Half-Projective TQFTs

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Structures on Tensor Categories and Topological Field Theories
Universität Erlangen-Nürnberg
Some (Naïve) TQFT Properties:
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Notations & Conventions

- $\text{Cob}_n$ category of $n$-dim (compact, oriented) cobordisms between $(n-1)$-dim closed (model) manifolds (mod homeomorphism type).
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- Composition = gluing \(\otimes = \sqcup\)

- Possibly additional geometric structures (framings, graphs, ...)

\[\text{R} = \text{comm. ring}, \quad \text{R-mod} = \text{category of (free...)}\]

\[\text{TQFT is tensor functor} \quad V : \text{Cob}_n \rightarrow \text{R-mod}\]

\[\text{Compatibility with symmetry constraint}\]

\[\gamma : \Sigma_1 \sqcup \Sigma_2 \rightarrow \Sigma_2 \sqcup \Sigma_1\]

\[V(\emptyset) = \text{R}\]
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Morphisms from Cylinder $\Sigma \times [0, 1]$
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For appropriate parametrization have

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Let $\Sigma$ be fixed $(n - 1)$-manifold. Can interpret $\Sigma \times [0, 1]$ with $\partial \Sigma \times [0, 1] \cong \Sigma \sqcup \Sigma$ as cobordism in several ways:

$id : \Sigma \to \Sigma$ or $ev_\Sigma : \Sigma \sqcup \Sigma \to \emptyset$ or $coev_\Sigma : \emptyset \to \Sigma \sqcup \Sigma$

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\[ id = (id \sqcup ev_\Sigma) \circ (coev_\Sigma \sqcup id) = (ev_\Sigma \sqcup id) \circ (id \sqcup coev_\Sigma) \]

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**Dimension formula for** $\mathbb{R} = \text{field}$
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Implies $ev_V \circ coev_V = \dim(V)$. 
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Hennings Invariant

Alternative to Reshetikhin Turaev Invariant.

Starting from ribbon Hopf algebra $\mathcal{H}$ (over $\mathbb{R}$) with ingredients

$R$-matrix $R = \sum_i e_i \otimes f_i \in \mathcal{H} \otimes \mathcal{H}$.

Integral: $\mu: \mathcal{H} \to \mathbb{R}$ such that $\mu \otimes \text{id}(\Delta(x)) = 1$ $\mu(x)$.

Co-integral: $\lambda \in \mathcal{H}$ such that $x\lambda = \epsilon(x)\lambda$.

$\mu(\lambda) = 1$.

Ribbon: $M = R' R = \Delta(v) v - 1 \otimes v - 1$ for $v \in z(\mathcal{H})$.

$M$ non-degenerate.

Lemma (Larson, Sweedler, ...): $\mathcal{H}$ finite dimensional $\lambda$ and $\mu$ exist and are unique up to scalar.

$\mathcal{H}$ is semi-simple iff $\epsilon(\Lambda) \neq 0$.

$\mathcal{H}$ is co-semi-simple iff $\mu(1) \neq 0$. 
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Starting from ribbon Hopf algebra \( \mathcal{H} \) (over \( \mathbb{R} \)) with ingredients

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**Naïve TQFT Axioms**

Hennings Invariant Paradox

Elementary Example
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**Hennings Functor or Algorithm**

For ribbon Hopf algebra $\mathcal{H}$ and closed, framed 3-manifold $M$ compute:
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For ribbon Hopf algebra $\mathcal{H}$ and closed, framed 3-manifold $M$ compute:

- Surgery presentation of manifold $M$ by framed link $\mathcal{L} \subset S^3$. 
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For ribbon Hopf algebra $\mathcal{H}$ and closed, framed 3-manifold $M$ compute:

- Surgery presentation of manifold $M$ by framed link $\mathcal{L} \subset S^3$.
- Turn projection of $\mathcal{L}$ into planar diagram with $\mathcal{H}$-decorations:

For example

```
\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw (-1,0) -- (1,0);
\draw (0,-1) -- (0,1);
\node at (0,0) {$e_i$};
\node at (0,0) {$f_i$};
\end{tikzpicture}
\end{figure}
```
**Hennings Functor or Algorithm**

For ribbon Hopf algebra $\mathcal{H}$ and closed, framed 3-manifold $M$ compute:

- Surgery presentation of manifold $M$ by framed link $L \subset S^3$.
- Turn projection of $L$ into planar diagram with $\mathcal{H}$-decorations:

  ![Diagram](attachment:planar_diagram.png)

  For example

  \[
  \begin{array}{c}
  \begin{array}{c}
  b \\
  a
  \end{array}
  \end{array}
  =
  \begin{array}{c}
  ab
  \end{array}
  \]

  \[
  \begin{array}{c}
  a
  \end{array}
  \quad = \quad
  \begin{array}{c}
  \downarrow
  \end{array}
  S(a)
  \]

- Collect $\mathcal{H}$-decorations along components using diagrammatic rules.

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  \[
  e_i \quad f_i
  \]

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  ![Diagram](image)

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- Evaluate collective element on each component against $\mu$. 

Theorem (Hennings)

The number $V_{\mathcal{H}}(M)$ resulting from the algorithm above is an invariant of closed, oriented 3-manifolds.
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Special case of Lyubashenko invariant.
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Theorem

The invariant extends to a TQFT $\mathcal{V}_\mathcal{H}$ between connected surfaces.
Improvements by Kauffman, Radford, Ohtsuki.
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**Theorem**

*The invariant extends to a TQFT $\mathcal{V}_H$ between connected surfaces.*

*It coincides with Reshetikhin-Turaev TQFT if $\mathcal{H}$ is semi-simple.*

At the same time compute $\mathcal{V}(S^1 \times S^2)$.
- $S^1 \times S^2$ surgery-represented by 0-framed unknot

$$\mu(1) = 0$$
Improvements by Kauffman, Radford, Ohtsuki.
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- $S^1 \times S^2$ surgery-represented by 0-framed unknot
- Evaluate $\mathcal{V}(S^1 \times S^2) = \mu(1)$.
- For naïve TQFT and $\mathcal{H}$ is non-(co-)semi-simple.

\[ \Rightarrow \dim(S^2) = 0 \quad \Rightarrow \quad \mathcal{V}_\mathcal{H} \equiv 0 \]
Integral TQFTs

- Consider Reshetikhin-Tuarev (or BHMV) TQFT for SO(3)-theory at \( \zeta \), a primitive \( p \)-th root of unity for prime \( p \).
Integral TQFTs

- Consider Reshetikhin-Tuarev (or BHMV) TQFT for $SO(3)$-theory at $\zeta$, a primitive $p$-th root of unity for prime $p$.
- Let $\mathbb{R} = \mathbb{Z}[\zeta]$ (or $\mathbb{Z}[i, \zeta]$).
**Integral TQFTs**

- Consider Reshetikhin-Turaev (or BHMV) TQFT for $SO(3)$-theory at $\zeta$, a primitive $p$-th root of unity for prime $p$.

- Let $R = \mathbb{Z}[\zeta]$ (or $\mathbb{Z}[i, \zeta]$).

- [Murakami, Ohtsuki, Le, …]
  For suitable normalization and homology constraints the invariant of closed manifolds is in $R$. 

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**Background and Motivation**

- Half Projective TQFTs
- Example, Non-Example, & Questions
- Naïve TQFT Axioms
- Hennings Invariant Paradox
- Elementary Example

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**Thomas Kerler**

- Half-Projective TQFTs
Integral TQFTs

- Consider Reshetikhin-Turaev (or BHMV) TQFT for $SO(3)$-theory at $\zeta$, a primitive $p$-th root of unity for prime $p$.

- Let $R = \mathbb{Z}[\zeta]$ (or $\mathbb{Z}[i, \zeta]$).

- [Murakami, Ohtsuki, Le, ...]
  For suitable normalization and homology constraints the invariant of closed manifolds is in $R$.

