The Relativistic Vlasov–Maxwell System with External Electromagnetic Fields

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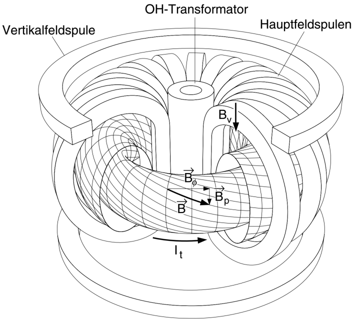
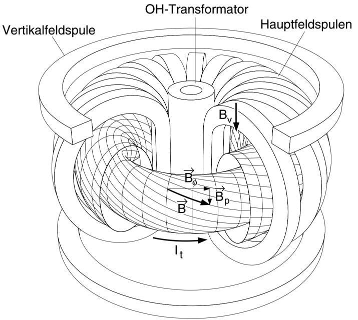
Set-up



source: M. Kaufmann, *Plasmaphysik und Fusionsforschung*, 2nd ed., Springer Spektrum, 2013.

- plasma in a container $\Omega \subset \mathbb{R}^3$
- external currents—may serve as control

Problems



source: M. Kaufmann, *Plasmaphysik und Fusionsforschung*, 2nd ed., Springer Spektrum, 2013.

- How to model such a situation by means of differential equations?
  ⇝ relativistic Vlasov–Maxwell system:
  - unknown: particle density f, electromagnetic fields E, H
  - given: initial conditions $\mathring{f}$, $\mathring{E}$, $\mathring{H}$, permittivity ε, permeability μ, external current density u
  - constitutive equations: $D = \varepsilon E$, $B = \mu H$
  - boundary conditions for plasma particles on $\partial\Omega$
- What about the solution theory?
- How to choose the external currents in order to confine the plasma as best as possible while having not too exhaustive control costs?
- Do confined steady states exist if an external magnetic field is adjusted suitably?

Outline

Vlasov equation

- particle density $f = f(t,x,v) \geq 0$: (roughly) how many particles at time t at position $x \in \Omega$ with momentum $v \in \mathbb{R}^3$
- Vlasov equation: continuous version of Newton's law of motion:

$$\partial_t f + \widehat{v} \cdot \partial_x f + (E + \widehat{v} \times H) \cdot \partial_v f = 0,$$

  where $\widehat{v} = \frac{v}{\sqrt{1+|v|^2}}$ relativistic velocity; nonlinearity
- introduce surface measure $d\gamma = |n(x) \cdot \widehat{v}|\, dv dS_x dt$ and decompose boundary:

$$\gamma^{\pm} = \left\{(t,x,v) \in [0,T] \times \partial\Omega \times \mathbb{R}^3 \mid n(x) \cdot v \gtrless 0\right\}$$
$$\gamma^{0} = \left\{(t,x,v) \in [0,T] \times \partial\Omega \times \mathbb{R}^3 \mid n(x) \cdot v = 0\right\}$$

- boundary condition:

$$f_-(t,x,v) = \left(aKf_+\right)(t,x,v) = a(t,x,v) f_+(t,x,v - 2(v \cdot n(x))n(x)), \quad (t,x,v) \in \gamma^+$$

  where $0 \leq a \leq a_0 < 1$
- initial condition: $f(0) = \mathring{f}$

Maxwell equations

- electric field $E = E(t,x)$, magnetic field $H = H(t,x)$, $x \in \mathbb{R}^3$
- permittivity $\varepsilon = \varepsilon(x)$, permeability $\mu = \mu(x)$; both matrix-valued, symmetric, $\varepsilon = \mu = 1$ on Ω; main assumption: $\sigma \leq \varepsilon, \mu \leq \sigma'$ for some $\sigma, \sigma' > 0$ in the sense of positive definiteness
- time-evolutionary Maxwell equations:

$$\varepsilon \partial_t E - \operatorname{curl}_x H = -4\pi j$$
$$\mu \partial_t H + \operatorname{curl}_x E = 0$$

