The limits of stabilizability for networks of strings

Martin Gugat, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)  
joint work with Stephan Gerster (RWTH Aachen), see On the limits of stabilizability for 
Waves in a nutshell, Erlangen, Monday, November 11, 2019. Video zeigen?
Motivation by application: Gas transportation through pipelines
   Example for a closed-loop system.

The example by Bastin and Coron: A single interval
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A tree of strings with a stabilizing and a destabilizing source term
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Application: Gas transportation through pipelines

The system dynamics in a pipe is described by

the isothermal Euler equations

\[
\begin{align*}
\rho_t + q_x &= 0 \\
q_t + \left( p + \frac{q^2}{\rho} \right)_x &= -\frac{f_g}{2\delta} \frac{q|q|}{\rho} - \rho g \sin(\alpha)
\end{align*}
\]

or a similar (linearized) model, see A. Osiadacz, M. Chazykowski, Comparison of isothermal and non-isothermal pipeline gas flow models, 2001.

See the results of DFG CRC 154:
A semilinear model for gas flow

In terms of the RIEMANN invariants the hyperbolic system has the \textit{quasilinear} form

\[
\begin{align*}
R_t + D_q(R)R_x &= F(R), \\
F(R) &= f(g\delta |R+ - R-|)\frac{1}{1-1}
\end{align*}
\]

with a diagonal matrix \(D_q(R)\) that contains the eigenvalues \(q\rho + c, q\rho - c\).

Replacing the eigenvalues by \(d+ = c, d- = -c\) yields a \textit{semilinear} model with a constant matrix \(D\).

Let a stationary state \(\bar{R}\) with \(D\bar{R}x = F(\bar{R})\) be given. Linearizing around \(\bar{R}\) with \(r = R - \bar{R}\) yields

\[
r_t + Dr_x = F'(\bar{R})r.
\]

For the ideal gas we get

\[
F'(\bar{R}) = -2f\frac{\delta |\bar{R}+ (x) - \bar{R}- (x)|}{1-1}.
\]

The matrix is positive semidefinite. (First go to RIEMANN invariants, then linearize!)

The pressure is given by \(p = \exp\left(\frac{1}{2}(r+ + r- + \bar{R}+ + \bar{R}-)\right) > 0\) and the gas velocity is proportional to \(r+ - r- + \bar{R}+ - \bar{R}-\).
A semilinear model for gas flow

In terms of the RIEMANN invariants the hyperbolic system has the \textit{quasilinear} form

\[ R_t + D_q(R) R_x = F(R), \quad F(R) = \frac{f_g}{\delta} |R_+ - R_-| (R_+ - R_-) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

with a diagonal matrix \( D_q(R) \) that contains the eigenvalues

\[ \frac{q}{\rho} + c, \quad \frac{q}{\rho} - c. \]
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In terms of the Riemann invariants the hyperbolic system has the quasilinear form

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Boundary control problems

We are interested in **boundary control** of the system!

For this purpose, we introduce a control $u(t)$ in the boundary condition.
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What happens if the matrix \( M \) in the pde

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    r_t + Dr_x = -Mr
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is *not* positive definite?

Can this cause difficulties?

A movie
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The solutions have the form \( U(t, x) = \alpha(t - x), \ V(t, x) = \beta(t + x). \)
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Together with the boundary conditions

\[ U(t, 0) = k \ V(t, 0), \quad V(t, L) = U(t, L) \]

where \( k \in (-1, 1) \) is a feedback parameter

(and initial data \( U(0, \cdot), \ V(0, \cdot) \))

we have a closed loop-system.
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In our example, with a \( 2 \times 2 \) matrix \( M \) we get

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The source term can cause instability!
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In *Stability and Boundary Stabilization of 1-D Hyperbolic Systems* (2016), BASTIN and CORON consider the diagonal system

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with a real parameter \( c > 0 \).
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Is it stable?
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Is it stable?