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  Restricts/extends to TQFT over $R$ with connectivity contraints.
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- Yield connectivity information (cut-numbers).
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- May be explained by connectivity anomaly in integral TQFT cases.
- Will need refinement of axioms for Hennings formalism
  - will explain from set-up.
Elementary Example

*Currents on Graphs*

\[ G = \text{finite abelian group}, \ \Gamma \text{ a graph with orientations}, \ E_\Gamma \text{ edge set}. \]
Elementary Example

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**Current Configuration:** \( \varphi : E_\Gamma \to G \) such that at each vertex \( p \)

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Let $G^*$ be the oriented **Graph Category**: 
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**Construction of Morphisms**

- Basis of free $\mathbb{R}$-module $\mathcal{V}(\epsilon)$ associated to $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ labeled by point configurations $g = (g_1, g_2, \ldots, g_k) \in G^k$.
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- Naturally functorial - that is, $\mathcal{V}(\Gamma_1) \mathcal{V}(\Gamma_2) = \mathcal{V}(\Gamma_1 \circ \Gamma_2)$
**Equivalent, algebraic topology construction**

Denote $C(X) = \mathbb{R}^X$ for ring $\mathbb{R}$ and finite set $X$. 

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- $\partial^{-1}((g, h)) = [\varphi] + H_1(\Gamma, G)$ if $(g, h) \in \text{im}(\overline{\partial})$.  


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$$\mathcal{V}(\Gamma) = |G|^{\beta_1(\Gamma)} \widehat{\mathcal{V}}(\Gamma).$$
Composition Rules for $\hat{\mathcal{V}}$

Functoriality of $\mathcal{V}$ implies

$$|G|\beta_1(\Gamma_1)|G|\beta_1(\Gamma_2)\hat{\mathcal{V}}(\Gamma_1)\hat{\mathcal{V}}(\Gamma_2) = |G|\beta_1(\Gamma_1 \circ \Gamma_2)\hat{\mathcal{V}}(\Gamma_1 \circ \Gamma_2)$$
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where

$$c(\Gamma_1, \Gamma_2) = \beta_1(\Gamma_1 \circ \Gamma_2) - \beta_1(\Gamma_1) - \beta_1(\Gamma_2).$$
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Mayer-Vietoris

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0 \rightarrow H_1(\Gamma_1) \oplus H_1(\Gamma_2) \xrightarrow{i_*} H_1(\Gamma_1 \circ \Gamma_2) \xrightarrow{\partial_*} H_0(\Gamma_1 \cap \Gamma_2)
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**Composition Rules for** \( \hat{V} \)

**Functoriality of** \( \forall \) **implies**

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\]

**Thus**

\[
c(\Gamma_1, \Gamma_2) = \text{rank}(\partial_*)
\]
**Composition Rules for** \( \hat{\mathcal{V}} \)

**Functoriality of** \( \mathcal{V} \) **implies**

\[
|G|\beta_1(\Gamma_1)|G|\beta_1(\Gamma_2) \hat{\mathcal{V}}(\Gamma_1) \hat{\mathcal{V}}(\Gamma_2) = |G|\beta_1(\Gamma_1 \circ \Gamma_2) \hat{\mathcal{V}}(\Gamma_1 \circ \Gamma_2)
\]

**Assume** \( |G| \in \mathbb{R} \) **is a unit and denote** \( x = |G|^{-1} \) **obtain**

\[
\hat{\mathcal{V}}(\Gamma_1 \circ \Gamma_2) = x^c(\Gamma_1, \Gamma_2) \hat{\mathcal{V}}(\Gamma_1) \hat{\mathcal{V}}(\Gamma_2)
\]

where

\[
c(\Gamma_1, \Gamma_2) = \beta_1(\Gamma_1 \circ \Gamma_2) - \beta_1(\Gamma_1) - \beta_1(\Gamma_2).
\]

