_0^{(m)} \cdot \left[ X_0^{(m)} - \frac{\eta}{\sigma^2} \cdot \frac{d^{(m),T}}{1 - d^{(m),T}} \right] \right.
\left. \left( 1 - \left( d^{(m),T} \right)^{\sigma^2 m t} \right) \right\}
\]

\[
+ x_0^{(m)} \frac{\eta}{\sigma^2} \cdot \left| \sigma^2 m t \right| + \zeta_{\left(\sigma^2 m t\right), 0}^{(m)} \cdot X_0^{(m)} + \frac{\vartheta_{\left(\sigma^2 m t\right), 0}^{(m)}}{\sigma^2 m t}
\]\n
\[
= \lim_{m \to \infty} \exp \left\{ m \cdot x_0^{(m)} \cdot \left[ \frac{X_0^{(m)}}{m} - \frac{\eta}{\sigma^2} \cdot \frac{d^{(m),T}}{1 - d^{(m),T}} \right] \right.
\left. \left( 1 - \left( d^{(m),T} \right)^{\sigma^2 m t} \right) \right\}
\]

\[
+ m \cdot x_0^{(m)} \frac{\eta}{\sigma^2} \cdot \left| \sigma^2 m t \right| + m \cdot \zeta_{\left(\sigma^2 m t\right), 0}^{(m)} \cdot X_0^{(m)} + \frac{\vartheta_{\left(\sigma^2 m t\right), 0}^{(m)}}{\sigma^2 m t}
\]\n
\[
= \exp \left\{ - \frac{\lambda - \kappa}{\sigma^2} \left( \bar{X}_0 - \frac{\eta}{\sigma^2} \cdot \frac{8 \sigma^2}{\lambda + \kappa} \right) \left( 1 - \left( e^{-\frac{1}{2} (\lambda + \kappa) t} \right) \right) - \frac{\lambda - \kappa}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \sigma^2 t
\]

\[
+ \frac{(\lambda - \kappa)^2}{\sigma^2} \cdot \left[ \frac{1}{\lambda + \kappa} \cdot e^{-\frac{1}{2} (\lambda + \kappa) t} - e^{-\lambda t} - \frac{1}{2 \lambda} \cdot e^{-\lambda t} \left( 1 - e^{-\lambda t} \right) \right] \cdot \bar{X}_0
\]

\[
+ \eta \frac{\lambda - \kappa}{\sigma^2} \cdot \left[ \frac{1}{\lambda + \kappa} \cdot e^{-\lambda t} - e^{-\lambda t} \cdot \frac{1}{\lambda + \kappa} \cdot \bar{X}_0 \right]
\]\n
\[
= \exp \left\{ - \frac{\lambda - \kappa}{\sigma^2} \left( \bar{X}_0 - \frac{\eta}{\sigma^2} \cdot \frac{8 \sigma^2}{\lambda + \kappa} \right) \left( 1 - \left( e^{-\frac{1}{2} (\lambda + \kappa) t} \right) \right) - \frac{\lambda - \kappa}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \sigma^2 t
\]

\[
- U_{\lambda}^{(1)}(t) \cdot \bar{X}_0 - \frac{\eta}{\sigma^2} \cdot U_{\lambda}^{(2)}(t)
\]\n
\[
\square
\]

### A.4. Proofs of Section 6

We start with two lemmas which will be useful for the proof of Theorem 6.1, and which can be easily seen by induction. They deal with the sequence \( \left( a_n^{(q_0)} \right)_{n \in \mathbb{N}} \) from (17).

**Lemma A.4.** For arbitrarily fixed parameter constellation \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in \mathcal{P}\), suppose that \( q_\lambda > 0 \) \((\lambda \in [0, 1])\) and \( \lim_{\lambda \to 1} q_\lambda = \beta_A \) holds. Then one gets the limit

\[
\lim_{m \to \infty} \exp \left\{ x_0^{(m)} \cdot \left[ X_0^{(m)} - \frac{\eta}{\sigma^2} \cdot \frac{d^{(m),T}}{1 - d^{(m),T}} \right] \right.
\left. \left( 1 - \left( d^{(m),T} \right)^{\sigma^2 m t} \right) \right\}
\]

\[
+ x_0^{(m)} \frac{\eta}{\sigma^2} \cdot \left| \sigma^2 m t \right| + \zeta_{\left(\sigma^2 m t\right), 0}^{(m)} \cdot X_0^{(m)} + \frac{\vartheta_{\left(\sigma^2 m t\right), 0}^{(m)}}{\sigma^2 m t}
\]\n
\[
= \exp \left\{ - \frac{\lambda - \kappa}{\sigma^2} \left( \bar{X}_0 - \frac{\eta}{\sigma^2} \cdot \frac{8 \sigma^2}{\lambda + \kappa} \right) \left( 1 - \left( e^{-\frac{1}{2} (\lambda + \kappa) t} \right) \right) - \frac{\lambda - \kappa}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \sigma^2 t
\]

\[
+ \frac{(\lambda - \kappa)^2}{\sigma^2} \cdot \left[ \frac{1}{\lambda + \kappa} \cdot e^{-\frac{1}{2} (\lambda + \kappa) t} - e^{-\lambda t} - \frac{1}{2 \lambda} \cdot e^{-\lambda t} \left( 1 - e^{-\lambda t} \right) \right] \cdot \bar{X}_0
\]

\[
+ \eta \frac{\lambda - \kappa}{\sigma^2} \cdot \left[ \frac{1}{\lambda + \kappa} \cdot e^{-\lambda t} - e^{-\lambda t} \cdot \frac{1}{\lambda + \kappa} \cdot \bar{X}_0 \right]
\]\n
\[
= \exp \left\{ - \frac{\lambda - \kappa}{\sigma^2} \left( \bar{X}_0 - \frac{\eta}{\sigma^2} \cdot \frac{8 \sigma^2}{\lambda + \kappa} \right) \left( 1 - \left( e^{-\frac{1}{2} (\lambda + \kappa) t} \right) \right) - \frac{\lambda - \kappa}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \sigma^2 t
\]

\[
- U_{\lambda}^{(1)}(t) \cdot \bar{X}_0 - \frac{\eta}{\sigma^2} \cdot U_{\lambda}^{(2)}(t)
\]
\[ \forall n \in \mathbb{N} : \lim_{\lambda \searrow 1} a_n^{(q_\lambda)} = 0. \]  

(97)

**Lemma A.5.** In addition to the assumptions of Lemma A.4, suppose that \( \lambda \to q_\lambda \) is continuously differentiable on \([0,1]\) and that the limit \( l := \lim_{\lambda \searrow 1} \frac{\partial q_\lambda}{\partial \lambda} \) is finite. Then one gets the limit

\[ \forall n \in \mathbb{N} : \lim_{\lambda \searrow 1} \frac{\partial a_n^{(q_\lambda)}}{\partial \lambda} = u_n := \begin{cases} \frac{l + \beta_n - \beta_A}{1 - \beta_A} \cdot (1 - (\beta_A)^n), & \text{if } \beta_A \neq 1, \\ n \cdot (l + \beta_H - 1), & \text{if } \beta_A = 1, \end{cases} \]

which is the unique solution of the linear recursion equation

\[ u_n = l + \beta_H - \beta_A + \beta_A \cdot u_{n-1}, \quad u_0 = 0. \]

Furthermore,

\[ \forall n \in \mathbb{N} : \sum_{k=1}^{n} \lim_{\lambda \searrow 1} \frac{\partial a_k^{(q_\lambda)}}{\partial \lambda} = \sum_{k=1}^{n} u_k := \begin{cases} \frac{l + \beta_n - \beta_A}{1 - \beta_A} \cdot \left[ n - \frac{\beta_A}{1 - \beta_A} (1 - (\beta_A)^n) \right], & \text{if } \beta_A \neq 1, \\ n \cdot (n+1) \cdot (l + \beta_H - 1), & \text{if } \beta_A = 1. \end{cases} \]