- current density:

$$j = j^{\text{int}} + u = \int_{\mathbb{R}^3} \widehat{v} f \, dv + u$$

- initial condition: $(E,H)(0) = \left(\mathring{E}, \mathring{H}\right)$

The whole system

$$\left.\begin{aligned}
\partial_t f + \widehat{v}\cdot\partial_x f + (E + \widehat{v}\times H)\cdot\partial_v f &= 0 && \text{on } [0,T]\times\Omega\times\mathbb{R}^3\\
f_- &= aKf_+ && \text{on } \gamma^-\\
f(0) &= \mathring{f} && \text{on } \Omega\times\mathbb{R}^3\\
\varepsilon\partial_t E - \mathrm{curl}_x H &= -4\pi\left(j^{\mathrm{int}} + u\right) && \text{on } [0,T]\times\mathbb{R}^3\\
\mu\partial_t H + \mathrm{curl}_x E &= 0 && \text{on } [0,T]\times\mathbb{R}^3\\
(E,H)(0) &= \left(\mathring{E},\mathring{H}\right) && \text{on } \mathbb{R}^3
\end{aligned}\right\}\quad \text{(VM)}$$

nonlinear, hyperbolic structure, boundary conditions—no satisfying solution theory!

Brief history

- always $u = 0$, $\varepsilon = \mu = 1$
- without boundary conditions, global existence of classical solutions:
    - electrostatic case—nonrelativistic Vlasov–Poisson:

      $$\widehat{v} \longrightarrow v, \quad H = 0, \quad E = \nabla\phi, \quad \Delta\phi = \pm 4\pi\rho$$

      Pfaffelmoser (1992), Lions, Perthame (1991)
    - continuation criterion: Glassey, Strauss (1986)
    - lower dimensional settings: Glassey, Schaeffer (1990,1997,1998)
- without boundary conditions, global existence of weak solutions: Di Perna, Lions (1989)
- with boundary conditions (including perfect conductor boundary conditions for electromagnetic fields), global existence of weak solutions: Guo (1993)

Weak formulation

- test function spaces:

$$\Psi := \Big\{\psi \in C^\infty\Big([0,T] \times \overline{\Omega} \times \mathbb{R}^3\Big) \mid \operatorname{supp}\psi \subset [0,T[\times \overline{\Omega} \times \mathbb{R}^3 \text{ compact}, \\ \operatorname{dist}(\operatorname{supp}\psi, \gamma^0) > 0, \operatorname{dist}(\operatorname{supp}\psi, \{0\} \times \partial\Omega \times \mathbb{R}^3) > 0\Big\}$$

$$\Theta := \Big\{\vartheta \in C^\infty\Big([0,T] \times \mathbb{R}^3; \mathbb{R}^3\Big) \mid \operatorname{supp}\vartheta \subset [0,T[\times \mathbb{R}^3 \text{ compact}\Big\}$$

- weak form: for all $\psi \in \Psi$, $\vartheta \in \Theta$

$$0 = -\int_0^T \int_\Omega \int_{\mathbb{R}^3} \big(\partial_t\psi + \widehat{v} \cdot \partial_x\psi + (E + \widehat{v} \times H) \cdot \partial_v\psi\big) f \, dv dx dt \\ + \int_{\gamma^+} f_+ \psi \, d\gamma - \int_{\gamma^-} a\big(K f_+\big)\psi \, d\gamma - \int_\Omega \int_{\mathbb{R}^3} \mathring{f} \psi(0) \, dv dx$$

$$0 = \int_0^T \int_{\mathbb{R}^3} \big(\varepsilon E \cdot \partial_t\vartheta - H \cdot \operatorname{curl}_x \vartheta - 4\pi j \cdot \vartheta\big) \, dx dt + \int_{\mathbb{R}^3} \varepsilon \mathring{E} \cdot \vartheta(0) \, dx$$

$$0 = \int_0^T \int_{\mathbb{R}^3} \big(\mu H \cdot \partial_t\vartheta + E \cdot \operatorname{curl}_x \vartheta\big) \, dx dt + \int_{\mathbb{R}^3} \mu \mathring{H} \cdot \vartheta(0) \, dx$$