**Mayer-Vietoris**

\[
0 \to H_1(\Gamma_1) \oplus H_1(\Gamma_2) \xrightarrow{i_*} H_1(\Gamma_1 \circ \Gamma_2) \xrightarrow{\partial_*} H_0(\Gamma_1 \cap \Gamma_2)
\]

**Thus**

\[
c(\Gamma_1, \Gamma_2) = \text{rank}(\partial_*) = \# \text{ of new 1-cycles}
\]
Half Projective TQFT Axiom

Mayer-Vietoris Sequence for Cobordisms:

Consider cobordisms \( M : \Sigma_i \rightarrow \Sigma_m \) and \( N : \Sigma_m \rightarrow \Sigma_o \).

\[
\begin{align*}
H_1(M \cap N) & \oplus H_1(N) = H_1(N \circ M) \\
H_0(M \cap N) & \oplus H_0(N) = H_0(N \circ M)
\end{align*}
\]

From exactness of lower row:

\[
\omega(N, M) = \beta_0(\Sigma_m) - (\beta_0(M) + \beta_0(N)) + \beta_0(N \circ M)
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Thomas Kerler

Half-Projective TQFTs
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$$
\begin{array}{c}
H_1(N \circ M) \leftarrow H_1(M) \oplus H_1(N) \leftarrow H_1(M \cap N) \\
\partial_* \downarrow \\
0 \rightarrow \text{im}(\partial_*) \rightarrow H_0(M \cap N) \rightarrow H_0(M) \oplus H_0(N) \rightarrow H_0(N \circ M) \rightarrow 0
\end{array}
$$

$\partial_* = \beta_0(\Sigma_m) - (\beta_0(M) + \beta_0(N)) + \beta_0(N \circ M)$. 

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For cobordism $M : \Sigma_{\text{in}} \to \Sigma_{\text{out}}$ set

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- However, $\omega$ non-negative, while $\alpha$ may be negative.
A half-projective TQFT $\mathcal{V} : \text{Cob}_n \to \mathbb{R}\text{-mod}$ with respect to an element $x \in \mathbb{R}$ fulfills all axioms of a standard TQFT.
A half-projective TQFT $\mathcal{V} : \text{Cob}_n \to \mathbf{R}-\text{mod}$ with respect to an element $x \in \mathbf{R}$ fulfills all axioms of a standard TQFT except that functoriality of composition is modified as

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Basic Properties

Cut-Numbers

For an oriented connected cobordism $M$ call an oriented co-dim=1 sub-manifold $\Sigma \subset M$ properly non-separating (p.n.s.) if $\Sigma$ is two-sided. $\Sigma \cap \partial M = \emptyset$. $M - \Sigma$ is connected.

Cut-Number: For connected cobordism as above defined by $\rho(M) = \max \{ \beta_0(\Sigma) : \Sigma \text{ is an p.n.s. sub-mfld in } M \}$.

For a connected $M$ call an epimorphism onto free group $f : \pi_1(M) \twoheadrightarrow F_k = \mathbb{Z} \ast \cdots \ast \mathbb{Z}$ an internal $k$-free projection if its restriction to all $i^* (\pi_1(\partial M))$ trivial.
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**Lemma**

*For $n = 3$ (and smooth manifolds in higher dim):*

$$\rho(M) = \varphi(M)$$
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Notes:

- Any cobordism can written as $M = \widehat{M} \circ C$ such that

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**Notes:**

- Any cobordism can written as $\mathcal{M} = \hat{M} \circ C$ such that
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- For $n = 3$ have
  $$\rho(S^1 \times \Sigma_g) = \max(g, 1).$$
Applications to half-projective TQFT and Dimensions

Theorem

Suppose $\mathcal{V}$ is a half-projective TQFT w.r.t. $x \in \mathbb{R}$, then

$$\mathcal{V}(M) \in x^{\rho(M)}\text{Hom}(\ldots) \quad \forall \text{cobordisms } M.$$ 

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Also $x = \mathcal{V}(S^1 \times S^2)$ under standard assumption $\mathcal{V}(S^2) = \mathbb{R}.$
Combining the previous

