An index theorem for Toeplitz operators with non-commutative quasicontinuous symbols and applications in solid state physics

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Contents

Abstract

This Master's thesis investigates new exact sequences of C^*-algebras for use in the bulk-boundary correspondence of solid state systems. The approach constructs half-space algebras using crossed products with dual actions and thereby generalizes the smoothed Toeplitz extension. By using weakly continuous actions and semifinite von Neumann algebras, this allows to extend bulk-boundary correspondence to elements with weaker regularity properties than before, which we call quasicontinuous in analogy to the classical case of Toeplitz operators on the torus.

We extend an index theorem by Lesch to Toeplitz operators with quasicontinuous symbols, which allows the computation of Breuer-Fredholm indices through the non-commutative winding number form. After generalizing the classical Besov spaces to the non-commutative case, we prove an analogue of Peller's trace class criterion for Hankel operators in our setting.

Using this constructive criterion we can then apply the formalism to the bulk-boundary correspondence of quasicontinuous elements over the disordered non-commutative torus. We use the index theorem to show that a chirally symmetric Hamilton operator with non-trivial winding numbers exhibits boundary states when restricted to a half-space. This extends known results for Hamilton operators with spectral gaps to the weaker assumption of a mobility- or pseudogap.

1 Introduction

The use of non-commutative geometry and operator K-theory has recently allowed much progress in the classification of topological quantum systems. This success is based on the identification of physically measurable topological invariants as index pairings over certain observable algebras, which shows a stability under perturbations that would be difficult to infer from the explicit formulas alone [37] [35]. Another strong point is a form of algebraic bulk-boundary correspondence, that allows to relate the topological invariants of a system with a boundary to those of an infinite, translationally invariant "bulk" system [36]. The idea is to link projections in the bulk, such as the Fermi-projection of a Hamilton operator, to unitary elements in some boundary algebra and vice versa using the connecting maps of K-theory. As the numerical topological invariants are given by pairings of cyclic cohomology and the K-groups of certain observable algebras, they posses stable homotopy invariance, i.e. to show numerical equalities one may deform projections and unitaries to find pairs for which one or both sides of an equation can be evaluated more easily.

The bulk-boundary correspondence takes place in an exact sequence of C^*-algebras

$$0 \to \mathcal{E}_d \to \hat{\mathcal{A}}_d \to \mathcal{A}_d \to 0,$$

where $\mathcal{A}_d$ describes operators in the bulk of a d-dimensional material and $\mathcal{E}_d$ operators supported on the boundary. The elements of the half-space-algebra $\hat{\mathcal{A}}_d$ describe observables on a semi-infinite system and can be split linearly into restrictions of bulk operators and a boundary term

$$\hat{a} = PaP^* + e \tag{1.1}$$

with P the projection to a half-space in some Hilbert space representation and $e \in \mathcal{E}_d$. Typically, the bulk algebra for a spatially homogeneous system is modeled as a crossed product

$$\mathcal{A}_d = \mathcal{A}_{d-1} \rtimes G$$

with $G = \mathbb{Z}$ for system with discrete degrees of freedom (i.e. on a lattice) and $G = \mathbb{R}$ for continuous systems [36] [23]. The corresponding boundary algebra is then given by some kind of stabilization such as $\mathcal{E}_d \simeq \mathcal{A}_{d-1} \otimes \mathbb{K}$ to describe operators of lower effective dimension.

In this work we pursue a more general approach, where we relate the exact sequences to Toeplitz operators on flows. The idea is to consider a flow $\alpha : \mathbb{R} \times \mathcal{A} \to \mathcal{A}$ on the bulk algebra, that is in some representation generated by a self-adjoint operator D chosen such that the projection to the half-space is given by

$$P := \chi_{\mathbb{R}^+}(D).$$

This approach to half-spaces in the commutative case was apparently pioneered by Coburn and Douglas [14]. Morally, the half-space algebra is then the C^*-algebra generated by the Toeplitz operators

$$\hat{\mathcal{A}} := C^*(P\mathcal{A}P),$$

but there are some subtleties involved, especially concerning whether $\hat{\mathcal{A}}$ admits a splitting of the form (1.1) with some boundary algebra $\mathcal{E}$. If α is a strongly continuous automorphic action, it is more natural to consider a smoothed version of this construction, which leads to an exact sequence for Toeplitz operators on the flow, that is also viable for bulk-boundary correspondence.

After recalling the necessary preliminaries, we review this so-called (smoothed) Toeplitz extension for a one-parameter flow, which was introduced by [25] [21]. These extensions give exact sequences with boundary algebras modeled as crossed products of the form $\mathcal{A} \rtimes \mathbb{R}$ or $\mathcal{A} \rtimes \mathbb{T}$. We investigate the relations of those extensions with the discrete Toeplitz and the Wiener-Hopf-extensions, which are more commonly used for bulk-boundary correspondence.

For some applications strong continuity of α is too restrictive. Assume that we have a bulk-boundary exact sequence based on such a strongly continuous action

$$0 \to \mathcal{E} \to \hat{\mathcal{A}} \to \mathcal{A} \to 0.$$

We are interested in classifying projections and unitary operators derived from Hamilton operators $h, \hat{h}$ in $\mathcal{A}$ and $\hat{\mathcal{A}}$, respectively. An example is the Fermi projection $p := \chi_{\mathbb{R}^+}(h)$ that encodes the even topological invariants of the system, but it is usually not in $\mathcal{A}$ unless h has a spectral gap around 0. Moreover, the orbit of p under α then can also fail to be norm-continuous such that one has to weaken the continuity requirements if one wants to choose a larger bulk algebra that contains p. For this reason we consider the von Neumann algebras generated by $\mathcal{A}$ and $\mathcal{E}$

$$L^\infty(\mathcal{A}) = \mathcal{A}'', \quad L^\infty(\mathcal{E}) = \mathcal{E}'' = L^\infty(\mathcal{A}) \rtimes_\alpha \mathbb{R}$$

with $\mathcal{E}, \hat{\mathcal{A}} \subset L^\infty(\mathcal{E})$. We then construct new exact sequences based on the now weakly continuous action α. Given a finite trace τ on $L^\infty(\mathcal{A})$ there is a canonical semifinite dual trace $\hat{\tau}$ on $L^\infty(\mathcal{E})$. The boundary observables are then conveniently given by the ideal $\mathcal{K}_{\hat{\tau}}$ of the $\hat{\tau}$-compact operators, which vanish at infinity in some sense.

If we have a subalgebra $B \subset L^\infty(\mathcal{A})$ such that

$$[P, b] \in \mathcal{K}_{\hat{\tau}}, \; \forall b \in B,$$

then the algebra generated by the Toeplitz operators $\mathscr{T}(B) = C^*(PBP)$ forms an exact sequence with the compact operators

$$0 \longrightarrow \mathcal{K}_{\hat{\tau}} \longrightarrow \mathscr{T}(B) + \mathcal{K}_{\hat{\tau}} \longrightarrow B \longrightarrow 0,$$

where the middle term splits as a linear space into $B \oplus \mathcal{K}_{\hat{\tau}}$. An element $a \in L^\infty(\mathcal{A})$ that satisfies the condition $[P, a] \in \mathcal{K}_{\hat{\tau}}$ will be called quasicontinuous in analogy to a characterization for the Toeplitz operators on the torus with discontinuous symbols [7].

This construction automatically links the exact sequences with the theory of Breuer-Fredholm indices [11], i.e. the bulk-boundary exact sequence can be embedded into the Calkin-extension

$$0 \longrightarrow \mathcal{K}_{\hat{\tau}} \longrightarrow L^\infty(\mathcal{E}) \longrightarrow L^\infty(\mathcal{E})/\mathcal{K}_{\hat{\tau}} \longrightarrow 0$$

and the connecting maps of K-theory relate the Breuer index of a Toeplitz operator

$$T_a = PaP \in \mathscr{T}(B)$$

with topological properties of its symbol $a \in B$. For the smoothed Toeplitz extension there is an index theorem by Lesch [25], which we extend to our more general setting. To be precise, we show that for certain Toeplitz operators the index can be computed in terms of their symbol as for the classical Toeplitz operators, i.e. by the non-commutative generalization of the winding number. This will then be interpreted in terms of K-theory and cyclic cohomology such that we can take advantage of the stable homotopy invariance.

For the construction of exact sequences we have to require that elements of the bulk algebra have τ-compact commutators with the half-space restriction map P and for the numerical bulk-boundary correspondence we further need them to be trace class with respect to $\hat{\tau}$. The analogous problem in the commutative theory is the question, when a Hankel operator with symbol in $L^\infty(\mathbb{T})$ is contained in the Schatten-von Neumann class $\mathcal{S}^p$. It was completely solved by Peller [31] in terms of the Besov spaces on the torus. Those Besov spaces are classical function spaces related to the fractional Sobolev spaces and are defined in terms of dyadic decompositions with certain smooth Fourier multipliers [43]. We provide a novel definition of Besov spaces for flows on a non-commutative von Neumann algebra and show that Peller's criterion for the commutator $[P, a]$, $a \in L^\infty(\mathcal{A})$, to be trace class also holds in the non-commutative case.