Physical laws

- search for solution in L^p-spaces
- Lorentz force is divergence free w.r.t. v
  $\Rightarrow$ characteristic flow of Vlasov equation is measure preserving
  $\Rightarrow$ L^p-estimates for f, f_+
- energy balance:

$$\frac{d}{dt}\left(\int_\Omega \int_{\mathbb{R}^3} \sqrt{1+|v|^2} f \, dv dx + \frac{1}{8\pi} \int_{\mathbb{R}^3} \left(\varepsilon E \cdot E + \mu H \cdot H\right) dx\right) \leq -\int_{\mathbb{R}^3} E \cdot u \, dx$$

$\Rightarrow$ L^1_{kin}-estimate for f, L^2-estimate for E, H after Gronwall argument

Construction

- consider cut-off system:
  - consider iteration scheme:
    - solve linear Vlasov equation with known force field
    - update

$$j_{k,R}^{\mathrm{int}} = \int_{B_R} \widehat{v} f_k \, dv$$

    - solve linear Maxwell equations
    - update E, H
  - extract weakly convergent subsequence: $j_{k,R}^{\mathrm{int}}$ *a priori* bounded in L^2
  - use compactness result

$$\int_{\mathbb{R}^3} \zeta f_k \, dv \text{ bounded in } H^{\frac{1}{4}} \text{ for any } \zeta \in C_c^\infty\left(\mathbb{R}^3\right)$$

  to
    - handle the nonlinearity in the Vlasov equation
    - show that

$$\lim_{k\to\infty} \int_0^T \int_\Omega \left(E_{k-1} \cdot j_{k,R}^{\mathrm{int}} - E_k \cdot j_{k,R}^{\mathrm{int}}\right) dx dt = 0$$

- drive $R \to \infty$

Results and problems

- assumptions: $\partial\Omega \in C^{1,\kappa}$, $0 < \sigma \leq \varepsilon, \mu \leq \sigma'$, $u \in L^1_t L^2_x$
- existence of a global-in-time weak solution $f \in L^\infty_t \left(L^1_{\text{kin}} \cap L^\infty\right)_x$, $f_+ \in L^1_{\text{kin}} \cap L^\infty$, $E, H \in L^\infty_t L^2_x$, $j^{\text{int}} \in L^\infty_t \left(L^1 \cap L^{\frac{4}{3}}\right)_x$
- L^p-estimates on f, f_+ and energy estimate carry over
- problems:
  - weak regularity of solutions
  - no uniqueness of solutions
  - no control-to-state operator
  - whether *any* weak solution satisfies such estimates, is not known
  - how formulate and tackle optimal control problem?

Redundancy for classical solutions

- divergence equations:

$$\operatorname{div}(\varepsilon E) = 4\pi\rho, \quad \operatorname{div}\big(\mu H\big) = 0$$

- well known: these equations are satisfied globally in time if
  - they hold initially
  - all functions are C^1
  - the time-evolutionary part of Maxwell's equations is satisfied, and
  - local conservation of charge holds, i.e.,

$$\partial_t \rho + \operatorname{div} j = 0$$

- more complicated in the context of a weak solution concept and in the presence of boundary conditions

Redundancy for weak solutions

- weak form: for all $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^3)$

$$0 = \int_0^T \int_{\mathbb{R}^3} (\varepsilon E \cdot \partial_x \varphi + 4\pi\rho\varphi)\, dx dt = \int_0^T \int_{\mathbb{R}^3} \mu H \cdot \partial_x \varphi \, dx dt$$

- trick: consider

$$\vartheta(t,x) = -\int_t^T \partial_x \varphi(s,x)\, ds$$

- easy: $\operatorname{div}(\mu H) = 0$ is redundant

Redundancy for weak solutions

not so easy: $\operatorname{div}(\varepsilon E) = 4\pi\rho$ is redundant:

- local conservation of charge necessary
- weak form of Vlasov equation should hold for ψ independent of v
- sufficient: f, f_+, E, H satisfy some integrability condition and $\partial\Omega$ is smooth (precisely: $f, f_+ \in L^1 \cap L^\infty$, $\sqrt{1+|v|^2} f^2 \in L^1$, $E, H \in L^q_t L^2_x$ for some $q > 2$, $\partial\Omega \in C^2$)
- what is ρ?

$$\rho = \rho^{\text{int}} + \rho^u + S_{\partial\Omega},$$

$$S_{\partial\Omega}\varphi = \int_0^T \int_{\partial\Omega} \varphi(t,x) \int_0^t n(x) \cdot \left(j^{\text{out}}_{\partial\Omega}(s,x) + j^{\text{in}}_{\partial\Omega}(s,x) \right) ds\, dS_x\, dt$$