**Corollary**

\[ x \dim(V(\Sigma_g)) \in x^{\max(g,1)} R \]

*If \( R \) is a domain thus have*

\[ \dim(V(\Sigma_g)) \in x^{g-1} R \text{ for } g \geq 1. \]
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Implies divisibility constraints for example for TQFTs over \( R = \mathbb{Z}[\zeta] \).
Construction from connected TQFTs

Given two \((n - 1)\)-manifolds \(\Sigma_1\) and \(\Sigma_2\) viewed as objects of \(\mathsf{Cob}_n\).
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- Analogously, construct \(\Pi_\Sigma^\dagger : \Sigma^\# \to \Sigma\).
**Construction from connected TQFTs**

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Consider connect sum \(\Sigma^\# = \Sigma_1 \# \Sigma_2\) also as object of \(\text{Cob}_n\).

Natural cobordism \(\Pi_\Sigma : \Sigma \to \Sigma^\#\) where \(\Sigma = \Sigma_1 \sqcup \Sigma_2\):

- \(\Pi_\Sigma\) given by \(\Sigma \times [0, 1]\) with 1-handle attached at target connecting components.

- or by \(\Sigma^\# \times [0, 1]\) with \((n - 1)\)-handle attached at source along separating \(S^{n-2} \subset \Sigma^\#\).

- Analogously, construct \(\Pi_\Sigma^\dagger : \Sigma^\# \to \Sigma\).

Composite Identities:

- \(\Pi_\Sigma^\dagger \Pi_\Sigma = \text{id}_{\Sigma_1} \# \text{id}_{\Sigma_2}\) (interior connect sum)
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Composite Identities (cont.): 

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Composite Identities (cont.):

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![Diagram of composite identities](image-url)
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- By induction/iteration obtain analogous maps/identities o for
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Composite Identities (cont.):

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\end{array}
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(except we have \((k - 1) = \beta_0(\Sigma)\) connected \((S^1 \times S^{n-1})\)-factors)
Main ingredient for construction \((n = 3)\).

**Lemma**

Every connected morphism \(M : \Sigma_{\text{in}} \rightarrow \Sigma_{\text{out}}\) can be expressed as

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M = \Pi_{\Sigma_{\text{out}}} \circ \hat{M} \circ \Pi_{\Sigma_{\text{in}}}
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where \(\hat{M} : \Sigma^\#_{\text{in}} \rightarrow \Sigma^\#_{\text{out}}\) is conn. cobordism of connected surfaces.

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$$i_{\Sigma} = \mathcal{V}(\Pi_{\Sigma}) : \mathcal{V}(\Sigma_1) \otimes \ldots \otimes \mathcal{V}(\Sigma_k) \to \mathcal{V}(\Sigma^\#)$$
List of Properties

- Expression independent of choice of $\hat{M}$.  
  \[ p_{\Sigma} i_{\Sigma} = \text{id} \]  
  so that  
  \[ P = p_{\Sigma} i_{\Sigma} \]  
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A few more general structure axioms.

Theorem

Suppose $\mathcal{V}^{\text{conn}}$ is a connected TQFT that admits maps with properties as listed. Then $\mathcal{V}^{\text{conn}}$ extends to a half-projective TQFT with respect to $x$ as above.
List of Properties

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Integral TQFT – \( p = 5 \) example

- Consider case \( p = 5 \) so that \( q = 1 \) and hence

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x = \mathcal{V}(S^1 \times S^2) = \mathcal{D} = u(\zeta - \zeta^{-1}).
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⇒ “rederivation” of Gilmer cut-number result from general axioms.
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  $\Rightarrow$ “rederivation” of Gilmer cut-number result from general axioms.