In the final part, we apply the formalism to concrete models in solid state physics. To be precise, we consider Hamilton operators with a chiral symmetry, i.e. they can be written in the form

$$h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix},$$

with the off-diagonal term $a \in \mathcal{A}$. We assume that h has no eigenvalue at 0 or even has a spectral gap around 0 such that one can assign to the Hamiltonian a unitary phase $u = \frac{a}{|a|}$, that encodes the chiral topological invariants of the system under consideration. The restriction to a half-space

$$\hat{h} = PhP$$

can still have a non-trivial kernel and the chiral symmetry implies that the projection onto the kernel must be diagonal

$$P_{\operatorname{Ker}\hat{h}} = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}$$

with a pair of projections $p_+, p_- \in \mathcal{K}_{\hat{\tau}}$.

We show that (under certain regularity conditions imposed on h and u) the K-theoretical index map links the bulk and boundary invariants via

$$\operatorname{Ind}[u]_1 = [p_+]_0 - [p_-]_0 \tag{1.2}$$

with the right hand side in $K_0(B)$ for some quasicontinuous bulk algebra B and

$$\hat{\tau}(p_+) - \hat{\tau}(p_-) = \operatorname{Wind}(u) \tag{1.3}$$

i.e. the left hand side, which is related to the density of the boundary states, can be computed in terms of some non-commutative winding number of the bulk unitary u. This is well-known if h has a spectral gap [36] and our analysis allows to extend the result to situations, where h has no spectral gap, but satisfies certain regularity assumptions.

The main problem in this setting is that if a is not invertible the phase u can only be computed by measurable functional calculus and is therefore not necessarily included in the bulk algebra. As an example, for periodic operators on a lattice a common choice would be $\mathcal{A} = \mathbb{C} \rtimes \mathbb{Z}^d = C(\mathbb{T}^d)$ such that $u = \frac{a}{|a|}$ is contained in $\mathcal{A}$ if and only if a is invertible, i.e. h has a spectral gap around 0. However, even without a gap, u will often still be quasicontinuous and using the formalism developed in this work, one may find a larger bulk algebra $B \subset L^\infty(\mathcal{A})$ and construct a bulk-boundary exact sequence for which (1.2) and (1.3) hold. We map out sufficient conditions for this in the cases, where h has a mobility gap (i.e. the states around zero are exponentially localized) or is a periodic operator with a pseudogap (i.e. the Fourier transform of a has only finitely many zeroes).

2 K-theory and Cyclic cohomology

K-theory

This section contains standard material assembled for the convenience of the reader, mostly drawing on [5] and [40].

Definition 2.1. *A local C^*-algebra A is a pre-C^*-algebra that is closed under the holomorphic functional calculus of its closure, i.e. for $a \in A$ and f a function holomorphic on a neighborhood of $\sigma(a)$ with $f(0) = 0$, we have $f(a) \in A$.*

If A is a local C^*-algebra with completion $\bar{A}$, the matrix algebra $M_n(A)$ with the natural C^*-norm is also a local C^*-algebra with completion $M_n(\bar{A})$.

For a local C^*- algebra (whether unital or not) we denote by $A^+ = A \oplus \mathbb{C}$ the unitization with the multiplication

$$(a + \lambda 1^+)(b + \mu 1^+) = ab + \mu a + \lambda b + \lambda\mu 1^+$$

and the usual C^*-norm.

A projection $p \in A$ is an element with $p^* = p = p^2$. We write $\mathcal{P}(A)$ for the set of projections of A and set

$$\mathcal{P}_n(A) := \mathcal{P}(M_n(A)).$$

The natural inclusion $M_n(A) \hookrightarrow M_{n+m}(A), a \mapsto a \oplus 0_m$ also embeds $\mathcal{P}_n(A)$ into $\mathcal{P}_{n+m}(A)$. In this sense we consider the union

$$\mathcal{P}_\infty(A) := \bigcup_{n \in \mathbb{N}} \mathcal{P}_n(A)$$

where we identify elements that differ only by dimension of the fiber, i.e. $p = p \oplus 0_n$. The set $\mathcal{P}_\infty(A)$ is a semigroup with the addition

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

for $p, q \in \mathcal{P}_\infty(A)$ and we write

$$p \sim_0 q$$

if one of the following equivalent conditions is fulfilled for some $n \in \mathbb{N}$:

1. $p = uu^*$ and $q = u^*u$ for some partial isometry $u \in M_n(A^+)$.
2. $p = uqu^*$ for some unitary element $u \in M_n(A^+)$.
3. There is a continuous path in $M_n(A)$ connecting p and q.

These conditions are equivalent, because we may consider p as a matrix of arbitrary dimension.

The equivalence classes $V(A) := \mathcal{P}_\infty(A)/\sim_0$ form an abelian semigroup, since $p \oplus q$ and $q \oplus p$ are homotopic. The Grothendieck group associated to $V(A)$ is denoted $K_{00}(A)$ and its elements are written as formal differences

$$[p]_0 - [q]_0$$

with some $p, q \in V(A)$ subject to the equivalence relation

$$\begin{aligned} &[p_1]_0 - [q_1]_0 = [p_2]_0 - [q_2]_0 \\ \iff &\exists r \in V(A) : [p_1]_0 \oplus [q_2]_0 \oplus [r]_0 = [p_2]_0 \oplus [q_2]_0 \oplus [r]_0. \end{aligned}$$

Given a homomorphism $\phi : A \to B$ of local C^*-algebras, we have an induced morphism

$$\phi_* : K_{00}(A) \to K_{00}(B), \quad \phi_*([p]_0 - [q]_0) := [\phi(p)]_0 - [\phi(q)]_0,$$

where the extension of ϕ to $M_n(A^+)$ is denoted by the same symbol.

The projection to the scalar part $s : A^+ \to \mathbb{C}, \quad s(a + \lambda 1^+) = \lambda$ induces a surjective homomorphism $s_* : K_{00}(A^+) \to K_{00}(\mathbb{C})$ and we define the K_0-group by

$$K_0(A) = \ker(s_*) \subset K_{00}(A^+).$$

If A is already unital, we can identify $K_0(A) \simeq K_{00}(A)$ as then $K_{00}(A^+) = K_{00}(A) \oplus K_{00}(\mathbb{C})$ and s_* strips away the second factor.

Any element of $K_0(A)$ can be written in the form

$$[e]_0 - [s(e)]_0$$

with some $e \in \mathcal{P}_\infty(A^+)$ and the group operation in this standard picture is given by

$$([e_1]_0 - [s(e_1)]_0) \oplus ([e_2]_0 - [s(e_2)]_0) = [e_1 \oplus e_2]_0 - [s(e_1 \oplus e_2)]_0.$$

The group $K_0(A)$ classifies the stable equivalency classes of projections over A; in particular we have

$$K_0(A) \simeq K_0(M_n(A)) \simeq K_0(\mathbb{K} \otimes A)$$

with $\mathbb{K}$ the algebra of compact operators on $\ell^2(\mathbb{N})$.

For A *unital* define the unitary $n \times n$-matrices

$$\mathcal{U}_n(A) := \{u \in M_n(A) : uu^* = 1_n = u^*u\}$$

and denote the connected component of the identity by $\mathcal{U}_n(A)_0$. There is the natural embedding $i_{n,n+m} : \mathcal{U}_n(A) \hookrightarrow \mathcal{U}_{n+m}(A)$, $u \mapsto u \oplus 1_m$ that also maps $\mathcal{U}_n(A)_0$ into $\mathcal{U}_{n+m}(A)_0$. Define

$$\mathcal{U}_\infty(A) := \bigcup_{n \in \mathbb{N}} \mathcal{U}_n(A)$$

where we identify elements u, v with $i_{n,k}(u) = v$ (more properly, we consider the inductive limit of $(\mathcal{U}_n(A))_{n \in \mathbb{N}}$ with the morphisms $(i_{n,k})_{\substack{n,k \in \mathbb{N} \\ n<k}}$).

For $u, v \in \mathcal{U}_n(A)$ write $u \sim_1 v$ if there is some $k \in \mathbb{N}$ such that there is a continuous path in $U_{n+k}(A)$ connecting $i_{n,n+k}(u)$ and $i_{n,n+k}(v)$. This equivalence relation extends to $\mathcal{U}_\infty(A)$ with the equivalence class of the identity given by

$$[1]_1 = \bigcap_{n \in \mathbb{N}} \mathcal{U}_n(A)_0.$$

For a local C^*-algebra A, whether unital or not, define

$$K_1(A) := U_\infty(A^+)/\sim_1$$

The elements of $K_1(A)$ can be written as equivalence classes $[u]_1, u \in \mathcal{U}_n(A^+)$ with $u \sim_1 v$ and form an abelian group with the multiplication defined through representatives $u, v \in \mathcal{U}_n(A^+)$ by

$$[u]_1 \oplus [v]_1 = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]_1 = \left[\begin{pmatrix} uv & 0 \\ 0 & 1_n^+ \end{pmatrix}\right]_1 = [uv]_1 = [vu]_1.$$

If A is already unital, we have an isomorphism

$$K_1(A) \simeq U_\infty(A)/\sim_1$$

induced by replacing 1^+ with 1 component-wise, i.e.

$$\tilde{u} \in \mathcal{U}_n(A^+) \mapsto \tilde{u} + s(\tilde{u})(1 - 1^+) \in \mathcal{U}_n(A),$$

such that we may instead consider matrices with coefficients in A.