We are now ready to prove the

**Proof of Theorem 6.1.**

(a) Recall that for the setup \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in (P_{NI} \cup P_{SP,1})\) we chose the intercept as \(p_\lambda := p_\lambda^E := \alpha_A \alpha_H^{1-\lambda}\) and the slope as \(q_\lambda := q_\lambda^E := \beta_A \beta_H^{1-\lambda}\), which in (20) lead to the exact value \(V_{\lambda,n}\) of the Hellinger integral. Because of \(\frac{\partial}{\partial q_\lambda} \beta_A - \alpha_A = 0\) as well as \(\lim_{\lambda \searrow 1} \beta_\lambda = \beta_A\), we obtain by using (19) and Lemma A.4 for all \(n \in \mathbb{N}\)

\[ \lim_{\lambda \searrow 1} V_{\lambda,n} := \lim_{\lambda \searrow 1} \exp \left\{ a_n^{(q_\lambda)} \cdot \omega_0 + \sum_{k=1}^{n} b_k^{(p_\lambda, q_\lambda)} \right\} = \lim_{\lambda \searrow 1} \exp \left\{ a_n^{(q_\lambda)} \cdot \omega_0 + \frac{\alpha_A}{\beta_A} \sum_{k=1}^{n} a_k^{(q_\lambda)} \right\} = 1, \quad n \in \mathbb{N}, \]

which leads by (71) to

\[ I(P_{A,n}||P_{H,n}) = \lim_{\lambda \searrow 1} \frac{1 - H_\lambda(P_{A,n}||P_{H,n})}{\lambda \cdot (1 - \lambda)} = \lim_{\lambda \searrow 1} \frac{1 - V_{\lambda,n}}{\lambda \cdot (1 - \lambda)} = \lim_{\lambda \searrow 1} \frac{\frac{\partial}{\partial \lambda} a_n^{(q_\lambda)} \cdot \omega_0 + \frac{p_\lambda}{q_\lambda} \sum_{k=1}^{n} a_k^{(q_\lambda)}}{\frac{\partial}{\partial \lambda} a_n^{(q_\lambda)} \cdot \omega_0 + \frac{p_\lambda}{q_\lambda} \sum_{k=1}^{n} a_k^{(q_\lambda)}}. \]

(98)

For further analysis, we use the obvious derivatives

\[ \frac{\partial p_\lambda}{\partial \lambda} = p_\lambda \log \left( \frac{\alpha_A}{\alpha_H} \right), \quad \frac{\partial p_\lambda}{\partial q_\lambda} = p_\lambda \log \left( \frac{\alpha_A \beta_H}{\alpha_H \beta_A} \right), \quad \frac{\partial q_\lambda}{\partial \lambda} = q_\lambda \log \left( \frac{\beta_A}{\beta_H} \right), \]

(99)

where the subcase \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in P_{NI}\) (with \(p_\lambda \equiv 0\)) is consistently covered. From (99) and Lemma A.5 we deduce

\[ \lim_{\lambda \searrow 1} \frac{\partial a_n^{(q_\lambda)}}{\partial \lambda} \cdot \omega_0 = \begin{cases} \left( \beta_A \log \left( \frac{\beta_A}{\beta_H} \right) - (\beta_A - \beta_H) \right) \cdot \frac{1 - (\beta_A)^n}{1 - \beta_A} \cdot \omega_0, & \text{if } \beta_A \neq 1, \\ n \cdot (\beta_A \log \left( \frac{\beta_A}{\beta_H} \right) - (\beta_A - \beta_H)) \cdot \omega_0, & \text{if } \beta_A = 1, \end{cases} \]

and by means of (97)

\[ \forall n \in \mathbb{N} : \lim_{\lambda \searrow 1} \left[ \frac{\partial}{\partial \lambda} p_\lambda^{(q_\lambda)} \cdot \sum_{k=1}^{n} a_k^{(q_\lambda)} \right] = 0. \]

For the last expression in (98) we again apply Lemma A.5 to end up with

\[ \lim_{\lambda \searrow 1} \frac{p_\lambda}{q_\lambda} \cdot \sum_{k=1}^{n} \frac{\partial a_k^{(q_\lambda)}}{\partial q_\lambda} = \begin{cases} \frac{\alpha_A \beta_A}{\alpha_H \beta_H} \cdot \left( \beta_A - \beta_H \right) \cdot \frac{1 - (\beta_A)^n}{1 - \beta_A} \cdot \left[ n - \frac{\beta_A}{1 - \beta_A} (1 - (\beta_A)^n) \right], & \text{if } \beta_A \neq 1, \\ n \cdot (n+1) \frac{\alpha_A}{\beta_H} \cdot \left( \beta_A \log \left( \frac{\beta_A}{\beta_H} \right) - (\beta_A - \beta_H) \right), & \text{if } \beta_A = 1, \end{cases} \]

(100)

which finishes the proof of part (a). To show part (b), for the corresponding setup \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in P_{SP} \setminus P_{SP,1}\) let us first choose – according to the Section 3.2 – the intercept as \(p_\lambda := p_\lambda^L := \alpha_A \alpha_H^{1-\lambda}\) and the
Combining these two limits we get slope as 
\[ \lambda p = \beta_A - \alpha_A \]
and hence
\[ \lim_{\lambda \to 1} \beta_A - \alpha_A = 0. \]

From this, (19), part (b) of Proposition 3.4 and Lemma A.4 we obtain

\[ \lim_{\lambda \to 1} B_{\lambda,n}^L = \lim_{\lambda \to 1} \exp \left( \frac{\rho_{\lambda} n}{\lambda} \sum_{k=1}^{n} a_k^{(\rho_{\lambda})} + n \cdot \left( \frac{\rho_{\lambda} \beta_A - \alpha_A}{\lambda} \right) \right) = 1 \]  

and hence
\[ I(P_{A,n}||P_{H,n}) \leq \lim_{\lambda \to 1} \frac{1 - B_{\lambda,n}^L}{\lambda (1 - \lambda)} = \lim_{\lambda \to 1} \frac{B_{\lambda,n}^L}{1 - 2\lambda} \cdot \frac{\partial}{\partial \lambda} \left[ \frac{\rho_{\lambda} n}{\lambda} \sum_{k=1}^{n} a_k^{(\rho_{\lambda})} + n \cdot \left( \frac{\rho_{\lambda} \beta_A - \alpha_A}{\lambda} \right) \right] \]

In the current setup, the first three expressions in (102) can be evaluated in exactly the same way as in (99) to (100), and for the last expression one has the limit
\[ \lim_{\lambda \to 1} \frac{\partial}{\partial \lambda} \left( \frac{\rho_{\lambda} \beta_A - \alpha_A}{\lambda} \right) = \frac{\rho_{\lambda}}{\lambda} \log \left( \frac{\alpha A}{\beta H} \right) \cdot \beta A + \frac{\rho_{\lambda}}{\lambda} \cdot (\beta A - \beta H) - (\alpha A - \alpha H) \]
which finishes the proof of part (b).