- local conservation of internal charge:

$$\partial_t \left(\rho^{\text{int}} + S_{\partial\Omega} \right) + \operatorname{div}_x j^{\text{int}} = 0$$

A prototype

- two aims: no hits on the boundary (they cause damage), low control costs
- consider $f_+ \in L^q$ $(1 < q < \infty)$, $E, H \in L^2$, $u \in H^1\left(]0,T[\times\Gamma;\mathbb{R}^3\right)$ (Γ open, bounded), and $f \in L^1_{\text{kin}} \cap L^\infty$ such that

$$\mathcal{N}(f) := \sup\left\|\partial_t(\eta f) + \widehat{v}\cdot\partial_x(\eta f)\right\|_{L^2([0,T]\times\Omega;H^{-1}(\mathbb{R}^3))} < \infty,$$

$$\eta \in C_c^\infty\left(]0,T[\times\Omega\times\mathbb{R}^3\right), \quad \|\eta\|_{H^1(]0,T[\times\Omega\times\mathbb{R}^3)} + \|\eta\|_{L^\infty(]0,T[\times\Omega;H^1(\mathbb{R}^3))} = 1.$$

- minimization problem:

$$\begin{aligned}
\min \quad & \frac{1}{q}\|f_+\|_{L^q}^q + \frac{\alpha}{2}\|u\|_{H^1}^2 \\
\text{s.t.} \quad & \left(f, f_+, E, H, j^{\text{int}} + u\right) \text{ solves (VM)} \\
& 0 \le f \le \left\|\mathring{f}\right\|_{L^\infty} \\
& \|f\|_{L^1_{\text{kin}}} + \frac{\sigma}{8\pi}\|(E,H)\|_{L^2}^2 \le \mathcal{I}(u) := 2T\left\|\mathring{f}\right\|_{L^1_{\text{kin}}} + \frac{T\sigma'}{4\pi}\left\|\left(\mathring{E},\mathring{H}\right)\right\|_{L^2}^2 + 2\pi T^2\sigma^{-1}\|u\|_{L^2}^2
\end{aligned}$$

Existence of minimizers

$$\begin{aligned} \min \quad & \frac{1}{q}\|f_+\|_{L^q}^q + \frac{\alpha}{2}\|u\|_{H^1}^2 \\ \text{s.t.} \quad & \left(f, f_+, E, H, j^{\text{int}} + u\right) \text{ solves (VM)} \\ & 0 \le f \le \left\|\mathring{f}\right\|_{L^\infty} \\ & \|f\|_{L^1_{\text{kin}}} + \frac{\sigma}{8\pi}\|(E,H)\|_{L^2}^2 \le \mathcal{I}(u) \end{aligned}$$

- consider minimizing sequence, extract weakly convergent subsequence
- objective function weakly lower semicontinuous
- inequality constraints preserved in the limit; $H^1(]0,T[\times\Gamma) \Subset L^2(]0,T[\times\Gamma)$ important
- passing to the limit in (VM) via compactness result as before; boundedness of $\left(f_k\right)$ and $\left((E_k + \widehat{v} \times H_k) f_k\right)$ in L^2 important—thus, the artificial constraints are added

Weak formulation—revisited

- idea: write (VM) as identity $\mathcal{G}\left(f,f_+,E,H,u\right)=0$ in dual space of some reflexive space
- recall weak form: for all $\psi\in\Psi$, $\vartheta\in\Theta$

$$
\begin{aligned}
0 = &-\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left(\partial_t\psi+\widehat{v}\cdot\partial_x\psi+(E+\widehat{v}\times H)\cdot\partial_v\psi\right)f\,dvdxdt\\
&+\int_{\gamma^+}f_+\psi\,d\gamma-\int_{\gamma^-}a\left(Kf_+\right)\psi\,d\gamma-\int_\Omega\int_{\mathbb{R}^3}\mathring{f}\psi(0)\,dvdx\\
0 = &\int_0^T\int_{\mathbb{R}^3}\left(\varepsilon E\cdot\partial_t\vartheta-H\cdot\operatorname{curl}_x\vartheta-4\pi j\cdot\vartheta\right)dxdt+\int_{\mathbb{R}^3}\varepsilon\mathring{E}\cdot\vartheta(0)\,dx\\
0 = &\int_0^T\int_{\mathbb{R}^3}\left(\mu H\cdot\partial_t\vartheta+E\cdot\operatorname{curl}_x\vartheta\right)dxdt+\int_{\mathbb{R}^3}\mu\mathring{H}\cdot\vartheta(0)\,dx
\end{aligned}
$$