Any homomorphism of local C^*-algebras $\phi : A \to B$ induces a group homomorphism

$$\phi_* : K_1(A) \to K_1(B), \quad \phi_*([u]_1) = [\phi(u)]_1.$$

The group $K_1(A)$ classifies stable homotopy classes of unitary elements over A and ,as for K_0, we have

$$K_1(A) = K_1(M_n(A)) = K_1(\mathbb{K} \otimes A).$$

The most important property of the K-groups is the six-term exact sequence. Let A be a local C^*-algebra and J a closed two-sided ideal in A, i.e. there is the short exact sequence

$$0 \longrightarrow J \xrightarrow{\ i\ } A \xrightarrow{\ \pi\ } A/J \longrightarrow 0.$$

To this sequence corresponds the long exact sequence of the K-groups

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{\ i_*\ } & K_0(A) & \xrightarrow{\ \pi_*\ } & K_0(A/J) \\ {\scriptstyle \mathrm{Ind}}\uparrow & & & & \downarrow{\scriptstyle \mathrm{Exp}} \\ K_1(A/J) & \xleftarrow[\ \pi_*\ ]{} & K_1(A) & \xleftarrow[\ i_*\ ]{} & K_1(J) \end{array}$$

with the connecting morphisms Ind and Exp defined as follows:

For any $[u]_1 \in K_1(A/J), u \in \mathcal{U}_n((A/J)^+)$ we can find some $v \in \mathcal{U}_{2n}(A^+)$ with

$$\pi(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in \mathcal{U}_{2n}((A/J)^+)$$

and set

$$\mathrm{Ind}([u]_1) := \left[v \begin{pmatrix} 1_n^+ & 0 \\ 0 & 0_n \end{pmatrix} v^* \right]_0 - \left[\begin{pmatrix} 1_n^+ & 0 \\ 0 & 0_n \end{pmatrix} \right]_0 \in K_1(J^+).$$

For any $[p]_0 - [s(p)]_0 \in K_0(A/J)$ with $p \in \mathcal{P}_n((A/J)^+)$ we can find a self-adjoint lift $\hat{p} \in M_n(A^+)$ and set

$$\mathrm{Exp}([p]_0 - [s(p)]_0) := [e^{-i2\pi\hat{p}}]_1 \in K_1(J)$$

with functional calculus in $M_n(A^+)$. It is a standard result that these maps do not depend on the representatives and the lifts used.

The index and exponential maps are natural in the following sense [40]:

Proposition 2.1. *If we have a commutative diagram of local C^*-algebras*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & A/J & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & J' & \longrightarrow & A' & \longrightarrow & A'/J' & \longrightarrow & 0 \end{array}$$

with exact rows, then the diagrams

$$\begin{array}{ccc} K_0(A/J) & \xrightarrow{\ \mathrm{Exp}\ } & K_1(J) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} \\ K_0(A'/J') & \xrightarrow{\ \mathrm{Exp}'\ } & K_1(J') \end{array}$$

and

$$\begin{array}{ccc} K_1(A/J) & \xrightarrow{\text{Ind}} & K_0(J) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} \\ K_1(A'/J') & \xrightarrow{\text{Ind}'} & K_0(J') \end{array}$$

are also commutative.

If A and A/J are unital, we adopt the following convention to simplify the connecting maps:

We identify $K_0(A) \simeq K_{00}(A)$ and $K_1(A) \simeq \mathcal{U}_\infty(A)/\sim_1$ and likewise for A/J such that we can consider matrices with entries in A respectively A/J. It is then convenient to also identify 1^+ from the unitization J^+ with the unit 1 of A so that we can formally eliminate the unitization from all expressions.

An important special case of the index map is the following

Proposition 2.2. *Let*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0$$

be an exact sequence of C^*-algebras with A and A/J unital. If the unitary $u \in \mathcal{U}_n(A/J)$ lifts to a partial isometry $\hat{u} \in M_n(A)$, $\pi(\hat{u}) = u$ then

$$\text{Ind}([u]_1) = [1_n - \hat{u}^*\hat{u}]_0 - [1_n - \hat{u}\hat{u}^*]_0.$$

Proof. Under the identifications above, the corresponding lift of $u \oplus u^*$ is given by

$$v = \begin{pmatrix} \hat{u} & 1_n - \hat{u}\hat{u}^* \\ 1_n - \hat{u}^*\hat{u} & \hat{u}^* \end{pmatrix}$$

such that

$$\text{Ind}([u]_1) = \left[v \begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix} v^* \right]_0 - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix} \right]_0 = \left[\begin{pmatrix} \hat{u}\hat{u}^* & 0 \\ 0 & 1_n - \hat{u}^*\hat{u} \end{pmatrix} \right] - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix} \right]_0$$

and this is equivalent to the image written above. □

C^*-algebras often don't contain many partial isometries but if a unitary element lifts to a contraction we can at least provide an explicit formula for the index map [40, Proposition 9.2.2]:

Proposition 2.3. *Let*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0$$

be an exact sequence of C^*-algebras with A and A/J unital. If the unitary $u \in \mathcal{U}_n(A/J)$ lifts to a contraction $a \in M_n(A)$, i.e. $\|a\| \leq 1$ and $\pi(a) = u$, we have

$$\text{Ind}([u]_1) = \left[v \begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix} v^* \right]_0 - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix} \right]_0.$$

with the unitary

$$v = \begin{pmatrix} a & \sqrt{1_n - aa^*} \\ -\sqrt{1_n - a^*a} & a^* \end{pmatrix} \in \mathcal{U}_{2n}(A)$$

Cyclic Cohomology

We recall some facts about cyclic cocycles and their pairings with K-theory, condensed from [16] and [17].

Definition 2.2. *A cyclic n-cocycle on an algebra A is an* $(n+1)$*-linear functional* $\phi : A^{n+1} \to \mathbb{C}$ *such that for all* $a_0, ..a_{n+1} \in A$ *one has*

1.
$$\phi(a_0, a_1, ..., a_n) = (-1)^n \phi(a_1, ..., a_n, a_0)$$

2.
$$b\phi(a_0, ..., a_{n+1}) := \sum_{j=0}^{n} (-1)^j \phi(a_0, ..., a_j a_{j+1}, .., a_{n+1}) + (-1)^{n+1} \phi(a_{n+1} a_0, ..., a_n) = 0$$

Any cyclic n-cocycle can be written as a trace on the universal differential algebra $\Omega^n(A) \simeq A^+ \otimes A^{\otimes n}$, which allows a canonical extension to a cyclic n-cocycle $\phi \# \mathrm{Tr}_k$ over $M_k(A^+)$ such that

$$(\phi \# \mathrm{Tr}_k)(a_0, a_1, ..., a_n) = (\phi \# \mathrm{Tr}_k)(a_0 - s(a_0), a_1 - s(a_1), ..., a_n - s(a_n))$$

and

$$(\phi \# \mathrm{Tr}_{m+k})(a_0 \oplus b_0, ..., a_n \oplus b_n) = (\phi \# \mathrm{Tr}_m)(a_0, a_1, ..., a_n) + (\phi \# \mathrm{Tr}_k)(b_0, b_1, ..., b_n).$$

Let A be a local C^*-algebra. In this work the most important property of cyclic cocycles is that they naturally pair with K-groups:

Theorem 2.1. *For* ϕ *a cyclic* $2n$*-cocycle over A there is the group homomorphism*

$$\langle \phi, \cdot \rangle : K_0(A) \to \mathbb{C}$$

defined by

$$\langle \phi, [e]_0 - [s(e)]_0 \rangle = (\phi \# \mathrm{Tr}_k)(e, e, ..., e) \tag{2.1}$$

for $[e]_0 - [s(e)]_0 \in K_0(A)$ *and* $e \in \mathcal{P}_k(A^+)$.

For ϕ *a cyclic* $(2n+1)$*-cocycle over A there is the group homomorphism*

$$\langle \phi, \cdot \rangle : K_1(A) \to \mathbb{C}$$

defined by

$$\langle \phi, [u]_1 \rangle = (\phi \# \mathrm{Tr}_k)(u, u^*, u, ..., u, u^*) \tag{2.2}$$

for $[u]_1 \in K_1(A)$ *and* $u \in \mathcal{U}_k(A^+)$.

Since this implies that the right hand sides of the pairings do not depend on the representatives, this gives us a convenient way to show numerical identities that would be difficult to prove by direct computation.

The cocycles usually involve unbounded traces or derivatives such that they will usually only be defined on some smooth subalgebra $\mathcal{A}$ that is dense with respect to the C^*-norm.

Proposition 2.4. *[15] If $\mathcal{A}$ is a local C^*-algebra and A its norm-closure, the maps $i_* : K_j(\mathcal{A}) \to K_j(A)$ induced by the inclusion $i : \mathcal{A} \to A$ are isomorphisms. Any element of $K_0(A)$ can be represented by a projection matrix over $\mathcal{A}^+$ and every element of $K_1(A)$ by a unitary matrix over $\mathcal{A}^+$.*

In this way the pairing with cyclic cocycles over $\mathcal{A}$ extends to pairings with $K_j(A)$.