**Proof of Theorem 6.3** Let us fix \((\beta_A, \beta_H, \alpha_A, \alpha_H) \in \mathcal{P}_{SP} \setminus \mathcal{P}_{SP,1}, \omega_0 \in \mathbb{N}, n \in \mathbb{N} \) and \(y \in [0, \infty]. \) The lower bound \( E_{\lambda,n}^{L,y} \) of the relative entropy is derived by using as a linear upper bound \( \phi_{\lambda}^T \) (cf. (15)) for \( \phi_{\lambda} \) \((\lambda \in [0,1]\)) the tangent line of \( \phi_{\lambda} \) at \( y. \) This corresponds to \( \phi_{\lambda}^T(x) := (p_{\lambda}^y - \alpha_A) + (q_{\lambda}^y - \beta_A)x \) \((x \in [0, \infty]\) with \( p_{\lambda} := p_{\lambda}(y) := \phi_{\lambda}(y) - y \phi_{\lambda}'(y) + \alpha_A \) and \( q_{\lambda} := q_{\lambda}(y) := \phi_{\lambda}'(y) + \beta_A, \) implying \( q_{\lambda} > 0 \) because of (p-xii). As a side remark, notice that this \( \phi_{\lambda}^T \) may become negative for some \( x \in [0, \infty], \) which is not always consistent with goal (Gc) for fixed \( \lambda, \) but leads to a tractable limit bound as \( \lambda \) tends to 1. Analogously to (101) and (102), we obtain from (19) and (21) the convergence limit \( \lim_{\lambda \to 1} B_{\lambda,n}^L = 1 \) and thus
\[ I(P_{A,n}||P_{H,n}) \geq \lim_{\lambda \to 1} \left[ \frac{\partial a_k^{(\rho_{\lambda})}}{\partial \lambda} \omega_0 + \left( \frac{\partial}{\partial \lambda} \frac{\rho_{\lambda}}{\lambda} \sum_{k=1}^{n} a_k^{(\rho_{\lambda})} + \frac{\rho_{\lambda}}{\lambda} \sum_{k=1}^{n} \frac{\partial a_k^{(\rho_{\lambda})}}{\partial \lambda} + n \cdot \frac{\partial}{\partial \lambda} \left( \frac{\rho_{\lambda} \beta_A - \alpha_A}{\lambda} \right) \right) \right]. \]

As before, we compute the involved derivatives. From (10) to (12) as well as (p-xii) we get
\[ \frac{\partial \rho_{\lambda}}{\partial \lambda} = \left( \frac{f_A(y)}{f_H(y)} \right) \lambda f_H(y) \int \log \left( \frac{f_A(y)}{f_H(y)} \right) - \beta_A \left( \frac{f_A(y)}{f_H(y)} \right) \lambda y - \beta_H \left( \frac{f_A(y)}{f_H(y)} \right) \lambda \log \left( \frac{f_A(y)}{f_H(y)} \right) \]

and
\[ \frac{\partial \lambda}{\partial \lambda} = \beta_A f_A(y) f_H(y) - \beta_H f_A(y) f_H(y) \]

Combining these two limits we get
\[
\frac{\partial}{\partial \lambda} \left( \frac{p_\lambda}{q_\lambda} \beta_\lambda - \alpha_\lambda \right) = \frac{q_\lambda}{q_\lambda} \left( \frac{\partial p_\lambda}{\partial \lambda} - \frac{\partial q_\lambda}{\partial \lambda} \right) \cdot \beta_\lambda + \frac{p_\lambda}{q_\lambda} \cdot (\beta_A - \beta_H) - (\alpha_A - \alpha_H)
\]
\[
\lambda \rightarrow 1 \quad \left[ y \cdot (\alpha_A \beta_H - \alpha_H \beta_A) - \frac{\partial \lambda}{\partial \lambda} \right] + \alpha_H - \frac{\alpha_A \beta_H}{\beta_A}.
\]

The above calculation also implies that \( \lim_{\lambda \rightarrow 1} \left( \frac{\partial p_\lambda}{\partial \lambda} \right) \) is finite and thus \( \lim_{\lambda \rightarrow 1} \left( \frac{\partial p_\lambda}{\partial \lambda} \right) + \sum_{k=1}^{n} \phi_k^{(q)} = 0 \) by means of Lemma A.4. The proof of \( I(P_{A,n} \mid P_{H,n}) \geq E_{k,n}^{L,\text{tan}} \) is finished by using Lemma A.5 with \( t \) defined in (105) and by plugging the limits (104) to (106) into (103).

To derive the lower bound \( E_{k,n}^{L,\text{sec}} \) (cf. (76)) for fixed \( k \in \mathbb{N}_0 \), we use as a linear upper bound \( \phi_\lambda^U \) for \( \phi_\lambda(\cdot) \) (\( \lambda \in [0,1] \)) the secant line of \( \phi_\lambda \) through the points \( k \) and \( k + 1 \), corresponding to the choices \( p_\lambda := p_\lambda(k) := (k + 1) \cdot \phi_\lambda(k) - k \cdot \phi_\lambda(k + 1) + \alpha_\lambda \) and \( q_\lambda := q_\lambda(k) := \phi_\lambda(k + 1) - \phi_\lambda(k) + \beta_\lambda \), implying \( q_\lambda > 0 \) because of (p-xiii) and (p-iv). As a side remark, notice that this \( \phi_\lambda^U(x) \) may become negative for some \( x \in [0,\infty) \) (which is not always consistent with goal (Gc) for fixed \( \lambda \), but leads to a tractable limit bound as \( \lambda \) tends to 1). Analogously to (101) and (102) we get again \( \lim_{\lambda \rightarrow 1} B_{\lambda,n}^U = 1 \), which leads to the lower bound given in (103) with appropriately plugged-in quantities. As in the above proof of the lower bound \( E_{k,n}^{L,\text{tan}} \), the inequality \( I(P_{A,n} \mid P_{H,n}) \geq E_{k,n}^{L,\text{sec}} \) follows straightforwardly from Lemma A.4, Lemma A.5 and the three limits

\[
\frac{\partial p_\lambda}{\partial \lambda} = \left( \frac{f_A(k)}{f_H(k)} \right)^{\lambda} \cdot f_H(k) \cdot (k+1) \cdot \log \left( \frac{f_A(k)}{f_H(k)} \right) - \left( \frac{f_A(k+1)}{f_H(k+1)} \right)^{\lambda} \cdot f_H(k+1) \cdot k \log \left( \frac{f_A(k+1)}{f_H(k+1)} \right),
\]

\[
\frac{\partial q_\lambda}{\partial \lambda} = \left( \frac{f_A(k+1)}{f_H(k+1)} \right)^{\lambda} \cdot f_H(k+1) \cdot (k+1) \cdot \log \left( \frac{f_A(k+1)}{f_H(k+1)} \right) - \left( \frac{f_A(k)}{f_H(k)} \right)^{\lambda} \cdot f_H(k) \log \left( \frac{f_A(k)}{f_H(k)} \right),
\]

\[
\frac{\partial p_\lambda}{\partial \lambda} = \left( \frac{f_A(k)}{f_H(k)} \right)^{\lambda} \cdot f_H(k) \cdot (k+1) \cdot \log \left( \frac{f_A(k)}{f_H(k)} \right) - \left( \frac{f_A(k+1)}{f_H(k+1)} \right)^{\lambda} \cdot f_H(k+1) \cdot k \log \left( \frac{f_A(k+1)}{f_H(k+1)} \right),
\]

To construct the third lower bound \( E_{n}^{L,\text{hor}} \) (cf. (77)), we start by using for each fixed \( \lambda \in [0,1] \) as an upper bound of \( \phi_\lambda \) the horizontal line through the intercept \( \sup_{x \in \mathbb{N}_0} \phi_\lambda(x) \). For \( P_{SP,3ab} \cup P_{SP,3c} \), this supremum is achieved at the finite integer point \( z_\lambda^H := \arg \max_{x \in \mathbb{N}_0} \phi_\lambda(x) \) (since \( \lim_{x \rightarrow \infty} \phi_\lambda(x) = -\infty \)) and one has \( \phi_\lambda(z_\lambda^H) \) as \( \phi_\lambda(z_\lambda^H) = \beta_\lambda \), \( p_\lambda = \phi_\lambda(z_\lambda^H) \), \( \alpha_\lambda = \alpha_\lambda \) to the Hellinger integral upper bound \( B_{\lambda,n}^U = \exp(\phi_\lambda(z_\lambda^H) \cdot n) < 1 \) (cf. (47)). To compute from this the required \( \lim_{\lambda \rightarrow 1} \frac{1-B_{\lambda,n}^U}{\lambda(1-\lambda)} \) is not straightforward since in general it seems to be intractable to express \( z_\lambda^H \) explicitly in terms of \( \lambda \). However, since \( \lim_{\lambda \rightarrow 1} \phi_\lambda(x) = 0 \) for all \( x \in [0,\infty) \), we obtain by l'Hospital's rule

\[
\lim_{\lambda \rightarrow 1} \phi_\lambda(x) = \alpha_\lambda + \beta_A x \left[ - \log \left( \frac{\alpha_\lambda + \beta_A x}{\alpha_H + \beta_H x} \right) + 1 \right] - (\alpha_H + \beta_H x).
\]