- From dimension corollary get $\dim(\mathcal{V}(\Sigma_g)) \in (\zeta - \zeta^{-1})^{g-1}\mathbb{R}$. 
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  $$5^{\alpha} \text{ divides } \dim(\mathcal{V}(\Sigma_g)) \text{ for } \alpha = \left\lfloor \frac{g - 1}{4} \right\rfloor.$$
**Integral TQFT – \( p = 5 \) example**

- Consider case \( p = 5 \) so that \( q = 1 \) and hence
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  x = \mathcal{V}(S^1 \times S^2) = D = u(\zeta - \zeta^{-1}).
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- Have \( \mathcal{V}(\Lambda_\Sigma) = x^{\beta_0(\Sigma)-1}P \), with integral projector \( P \).

- Thus \( (\zeta - \zeta^{-1})^{\rho(M)} \) divides \( \mathcal{V}(M) \) for closed manifold.

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  5^a \text{ divides } \dim(\mathcal{V}(\Sigma_g)) \text{ for } a = \left\lfloor \frac{g-1}{4} \right\rfloor.
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- Indeed have \( \left\lfloor \frac{g}{2} \right\rfloor \text{ divides } \dim(\mathcal{V}(\Sigma_g)) \).
Hennings TQFTs

- Vector spaces $\mathcal{V}(\Sigma_g) = \text{Inv}_H(H \otimes g)$ w.r.t. to adjoint action of $H$.
- Denote $\lambda_\alpha$ the adjoint action of $\lambda$ on $H \otimes \alpha$. 
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- Vector spaces \( \mathcal{V}(\Sigma_g) = \text{Inv}_\mathcal{H}(\mathcal{H}^{\otimes g}) \) w.r.t. to adjoint action of \( \mathcal{H} \).
- Denote \( \overline{\lambda}_a \) the adjoint action of \( \lambda \) on \( \mathcal{H}^{\otimes a} \).
- For \( \Sigma = \Sigma_{g_1} \sqcup \Sigma_{g_2} \) have

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  acting on $\text{Inv}(\mathcal{H} \otimes g_1 \otimes \mathcal{H} \otimes g_2)$.

- Similarly, for $\Sigma = \Sigma g_1 \sqcup \Sigma g_2 \sqcup \ldots \Sigma g_k$ (with movable id insertion)

  $\overline{\Lambda}_\Sigma = \text{Inv}(\overline{\lambda}_{g_1} \otimes \ldots \otimes \overline{\lambda}_{g_{k-1}} \otimes \text{id}_{g_k})$.

- For non-semisimple $\mathcal{H}$ have $\epsilon(\lambda) = 0$ and hence $\lambda^2 = 0$ implying
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- For non-semisimple $\mathcal{H}$ have $\epsilon(\lambda) = 0$ and hence $\lambda^2 = 0$ implying

  - $\text{im}(\overline{\Lambda}_\Sigma) \subseteq \text{Inv}(\mathcal{H}^\otimes g_1) \otimes \ldots \otimes \text{Inv}(\mathcal{H}^\otimes g_k)$.

  - $\text{Inv}(\mathcal{H}^\otimes g_1) \otimes \ldots \otimes \text{Inv}(\mathcal{H}^\otimes g_k) \subseteq \ker(\overline{\Lambda}_\Sigma)$. 
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Natural inclusion

$$i_\Sigma : \text{Inv}(\mathcal{H}^{\otimes g_1}) \otimes \ldots \otimes \text{Inv}(\mathcal{H}^{\otimes g_k}) \rightarrow \text{Inv}(\mathcal{H}^{\otimes g_1} \otimes \ldots \otimes \mathcal{H}^{\otimes g_k}).$$
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\]

Natural opposite map $\tilde{p}_\Sigma = i^{-1}_\Sigma \Lambda_\Sigma$:
\[
\text{Inv}(\mathcal{H}^{\otimes g_1} \otimes \ldots \otimes \mathcal{H}^{\otimes g_k}) \to \text{Inv}(\mathcal{H}^{\otimes g_1}) \otimes \ldots \otimes \text{Inv}(\mathcal{H}^{\otimes g_k})
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Thus, instead of a multiple of a projector have \( \overline{\Lambda}^2 = 0 \)
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Natural opposite map \( \tilde{p}_\Sigma = i_{\Sigma}^{-1} \overline{\Lambda} \) :
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\]
\[
i_\Sigma \tilde{p}_\Sigma = \text{id} \quad \text{but} \quad \tilde{p}_\Sigma i_\Sigma = 0.
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Thus, instead of a multiple of a projector have \( \Lambda_\Sigma^2 = 0 \)
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**Strategies**

