Finite volume calculation of K-theory invariants

Hermann Schulz-Baldes, Erlangen

collaborator:

Terry Loring, Alberquerque

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Plan of the talk

- Classical topological invariants and index theorem
- Construction of associated Bott operator (matrix)
- Main result: invariant as signature of Bott operator
- Connection to η-invariant
- Elements of proof based on K-theory
- Implementation of symmetries
- Application to topological insulators
- Even dimensional case

Motivating example: higher winding numbers

$\mathbb{T}^d$ torus of odd dimension d

Given: smooth function $k \in \mathbb{T}^d \mapsto A(k) \in \mathrm{Gl}(N, \mathbb{C})$

Higher winding number (also called odd Chern number):

$$\mathrm{Ch}_d(A) = \frac{(\frac{1}{2}(d-1))!}{d!} \left(\frac{i}{2\pi}\right)^{\frac{d+1}{2}} \int_{\mathbb{T}^d} \mathrm{Tr}\left(\left(A^{-1}\mathbf{d}A\right)^d\right)$$

Faithful irrep $\Gamma_1, \ldots, \Gamma_d$ of complex Clifford $\mathbb{C}_d$ on $\mathbb{C}^N$

(possibly given only after augmenting N)

Selfadjoint Dirac operator on $L^2(\mathbb{T}^d, \mathbb{C}^N)$:

$$D = \sum_{j=1}^{d} \Gamma_j \, \partial_{k_j}$$

Positive spectral (Hardy) projection $\Pi = \chi(D \geq 0)$

Theorem

Viewing A as multiplication operator on $L^2(\mathbb{T}^d, \mathbb{C}^N)$, the operator $\Pi A \Pi + (\mathbf{1} - \Pi)$ is Fredholm and:

$$\mathrm{Ch}_d(A) \;=\; \mathrm{Ind}\big(\Pi A \Pi + (\mathbf{1} - \Pi)\big)$$

Case $d = 1$: Fritz Noether 1921 and Gohberg-Krein 1960

Case $d \geq 3$: probably follows from Atiyah-Singer 1960's and 1970's

Extension to covariant operators with Prodan 2016

Aim: express $\mathrm{Ch}_d(A)$ as signature of a finite dimensional matrix

Also extend to situations where no differential calculus available

This makes invariants numerically calculable

Extension to local operators on lattice

After Fourier transform $\mathcal{F} : L^2(\mathbb{T}^d, \mathbb{C}^N) \to \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$

$$(\mathcal{F}\psi)(x) \;=\; \int_{\mathbb{T}^d} \frac{dk}{(2\pi)^d}\, e^{-ikx}\, \psi(k)$$

Dirac $\widehat{D} = \mathcal{F} D \mathcal{F}^* = \sum_{j=1}^{d} X_j \Gamma_j$ with position operators X_j

$\widehat{A} = \mathcal{F} A \mathcal{F}^*$ convolution operator

Differentiability satisfied if locality condition holds:

$$\|[\widehat{A}, X_j]\| \;\leq\; C \;\; \forall\, j = 1, \ldots, d \qquad \Longleftrightarrow \qquad \|[\widehat{A}, D]\| \;\leq\; C'$$

From now on only local operators on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$, so let's drop hats

Fact: If A invertible local operator, $\Pi A \Pi + \mathbf{1} - \Pi$ is Fredholm

Fact: If A covariant, index is still given by a Chern number

Aim: calculate index as signature of finite matrix

Bott operator

For tuning parameter $\kappa > 0$ and invertible local A:

$$B_\kappa = \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix} = \kappa\, D \otimes \sigma_3 + H$$

where $H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$. Clearly B_κ selfadjoint

D unbounded with discrete spectrum, A viewed as perturbation

A may lead to spectral asymmetry of B_κ, but not for $A = \mathbf{1}$

Measured by signature, already on finite volume approximation!