3 Crossed products of C^*- and von Neumann-Algebras

C^*-dynamical systems

As alluded to in the introduction, many algebras in the applications can be written as crossed products with $\mathbb{Z}^d$ or $\mathbb{R}^d$ as a consequence of a discrete or continuous translational symmetry. Those crossed products have a dual action that can be used to define a non-commutative differential calculus. Furthermore, we will later construct boundary algebras as a crossed product with a one-parameter subgroup of this dual group.

Definition 3.1. *A C^*-dynamical system is a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G and a strongly continuous action $\alpha : G \to Aut(A)$, i.e. $t \mapsto \alpha_t(a)$ is norm-continuous for each $a \in A$.*

A covariant representation of a C^*-dynamical system is a pair (π, U) with π a non-degenerate representation of A on a Hilbert space H and U a strongly continuous unitary representation of G on H such that

$$\pi(\alpha_g(a)) = U(g)\pi(a)U(g)^*, \quad \forall a \in A, g \in G.$$

For simplicity we will only consider abelian groups and therefore write the group operation as addition. A dynamical system determines a unique crossed product algebra that contains A and G as multipliers and for which any covariant representation extends to a $*$-representation.

A convenient definition for crossed products is given by [38]

Definition 3.2. *A crossed product for the dynamical system (A, G, α) is a C^*-algebra B with a homomorphism $i_A : A \to M(B)$ and a strictly continuous homomorphism $i_G : G \to \mathcal{U}(M(B))$ such that*

1. $i_A(\alpha_s(a)) = i_G(s)i_A(a)i_G(s)^*, \quad \forall a \in A, g \in G$

2. *If (π, U) is a covariant representation, there is a unique non-degenerate representation $\pi \times U$ with $\pi = (\pi \times U) \circ i_A$ and $U = (\pi \times U) \circ i_G$.*

3. *i_G extends to a representation of $C_c(G)$ via Riemann-integration and products of the form $i_A(a)i_G(f), a \in A, f \in C_c(G)$ are dense in B.*

These universal properties determine an algebra $A \rtimes_\alpha G$ uniquely up to isomorphism (as one can always map the generators of one crossed product to those of another). In particular i_A and i_G must be injective due to non-degeneracy.

For functions $f, g \in C_c(G,A)$ define the operations

$$\begin{aligned} (f * g)(s) &= \int_G f(t)\alpha_s(g(t-s))\mathrm{d}t \\ (f^*)(s) &= \alpha_s(f(-s)^*), \end{aligned} \tag{3.1}$$

where we consider integration with respect to a Haar measure of G. Properties 1 and 3 imply that $C_c(G,A)$ with this multiplication and involution is a dense subalgebra of $A \rtimes_\alpha G$.

Any covariant representation (π, U) on some Hilbert space H can be integrated to a representation $\pi \times U$ of the crossed product on H by defining densely for $f \in C_c(G,A)$:

$$(\pi \times U)(f) := \int_G \pi(f(s))U(s)\mathrm{d}s \tag{3.2}$$

Assuming $A \subset \mathcal{B}(H)$ for some Hilbert space H we define a covariant representation (π, λ) on $L^2(G,H)$ by

$$\begin{aligned} (\pi(a)\xi)(s) &= \alpha_s^{-1}(a)\xi(s) \\ (\lambda(t)\xi)(s) &= \xi(s-t) \end{aligned} \tag{3.3}$$

The integrated representation corresponding to (π, λ) is faithful if G is amenable (as all abelian groups are) and is called the regular representation [6] . The operator norm on $L^2(G,H)$ defines the C^*-norm on $A \rtimes_\alpha G$. In this sense the crossed product can be considered the C^*-completion of the convolution algebra $C_c(G,A)$.

Definition 3.3. *On a crossed product $A \rtimes_\alpha G$ we define a dual action $\hat{\alpha} : \hat{G} \to Aut(A \rtimes_\alpha G)$ of the dual group $\hat{G}$ by*

$$\hat{\alpha}_\gamma(i_A(a)i_G(f)) = i_A(a)i_G(\langle \gamma, \cdot \rangle f), \quad \gamma \in \hat{G}, a \in A, f \in C_c(G),$$

where $\langle \gamma, s \rangle$ is the dual pairing of $\hat{G}$ and G.

This action defines a group of automorphisms (which follows largely from the uniqueness of the crossed product) and is strongly continuous. Therefore one can construct the crossed product $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$ leading to the periodicity result:

Theorem 3.1 (Takai duality)**.** *The second crossed product $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$ is isomorphic to $A \otimes \mathbb{K}(L^2(G))$. The isomorphism can be chosen in such a way that the second dual action $\hat{\hat{\alpha}}$ acts as $\alpha \otimes Ad\rho_G$, with ρ_G the regular representation of G on $L^2(G)$, i.e. acting by translation.*

W^*-dynamical systems

A von Neumann algebra M is a C^*-algebra that is (as a Banach space) isomorphic to the dual space of a Banach space M_*, its pre-dual. When considering actions on von Neumann algebras, strong continuity is usually too strong to be useful, which means that we have to weaken the continuity requirements.

Assuming that M acts on a Hilbert space H, there are several possible choices as we recall:

Definition 3.4. *A net x_β in M converges to* 0

1. *σ-weakly if and only if $\phi(x_\beta) \to 0$ for all $\phi \in M_*$*
2. *σ-strongly if and only if $\phi(x_\beta^* x_\beta) \to 0$ for all $\phi \in M_*$*
3. *weakly if and only if $\langle v, x_\beta w\rangle \to 0$ for all $v, w \in H$*
4. *strongly if and only if $x_\beta v \to 0$ for all $v \in H$.*

Since all of these topologies coincide on the unitary group of M and any element of M can be written as a linear combination of four unitaries, they define the same notion of weak continuity for automorphisms [29]:

Definition 3.5. *A W^*-dynamical system is a triple (M, G, α) consisting of a von Neumann algebra M, a locally compact group G and a weakly continuous action $\alpha : G \to Aut(M)$, i.e. the map*

$$t \mapsto \alpha_t(a)$$

is continuous with respect to any and therefore all of the topologies in Definition 3.4.

A covariant representation of a W^*-dynamical system is a pair (π, U) with π a non-degenerate normal (i.e. σ-weakly continuous) representation of M on a Hilbert space H and U a strongly continuous unitary representation of G on H, such that

$$\pi(\alpha_s(a)) = U(s)\pi(a)U(s)^*, \quad \forall a \in M, g \in G.$$

Without loss of generality we consider M to be acting on a Hilbert space H. If there is a strongly continuous unitary representation of G on H that satisfies

$$U_s M U_s^* = M, \quad \forall s \in G,$$

then

$$\alpha_s(x) := U_s x U_s^*$$

defines a weakly continuous action [42]. A general W^*-dynamical system (M, G, α) can always be written in this form by considering the regular representation (π, λ) on $L^2(G, H)$ defined for all $a \in M, t \in G$ by

$$\begin{aligned} (\pi(a)\xi)(s) &= \alpha_s^{-1}(a)\xi(s) \\ (\lambda(t)\xi)(s) &= \xi(s - t). \end{aligned} \tag{3.4}$$

It can be shown [42] that the following definition does not depend on the choice of Hilbert space representation.

Definition 3.6. *The crossed product $M \rtimes_\alpha G$ corresponding to a W^*-dynamical system (M, G, α) is the smallest von Neumann algebra in $\mathcal{B}(L^2(G,H))$ containing $\pi(M)$ and $\lambda(G)$.*

If (A, G, α) is a C^*-dynamical system with a non-degenerate covariant representation $\pi \times U$, then $\pi(A)''$ is a von Neumann algebra on which we have a weakly continuous G-action defined by

$$\tilde{\alpha}_t(\pi(a)) = U(t)\pi(a)U(t)^*, \quad \forall t \in G.$$

The W^*-dynamical system $(\pi(A)'', G, \tilde{\alpha})$ can be considered the W^*-completion of the dynamical system. One can then show

$$\pi(A)'' \rtimes_{\tilde{\alpha}} G = (\pi \times U)(A \rtimes_\alpha G)''$$

i.e. that C^*- and W^*-crossed products are compatible.