BASTIN & CORON construct *product solutions* of the form (separation ansatz)

\[
\begin{pmatrix} U(t, x) \\ V(t, x) \end{pmatrix} = \exp(\sigma t) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.
\]

If such a solution can be found with \( \sigma > 0 \), the system is *exponentially unstable* and cannot be stabilized.
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The example by Bastin and Coron

In fact, the following proposition is already proved by Bastin and Coron:

**Proposition:**

If \( cL \geq \pi \),

there is no real value of \( k \) such that the closed loop system with the pde

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Boundary stabilization becomes **impossible** if the length or the (negative eigenvalue of the) source term is too large!
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If $\lambda > 0$, we have $\lambda L < \frac{\pi}{2}$, and for $\epsilon = \frac{\pi}{4} - \frac{\lambda L}{2}$ we have $|k| \leq \tan^2(\epsilon)$ and $cL < \lambda \tan^2(\epsilon)$

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A sufficient condition for stabilizability

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This can be seen considering the quadratic LYAPUNOV function

$$\mathcal{L}(t) = \frac{1}{2} \int_{0}^{L} A \cot(\epsilon + \lambda x) U^2(t, x) + A^{-1} \tan(\epsilon + \lambda x)) V^2(t, x) \, dx.$$
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- Here the trigonometric weights give a better result (\( cL \leq 0.177.. \)) than exponential weights.
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How is the situation on networks?
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A tree of strings

Now we consider a star-shaped networks of strings.

Figure: A star-shaped network with $N = 4$ edges.
A tree of strings

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Figure: A star-shaped network with $N = 4$ edges.

We consider feedback control at all boundary nodes except one. Let $N \in \{2, 3, 4, \ldots\}$ denote the number of strings.
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Figure: A star-shaped network with $N = 4$ edges.

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For $i \in \{1, 2, \ldots, N\}$ let $c_i > 0$ and $\varepsilon_i \geq 0$ be given and consider the wave equation

$$ U_{tt}^i = U_{xx}^i - 2 \varepsilon_i U_t^i - (\varepsilon_i^2 - c_i^2) U^i = 0 \tag{1} $$

on the space interval $[0, L_i]$.  

A tree of strings

Now we consider a star-shaped networks of strings.

We consider feedback control at all boundary nodes except one. Let $N \in \{2, 3, 4, \ldots\}$ denote the number of strings.

For $i \in \{1, 2, \ldots, N\}$ let $c_i > 0$ and $\varepsilon_i \geq 0$ be given and consider the wave equation

\[
U_{tt}^i = U_{xx}^i - 2\varepsilon_i U_t^i - (\varepsilon_i^2 - c_i^2) U^i = 0
\]  

(1)

on the space interval $[0, L_i]$. The edges are coupled at $x = 0$ by node conditions:
At the central node:

For $i, j \in \{1, ..., N\}$

\[
U^i(t, 0) - U^j(t, 0) = 0,
\]

\[
\sum_{k=1}^{N} U^{k}_{x}(t, 0) = 0.
\]
A tree of strings

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### A tree of strings

**At the central node:**

For \( i, j \in \{1, ..., N\} \)

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U^i(t, 0) - U^j(t, 0) = 0,
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**At the boundary node of edge number 1 at \( x = L_1 \) we have a homogeneous DIRICHLET condition**

\[
U_1(t, L_1) = 0
\]

and **at the other boundary nodes** for \( j \in \{2, ..., N\} \) at \( x = L_j \) we have a NEUMANN velocity feedback

\[
U^j_x(t, L_j) = K_j U^j_t(t, L_j)
\]

with a real **feedback** parameter \( K_j \).
The wave equation (1) can be transformed to a $2 \times 2$ system:

For $i \in \{1, \ldots, N\}$ define $V_i = -\frac{1}{c_i} (U_i t + U_i x + \varepsilon_i U_i)$. Then due to the definition of $V_i$ and (1), the function $(U_i, V_i)$ solve

$$
\begin{pmatrix}
U_i \\
V_i
\end{pmatrix} t +
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
U_i \\
V_i
\end{pmatrix} x +
\begin{pmatrix}
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\begin{pmatrix}
0 \\
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\end{pmatrix}.
$$

Also the node conditions and boundary conditions can be transformed similarly: For example, at $x = L_1$, we have $V_1(t, L_1) = -\frac{1}{c_1} (U_1 x + U_1 t)$. For $i \in \{2, \ldots, N\}$, at $x = L_i$, we have $V_i(t, L_i) = -\frac{1}{c_i} (\varepsilon_i U_i + (K_i + 1) U_i t)$. 


A tree of strings

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$$
(U^i_t V^i_t) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (U^i_t V^i_x) + \begin{pmatrix} \varepsilon_i c_i c_i \varepsilon_i \\ -1 \end{pmatrix} (U^i_t V^i) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
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A tree of strings

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$$V^1(t, L_1) = -\frac{1}{c_1} (U^1_x + U^1_t).$$

For $i \in \{2, ..., N\}$, at $x = L_i$, we have

$$V^i(t, L_i) = -\frac{1}{c_i} (\varepsilon_i U^i + (K_i + 1) U^i_t).$$
Aim of this talk

• Also for our star of strings, *boundary feedback stabilization* is *not always possible*!