Accordingly, let us define \( z := \arg \max_{x \in \mathbb{N}_0} \left\{ \alpha_\lambda + \beta_A x \left[ - \log \left( \frac{\alpha_\lambda + \beta_A x}{\alpha_H + \beta_H x} \right) + 1 \right] - (\alpha_H + \beta_H x) \right\} \) (note that the maximum exists since \( \lim_{x \rightarrow \infty} \{ \alpha_\lambda + \beta_A x \left[ - \log \left( \frac{\alpha_\lambda + \beta_A x}{\alpha_H + \beta_H x} \right) + 1 \right] - (\alpha_H + \beta_H x) \} = -\infty \). Due to continuity of the function \( (\lambda, x) \rightarrow \phi_\lambda(x) \), there exists an \( \epsilon > 0 \) such that for all \( \lambda \in [1 - \epsilon, 1] \) it holds \( z_\lambda^H = z^* \). Applying these considerations, we get with l'Hospital’s rule

\[
I(P_{A,n} \mid P_{H,n}) \geq \lim_{\lambda \rightarrow 1} \frac{1 - \exp(\phi_\lambda(z^*) \cdot n)}{\lambda(1-\lambda)} = \left[ f_A(z^*) \cdot \left( \log \left( \frac{f_A(z^*)}{f_H(z^*)} \right) - 1 \right) + f_H(z^*) \right] \cdot n \geq 0.
\]

In fact, in the current parameter constellation \( P_{SP,3ab} \cup P_{SP,3c} \), we have \( \phi_\lambda(x) < 0 \) for all \( \lambda \in [0,1] \) and all \( x \in \mathbb{N}_0 \) which implies \( f_A(z^*) \neq f_H(z^*) \) by Lemma A.1; thus, we even get \( E_{n}^{L,\text{hor}} > 0 \) for all \( n \in \mathbb{N} \) by virtue of the inequality \( -\log \left( \frac{f_A(z^*)}{f_H(z^*)} \right) > -\frac{f_H(z^*)}{f_A(z^*)} + 1 \).
For the case $\mathcal{P}_{SP,2}$, the abovementioned procedure leads to $z^*_\lambda = 0 = z^*(\lambda \in [0,1])$ which implies $\phi_\lambda(z^*_\lambda) = 0$, $B^U_{\lambda,n} = 1$ and thus the trivial lower bound $E^{L,hor\_1}_{n} = \lim_{\lambda \to 1} B^L_{\lambda,n} = 0$ follows. In contrast, for the case $\mathcal{P}_{SP,3d}$ one gets $z^*_\lambda = \frac{\alpha - \alpha_\lambda}{\beta - \beta_\lambda} = z^*(\lambda \in [0,1])$ which nevertheless also implies $\phi_\lambda(z^*_\lambda) = 0$ and hence $E^{L,hor\_1}_{n} = 0$. On $\mathcal{P}_{SP,4}$, we have $\sup_{x \in \mathfrak{N}_0} \phi_\lambda(x) = \phi_\lambda(\infty) = 0$ and hence we set $E^{L,hor\_1}_{n} = 0$.

To show the strict positivity $E^{L}_{n} > 0$ in the parameter case $\mathcal{P}_{SP,2}$, we inspect the bound $E^{L,sec\_1}_{0,n}$. With the help of $\alpha := \alpha_* := \alpha_\lambda = \alpha_\Omega$ (the bullet will be omitted in this proof) and the auxiliary variable $x := \frac{\beta y}{\beta_\lambda} > 0$, the definition (76) respectively its special case (79) rewrites as

$$E^{L,sec\_1}_{0,n} = E^{L,sec\_1}_{0,n}(x) = \begin{cases} \left[ - (\alpha + \beta_\lambda) \cdot \log \left( \frac{\alpha + \beta_\lambda x}{\alpha + \beta A} \right) + \beta_\lambda (x - 1) \right] \cdot \frac{1 - (\beta A)}{1 - \beta_\lambda} \cdot \left( \omega_0 - \frac{\alpha}{1 - \beta_\lambda} \right) \\
+ \left[ \frac{\alpha}{\beta_\lambda (1 - \beta_\lambda)} \left( - (\alpha + \beta_\lambda) \cdot \log \left( \frac{\alpha + \beta_\lambda x}{\alpha + \beta A} \right) + \beta_\lambda (x - 1) \right) \right] \cdot n, & \text{if } \beta_\lambda \neq 1, \end{cases} \tag{108}$$

To prove that $E^{L,sec\_1}_{0,n} > 0$ for all $\omega_0 \in \mathbb{N}$ and all $n \in \mathbb{N}$ it suffices to show that $E^{L,sec\_1}_{0,n}(1) = \left( \frac{\partial}{\partial x} E^{L,sec\_1}_{0,n} \right)(1) = 0$ and $\left( \frac{\partial^2}{\partial x^2} E^{L,sec\_1}_{0,n} \right)(x) > 0$ for all $x \in [0, \infty) \setminus \{1\}$. The assertion $E^{L,sec\_1}_{0,n}(1) = 0$ is trivial from (108). Moreover, we obtain

$$\left( \frac{\partial}{\partial x} E^{L,sec\_1}_{0,n} \right)(x) = \begin{cases} \beta_\lambda \cdot \left[ 1 - \frac{\alpha + \beta_\lambda}{\alpha + \beta A} \right] \cdot \frac{1 - (\beta A)}{1 - \beta_\lambda} \cdot \left( \omega_0 - \frac{\alpha}{1 - \beta_\lambda} \right) \\
+ \alpha \cdot \left[ 1 - \frac{\alpha + \beta_\lambda}{\alpha + \beta A} \right] \cdot \frac{\beta A}{1 - \beta_\lambda} \cdot n, & \text{if } \beta_\lambda \neq 1, \end{cases} \tag{109}$$

which immediately yields $\left( \frac{\partial}{\partial x} E^{L,sec\_1}_{0,n} \right)(1) = 0$. For the second derivative we get

$$\left( \frac{\partial^2}{\partial x^2} E^{L,sec\_1}_{0,n} \right)(x) = \begin{cases} \frac{\alpha + \beta_\lambda}{\alpha + \beta A} \cdot \frac{1 - (\beta A)}{1 - \beta_\lambda} \cdot \left( \omega_0 - \frac{\alpha}{1 - \beta_\lambda} \right) \\
+ \alpha \cdot \frac{\beta A}{\alpha + \beta A} \cdot \frac{1 - (\beta A)}{1 - \beta_\lambda} \cdot n, & \text{if } \beta_\lambda \neq 1, \end{cases} \tag{110}$$

where the strict positivity in the case $\beta_\lambda \neq 1$ follows immediately by replacing $\omega_0$ with 1 and by using the obvious relation $\left[ \omega_0 - \frac{\alpha}{1 - \beta_\lambda} \right] = \frac{1}{1 - \beta_\lambda} \cdot \left[ n - \frac{1 - \beta_\lambda}{1 - \beta_\lambda} \right] = \frac{1}{1 - \beta_\lambda} \cdot \sum_{k=0}^{n-1} (1 - \beta_\lambda^k) > 0$.