Definition 3.7. *Define a unitary representation σ of $\hat{G}$ on $L^2(H, M)$ by*

$$(\sigma(\gamma)\xi)(s) = \langle \gamma, s \rangle \xi(s), \quad \xi \in L^2(G,H)$$

Then the dual action $\hat{\alpha} : \hat{G} \to Aut(M \rtimes_\alpha G)$ on the W^*-crossed product $M \rtimes_\alpha G$ is given by

$$\hat{\alpha}_\gamma(m) = \mu(\gamma) x \, \mu(\gamma)^*, \quad x \in M \rtimes_\alpha G.$$

The dual action also defines a W^*-dynamical system $(M \rtimes_\alpha G, \hat{G}, \hat{\alpha})$ and

Theorem 3.2 (Takesaki duality [42])**.** *The second crossed product $M \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$ is isomorphic to $M \otimes \mathcal{B}(L^2(G))$, with the second dual action $\hat{\hat{\alpha}}$ again acting as $\alpha \otimes Ad\rho_G$.*

Crossed products for one-parameter groups

In this section let $G = \mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$ or $G = \mathbb{R}$. Let (A, G, α) be a C^*-dynamical system. If we identify $A \rtimes_\alpha G$ with the image of its regular representation $\pi \times U$ on a Hilbert space $L^2(G,H)$, there is a convenient alternative description (see e.g. [25]):

Denote by D the self-adjoint, densely defined generator of $U(t)$ such that

$$U(t) = e^{i2\pi Dt}$$

as operators on $L^2(G,H)$. Since $e^{i2\pi Dt}$ is just left translation along the base space G, the generator D is the derivative

$$(D\xi)(t) = \frac{i}{2\pi}(\partial_t \xi)(t).$$

The spectrum of D is given by $\sigma(D) = \mathbb{Z}$ respectively $\sigma(D) = \mathbb{R}$ and can therefore be identified with the dual group $\hat{G}$.

Hence, the generators of $A \rtimes_\alpha G$ can be written for $a \in A$, $t \in G$, $f \in C_c(G)$

$$\begin{aligned} i_A(a) &= \pi(a) \\ i_G(t) &= e^{i2\pi D t} \\ i_A(a) i_G(f) &= \pi(a)(\mathcal{F}f)(D) \end{aligned} \tag{3.5}$$

with the last line a consequence of continuous functional calculus with the Fourier transform $\mathcal{F} : C_c(G) \to C_0(\hat{G})$. We can therefore describe $A \rtimes_\alpha G$ as the C^*-algebra generated by polynomials in

$$\{\pi(a)\hat{f}(D) : a \in A, \hat{f} \in C_0(\hat{G})\}.$$

If we take instead the representation of $A \rtimes_\alpha G$ as convolution operators generated by $f \in C_c(G,A)$, the integrated form of the representation can be written

$$(\pi \times U)(f) = \int_G \pi(f(t)) e^{i2\pi D t} \mathrm{d}t,$$

which is consistent with the multiplication laws (3.1).

Given a different covariant representation (ψ, V) with

$$V(t) = e^{i2\pi \tilde{D} t}$$

the integrated representation $\psi \times V$ is given by

$$(\psi \times V)(f) = \int_G \psi(f(t)) e^{i2\pi \tilde{D} t} \mathrm{d}t$$

and therefore $\psi \times V$ can be equivalently defined on the generators by the relation

$$\pi(a)\hat{f}(D) \mapsto \psi(a)\hat{f}(\tilde{D}).$$

For a W^*-dynamical system (M, G, α) there is an analogous description of the regular representation on $L^2(G, H)$. The crossed product $M \rtimes_\alpha G$ is the weak closure of the $*$-algebraic span of

$$\{\pi(a)\hat{f}(D) : a \in M, \hat{f} \in B(\hat{G})\},$$

with $B(\hat{G})$ the bounded Borel-measurable functions (the set clearly contains the generators $\pi(M)$ and $\lambda(G)$). A general element of $M \rtimes_\alpha G$ cannot be represented as a convolution, however, for a function $f : G \to M$ the integral

$$\int_G \pi(f(t)) e^{i2\pi D t} \mathrm{d}t$$

defines an element of $M \rtimes_\alpha G$ if it exists in the weak sense.

4 Traces and non-commutative L^p-spaces

4.1 Traces on C^*- and von Neumann algebras

In this section we recall basic facts about traces on C^*- and von Neumann algebras based on [18] and construct a dual trace on crossed product algebras using the approach of [25] [32]. Let A be a C^*-algebra and denote by A_+ its positive cone. If A is not already unital, then let $\tilde{A}$ be its unitization.

Definition 4.1. *A weight ϕ on A is a function $\phi : A_+ \to [0, \infty]$, such that*

1. $\phi(\lambda x) = \lambda\phi(x), \forall x \in A_+, \lambda \in [0, \infty)$
2. $\phi(x + y) = \phi(x) + \phi(y), \forall x, y \in A_+$

A weight ϕ is called

- *a trace, if in addition $\phi(uxu^{-1}) = \phi(x)$ for all $x \in A$ and unitaries $u \in \tilde{A}$.*
- *faithful, if $\phi(x) = 0$ implies $x = 0$ for all positive $x \in A_+$.*
- *finite, if $\phi(x) < \infty, \forall x \in A_+$.*
- *lower semi-continuous, if $\phi(x) \leq \liminf \phi(x_n)$ for $x_n \to x$ in A_+.*

A weight ϕ on a von Neumann algebra N is called semifinite, if for all $x \in N_+$

$$\phi(x) = \sup\{\phi(y) | y \in N_+, y \leq x, \phi(y) < \infty\}$$

and it is called normal if for any increasing net $(x_\lambda)_{\lambda\in\Lambda}$ with supremum in N

$$\phi(\sup x_\lambda) = \sup \phi(x_\lambda).$$

For a weight ϕ on A we denote by A_+^ϕ the set of all positive elements $a \in A_+$ such that $\phi(a) < \infty$ and call ϕ densely defined if A_+^ϕ is dense in A_+. The linear span of A_+^ϕ is denoted A^ϕ and forms a $*$-subalgebra of A. Any weight ϕ extends to a linear functional on A^ϕ which will be denoted by the same letter. We define

$$A_2^\phi = \{a \in A | \phi(a^*a) < \infty\},$$

which is a left ideal of A containing A^ϕ and a two-sided ideal if ϕ is a trace. Note that for $x, y \in A_2^\phi$ we have

$$\tau(xy) = \tau(yx),$$

since any element of A can be written as a linear combination of four unitary elements in $\tilde{A}$.

For extending traces on C^*-algebras to their von Neumann closure and semidirect products we use an approach similar to the one in [32] and described in the following. The main tool one uses for the construction of traces are Hilbert algebras.

Definition 4.2. *Let U be a $*$-algebra over $\mathbb{C}$ with a scalar product $(x|y)$ which makes it into a pre-Hilbert space whose completion we denote by H. Then U is called a Hilbert algebra, if*

1. $(x|y) = (y^*|x^*), \forall x, y \in U$
2. $(xy|z) = (y|x^*z), \forall x, y, z \in U$
3. *For each $x \in U$ the maps $u_x : U \to U, y \mapsto xy$ and $v_x : U \to U, y \mapsto yx$ are continuous.*
4. *The set of all products xy is total in U.*

The maps u_x,v_x can be continued uniquely to operators on H and an element $a \in H$ is called bounded if there is a bounded operator $u_a \in \mathcal{L}(H)$ such that $u_a x = v_x a$ for all $x \in U$.

The algebraic rules can be motivated by the fact that for a C^*-algebra with a (densely defined) trace τ the image under the (unbounded) GNS-representation π_τ defines a Hilbert algebra with the usual scalar product $(\pi_\tau(x)|\pi_\tau(y)) = \tau(x^*y)$.

The main result that we use is

Theorem 4.1. *[18] Let U be a Hilbert algebra, H its completion and N the von Neumann algebra generated by the bounded elements of H. We define a weight for $S \in N_+$ through*

$$\phi(S) = \begin{cases} (a|a) & \text{if } \sqrt{S} = U_a \text{ for some bounded } a \in H \\ \infty & \text{otherwise.} \end{cases} \tag{4.1}$$

Then ϕ is a faithful, semifinite normal trace on H such that

$$N_2^\phi = \{x \in N | \phi(x^*x) < \infty\} = \{x \in N | x = U_a \text{ for some bounded } a \in H\}.$$

Moreover

$$\phi(U_a^* U_b) = (a|b)$$

by polarization.

We use this theorem to extend traces on C^*-algebras to the von Neumann algebras that they generate:

Proposition 4.1. *Let (A, G, α) be a C^*-dynamical system and τ be a finite, faithful and continuous trace on A. If τ is α-invariant, i.e.*

$$\tau(\alpha_t(x)) = \tau(x), \quad \forall x \in A, t \in G,$$

then A embeds covariantly into a von Neumann algebra $L^\infty(A)$ such that $(L^\infty(A), G, \tilde{\alpha})$ is a W^*-dynamical system with a finite, faithful and normal trace on $L^\infty(A)$ extending τ and invariant under $\tilde{\alpha}$.

Proof. Define a scalar product on A_2^τ through

$$(a|b) := \tau(a^* b),$$

which makes A_2^τ into a Hilbert algebra. Clearly, the completion of A_2^τ coincides with the GNS-representation (π_τ, H_τ) of A, which is faithful since τ is faithful. We put $L^\infty(A) = \pi_\tau(A)'' = \pi_\tau(A_2^\tau)''$ which coincides with the von Neumann algebra generated by the bounded elements of H_τ. Therefore Theorem 4.1 allows to extend τ to $L^\infty(A)$ in the way described. The extension of the trace is again finite as $\tau(x) \leq \|x\|\tau(1) < \infty$ for all $x \in L^\infty(A)$.