A_ρ restriction of A (Dirichlet b.c.) to $\mathbb{D}_\rho = \{x \in \mathbb{Z}^d \,:\, |x| \leq \rho\}$

$$B_{\kappa,\rho} = \begin{pmatrix} \kappa D_\rho & A_\rho \\ A_\rho^* & -\kappa D_\rho \end{pmatrix}$$

Main Result

Theorem

Let $g = \|A^{-1}\|^{-1}$ be the invertibility gap. Provided that

$$\|[D,A]\| \;\leq\; \frac{g^3}{18\,\|A\|\,\kappa} \qquad (*)$$

and

$$\frac{2\,g}{\kappa} \;\leq\; \rho \qquad (**)$$

the matrix $B_{\kappa,\rho}$ is invertible and

$$\tfrac{1}{2}\,\mathrm{Sig}(B_{\kappa,\rho}) \;=\; \mathrm{Ind}\big(\Pi A \Pi + (\mathbf{1}-\Pi)\big)$$

How to use: form (*) infer κ, then ρ from (**)

If A unitary, $g = \|A\| = 1$ and $\kappa = (18\|[D,A]\|)^{-1}$ and $\rho = 2/\kappa$

Hence **small** matrix of size $\leq$ 100 sufficient! Great for numerics!

Why it can work:

Proposition

If (*) and (**) hold,

$$B^2_{\kappa,\rho} \;\geq\; \frac{g^2}{2}$$

Proof:

$$B^2_{\kappa,\rho} \;=\; \begin{pmatrix} A^*_\rho A_\rho & 0 \\ 0 & A_\rho A^*_\rho \end{pmatrix} + \kappa^2 \begin{pmatrix} D^2_\rho & 0 \\ 0 & D^2_\rho \end{pmatrix} + \kappa \begin{pmatrix} 0 & [D_\rho, A_\rho] \\ [D_\rho, A_\rho]^* & 0 \end{pmatrix}$$

Last term is a perturbation controlled by (*)

First two terms positive (indeed: close to origin and away from it)

Now $A^*A \geq g^2$, but $(A^*A)_\rho \neq A^*_\rho A_\rho$

This issue can be dealt with by tapering argument:

Proposition (Bratelli-Robinson)

For $f : \mathbb{R} \to \mathbb{R}$ with Fourier transform defined without $\sqrt{2\pi}$,

$$\|[f(D), A]\| \ \le \ \|\widehat{f'}\|_1 \ \|[D, A]\|$$

Lemma

$\exists$ even function $f : \mathbb{R} \to [0, 1]$ with $f(x) = 0$ for $|x| \ge \rho$ and $f(x) = 1$ for $|x| \le \frac{\rho}{2}$ such that $\|\widehat{f'}\|_1 = \frac{8}{\rho}$

With this, $f = f(D) = f(|D|)$ and $\mathbf{1}_\rho = \chi(|D| \le \rho)$:

$$\begin{aligned} A_\rho^* A_\rho \ &= \ \mathbf{1}_\rho A^* \mathbf{1}_\rho A \mathbf{1}_\rho \ \ge \ \mathbf{1}_\rho A^* f^2 A \mathbf{1}_\rho \\ &= \ \mathbf{1}_\rho f A^* A f \mathbf{1}_\rho \ + \ \mathbf{1}_\rho \big([A^*, f] f A + f A^* [f, A]\big) \mathbf{1}_\rho \\ &\ge \ g^2 f^2 \ + \ \mathbf{1}_\rho \big([A^*, f] f A + f A^* [f, A]\big) \mathbf{1}_\rho \end{aligned}$$

So indeed $A_\rho^* A_\rho$ positive close to origin

Then one can conclude... but TEDIOUS $\square$

η-invariant (Atiyah-Patodi-Singer 1977)

Definition

$B = B^*$ invertible operator on $\mathcal{H}$ with compact resolvent. Then

$$\eta(B) = \mathrm{Tr}(B|B|^{-s-1})|_{s=0} = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty dt\; t^{\frac{s-1}{2}}\; \mathrm{Tr}(B\, e^{-tB^2})\Big|_{s=0}$$

provided it exists!