For the constellation $\mathcal{P}_{SP,4}$ with parameters $\beta := \beta_* := \beta_\lambda = \beta_\Omega$, $\alpha_\lambda \neq \alpha_\Omega$, the strict positivity of $E^n_L > 0$ follows by showing that $E^n_{y\_,tan\_1}$ converges from above to zero as $y$ tends to infinity. In fact, there holds $\lim_{y \to \infty} y \cdot E^n_{y\_,tan\_1} \in [0, \infty]$. To see this, let us first observe that by l’Hospital’s rule we get

$$\lim_{y \to \infty} y \cdot \log \left( \frac{\alpha_\lambda + \beta y}{\alpha_\Omega + \beta y} \right) = \frac{\alpha_\lambda - \alpha_\Omega}{\beta} \quad \text{as well as} \quad \lim_{y \to \infty} y \cdot \left( 1 - \frac{\alpha_\lambda + \beta y}{\alpha_\Omega + \beta y} \right) = - \frac{\alpha_\lambda - \alpha_\Omega}{\beta}.$$ 

From this and (75), we obtain $\lim_{y \to \infty} y \cdot E^n_{y\_,tan\_1} = \frac{(\alpha_\lambda - \alpha_\Omega)^2}{\beta} \cdot n > 0$ in both cases $\beta \neq 1$ and $\beta = 1$.

Finally, in the parameter case $\mathcal{P}_{SP,3d}$ we consider the bound $E^n_{y\_,tan\_1}$, with $y^* = \frac{\alpha + \alpha_{\Omega}}{\beta_\lambda - \beta_\Omega}$. Since $\alpha_\lambda + \beta A y^* = \alpha_\Omega + \beta_\Omega y^*$ holds, it is easy to see that $E^n_{y\_,tan\_1} = 0$ for all $n \in \mathbb{N}$. However, the condition $\left( \frac{\partial}{\partial y} E^n_{y\_,tan\_1} \right)(y^*) \neq 0$ implies that $\sup_{y \geq 0} E^n_{y\_,tan\_1} > 0$. The explicit form (78) of this condition follows from

$$\left( \frac{\partial}{\partial y} E^n_{y\_,tan\_1} \right)(y) = \begin{cases} \frac{(\alpha A + \beta A)^2 (\beta A - \beta_\lambda)}{f_A(y)(f_A(y))^2} \cdot \frac{1 - (\beta A)}{1 - \beta_\lambda} \cdot \left( \omega_0 - \frac{\alpha A}{1 - \beta_\lambda} \right) \\
+ \frac{\alpha A (\beta A - \beta_\lambda)}{f_A(y)(f_A(y))^2} \cdot \left( \frac{\alpha A}{\beta A (1 - \beta_\lambda)} - \frac{\alpha A (\beta A - \beta_\lambda)}{\beta_\lambda} \right) \cdot n, & \text{if } \beta_\lambda \neq 1, \end{cases} \tag{108}$$

where $\beta_\lambda = 1$. 

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y \geq 0$, by using the particular choice $y = y^*$ together with $f_A(y^*) = f_H(y^*) = \frac{\sigma_A \partial H - \partial H \partial A}{\partial A - \partial H}$.

The next lemma (and parts of its proof) will be useful for the verification of Theorem 6.6:

**Lemma A.6.** Recall the bounds on the Hellinger integral $m$–limit given in (66) and (66) of Theorem 5.3, in terms of $L(1, t)$ and $U(1, t)$ $(i = 1, 2)$ defined by (67) to (70). Correspondingly, one gets the following $\lambda$–limits for all $t \in [0, \infty[$:

(a) for all $\kappa_A \in [0, \infty]$ and all $\kappa_H \in [0, \infty]$ with $\kappa_A \neq \kappa_H$

$$
\lim_{\lambda \to 1} \frac{\partial L(1, t)}{\partial \lambda} = \lim_{\lambda \to 1} \frac{\partial L(2, t)}{\partial \lambda} = \lim_{\lambda \to 1} \frac{\partial U(1, t)}{\partial \lambda} = \lim_{\lambda \to 1} \frac{\partial U(2, t)}{\partial \lambda} = 0. 
$$

(b) for $\kappa_A = 0$ and all $\kappa_H \in [0, \infty[$

$$
\lim_{\lambda \to 1} \frac{\partial L(1, t)}{\partial \lambda} = -\frac{\kappa_A^2 \cdot t}{2 \sigma^2}.
$$

Proof of Lemma A.6 For all $\kappa_A, \kappa_H \in [0, \infty]$ with $\kappa_A \neq \kappa_H$ one can deduce from (64) as well as (67) to (70) the following derivatives:

$$
\frac{\partial L(1, t)}{\partial \lambda} = \frac{1}{2 \sigma^2} \left\{ \frac{t}{2} \left( \frac{\Lambda - \kappa_A}{\Lambda} \right)^2 \left( \kappa_A^2 - \kappa_H^2 \right) \left[ 2 e^{-2 \lambda_A t} - e^{-2 \lambda_H t} \right] 
+ e^{-\lambda_A t} \frac{1 - e^{-\lambda_A t}}{\Lambda} \left[ \frac{\Lambda - \kappa_A}{\Lambda} \left( \kappa_A^2 - \kappa_H^2 - 2 \lambda_A (\kappa_A - \kappa_H) \right) - \frac{\lambda_A - \kappa_A}{\Lambda} \right] \right\} 
+ \frac{1}{2} \left( \frac{\Lambda - \kappa_A}{\Lambda} \right)^2 \left[ \frac{t}{2} e^{-\lambda_A t} \left( \kappa_A^2 - \kappa_H^2 \right) - \frac{\lambda_A - \kappa_A}{\Lambda} \right] 
+ \frac{1}{4} \left( \frac{\Lambda - \kappa_A}{\Lambda} \right)^2 \left[ \frac{t}{2} e^{-\lambda_A t} \left( \kappa_A^2 - \kappa_H^2 \right) + 2 \lambda_A (\kappa_A - \kappa_H) \right] 
\right\}.
$$

$$
\frac{\partial L(2, t)}{\partial \lambda} = \frac{1}{4} \left\{ \frac{t}{2} e^{-\lambda_A t} \left( \kappa_A^2 - \kappa_H^2 \right) \cdot \frac{1 - e^{-\lambda_A t}}{\Lambda} \right\}.
$$

$$
\frac{\partial U(1, t)}{\partial \lambda} = \frac{1}{2 \sigma^2} \left\{ \frac{t}{2} \left( \frac{\Lambda - \kappa_A}{\Lambda} \right)^2 \left( \kappa_A^2 - \kappa_H^2 \right) \left[ 2 e^{-2 \lambda_A t} - e^{-2 \lambda_H t} \right] 
+ e^{-\lambda_A t} \frac{1}{\Lambda} \left[ \frac{\Lambda - \kappa_A}{\Lambda} \left( \kappa_A^2 - \kappa_H^2 - 2 \lambda_A (\kappa_A - \kappa_H) \right) - \frac{\lambda_A - \kappa_A}{\Lambda} \right] \right\} 
+ \frac{1}{4} \left( \frac{\Lambda - \kappa_A}{\Lambda} \right)^2 \left[ \frac{t}{2} e^{-\lambda_A t} \left( \kappa_A^2 - \kappa_H^2 \right) + 2 \lambda_A (\kappa_A - \kappa_H) \right] 
\right\}.
$$

$$
\frac{\partial U(2, t)}{\partial \lambda} = \frac{1}{2 \sigma^2} \left\{ \frac{t}{2} \left( \frac{\Lambda - \kappa_A}{\Lambda} \right)^2 \left( \kappa_A^2 - \kappa_H^2 \right) \left[ 2 e^{-2 \lambda_A t} - e^{-2 \lambda_H t} \right] 
+ e^{-\lambda_A t} \frac{1}{\Lambda} \left[ \frac{\Lambda - \kappa_A}{\Lambda} \left( \kappa_A^2 - \kappa_H^2 - 2 \lambda_A (\kappa_A - \kappa_H) \right) - \frac{\lambda_A - \kappa_A}{\Lambda} \right] \right\}.
$$
\[
\frac{\partial U^{(2)}(t)}{\partial \lambda} = \begin{cases} 
\left(\frac{\Lambda - \kappa\lambda}{3\Lambda + \kappa\lambda}\right)^2 \cdot \frac{t}{2} e^{-\frac{\Lambda}{4}(3\Lambda + \kappa\lambda)t} \left(\frac{3\Lambda^2 - \kappa^2\lambda^2}{2\Lambda} + \kappa\Lambda - \kappa\H\right) \\
\cdot \frac{1 - e^{-\frac{\Lambda}{2}(3\Lambda + \kappa\lambda)t}}{3\Lambda + \kappa\lambda} \cdot \left(\frac{3\Lambda^2 - \kappa^2\lambda^2}{2\Lambda} + \kappa\Lambda - \kappa\H\right)
\end{cases}
\]