The action α restricts to A_2^τ and the maps $\alpha_s : A_2^\tau \to A_2^\tau$ are isometries and therefore extend to unitary operators U_s on H_τ. As $s \mapsto \alpha_s$ is strongly continuous and τ is finite, $s \mapsto \alpha_s$ is strongly continuous as a map on H_τ since

$$\|a - \alpha_s(a)\|_{H_\tau}^2 = \tau((a - \alpha_s(a))^*(a - \alpha_s(a))) \leq \|a - \alpha_s(a)\|^2 \tau(1).$$

We can therefore extend the action α to $L^\infty(A)$ through

$$\tilde{\alpha}_s(a) := U_s a U_s^*,$$

which is also weakly continuous since it is already defined through a covariant representation. As $\tilde{\alpha}$ preserves the scalar product of H_τ, the extension of τ is also $\tilde{\alpha}$-invariant. □

We apply a similar construction to define a dual trace on the crossed product $A \rtimes_\alpha G$.

Proposition 4.2. *Let (A, G, α) be a C^*-dynamical system with a finite, faithful α-invariant trace τ. The crossed product $L^\infty(A) \rtimes_\alpha G$ has a semifinite, faithful normal trace $\hat{\tau}$ that is left invariant by the dual action $\hat{\alpha}$.*

Proof. Consider the regular representation $\pi \times U$ on $L^2(G, H_\tau)$. We define for $f, g \in C_c(G, A_2^\tau)$ a scalar product through

$$(f|g)_G = \tau((f^* g)(0)) = \int_G \mathrm{d}s\, \tau(f(s)^* g(s)) \tag{4.2}$$

with the integral with respect to a Haar-measure on G.

This makes $C_c(G, A_2^\tau)$ into a Hilbert algebra and the Hilbert space completion of its image under $\pi \times U$ is given by $L^2(G, H_\tau)$. Theorem 4.1 then defines a trace $\hat{\tau}$ on the von Neumann algebra generated by the bounded elements of $L^2(G, H_\tau)$, which we denote by $L^\infty(A \rtimes_\alpha G)$.

As the Hilbert algebra is defined through the regular representation of the crossed product, we have the obvious inclusion $L^\infty(A) \rtimes_\alpha G \subset L^\infty(A \rtimes_\alpha G)$ and indeed one can show that the two algebras are equal [42, Theorem 5.12].

The dual action $\hat{\alpha}$ acts on $f \in C_c(G, A_2^\tau)$ as

$$(\hat{\alpha}_\gamma f)(s) = \langle \gamma, f(s) \rangle$$

and the scalar product $(f|g)_G$ is $\hat{\alpha}$-invariant on a dense subset, hence this also holds for the dual trace $\hat{\tau}$. □

The dual trace depends on the choice of Haar measure, i.e. it is only determined up to a multiplicative constant that we won't fix at this point.

In order to compute the trace of an element $f \in L^\infty(A \rtimes_\alpha G)$ we have to find a factorization $f = h^* g$ with $g, h \in L^2(G, H_\tau) \cap L^\infty(A \rtimes_\alpha G)$, which then implies

$$\hat{\tau}(f) = (g|h).$$

If G is a one-parameter group this can be done canonically [25]:

Proposition 4.3. *Let $G = \mathbb{T}$ or $G = \mathbb{R}$, (A, G, α) and τ as above. Then we have for the generators of the crossed product*

$$\hat{\tau}(\pi(a)f(D)) = \tau(a) \cdot \int_{\hat{G}} f(x)\mathrm{d}x \tag{4.3}$$

for $a \in L^\infty(A)^\tau$, $f \in L^1(\hat{G})$ and a Haar measure on the right hand side that is fixed by the normalization of (4.2).

Proof. We may write $f = \overline{g}h$ with $g, h \in L^2(\hat{G})$ which gives us the factorization $af(D) = (a\overline{g}(D))h(D) =: a_1^* a_2$. As vectors in $L^2(G, H_\tau)$ we have the representations

$$a_1^*(s) = \pi(\alpha_s(a))\mathcal{F}(\overline{g})(s)$$

and

$$a_2(s) = \mathcal{F}(h)(s).$$

One can now evaluate the scalar product

$$\begin{aligned}
\hat{\tau}(\pi(a)f(D)) = (a_1|a_2) &= \int_G \mathrm{d}s\,\tau(\alpha_s(a)\mathcal{F}(\overline{g})(s)\mathcal{F}(h)(s)) \\
&= \int_G \mathrm{d}s\,\tau(\alpha_s(a))\mathcal{F}(\overline{g})(s)\mathcal{F}(h)(s)) \\
&= \tau(a)\int_{\hat{G}} \mathrm{d}x\,\overline{g}(x)h(x) = \tau(a)\int_{\hat{G}} f(x)\mathrm{d}x,
\end{aligned}$$

where we used α-invariance of the trace and the Plancherel theorem. □

In this picture $\hat{\alpha}$ generates translations $\hat{\alpha}_\gamma(f(D)) = f(D+\gamma)$ such that the invariance of the trace becomes obvious.

4.2 Non-commutative L^p-spaces

In the previous section we already introduced von Neumann algebras of the form $L^\infty(A)$ and their representation spaces H_τ can be interpreted as corresponding L^2-spaces. In this section we outline the more general theory of non-commutative L^p-spaces based on the review [33].

Let M be a von Neumann algebra with a normal semifinite faithful trace. We write $L^\infty(M) = M$ for convenience and assume that M acts on a Hilbert space H.

For an element $x \in M$ define the support projection

$$\mathrm{supp}(x) = \inf\{e \in \mathcal{P}(M) | \, xe = x\}.$$

The positive elements with finite support are denoted

$$S_+ := \{x \in M_+ : \tau(\mathrm{supp}\ x) < \infty\} \in M. \tag{4.4}$$

and let S be the linear span of S_+. As τ is semifinite and normal, any projection is the increasing strong limit of τ-finite projections and as projections generate M, the set S is weakly dense in M.

Note that $x \in S$ implies $|x|^p \in S_+$ and hence

$$\|x\|_p := \tau(|x|^p)^{1/p}$$

defines a norm on S for $1 \le p < \infty$. The non-commutative L^p-space is defined as the completion of S in the p-norm and is denoted $L^p(M)$.

A (possibly unbounded) operator x on H is called affiliated to M if the partial isometry u of its polar decomposition $x = u\,|x|$ is contained in M as well as all spectral projections of $|x|$. It is further called τ-measurable if there is a spectral projection $p_\lambda = \chi_{(\lambda,\infty)}(|x|)$ with $\tau(p_\lambda) < \infty$. The space of τ-measurable operators is denoted $L^0(M)$ and there is a unique extension of τ to $L^0(M)_+$. The L^p-spaces can be realized as subspaces of $L^0(M)$, i.e. in particular every Cauchy sequence in $L^p(M) \cap M$ converges to a densely defined operator on H.

The Hölder-inequality holds in the non-commutative case in the form

$$x \in L^p(M), y \in L^q(M) \implies xy \in L^r(M)$$

and

$$\|xy\|_r \le \|x\|_p \|y\|_q,$$

where as usual $1/r = 1/p + 1/q, 0 < r, p, q \le \infty$.

This allows us to make sense of products involving factors in L^p-spaces and the density of $L^p(M) \cap M$ implies that the cyclicity of the trace extends to

$$\tau(xy) = \tau(yx), \quad \forall x \in L^p(M), y \in L^q(M).$$

For $1 \le p < \infty$ one still has the natural duality

$$L^p(M)^* = L^q(M),$$

i.e. $L^p(M)$ is reflexive for $1 < p < \infty$ and $L^1(M)$ is the pre-dual M_* of M. The Hilbert space $L^2(M)$ coincides with the representation space associated to the semifinite faithful normal trace τ and hence the representation by left multiplication is faithful and normal.

Many other results especially concerning convexity and interpolation also generalize to the non-commutative case.

We prove a small lemma that will help with showing convergence in L^1:

Lemma 4.1. *i) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in M. Then a_n converges in the strong topology of M on H if and only if it converges in the strong topology of M on $L^2(M)$.*

ii) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $M \cap L^1(M)$ that converges in the L^1-norm to some $a \in L^\infty \cap L^1(M)$ and is uniformly bounded in $L^\infty(M)$. Then $a = s\text{-}\lim_{n\to\infty} a_n$.

Proof. i) A strongly convergent sequence is bounded and therefore also σ-strongly convergent, as the strong and σ-strong topologies coincide on bounded sets. The representation of M on $L^2(M)$ is normal and hence σ-strongly continuous, i.e. a_n converges in the σ-strong topology of $\mathcal{B}(L^2(M))$ and because it is bounded, also in the strong topology of $\mathcal{B}(L^2(M))$. Exchanging the roles of H and $L^2(M)$ shows the other direction.

ii) By i) it is sufficient to show strong convergence on $L^2(M)$ and as there is a uniform bound on the operator norm, it is enough to show

$$a_n x \to ax$$

for x in the dense subset $M \cap L^2(M)$. We have

$$\|(a-a_n)x\|_2^2 = \|((a-a_n)x)^*(a-a_n)x\|_1 \leq \|(a-a_n)x\|\, \|a-a_n\|_1\, \|x\|$$

which converges to zero as a_n is uniformly bounded. □

The first statement also holds if one replaces strong with weak convergence and the second if one replaces L^1 with L^2 with the obvious modifications of the argument.