If $\dim(\mathcal{H}) < \infty$, then $\eta(B) = \mathrm{Sig}(B)$

Usually existence of η-invariant for ψ-Diffs difficult issue

Proposition

If (*) holds, B_κ has well-defined η-invariant

Proof. Integral for large t controlled by gap (Proposition above)

For small t appeal to Dyson series (iteration of DuHamel):

$$e^{-tB_\kappa^2} \;=\; e^{-t\Delta} \;+\; t\int_0^1 dr\, e^{-(1-r)t\Delta} R e^{-rtB_\kappa^2}$$

where $B_\kappa^2 = \Delta + R$ with

$$\Delta \;=\; \kappa^2 \begin{pmatrix} D^2 & 0 \\ 0 & D^2 \end{pmatrix} \quad , \quad R \;=\; \begin{pmatrix} AA^* & \kappa[D,A] \\ \kappa[D,A]^* & A^*A \end{pmatrix}$$

Now replacing $B_\kappa = \kappa\, D \otimes \sigma_3 + H$

$$\mathrm{Tr}(B_\kappa e^{-t\Delta}) = \kappa\,\mathrm{Tr}\left(\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} e^{-t\Delta}\right) + \mathrm{Tr}\left(\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} e^{-t\Delta}\right) \;=\; 0$$

Second term has supplementary factor t $\square$

Theorem (follows from Getzler 1993, Carey-Phillips 2004)

Suppose (*) so that B_κ has well-defined η-invariant

For path $\lambda \in [0,1] \mapsto B_\kappa(\lambda) = \kappa D \otimes \sigma_3 + \lambda H$ of selfadjoints

$$2\,\mathrm{SF}\big(\lambda \in [0,1] \mapsto B_\kappa(\lambda)\big) \;=\; \eta(B_\kappa(1)) \;-\; \eta(B_\kappa(0)) \;=\; \eta(B_\kappa)$$

Consequence: As spectral flow homotopy invariant, so is $\eta(B_\kappa)$

Using this, **first proof of Main Result** for dimension $d = 1$:

By homotopy invariance sufficient: $A = S^n$ for $n \in \mathbb{Z}$ and S shift

Then calculate spectrum of $B_\kappa(\lambda)$ explicity using $XS = (X+1)S$:

$$\sigma(B_\kappa(\lambda)) \;=\; \left\{ \frac{\kappa}{2}\left(n \pm \left((n-2k)^2 + \frac{4\lambda^2}{\kappa^2}\right)^{\frac{1}{2}}\right) \;:\; k \in \mathbb{Z} \right\}$$

Now carefully follow eigenvalues to calculate spectral flow $\square$

Preparations for K-theoretic argument for other d

Unitization $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ of C*-algebra $\mathcal{A}$ by

$$(A,t)(B,s) \;=\; (AB + As + Bt, ts) \qquad , \qquad (A,t)^* = (A^*, \overline{t})$$

Natural C*-norm $\|(A,t)\| = \max\{\|A\|, |t|\}$. Unit $\mathbf{1} = (0,1) \in \mathcal{A}^+$

Exact sequence of C*-algebras $0 \to \mathcal{A} \overset{i}{\hookrightarrow} \mathcal{A}^+ \overset{\rho}{\to} \mathbb{C} \to 0$

ρ has inverse $i'(t) = (0,t)$, then $s = i' \circ \rho : \mathcal{A}^+ \to \mathcal{A}^+$ scalar part

$$\mathcal{V}_0(\mathcal{A}) \;=\; \left\{ V \in \cup_{n \geq 1} M_{2n}(\mathcal{A}^+) \;:\; V^* \;=\; V \,,\;\; V^2 \;=\; \mathbf{1} \,,\;\; s(V) \sim_0 E_{2n} \right\}$$

where homotopic to $E_{2n} = E_2^{\oplus^n}$ with $E_2 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Equivalence relation $\sim_0$ on $\mathcal{V}_0(\mathcal{A})$ by homotopy and $V \sim_0 \begin{pmatrix} V & 0 \\ 0 & E_2 \end{pmatrix}$

Then $K_0(\mathcal{A}) = \mathcal{V}_0(\mathcal{A}) / \sim_0$ abelian group via $[V] + [V'] = [\begin{pmatrix} V & 0 \\ 0 & V' \end{pmatrix}]$

Definition of $K_0(\mathcal{A})$ is equivalent standard one via $V = 2P - \mathbf{1}$:

$$K_0(\mathcal{A}) \;=\; \{[P] - [s(P)] \;:\; \text{projections in some } M_n(\mathcal{A}^+)\}$$

For definition of $K_1(\mathcal{A})$ set

$$\mathcal{V}_1(\mathcal{A}) \;=\; \left\{ U \in \cup_{n\geq 1} M_n(\mathcal{A}^+) \;:\; U^{-1} \;=\; U^* \right\}$$