If \(\kappa\Lambda \in [0, \infty]\) and \(\kappa\H \in [0, \infty]\) with \(\kappa\Lambda \neq \kappa\H\), then one gets \(\lim_{\lambda \to 1} \Lambda = \lim_{\lambda \to 1} \kappa\Lambda = \kappa\Lambda > 0\) which implies (111) from (115) to (116). For the proof of part (b), let us correspondingly assume \(\kappa\Lambda = 0\) and \(\kappa\H \in [0, \infty]\), which by (64) leads to \(\kappa\Lambda = \kappa\H \cdot (1 - \lambda)\), \(\Lambda = \kappa\H \cdot \sqrt{1 - \lambda}\) and the convergences \(\lim_{\lambda \to 1} \Lambda = \lim_{\lambda \to 1} \kappa\Lambda = 0\). From this, the assertions (112), (113), (114) follow in a straightforward manner from (115), (116), (116) – respectively – by using (parts of) the obvious relations

\[
\lim_{\lambda \to 1} \frac{\kappa\Lambda}{\Lambda} = 0, \quad \lim_{\lambda \to 1} \frac{\Lambda \pm \kappa\lambda}{\Lambda} = \frac{\Lambda - \kappa\Lambda}{\Lambda + \kappa\lambda} = 1,
\]

\[
\lim_{\lambda \to 1} \frac{1 - e^{-\frac{\Lambda}{4}(3\Lambda + \kappa\lambda)t}}{\Lambda} = t \quad \text{for all } c_\lambda \in \left\{\frac{\Lambda\lambda + \kappa\lambda}{2}, \frac{3\Lambda\lambda + \kappa\lambda}{2}\right\}.
\]

In order to get the last assertion (114) we make use of the following limits

\[
\lim_{\lambda \to 1} \frac{1}{\Lambda\lambda - \kappa\lambda} = \frac{3}{3\Lambda + \kappa\lambda} = \frac{4\kappa\H}{\lambda - \kappa\lambda} = \frac{4}{3\kappa\H},
\]

and

\[
\lim_{\lambda \to 1} \frac{1}{\Lambda\lambda} \left[1 - \frac{1}{3}\frac{e^{-\frac{\Lambda}{2}(3\Lambda + \kappa\lambda)t}}{\Lambda\lambda - \kappa\lambda} + \frac{1}{3\Lambda\lambda + \kappa\lambda} \right] = 0.
\]

To see (120), let us first observe that the involved limit can be rewritten as

\[
\lim_{\lambda \to 1} \frac{1}{\Lambda\lambda(\Lambda\lambda - \kappa\lambda)} \left[\frac{1}{3} - \frac{1}{3}\frac{e^{-\frac{\Lambda}{2}(3\Lambda + \kappa\lambda)t}}{\Lambda\lambda - \kappa\lambda} - \frac{1}{3\Lambda\lambda + \kappa\lambda} \right] = \left[\frac{1}{3} \left(\frac{1}{\Lambda\lambda} - \frac{1}{3(\Lambda\lambda - \kappa\lambda)} \right) \right] - \frac{1}{3}\frac{e^{-\frac{\Lambda}{2}(3\Lambda + \kappa\lambda)t}}{\Lambda\lambda - \kappa\lambda}.
\]

Substituting \(\lambda := \sqrt{1 - \lambda}\) and applying l’Hospital’s rule twice, we get for the first limit (121)

\[
\lim_{\lambda \to 0} \frac{1}{\kappa\H \cdot (x^2 - x^3)} = \lim_{\lambda \to 0} \frac{\kappa\H \cdot (3 + 2x)}{\kappa\H \cdot (2x - 3x^2)}
\]

\[
= \lim_{\lambda \to 0} \frac{\kappa\H \cdot (3 + 2x)}{\kappa\H \cdot (2x - 3x^2)} \cdot e^{-\kappa\H t} \cdot e^{-\frac{\kappa\H t}{3}\left(3x + x^2\right)} = \frac{e^{-\frac{\kappa\H t}{3}\left(3x + x^2\right)} + \kappa\H \cdot (1 + 2x) e^{-\frac{\kappa\H t}{3}(x + x^2)}}{\kappa\H \cdot (2 - 6x)}
\]

\[
= \frac{1}{2\kappa\H^2} \left[-3\kappa\H^2 t^2 + 3\kappa\H t + \kappa\H^2 t^2 - \kappa\H^2 t^2 + \kappa\H t \cdot (2 - 6x) - \frac{4\kappa\H}{3\kappa\H} \right] = \frac{2t}{3}\kappa\H.
\]

The second limit (122) becomes

\[
\lim_{\lambda \to 0} \frac{1}{3\Lambda\lambda + \kappa\lambda} = \frac{4\kappa\H}{3(\kappa\H + \sqrt{1 - \kappa\H})(3\kappa\H + 3\sqrt{1 - \kappa\H})} = \frac{2t}{3}\kappa\H.
\]
and consequently (120) follows. To proceed with the proof of (114), we rearrange

\[
\lim_{\lambda' \to \lambda} \frac{\partial U_2^{(2)}(t)}{\partial \lambda} = \lim_{\lambda' \to \lambda} \left\{ \left( \frac{A_{\lambda} - \kappa_{\lambda}}{A_{\lambda}} \right)^2 \left[ \frac{A_{\lambda}}{3A_{\lambda} + \kappa_{\lambda}} \left( \frac{t}{2} e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t} \left( -\frac{3\kappa_{\lambda}^2}{2A_{\lambda}} + \kappa_{\lambda} t \right) \right) \right. \right.
\]
\[
- \frac{A_{\lambda}}{3A_{\lambda} + \kappa_{\lambda}} \cdot 1 - e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t} \left( -\frac{3\kappa_{\lambda}^2}{2A_{\lambda}} - \kappa_{\lambda} t \right) + \frac{A_{\lambda}}{A_{\lambda} - \kappa_{\lambda}} \cdot \frac{e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t} - e^{-\kappa_{\lambda} t}}{A_{\lambda}(A_{\lambda} - \kappa_{\lambda})} \left( -\frac{3\kappa_{\lambda}^2}{2A_{\lambda}} + \kappa_{\lambda} t \right) \right]
\]
\[
+ \left. \left[ \frac{A_{\lambda} - \kappa_{\lambda}}{A_{\lambda}} (-\kappa_{\lambda}^2 + 2A_{\lambda} \kappa_{\lambda}) + \left( \frac{A_{\lambda} - \kappa_{\lambda}}{A_{\lambda}} \right)^2 \left( \frac{t}{2} e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t} \left( -\frac{3\kappa_{\lambda}^2}{2A_{\lambda}} - \kappa_{\lambda} t \right) \right) \right] \cdot \left[ \frac{1 - e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t} - e^{-\kappa_{\lambda} t}}{A_{\lambda}(3A_{\lambda} + \kappa_{\lambda})} - \frac{e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t} - e^{-\kappa_{\lambda} t}}{A_{\lambda}(A_{\lambda} - \kappa_{\lambda})} \right] \right\} \right\}
\]