4.3 Compact and Breuer-Fredholm operators

We outline the theory of Breuer-Fredholm operators with respect to semifinite von Neumann algebras, which originated from [11] and was extended to real-valued indices by [32]. A more recent review of the theory can be found in [4].

A projection $p \in M$ is called finite if it is in $L^1(M)$ and we denote by $\mathcal{F}_\tau$ the smallest algebraic ideal of M that contains the finite projections. Its norm-closure $\mathcal{K}_\tau$ is the ideal of the so-called τ-compact elements. While $\mathcal{K}_\tau$ is always a C^*-algebra, it is not necessarily separable.

Another well-known characterization of $\mathcal{K}_\tau$ is given by [25]

$$\mathcal{K}_\tau = \overline{M \cap L^p(M)}, \forall 1 \leq p < \infty$$

with the closure in operator norm. One of the inclusions can be seen from the fact that the set on the right hand side contains all finite projections and is an algebraic ideal in M, hence it contains the generator $\mathcal{F}_\tau$. For the other inclusion we just need to consider operator norm-convergent approximation through spectral projections, which themselves are finite by majorization.

There is a nice result concerning projections in $\mathcal{K}$ (we will drop the subscript τ for the remainder of the section):

Proposition 4.4. *[22]*

- *Let $p \in \mathcal{K}$ be a projection. Then p is already finite, i.e. in $\mathcal{F}$.*
- *The algebra $\mathcal{F}$ is closed under the holomorphic functional calculus of its closure $\mathcal{K}$. Hence it is a local C^*-algebra and the inclusion into $\mathcal{K}$ induces an isomorphism*

$$i_* : K_0(\mathcal{F}) \to K_0(\mathcal{K})$$

Since τ defines a 0-cycle on $\mathcal{F}$, we therefore have an induced homomorphism

$$\tau_* = \langle \tau, \cdot \rangle : K_0(\mathcal{F}) \to \mathbb{C}$$

that extends to $K_0(\mathcal{K}) \simeq K_0(\mathcal{F})$.

The Calkin-algebra is defined as $\mathcal{Q} := M/\mathcal{K}$ and is also a C^*-algebra. An element $T \in M$ is called τ-Fredholm (or Breuer-Fredholm with respect to $\mathcal{K}_\tau$) if its image

$$T + \mathcal{K} \in M/\mathcal{K}$$

is invertible in the Calkin-algebra.

For any $T \in M$ denote by N_T the projection onto the kernel of T and by R_T the projection onto the closure of the range of T. Both of these projections are elements of M since they can be written as strong limits and there are the convenient expressions

$$\begin{aligned} N_T &= \sup\{E \in \mathcal{P}(M) : TE = 0\} \\ R_T &= \inf\{E \in \mathcal{P}(M) : ET = T\}. \end{aligned} \tag{4.5}$$

There is the following generalization of Atkinson's theorem:

Theorem 4.2. *[32] $T \in M$ is τ-Fredholm if and only if there exists a projection $E \in \mathcal{K}$ such that*

1. $N_T \in \mathcal{K}$
2. $\mathrm{range}(1 - E) \subset \mathrm{range}(T)$

The second condition implies $1 - R_T \leq E \in \mathcal{K}$ and is strictly stronger than this, as τ-Fredholm elements in general need not have a closed range. As an easy consequence we also have

$$N_{T^*} = 1 - R_T \in \mathcal{K}.$$

To a Fredholm element T we can associate two indices: The K_0-index

$$\mathrm{Ind}(T) = [N_T]_0 - [N_{T^*}]_0 \in K_0(\mathcal{K})$$

and the numerical index

$$\tau\text{-}\mathrm{Ind}(T) = \tau(N_T) - \tau(N_{T^*}) = \tau_*(\mathrm{Ind}(T)) \in \mathbb{R}.$$

The K_0-index is closely related to the index map $\mathrm{Ind} : K_1(\mathcal{Q}) \to K_0(\mathcal{K})$ corresponding to the extension

$$0 \to \mathcal{K} \to M \to \mathcal{Q} \to 0. \tag{4.6}$$

Consider the polar decomposition

$$T = U|T|$$

with $U \in M$ the unique partial isometry that satisfies this formula and for which

$$\begin{aligned} 1 - U^*U &= N_T \\ 1 - UU^* &= N_{T^*}. \end{aligned}$$

The image $U + \mathcal{K}$ is a unitary element of $\mathcal{Q}$ and since it lifts to the partial isometry U we have by Proposition 2.2

$$\mathrm{Ind}([U + \mathcal{K}]_1) = [1 - U^*U]_0 - [1 - UU^*]_0 = [N_T]_0 - [N_{T^*}]_0 = \mathrm{Ind}(T).$$

If T is Breuer-Fredholm and $k \in \mathcal{K}$ then $T + k$ clearly is also Breuer-Fredholm and with the same K-theoretical index and therefore with the same τ-index.

We close this section with a useful criterion and formula, which is certainly known but for which a proof in this exact setting was not readily available in the literature. The case $n = 1, m = 1$ can be found in [25] and for the general case we reproduce the argument from [19].

Theorem 4.3 (Fedosov-Calderon formula)**.** *Let $T \in M$ such that*

$$(1 - TT^*)^n \in L^1(M)$$

and

$$(1 - T^*T)^n \in L^1(M)$$

for some $n \in \mathbb{N} \setminus \{0\}$. Then T is τ-Fredholm and

$$\tau\text{-}\mathrm{Ind}(T) = \tau((1 - T^*T)^m) - \tau((1 - TT^*)^m) \tag{4.7}$$

for all $m \geq n$.

Proof. Trace class and bounded implies compact and therefore

$$((1-TT^*)^n)^*(1-TT^*)^n = (|1-TT^*|^2)^n \in \mathcal{K}.$$

As $\mathcal{K}$ is closed under functional calculus $|1-TT^*| \in \mathcal{K}$ and since $\mathcal{K}$ is an ideal, polar decomposition implies $1-TT^* \in \mathcal{K}$. Hence T^* is an inverse for T up to compacts and therefore T is Fredholm.

Consider now again the polar decomposition

$$T = U|T|$$

with $U \in N$ unique partial isometry with

$$UU^* = R_T = 1 - N_{T^*}, \quad U^*U = R_{T^*} = 1 - N_T$$

Clearly

$$\tau\text{-Ind}(T) = \tau(1-U^*U) - \tau(1-UU^*).$$

Writing $a = 1-TT^*$ as a matrix with respect to the decomposition $1 = R_T + N_{T^*}$, we get using the relation between U and T and the support conditions

$$a = \begin{pmatrix} R_T a R_T & R_T a N_{T^*} \\ N_{T^*} a R_T & N_{T^*} a N_{T^*} \end{pmatrix} = \begin{pmatrix} U(1-|T|^2)U^* & 0 \\ 0 & 1-UU^* \end{pmatrix}.$$

Noting that $|T|U^*U = |T| = U^*U|T|$ this implies

$$(1-TT^*)^n = U(1-|T|^2)^n U^* + 1 - UU^*.$$

The same relation shows that

$$U^*U(1-|T|^2)^n = U^*U - 1 + (1-|T|^2)^n = U^*U - 1 + (1-T^*T)^n,$$

since U^*U only acts nontrivially on the constant term.

Using cyclicity of the trace then gives

$$\begin{aligned}
&\tau((1-T^*T)^n) - \tau((1-TT^*)^n) \\
&= \tau((1-T^*T)^n) - \tau(1-UU^*) - \tau(U(1-|T|^2)^n U^*) \\
&= \tau((1-T^*T)^n) - \tau(1-UU^*) - \tau(U^*U(1-|T|^2)^n) \\
&= \tau(1-U^*U) - \tau(1-UU^*) = \tau\text{-Ind}(T)
\end{aligned}$$

□

5 Flows, Generators and Derivations

This section presents some material about $\mathbb{R}$-actions on Banach spaces based on [8], [10], with an emphasis on C^*- and von Neumann algebras. Any such action defines some notion of a derivative, which gives a derivation in the algebraic case.

Definition 5.1. *Let X be a Banach space and $\alpha : G \times X \to X$ an action of a locally compact group G, that acts on X as a group of isometries, i.e. $\alpha_g : X \to X$ is a linear isometry for each g and $\alpha : G \to L(X)$ is a homomorphism.*

Such an action is called strongly continuous if $g \mapsto \alpha_g(x)$ is norm-continuous for each $x \in X$.

If X is a von Neumann algebra, we call the automorphic action α weakly continuous if (X, G, α) is a W^*-dynamical system.

For $G = \mathbb{R}$ the generator δ of a group of isometries defined as

$$\delta(a) = \lim_{t \to 0} \frac{\alpha_t(a) - a}{t}$$

on the subspace

$$D(\delta) = \{a \in X : \lim_{t \to 0} \frac{\alpha_t(a) - a}{t} \text{ exists}\}.$$

In the von Neumann case with a weakly continuous action we instead ask for the weaker notion

$$\phi(\delta(a)) = \lim_{t \to 0} \phi\left(\frac{\alpha_t(a) - a}{t}\right), \quad \forall \phi \in M_*$$

and define $D(\delta)$ accordingly. If X is a Banach-$*$-algebra and $\alpha_g(a)^* = \alpha_g(a^*)$, the generator is a $*$-derivation, i.e. we have

$$\delta(ab) = \delta(a)b + a\delta(b)$$
$$\delta(a^*) = \delta(a)^*$$

for all $a, b \in D(\delta)$.