Equivalence relation $\sim_1$ by homotopy and $[U] = [\left(\begin{smallmatrix} U & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)]$

Then $K_1(\mathcal{A}) = \mathcal{V}_1(\mathcal{A})/\sim_1$ with addition $[U] + [U'] = [U \oplus U']$

If $\mathcal{A}$ unital, one can work with $M_n(\mathcal{A})$ instead of $M_n(\mathcal{A}^+)$ in $\mathcal{V}_1(\mathcal{A})$

Example 1: $K_0(\mathbb{C}) = \mathbb{Z}$ with invariant $\dim(P)$

Example 2: $K_1(C(\mathbb{S}^1)) = \mathbb{Z}$ with invariant "winding number"

Index map

Example 3: Calkin's exact sequence over a Hilbert space:

$$0 \;\to\; \mathcal{K} \;\to\; \mathcal{B} \;\overset{\pi}{\to}\; \mathcal{Q} = \mathcal{B}/\mathcal{K} \;\to\; 0$$

For Calkin algebra $K_1(\mathcal{Q}) = \mathbb{Z}$ with invariant $=$ index of Fredholm

Also $K_0(\mathcal{B}) = K_1(\mathcal{B}) = 0$ and $K_0(\mathcal{K}) = \mathbb{Z}$

Isomorphism $K_1(\mathcal{Q}) \cong K_0(\mathcal{K})$ given by index map (Rordam *et. al.*):

Unitary $U = \pi(B) \in \mathcal{V}_1(\mathcal{Q})$, with contraction lift $B \in \mathcal{B}$,

$$\mathrm{Ind}[U]_1 \;=\; \left[\begin{pmatrix} 2BB^* - \mathbf{1} & 2B(\mathbf{1} - B^*B)^{\frac{1}{2}} \\ 2(\mathbf{1} - B^*B)^{\frac{1}{2}} B^* & \mathbf{1} - 2B^*B \end{pmatrix}\right]_0$$

where for r.h.s. $V \in \mathcal{K}^+$: $V^2 = \mathbf{1}$ and $s(V) \sim_0 E_2$ up to compact

Index map versus index of Fredholm operator

B unitary up to compact $\iff$ $\mathbf{1}-B^*B\in\mathcal{K}$ and $\mathbf{1}-BB^*\in\mathcal{K}$

$\implies$ B Fredholm operator and $U=\pi(B)\in\mathcal{Q}$ unitary

Fedosov formula if $\mathbf{1}-B^*B$ and $\mathbf{1}-BB^*$ are traceclass:

$$\begin{aligned}
\mathrm{Ind}(B) &= \dim(\mathrm{Ker}(B)) - \dim(\mathrm{Ker}(B^*)) \\
&= \mathrm{Tr}(\mathbf{1}-B^*B) - \mathrm{Tr}(\mathbf{1}-BB^*) \\
&= \mathrm{Tr}\begin{pmatrix} BB^*-\mathbf{1} & B(\mathbf{1}-B^*B)^{\frac{1}{2}} \\ (\mathbf{1}-B^*B)^{\frac{1}{2}}B^* & \mathbf{1}-B^*B \end{pmatrix} \\
&= \mathrm{Tr}\big(\tfrac{1}{2}(V-\mathbf{1})\big) \\
&= \tfrac{1}{2}\,\mathrm{Sig}(V) \qquad\qquad \text{if } \mathbf{1}-B^*B,\ \mathbf{1}-BB^* \text{ projections} \\
&= \mathrm{Tr}\big(\tfrac{1}{2}(\mathrm{Ind}[U]-\mathbf{1})\big) \\
&= \mathrm{Tr}\big(\mathrm{Ind}^{\sim}[U]\big)
\end{aligned}$$

if $\mathrm{Ind}^{\sim}[U]$ is the projection-valued version of index map

Localizing index map for index pairings

Suppose now $U = \pi\big(\Pi A \Pi + (\mathbf{1} - \Pi)\big) \in \mathcal{Q}$ as in Main Theorem

but first A unitary. Then contraction lift $B = \Pi A \Pi + (\mathbf{1} - \Pi)$

Modify Π and $\mathbf{1} - \Pi$ to $p = p(D)$ smooth and $n = n(D)$ where

$$p(x) = \begin{cases} 0\,, & x \leq -\rho \\ p(x)\,, & |x| \leq \rho \\ 1\,, & x \geq \rho \end{cases} \quad , \quad n(x) = \begin{cases} 1\,, & x \leq -\rho \\ 0\,, & x \geq -\rho \end{cases}$$