• If one of the strings is too long, it can become *impossible* for all feedback parameters!
Motivation by application: Gas transportation through pipelines  
Example for a closed-loop system.

The example by Bastin and Coron: A single interval  
A sufficient condition for non-stabilizability  
A sufficient condition for stabilizability

A tree of strings with a stabilizing and a destabilizing source term  
Sufficient conditions for non-stabilizability  
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A tree of strings

Limits of stabilizability: Assume that for all \(i \in \{1, \ldots, N\}\) we have \(c_i > \varepsilon_i\).
A tree of strings

Limits of stabilizability: Assume that for all \( i \in \{1, \ldots, N\} \) we have \( c_i > \varepsilon_i \).

1. Instability if ALL edges are sufficiently long:

   If

   \[
   c_1^2 \geq \varepsilon_1^2 + \frac{\pi^2}{L_1^2}
   \]

   and for all \( i \in \{2, \ldots, N\} \) we have

   \[
   c_i = c_1, \quad \varepsilon_i = \varepsilon_1, \quad L_i = L_1, \quad K_i = K_2
   \]

   there are no values of \( K_2 \in (-\infty, \infty) \) such that the closed loop system with the wave equation (1), the node conditions and the boundary conditions is asymptotically stable.

   In fact, there are solutions with exponentially increasing norms in \( X_{i=1}^N L^2(0, L_i) \).
A tree of strings

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   In fact, there are solutions with exponentially increasing norms in \( X_{i=1}^N L^2(0, L_i) \).

2. Instability if ONE edge is sufficiently long:

   If
   \[
   c_1^2 > \varepsilon_1^2 + \frac{9}{4} \frac{\pi^2}{L_1^2} \quad \text{and} \quad c_1 - \varepsilon_1 \leq \min_{i \in \{2, \ldots, N\}} \{ c_i - \varepsilon_i \}, \tag{2}
   \]

   there are no values of \( K_2, \ldots, K_N \in (-\infty, 0] \) such that the closed loop system is asymptotically stable.
Limits of stabilizability: Assume that for $i \in \{1, ..., N\}$ we have $c_i > \varepsilon_i = 0$ and one of the values of $L_i > 0$ is sufficiently large.

- Due to the POINCARÉ–inequality, if the $c_i > 0$ are sufficiently small, we can use the energy

$$E(t) = \frac{1}{2} \sum_{i=1}^{N} \int_0^{L_i} \left( U^i_x(t, x) \right)^2 + \left( U^i_t(t, x) \right)^2 - c_i^2 \left( U^i(t, x) \right)^2 \, dx.$$
A tree of strings

Limits of stabilizability: Assume that for \( i \in \{1, \ldots, N\} \) we have \( c_i > \varepsilon_i = 0 \) and one of the values of \( L_i > 0 \) is sufficiently large.

- Due to the Poincaré–inequality, if the \( c_i > 0 \) are sufficiently small, we can use the energy

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\]

- We have

\[
E'(t) = \sum_{i \in I_F} K_i \left( U^i_t(t, L_i) \right)^2.
\]

Thus if \( K_i \geq 0 \), we have \( E'(t) \geq 0 \) and thus the energy does not decay.
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\[
E(t) = \frac{1}{2} \sum_{i=1}^{N} \int_{0}^{L_i} \left( U_x^i(t, x) \right)^2 + \left( U_t^i(t, x) \right)^2 - c_i^2 \left( U^i(t, x) \right)^2 \, dx.
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  \]

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- Thus there are no parameter vectors with components $K_i \geq 0$ such that the system is asymptotically stable.

- With the Result 2. above, this implies that there are no parameter vectors with components of equal sign such that the system is asymptotically stable.
A tree of strings

Limits of stabilizability - Result 3.: Instability for a large number of short edges

Assume that for \( i \in \{1, \ldots, N\} \) we have \( c_i > \varepsilon_i \) and

\[
\begin{align*}
  c_i &= c_1, \quad \varepsilon_i = \varepsilon_1, \\
  L_i &= L_1, \quad K_i = K_2.
\end{align*}
\]
A tree of strings

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c_i = c_1, \quad \varepsilon_i = \varepsilon_1, \quad L_i = L_1, \quad K_i = K_2.
\]

If

\[
\sin^2\left(\sqrt{c_1^2 - \varepsilon_1^2 L_1}\right) = \frac{1}{N}
\]

there are no real values of \( K_2 \in (-\infty, \infty) \) \((j \in \{2, \ldots, N\})\) such that the closed loop system is asymptotically stable.