- linear operators in ψ and ϑ; bounded w.r.t. $W^{1,p,\tilde{q}}$- and H^1-norm, where $q>2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{q}+\frac{1}{\tilde{q}}+\frac{1}{2}=1$,

$$
\left\|\psi\right\|_{W^{1,p,\tilde{q}}}:=\left(\int_{\mathbb{R}^3}\left(\int_0^T\int_\Omega\left(\left|\psi\right|^{\tilde{q}}+\left|\partial_t\psi\right|^{\tilde{q}}+\left|\partial_x\psi\right|^{\tilde{q}}+\left|\partial_v\psi\right|^{\tilde{q}}\right)dxdt\right)^{\frac{p}{\tilde{q}}}dv\right)^{\frac{1}{p}}
$$

Weak formulation—revisited

- denote $\Lambda := \overline{\Psi} \times \overline{\Theta}^2$ (closure in $W^{1,p,\tilde{q}}$ and H^1)
- weak form can be written as $\mathcal{G}(f, f_+, E, H, u) = 0$ in Λ^*
- $\mathcal{G}$ is differentiable
- Λ is uniformly convex, reflexive Banach space
- norm of dual space Λ^* is differentiable

Approximate optimization problem

- idea: introduce penalization parameter $s > 0$, later drive $s \to \infty$
- consider penalized problem:

$$\begin{aligned} \min \quad & \frac{1}{q}\|f_+\|_{L^q}^q + \frac{\alpha}{2}\|u\|_{H^1}^2 + \frac{s}{2}\|\mathcal{G}(f, f_+, E, H, u)\|_{\Lambda^*}^2 \\ \text{s.t.} \quad & 0 \le f \le \left\|\mathring{f}\right\|_{L^\infty} \\ & \|f\|_{L^1_{\text{kin}}} + \frac{\sigma}{8\pi}\|(E, H)\|_{L^2}^2 \le \mathcal{I}(u) \\ & \mathcal{N}(f) \le \mathcal{L} \end{aligned}$$

- the additional, artificial constraint
  - is automatically satisfied for any minimizer of original problem (choose $\mathcal{L} > 0$ appropriately)
  - ensures a certain weak lower semicontinuity of $\|\mathcal{G}\|_{\Lambda^*}$
  - thus ensures existence of minimizers

Approximate optimization problem—optimality conditions

- if s is sufficiently large, first order optimality conditions can be established:
- adjoint equation:

$$\begin{aligned}
\partial_t \psi_s + \widehat{v} \cdot \partial_x \psi_s + (E_s + \widehat{v} \times H_s) \cdot \partial_v \psi_s &= 4\pi \widehat{v} \cdot \vartheta_s^e + \nu_s \sqrt{1+|v|^2} + \tau_s \\
K a K \psi_{s,-} &= \psi_{s,+} + \operatorname{sign}(f_{s,+}) |f_{s,+}|^{q-1} \\
\varepsilon \partial_t \vartheta_s^e + \operatorname{curl}_x \vartheta_s^h &= -\int_{\mathbb{R}^3} f_s \partial_v \psi_s \, dv - \frac{\nu_s \sigma}{4\pi} E_s \\
\mu \partial_t \vartheta_s^h - \operatorname{curl}_x \vartheta_s^e &= -\int_{\mathbb{R}^3} f_s (\partial_v \psi_s \times \widehat{v}) \, dv - \frac{\nu_s \sigma}{4\pi} H_s \\
\psi_s(T) = \vartheta_s^e(T) = \vartheta_s^h(T) = 0
\end{aligned}$$