- For closed \( M \) with \( \rho(M) > 0 \) have \( \mathcal{V}(M) = \text{trace}(\Lambda f) = 0. \)
Thus, instead of a multiple of a projector have \( \Lambda^2 = \lambda \Sigma = 0 \) (but generally \( \lambda \Sigma \neq 0 \)).

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**Strategies**

For closed \( \mathcal{M} \) with \( \rho(\mathcal{M}) > 0 \) have \( \mathcal{V}(\mathcal{M}) = \text{trace}(\lambda f) = 0 \).

Note that \( \mathcal{V}(N \circ \mathcal{M}) = \text{Inv}_\mathcal{H}(\mathcal{N} \circ \lambda \circ \mathcal{M}) \) where \( \lambda \) is tensor of \( c(N, \mathcal{M}) \) factors \( \lambda_{g_i} \).
Thus, instead of a multiple of a projector have $\Lambda^2 = 0$ (but generally $\Lambda \neq 0$).

Natural inclusion

$$i_\Sigma : \text{Inv}(H^{\otimes g_1}) \otimes \ldots \otimes \text{Inv}(H^{\otimes g_k}) \to \text{Inv}(H^{\otimes g_1} \otimes \ldots \otimes H^{\otimes g_k})$$

Natural opposite map $\tilde{p}_\Sigma = i^{-1}\Lambda :$

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**Strategies**

For closed $M$ with $\rho(M) > 0$ have $\mathcal{V}(M) = \text{trace}(\lambda f) = 0$.

Note that $\mathcal{V}(N \circ M) = \text{Inv}_H(\tilde{N} \circ \lambda \circ \tilde{M})$ where $\lambda$ is tensor of $c(N, M)$ factors $\bar{\lambda}_{g_i}$.

In some sense “categorify” $x \sim$ insertion of $\bar{\lambda}$-factors.
Other Situations

*Invariants from Character Varieties*

- Variety \( J(X) = \text{Hom}(\pi_1(X), G)/G \) for Lie group \( G \).
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- Variety \( J(X) = \text{Hom}(\pi_1(X), G)/G \) for Lie group \( G \).
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- Correspondence – using top form evaluation
  $\mathcal{V}(\Sigma_{in}) \otimes \mathcal{V}(\Sigma_{out}) \longrightarrow \mathcal{V}(M)$. 

$G = U(1)$ is Frohman-Nicas TQFT = Hennings TQFT.

Casson and other finite types Invariants.

More "exotic" examples with vanishing/connectivity properties

[Wehrheim & Woodward]: Floer Field & Connected Cerf Theory.

Heegaard-Floer Homology.
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Other Situations

*Invariants from Character Varieties*

- Variety \( J(X) = \text{Hom}(\pi_1(X), G)/G \) for Lie group \( G \).
- Linearize \( \mathcal{V}(X) = H^*(J(X), ...) \)...
- Correspondence – using top form evaluation
  \[ \mathcal{V}(\Sigma_{in}) \otimes \mathcal{V}(\Sigma_{out}) \rightarrow \mathcal{V}(M) \rightarrow \mathbb{R} \]
- Too much homology in \( M \) ⇒ degree mismatches ⇒ 0.
- \( G = \mathbb{U}(1) \) is Frohman-Nicas TQFT
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- Casson and other finite types Invariants.
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*More “exotic” examples with vanishing/connectivity properties*
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*More “exotic” examples with vanishing/connectivity properties*

- [Wehrheim & Woodward]: Floer Field & Connected Cerf Theory.
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- [Wehrheim & Woodward]: Floer Field & Connected Cerf Theory.
- Heegaard-Floer Homology.
Thank You!