By means of (117) to (119), the limit of the expression after the squared brackets in (123) becomes

\[
\lim_{\lambda' \to \lambda} \left\{ \frac{\kappa_{\lambda}^2 t}{4} \left[ \frac{1 - e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t}}{A_{\lambda} - \kappa_{\lambda}} - 2 \frac{1 - e^{-\kappa_{\lambda} t}}{A_{\lambda} - \kappa_{\lambda}} + 3 \frac{e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t}}{3A_{\lambda} + \kappa_{\lambda}} + \frac{1}{A_{\lambda} - \kappa_{\lambda}} - \frac{3}{3A_{\lambda} + \kappa_{\lambda}} \right] = \frac{\kappa_{\lambda} t}{3}, \right. \]

and the limit of the expression in (124) becomes with (120)

\[
\lim_{\lambda' \to \lambda} \left\{ \frac{\lambda_{\lambda} \cdot \kappa_{\lambda}^2}{2A_{\lambda}} \cdot \left( \frac{1 - e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t}}{3A_{\lambda} + \kappa_{\lambda}} - \frac{1 - e^{-\kappa_{\lambda} t}}{A_{\lambda} - \kappa_{\lambda}} + \frac{1 - e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t}}{A_{\lambda} - \kappa_{\lambda}} \right) \right. \right.
\]
\[
- \frac{\kappa_{\lambda}^2}{2} \cdot \frac{1 - e^{-\frac{1}{2}(3A_{\lambda} + \kappa_{\lambda})t}}{3A_{\lambda} + \kappa_{\lambda}} \cdot \frac{1}{A_{\lambda} - \kappa_{\lambda}} - \frac{3}{3A_{\lambda} + \kappa_{\lambda}} = -\frac{\kappa_{\lambda} t}{3}. \right. \]

By putting (125), (126), (127) together with (120) we finally end up with

\[
\lim_{\lambda' \to \lambda} \frac{\partial U_2^{(2)}(t)}{\partial \lambda} = \left[ \frac{\kappa_{\lambda} t}{3} - \frac{\kappa_{\lambda} t}{3} \right] + \kappa_{\lambda} \left( -\frac{t}{6} + \frac{t}{2} + t + \frac{t}{2} \right) + \left[ -\frac{\kappa_{\lambda}^2}{2} + \frac{\kappa_{\lambda}^2}{2} \right], \quad 0 = 0, \]

which finishes the proof of Lemma A.6.

\[\Box\]

**Proof of Theorem 6.6** Recall from (59) the approximative offspring-distribution-parameter \( \beta_{\lambda}^{(m)} := 1 - \frac{\kappa_{\lambda}}{2\sigma^2} \), and immigration-distribution parameter \( \alpha_{\lambda}^{(m)} := \beta_{\lambda}^{(m)} \cdot \frac{n}{\kappa_{\lambda}} \), which is a special case of \( P_{\lambda} \cup P_{\lambda} \).

Let us first calculate \( \lim_{m \to \infty} \int f_{\lambda_{\lambda}^{(m)}(\sigma^2 m)} P_{\lambda_{\lambda}^{(m)}\sigma^2 m} \) by starting from Theorem 6.1(a). Correspondingly, we evaluate for all \( \kappa_{\lambda} \geq 0, \kappa_{\lambda} \geq 0 \) with \( \kappa_{\lambda} \neq \kappa_{\lambda} \)

\[
\lim_{m \to \infty} m^2 \cdot \left[ \frac{\beta_{\lambda}^{(m)}}{\lambda_{\lambda}^{(m)}} \cdot \log \left( \frac{\beta_{\lambda}^{(m)}}{\beta_{\lambda}^{(m)}} \right) - 1 \right] = \lim_{m \to \infty} \frac{-m}{2\sigma^2} \cdot \left[ \kappa_{\lambda} \log \left( \frac{\beta_{\lambda}^{(m)}}{\beta_{\lambda}^{(m)}} \right) + \kappa_{\lambda} \left( 1 - \frac{\beta_{\lambda}^{(m)}}{\beta_{\lambda}^{(m)}} \right) \right]
\]
\[
= \frac{1}{2\sigma^4} \cdot \lim_{m \to \infty} \frac{\beta_{\lambda}^{(m)} \cdot \kappa_{\lambda} - \beta_{\lambda}^{(m)} \cdot \kappa_{\lambda}}{\left( \beta_{\lambda}^{(m)} \right)^2} \cdot \left( \kappa_{\lambda} \cdot \beta_{\lambda}^{(m)} - \kappa_{\lambda} \right) = \frac{\left( \kappa_{\lambda} - \kappa_{\lambda} \right)^2}{2\sigma^4}. \quad (128)
\]
Additionally there holds
\[
\lim_{m \to \infty} m \cdot (1 - \beta_A^{(m)}) = \frac{\kappa_A}{\sigma^2} \quad \text{and} \quad \lim_{m \to \infty} \left(\beta_A^{(m)}\right)^{\sigma^2 mt} \cdot \frac{1}{m} = \lim_{m \to \infty} \left[ \left(1 - \frac{\kappa_A}{\sigma^2 m}\right)^{\frac{1}{m}} \right]^{\sigma^2 mt} = e^{-\kappa_A t} . \tag{129}
\]
For \(\kappa_A > 0\), we apply the upper part of formula (72) as well as (128), (129) to derive
\[
\lim_{m \to \infty} I_\lambda \left( P_A^{(m)}_{\lambda, \sigma^2 m t} \right) = \lim_{m \to \infty} \frac{m \cdot \left[ \beta_A^{(m)} \cdot \left( \log \left( \frac{\beta_A^{(m)}}{\beta_H^{(m)}} \right) - 1 \right) + \beta_H^{(m)} \right]}{m \cdot \left(1 - \beta_A^{(m)}\right)}.
\]
For \(\kappa_A = 0\) (and thus \(\kappa_H > 0\), \(\beta_A^{(m)} \equiv 1, \alpha_A^{(m)} \equiv \eta/\sigma^2\)), we apply the lower part of formula (72) as well as (128), (129) to obtain
\[
\lim_{m \to \infty} I_\lambda \left( P_A^{(m)}_{\lambda, \sigma^2 m t} \right) = \lim_{m \to \infty} \frac{m \cdot \left[ \beta_H^{(m)} - \log \beta_H^{(m)} - 1 \right]}{m \cdot \left(1 - \beta_A^{(m)}\right)}.
\]
Let us now calculate the “converse” double limit
\[
\lim_{\lambda \to 1} \lim_{m \to \infty} I_\lambda \left( P_A^{(m)}_{\lambda, \sigma^2 m t} \right) = \lim_{\lambda \to 1} \frac{1 - H_\lambda \left( P_A^{(m)}_{\lambda, \sigma^2 m t} \right)}{\lambda \cdot (1 - \lambda)} .
\]
This will be achieved by evaluating for each \(t > 0\) the two limits
\[
\lim_{\lambda \to 1} \frac{1 - D_{\lambda, t}}{\lambda \cdot (1 - \lambda)} \quad \text{and} \quad \lim_{\lambda \to 1} \frac{1 - D'_{\lambda, t}}{\lambda \cdot (1 - \lambda)} , \tag{130}
\]
which will turn out to coincide; the involved lower and upper bound \(D_{\lambda, t}^{L}, D_{\lambda, t}^{U}\) defined by (66) and (66) satisfy \(\lim_{\lambda \to 1} D_{\lambda, t}^{L} = \lim_{\lambda \to 1} D_{\lambda, t}^{U} = 1\) as an easy consequence of the limits (cf. 64)
\[
\lim_{\lambda \to 1} \Lambda_{\lambda} = \kappa_{\lambda} \geq 0 \quad \text{and} \quad \lim_{\lambda \to 1} \kappa_{\lambda} = \kappa_{\Lambda} \geq 0 , \tag{131}
\]
as well as the formulae (117), (118) for the case \(\kappa_{\Lambda} = 0\). Accordingly, we compute
\[
\lim_{\lambda \to 1} \frac{1 - D_{\lambda, t}^{L}}{\lambda \cdot (1 - \lambda)} = \lim_{\lambda \to 1} \frac{-D_{\lambda, t}^{L}}{1 - 2\lambda} \frac{\partial}{\partial \lambda} \left[ -\frac{\lambda - \kappa_{\lambda}}{\sigma^2} \cdot \left( \tilde{X}_0 - \frac{\eta}{\Lambda_{\lambda}} \right) \cdot (1 - e^{-\Lambda_{\lambda} \cdot t}) - \frac{\eta}{\sigma^2} \cdot (\Lambda_{\lambda} - \kappa_{\lambda}) \cdot t \right.
\]
\[
+ \left. L_{\lambda}^{(1)}(t) \cdot \tilde{X}_0 + \frac{\eta}{\sigma^2} \cdot L_{\lambda}^{(2)}(t) \right] \]
\[
= \lim_{\lambda \to 1} \left\{ -\frac{\lambda - \kappa_{\lambda}}{\sigma^2} \cdot \left[ \left( \tilde{X}_0 - \frac{\eta}{\Lambda_{\lambda}} \right) \cdot t e^{-\Lambda_{\lambda} \cdot t} + \frac{\partial \Lambda_{\lambda}}{\partial \lambda} \right. \right.
\]
\[
\frac{1}{\sigma^2} \cdot \left. \frac{\partial}{\partial \lambda} (\Lambda_{\lambda} - \kappa_{\lambda}) \cdot \left( \tilde{X}_0 - \frac{\eta}{\Lambda_{\lambda}} \right) \cdot (1 - e^{-\Lambda_{\lambda} \cdot t}) - \frac{\eta t}{\sigma^2} \cdot \frac{\partial (\Lambda_{\lambda} - \kappa_{\lambda})}{\partial \lambda} \right]
\]
\[
+ \left. \tilde{X}_0 \cdot \frac{\partial L_{\lambda}^{(1)}(t)}{\partial \lambda} + \frac{\eta}{\sigma^2} \cdot \frac{\partial L_{\lambda}^{(2)}(t)}{\partial \lambda} \right\} \quad \text{with} \quad \frac{\partial \Lambda_{\lambda}}{\partial \lambda} = -\frac{\kappa_{\Lambda}^2 - \kappa_{\lambda}^2}{2\Lambda_{\lambda}} \quad \text{and} \quad \frac{\partial \kappa_{\lambda}}{\partial \lambda} = \kappa_{\Lambda} - \kappa_{\lambda} . \tag{132}
\]
For the case \( \kappa_A > 0 \), one can combine this with (131) and (111) to end up with
\[
\lim_{\lambda \to 1} \frac{1 - D_{\lambda,t}^U}{\lambda - (1 - \lambda)} = \frac{(\kappa_A - \kappa_H)^2}{2\sigma^2 \cdot \kappa_A} \cdot \left[ \left( \tilde{X}_0 - \frac{\eta}{\kappa_A} \right) \cdot (1 - e^{-\kappa_A \cdot t}) + \eta \cdot t \right].
\]  
(134)