- stationarity condition:

$$\begin{aligned}
\partial_t^2 u_s + \Delta_x u_s &= -4\pi \alpha^{-1} \vartheta_s^e + (\beta \nu_s + 1) u_s \\
\partial_t u_s(0) = \partial_t u_s(T) = \partial_{n_\Gamma} u_s &= 0
\end{aligned}$$

- duality:

$$\begin{aligned}
\left\| \left(\psi_s, \vartheta_s^e, \vartheta_s^h \right) \right\|_\Lambda &= s \left\| \mathcal{G}(f_s, f_{s,+}, E_s, H_s, u_s) \right\|_{\Lambda^*} \\
\mathcal{G}(f_s, f_{s,+}, E_s, H_s, u_s) \left(\psi_s, \vartheta_s^e, \vartheta_s^h \right) &= s \left\| \mathcal{G}(f_s, f_{s,+}, E_s, H_s, u_s) \right\|_{\Lambda^*}^2
\end{aligned}$$

Passing to the limit $s \longrightarrow \infty$

- natural approach: try to pass to the limit in the optimality conditions directly; fails (among other reasons) because of appearance of derivatives of adjoint state in some source terms
- consider sequence $(f_s, f_{+,s}, E_s, H_s, u_s)$ of optimal points
- decay for error term of (VM):

$$\left\|\mathcal{G}(f_s, f_{+,s}, E_s, H_s, u_s)\right\|_{\Lambda^*} = \mathcal{O}\left(s^{-\frac{1}{2}}\right)$$

- along a suitable sequence $s \longrightarrow \infty$:
  - weak convergence of $(f_s, f_{+,s}, E_s, H_s, u_s)$ to some point which is optimal for the original problem
  - even

$$\left\|\mathcal{G}(f_s, f_{+,s}, E_s, H_s, u_s)\right\|_{\Lambda^*} = o\left(s^{-\frac{1}{2}}\right)$$

  - convergence of $f_{+,s}$ and u_s is strong

Setup and ansatz

- infinitely long cylinder
- symmetry assumption: only one important spatial variable r
- confining external magnetic field B^{ext}
- introduce electromagnetic potentials

$$E = -\partial_x \phi - \partial_t A, \quad B = \operatorname{curl}_x A, \quad B^{\mathrm{ext}} = \operatorname{curl}_x A^{\mathrm{ext}}$$

- ansatz

$$f = \eta(\mathcal{E}, \mathcal{F}, \mathcal{G}), \qquad \mathcal{E} = \sqrt{1 + |v|^2} + \phi, \quad \mathcal{F} = r\left(v_\varphi + A_\varphi^{\mathrm{tot}}\right), \quad \mathcal{G} = v_3 + A_3^{\mathrm{tot}}$$

- $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are invariant quantities
- η “suitable” ansatz function

Construction

- semilinear elliptic equations for the potentials

$$-\Delta_x \phi = 4\pi\rho = 4\pi \int_{\mathbb{R}^3} f\, dv, \quad -\Delta_x A = 4\pi j = 4\pi \int_{\mathbb{R}^3} \widehat{v} f\, dv,$$

symmetry: $$-\frac{1}{r}(r\phi')' = 4\pi\rho, \quad -\left(\frac{1}{r}(rA_\varphi)'\right)' = 4\pi j_\varphi, \quad -\frac{1}{r}(rA_3')' = 4\pi j_3$$

- integrate equations; leads to a fixed point problem $(\phi, A_\varphi, A_3) = \mathcal{M}(\phi, A_\varphi, A_3)$
- derive a priori bounds for potentials, *independent* of A^{ext}
- solve equations via fixed point argument or via approximating sequence $(\phi^{k+1}, A_\varphi^{k+1}, A_3^{k+1}) = \mathcal{M}(\phi^k, A_\varphi^k, A_3^k)$
- explicit conditions on A_φ^{ext} ($\rightsquigarrow$ "θ-pinch") or A_3^{ext} ($\rightsquigarrow$ "z-pinch") to ensure confinement of corresponding steady state

Conclusion

- mathematical model of a hot plasma inside a container; external currents that may serve as a control
- existence of global-in-time weak solutions
- redundancy of the divergence part of Maxwell's equations
- existence of minimizers for an optimal control problem
- approach to derive first order optimality conditions
- confined steady states in an infinitely long cylinder

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Thank you for your attention!