Now $p - \Pi$, $n - (\mathbf{1} - \Pi)$ compact, $np = pn = 0$ and $n + p|_{\mathbb{D}_\rho^c} = \mathbf{1}_{\mathbb{D}_\rho^c}$

With notation $A_p = pAp$ acting only on $\ell^2(\mathbb{D}_\rho) \otimes \mathbb{C}^N$:

$$\begin{aligned} \mathrm{Ind}[U] &= \mathrm{Ind}[pAp + n] = \mathrm{Ind}[A_p + n] \\ &= \left[\begin{pmatrix} 2A_pA_p^* - \mathbf{1} & 2A_p(\mathbf{1} - A_p^*A_p)^{\frac{1}{2}} \\ 2(\mathbf{1} - A_p^*A_p)^{\frac{1}{2}} A_p^* & \mathbf{1} - 2A_p^*A_p \end{pmatrix} \oplus \begin{pmatrix} \mathbf{1}_{\mathbb{D}_\rho^c} & 0 \\ 0 & -\mathbf{1}_{\mathbb{D}_\rho^c} \end{pmatrix}\right] \end{aligned}$$

Summand on $\mathbb{D}_\rho^c$ trivial (as equal to E_2). Thus:

$$\mathrm{Ind}[U] \;=\; \left[\begin{pmatrix} 2A_pA_p^* - \mathbf{1} & 2A_p(\mathbf{1} - A_p^*A_p)^{\frac{1}{2}} \\ 2(\mathbf{1} - A_p^*A_p)^{\frac{1}{2}}A_p^* & \mathbf{1} - 2A_p^*A_p \end{pmatrix}\right]$$

Numerical index is signature of this finite-dimensional matrix!

Modify to self-adjoint matrix without spoiling invertibility

$$\begin{aligned} \|A_pA_p^* - p^4\| \;&=\; \|pAp^2A^*p - p^3AA^*p\| \;\leq\; \|[p^2, A]\| \\ &\leq\; \frac{C}{\rho}\|[D, A]\| \;<\; \frac{1}{4} \end{aligned}$$

by the smoothness of p and for ρ sufficiently large. Similarly

$$\|A_p(\mathbf{1} - A_p^*A_p)^{\frac{1}{2}} - (\mathbf{1} - p^4)^{\frac{1}{4}}pAp(\mathbf{1} - p^4)^{\frac{1}{4}}\| \;\leq\; \frac{C}{\rho}\|[D, A]\| \;<\; \frac{1}{4}$$

Thus just replace matrix entries without changing signature!

Proposition

If (*) and (**) hold,

$$\mathrm{Ind}\big(\Pi A \Pi + (\mathbf{1} - \Pi)\big) = \mathrm{Sig}\begin{pmatrix} 2p^4 - \mathbf{1} & 2(\mathbf{1} - p^4)^{\frac{1}{4}} pAp (\mathbf{1} - p^4)^{\frac{1}{4}} \\ 2(\mathbf{1} - p^4)^{\frac{1}{4}} pA^* p (\mathbf{1} - p^4)^{\frac{1}{4}} & \mathbf{1} - 2p^4 \end{pmatrix}$$

Last tasks:

1) replace $2p^4 - \mathbf{1}$ by κD_ρ

2) replace $\sqrt{2}(\mathbf{1} - p^4)^{\frac{1}{4}} p$ by $\mathbf{1}_\rho$ indicator on $\mathbb{D}_\rho$. Then $\mathbf{1}_\rho A \mathbf{1}_\rho = A_\rho$

Both follows again by a tapering argument **UUuuuffff**

Implementation of real symmetries

Fix a real structure on complex Hilbert space, denoted by overline

There is irrep $\Gamma_1, \ldots, \Gamma_d$ and real unitary matrix Σ

$d \bmod 8$	1	3	5	7
$\Sigma^* \overline{D} \Sigma =$	D	$-D$	D	$-D$
$\Sigma^2 =$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$
$\Sigma^* \overline{\Pi} \Sigma =$	Π	$\mathbf{1} - \Pi$	Π	$\mathbf{1} - \Pi$