So here the total length of the strings \( N \, L_1^2 \) must be sufficiently large!
A tree of strings

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If

$$\sin^2(\sqrt{c_1^2 - \varepsilon_1^2} L_1) = \frac{1}{N}$$

there are no real values of $K_2 \in (-\infty, \infty)$ ($j \in \{2, \ldots, N\}$) such that the closed loop system is asymptotically stable. This implies

$$L_1^2 (c_1^2 - \varepsilon_1^2) = \left(\arcsin(\frac{1}{\sqrt{N}})\right)^2.$$ 

Since we have $\lim_{N \to \infty} \arcsin(\frac{1}{\sqrt{N}}) = 0$, for $N$ sufficiently large we obtain arbitrarily small values of the lengths $L_i > 0$, for which the system is not exponentially stable!

So here the total length of the strings $N L_1^2$ must be sufficiently large!
Motivation by application: Gas transportation through pipelines
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Assume that $c_i = c_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and

$$\varepsilon_1 \geq c_1 \geq 0$$

with $K_2 = 0$ the closed loop system is exponentially stable.

Thank you for your attention!
A sufficient condition for stabilizability

Assume that $c_i = c_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and $\varepsilon_1 \geq c_1 \geq 0$

with $K_2 = 0$ the closed loop system is exponentially stable.

- This can be shown by the analysis of the *eigenfunctions* of the system.
- For $\varepsilon_1 \geq c_1 \geq 0$, the matrix of the source term is positive semidefinite.
A sufficient condition for stabilizability

Assume that $c_i = c_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and $\varepsilon_1 \geq c_1 \geq 0$ with $K_2 = 0$ the closed loop system is exponentially stable.

• This can be shown by the analysis of the eigenfunctions of the system.
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Semilinear Gas: $R_t + D R_x = F(R)$, $R_+(0) = f_0 R_-(0)$, $R_-(L) = f_L R_+(L)$

If $|f_0| \leq 1$ and $|f_L| \leq 1$, $\mathcal{L}(t) = \int_0^L (R_+)^2 + (R_-)^2 \, dx$ is decreasing.
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Assume that $c_i = c_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and

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Moreover, $\tilde{\mathcal{L}}(t) = \mathcal{L}(t) + \theta \int_0^t \int_0^L |R_+ - R_-|^3 \, dx \, dt$ is decreasing. This implies

$$\int_0^\infty \int_0^L |R_+ - R_-|^3 \, dx \, dt < \infty.$$

The source term has an $L^3$-regularizing effect!
A sufficient condition for stabilizability

Assume that $c_i = c_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and

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Similarly, for $H^1$ initial data, $\tilde{\mathcal{L}}_1(t) = \int_0^L (\partial_x R_+)^2 + (\partial_x R_-)^2 \, dx$ is decreasing, if $|f_0| < 1$, $|f_L| < 1$ and the factor in $F$ is sufficiently small.
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Assume that \( c_i = c_1, \varepsilon_i = \varepsilon_1, L_i = L_1, K_i = K_2. \) If \( \varepsilon_1 > 0 \) and

\[
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With \( f_0 = f_L = 0 \), the system is finite-time stable. This can be shown with

\[
\tilde{\mathcal{L}}(t) = \int_0^L e^{-\mu(x-L)}(R_+)^2 + e^{\mu(x-L)}(R_-)^2 \, dx.
\]
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Assume that $c_i = c_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and

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Moreover, $\tilde{L}(t) = L(t) + \theta \int_0^t \int_0^L |R_+ - R_-|^3 \, dx \, dt$ is decreasing. This implies

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Similarly, for $H^1$ initial data, $\tilde{L}_1(t) = \int_0^L (\partial_x R_+)^2 + (\partial_x R_-)^2 \, dx$ is decreasing, if $|f_0| < 1$, $|f_L| < 1$ and the factor in $F$ is sufficiently small.

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$$\hat{L}(t) = \int_0^L e^{-\mu(x-L)} (R_+)^2 + e^{\mu(x-L)} (R_-)^2 \, dx.$$

What happens for $cL \in (0.177..., \pi)$?

Thank you for your attention!