For the case \( \kappa_A = 0 \), we continue the calculation (132) by rearranging terms and by employing the formulæ (112), (113), (117), (118) as well as the obvious relation \( \frac{t}{\Lambda} - \frac{\Lambda - \kappa_A}{\kappa_A^2} = \frac{\sigma}{\kappa_H} \) to obtain
\[
\lim_{\lambda \to 1} \frac{1 - D_{\lambda,t}^U}{\lambda - (1 - \lambda)} = \lim_{\lambda \to 1} \left\{ \frac{\kappa_A^2 \cdot \tilde{X}_0}{2\sigma^2} \left[ \frac{\Lambda - \kappa_A}{\Lambda^2} \cdot t \cdot e^{-\Lambda \cdot t} + \frac{1 - e^{-\Lambda \cdot t}}{\Lambda} \right] 
+ \frac{\eta \cdot \kappa_A^2 \cdot t}{2\sigma^2} \left[ \frac{1}{\Lambda} - \frac{\Lambda - \kappa_A}{\Lambda^2} - \frac{1 - e^{-\Lambda \cdot t}}{\Lambda} \right] 
- \frac{\kappa_H \cdot \tilde{X}_0}{t} \cdot (1 - e^{-\Lambda \cdot t}) \right) + \left[ 1 - e^{-\frac{t}{2}(\Lambda + \kappa_A) \cdot t} \right] + \frac{\eta \cdot \kappa_A}{\sigma^2} \cdot \left[ \partial U^{(1)}_\lambda(t) \right] \cdot \tilde{X}_0 + \frac{\eta}{\sigma^2} \cdot \left[ \frac{\eta}{\sigma^2} \cdot t \cdot 2 + \tilde{X}_0 \cdot t \right].
\]  
(135)

Let us now turn to the second limit (130) for which we compute analogously to (132)
\[
\lim_{\lambda \to 1} \frac{1 - D_{\lambda,t}^U}{\lambda - (1 - \lambda)} = \lim_{\lambda \to 1} \frac{U^{(1)}_\lambda(t)}{2\sigma^2} \cdot \partial \left[ \frac{\Lambda - \kappa_A}{\Lambda^2} \cdot \left( \tilde{X}_0 - \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \cdot \frac{\eta}{\sigma^2} \right) \cdot (1 - e^{-\frac{t}{2}(\Lambda + \kappa_A) \cdot t}) \right] 
- \frac{\eta}{\sigma^2} \cdot \left( \Lambda - \kappa_A \right) \cdot t + \frac{\partial U^{(1)}_\lambda(t)}{\partial \Lambda} \cdot \tilde{X}_0 \right) 
\]  
(136)

For the case \( \kappa_A > 0 \), one can combine this with (131), (133) and (111) to end up with
\[
\lim_{\lambda \to 1} \frac{1 - D_{\lambda,t}^U}{\lambda - (1 - \lambda)} = \frac{(\kappa_A - \kappa_H)^2}{2\sigma^2 \cdot \kappa_A} \cdot \left[ \left( \tilde{X}_0 - \frac{\eta}{\kappa_A} \right) \cdot (1 - e^{-\kappa_A \cdot t}) + \eta \cdot t \right].
\]  
(137)

For the case \( \kappa_A = 0 \), we continue the calculation of (136) by rearranging terms and by employing the formulæ (114), (117), (118) as well as the obvious relation \( \frac{t}{\Lambda} - \frac{\Lambda - \kappa_A}{\kappa_A^2} = \frac{2}{\kappa_H} \) to obtain
\[
\lim_{\lambda \to 1} \frac{1 - D_{\lambda,t}^U}{\lambda - (1 - \lambda)} = \lim_{\lambda \to 1} \left\{ \frac{t}{\sigma^2} \cdot \tilde{X}_0 \cdot \left[ \frac{\Lambda - \kappa_A}{\Lambda} \cdot e^{-\frac{t}{2}(\Lambda + \kappa_A) \cdot t} \left( \kappa_A^2 + 2\Lambda \kappa_H \right) \right] 
+ \frac{\kappa_H}{\sigma^2} \cdot \left( \frac{1}{\Lambda} - \frac{\Lambda - \kappa_A}{\Lambda^2} + \frac{1 - e^{-\frac{t}{2}(\Lambda + \kappa_A) \cdot t}}{\Lambda} \right) \right) 
+ \frac{\eta}{\sigma^2} \cdot \left[ \frac{\Lambda - \kappa_A}{\Lambda + \kappa_A} \cdot \left( 1 + \frac{\Lambda - \kappa_A}{\Lambda + \kappa_A} \right) - \frac{\kappa_H}{\sigma^2} \right] 
\right) 
\]  
(138)
Since (134) coincides with (137) and (135) coincides with (138), we have finished the proof.

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References