For $d = 3$: $D = X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3$ and $\Sigma = i\sigma_2$

Furthermore given real unitary S with $[S, \Sigma] = [S, D] = 0$:

$j \bmod 8$	2	4	6	8
$S^* \overline{A} S =$	A^*	A	A^*	A
$S^2 =$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$

Symmetries of $T = \Pi A \Pi + (\mathbf{1} - \Pi)$ such that index pairings are:

$\mathrm{Ind}_{(2)}(T)$	$j = 2$	$j = 4$	$j = 6$	$j = 8$
$d = 1$	0	$2\,\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$
$d = 3$	$2\,\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$d = 5$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$2\,\mathbb{Z}$
$d = 7$	$\mathbb{Z}$	0	$2\,\mathbb{Z}$	$\mathbb{Z}_2$

where $\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2 \in \mathbb{Z}_2$

For Bott operator follows $R^* \, \overline{B_\kappa} \, R = s \, B_\kappa$ and $R^2 = s' \mathbf{1}$ with

$s = \;,\; s' =$	$j = 2$	$j = 4$	$j = 6$	$j = 8$
$d = 1$	$-1 \,,\, -1$	$1 \,,\, -1$	$-1 \,,\, 1$	$1 \,,\, 1$
$d = 3$	$1 \,,\, -1$	$-1 \,,\, 1$	$1 \,,\, 1$	$-1 \,,\, -1$
$d = 5$	$-1 \,,\, 1$	$1 \,,\, 1$	$-1 \,,\, -1$	$1 \,,\, -1$
$d = 7$	$1 \,,\, 1$	$-1 \,,\, -1$	$1 \,,\, -1$	$-1 \,,\, 1$

Same pattern!

Thus Ind and Ind_2 can be calculated from Bott operator using:

Proposition

$B = B^*$ invertible complex matrix. $R = \overline{R}$ real unitary such

$$R^* \, \overline{B} \, R \;=\; s \, B \,, \qquad R^2 \;=\; s' \, \mathbf{1}$$

(i) *If $s = 1$ and $s' = 1$, then $\mathrm{Sig}(B) \in \mathbb{Z}$ arbitrary*

(ii) *If $s = 1$ and $s' = -1$, then $\mathrm{Sig}(B) \in 2\,\mathbb{Z}$ arbitrary*

(iii) *If $s = -1$ and $s' = 1$, then $\mathrm{Sig}(B) = 0$, but setting $M = R^{\frac{1}{2}}$ one obtains real antisymmetric matrix $iMBM^*$ with invariant $\mathrm{sgn}(\mathrm{Pf}(iMBM^*)) \in \mathbb{Z}_2$*

(iv) *If $s = -1$ and $s' = -1$, then $\mathrm{Sig}(B) = 0$*

Application to topological insulators

$$B_\kappa = \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix} = \kappa\, D \otimes \sigma_3 + H \quad , \quad H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

Data: $H = -J^* H J$ chiral quantum Hamiltonian where $J = \left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{smallmatrix}\right)$

Invertibility of H (and hence A) means: H describes insulator

Non-trivial higher winding numbers make it a topological insulator

Main Theorem allows to efficiently calculate this topology

As calculation local, one can determine quantum phase transitions

Implementation of physical symmetries on H (like TRS and PHS)

lead to symmetries of $A \implies \mathbb{Z}_2$ invariants calculable

Now: not every H is chiral & dimension not always even...

Even dimensional pairings

Consider projection P on $\ell^2(\mathbb{Z}^d, \mathbb{C}^{2N})$ with d even

Even-dimensional Dirac operator has grading $\Gamma_{d+1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Dirac phase F is unitary operator in $D|D|^{-1} = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}$

Fredholm operator $PFP + (\mathbf{1} - P)$ has index equal to $\mathrm{Ch}_d(P)$

Associated Bott operator

$$B_\kappa \;=\; \kappa\, D \;+\; (2P - \mathbf{1})\Gamma_{d+1}$$

Theorem

Suppose $\|[P, D]\| < \infty$ and that κ is sufficiently small

For ρ sufficiently large,

$$\mathrm{Ind}\big(PFP + (\mathbf{1} - P)\big) \;=\; \mathrm{Sig}(B_{\kappa,\rho})$$

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