For $\alpha : \mathbb{R} \to \mathrm{Aut}(X)$ a strongly continuous action on a Banach space or a weakly continuous action on a von Neumann algebra denote the generator by δ and define the subspaces of differentiable elements

$$X_n := \{x \in X : x \in D(\delta^n)\}$$
$$X_\infty := \bigcap_{n>0} X_n.$$

It is known [9], [8]

Proposition 5.1. *If α is strongly continuous, then X_∞ and thus X_n are norm-dense in X.*

If α is weakly continuous, then X_∞ and thus X_n are weakly dense in X.

The main argument is that for $f \in C_c^\infty(\mathbb{R})$ and $x \in X$ the element

$$\alpha_f(x) := \int_{\mathbb{R}} \alpha_t(x) f(t) \mathrm{d}g$$

is smooth with respect to the action, where we interpret the integral in the strong respectively weak sense.

Let us now consider the special case that α is a weakly continuous $\mathbb{R}$-action on a von Neumann algebra M. We assume that there is a normal semifinite faithful trace on τ that is left invariant under α. As α acts isometrically, the action extends from $L^\infty \cap L^p(M)$ to a group of isometries on $L^p(M)$.

There is the technical result

Proposition 5.2. *[26, Lemma 13.4] The action $\alpha : G \times L^p(M) \to L^p(M)$ is strongly continuous with respect to the L^p-norm for $1 \leq p < \infty$.*

In particular, this implies that the smooth elements with respect to the action are dense in $L^p(M)$ and hence also the Sobolev spaces

$$W_{s,k}(M) := \{x \in L^s(M) : \delta^n x \in L^s(M), \quad \forall 0 \leq n \leq k\}$$

are dense in $L^p(M)$.

Proposition 5.3. *Let $M = L^\infty(M)$ be as above. Let B be a C^*-subalgebra of $L^\infty(M)$. If for some $p \in [1, \infty)$ the space $\mathscr{B} := B \cap W_{p,1}(M)$ is norm-dense in B, then $\mathscr{B}$ is a local C^*-subalgebra of B.*

Proof. We first note that by a simple computation the Leibniz rule holds for $a, b \in L^\infty(M) \cap W_{p,1}(M)$, i.e.

$$a\delta(b) + \delta(a)b = \lim_{t \to 0} \frac{\alpha_t(ab) - ab}{t}$$

in $L^p(M)$. Hence $\mathscr{B}$ is an algebra and one checks that it is a Banach-$*$-algebra with respect to the norm $\|\cdot\| + \|\cdot\|_{p,1}$. The only property that is not obvious is completeness. For a Cauchy-sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathscr{B}$ we have

$$\begin{aligned} a_n &\to a \in B \\ a_n &\to b \in L^p(M) \\ \delta(a_n) &\to c \in L^p(M) \end{aligned}$$

Clearly $|a_n|^p \to |a|^p$ in operator norm as a consequence of continuous functional calculus. As the trace τ is semifinite and normal, it is σ-weakly lower semicontinuous and hence $\|a\|_p = \tau(|a|^p) \leq \liminf \tau(|a_n|^p) = \|b\|_p$ showing $a \in L^p(M)$. Both a and b define bounded functionals on $L^q(M)$,$p^{-1} + q^{-1} = 1$ via

$$\phi_a := \tau(a\cdot), \quad \phi_b := \tau(b\cdot)$$

and for $x \in S$ (as defined in (4.4)) we have

$$\begin{aligned} |\phi_a(x) - \phi_b(x)| &= \lim_{n\to\infty} |\phi_a(x) - \phi_{a_n}(x)| \\ &= \lim_{n\to\infty} |\tau((a - a_n)x)| \leq \lim_{n\to\infty} \|a_n - a\| \, \|x\|_1 = 0. \end{aligned}$$

As ϕ_a and ϕ_b coincide on a dense subspace (σ-weakly dense for $p = 1$), we get $a = b$ since $\phi_.$ is just the embedding $L^p \hookrightarrow (L^q)^*$. The equality $c = \delta(b)$ follows from the fact that $\delta : W_{p,1}(M) \to L^p(M)$ is bounded linear and hence continuous with respect to the Sobolev norm.

We need to show that $\mathscr{B}$ is local, i.e. closed under the holomorphic functional calculus of its closure B. For this we first note that the embedding $i : \mathscr{B} \to B$ is a homomorphism and continuous with respect to the Banach algebra norm on $\mathscr{B}$. Hence it follows for all $a \in \mathscr{B}$ that $\sigma_B(a) \subseteq \sigma_{\mathscr{B}}(a)$. We will now show that the spectra are actually equal. As it is the more complicated case, we assume that B and $\mathscr{B}$ have no unit.

Recall that the resolvent set of an element a in a non-unital Banach algebra is given by the set of all $\lambda \in \mathbb{C} \setminus \{0\}$, for which a/λ has a quasi-inverse b that satisfies [28]

$$a\lambda^{-1} + b - ab\lambda^{-1} = 0 = a\lambda^{-1} + b - ba\lambda^{-1}.$$

It is therefore sufficient to show that if $a \in \mathscr{B}$ has a quasi-inverse $b \in B$, then already $b \in \mathscr{B}$. Since $\mathscr{B}$ is dense in B and the set of quasi-invertible elements is open, we can always translate the problem to a neighborhood of the neutral element 0 of the quasi-multiplication. Hence we may assume $\|a\|_B < 1$ such that the quasi-inverse b is given by the B-norm convergent series

$$b = -\sum_{k=1}^{\infty} a^k.$$

Now note that

$$\|a^k\|_{\mathscr{B}} = \|a^k\|_B + \|a^k\|_p + \|\delta(a^k)\|_p \leq \|a\|_B^{k-1} \left(\|a\|_B + \|a\|_p + k\|\delta(a)\|_p\right),$$

from which we conclude that the sum also converges in $\mathscr{B}$-norm. Hence we have $\sigma_{\mathscr{B}}(a) = \sigma_B(a)$ for all $a \in \mathscr{B}$.

As the holomorphic functional calculus in a Banach algebra is unique, it commutes with the embedding

$$i(f(a)) = f(i(a))$$

for all $a \in \mathscr{B}$ and f holomorphic on a neighbourhood of $\sigma_{\mathscr{B}}(a) = \sigma_B(a)$ with $f(0) = 0$. Therefore $\mathscr{B}$ is stable under the calculus in B. The unital case follows by an almost identical argument. □

This proof can be adapted for various similar algebras; in particular we note that if one ignores the parts about the Sobolev-spaces, we have that $B \cap L^1(M)$ dense in B implies that the set of τ-finite elements of B is closed under holomorphic functional calculus and therefore a local C^*-subalgebra of B.

The Winding number

Let M be a von Neumann algebra with a normal semifinite faithful trace τ together with an $\mathbb{R}$-action α and denote the associated derivation as δ. We assume that the action leaves τ invariant, which implies $\tau(\delta a) = 0$ for all $a \in W_{1,1}(M)$. Let B be a C^*-subalgebra of M such that $\mathscr{B} = B \cap W_{1,1}(M)$ is norm-dense in B, i.e. it satisfies the conditions of Proposition 5.3.

On $\mathscr{B}$ we define the 1-cycle

$$\eta(a,b) := i\tau(a\delta(b))$$

and we check explicitly that it is a cyclic cocycle. For cyclicity we have

$$\eta(a,b) = i\tau(a\delta(b) + \delta(a)b) - i\tau(\delta(a)b) = i\tau(\delta(ab)) - i\tau(\delta(a)b) = -\eta(b,a).$$

For the boundary property we note that

$$\begin{aligned}(b\eta)(a,b,c) &= \eta(ab,c) - \eta(a,bc) + \eta(ca,b) \\ &= \tau(ab\delta(c)) - \tau(a\delta(b)c + ab\delta(c)) + \tau(ca\delta(b)) = 0\end{aligned}$$

by the cyclicity of τ.

The canonical extension of η to $M_n(\mathscr{B}^+)$ is given by

$$(\eta\#\mathrm{Tr}_n)(a,b) = \tau(\mathrm{Tr}_n((a - s(a))\delta(b - s(b)))$$

where $s : M_n(\mathscr{B}^+) \to M_n(\mathbb{C})$ is again the scalar part and δ acts component-wise.

As η is an odd cocycle, it pairs with $K_1(\mathscr{B}) \simeq K_1(B)$ via

$$\langle \eta, [u]_1 \rangle = \eta(u, u^*) \in \mathbb{C}.$$

If $\mathscr{B}$ is already unital, it is usual to consider unitary matrices $u \in M_n(\mathscr{B})$ with coefficients in $\mathscr{B}$ instead of $\mathscr{B}^+$. We may of course reinterpret those as $u - 1_n + 1_n^+$ such that the pairing is given by

$$\langle \eta, [u - 1_n + 1_n^+]_1 \rangle = \eta(u - 1_n, u^* - 1_n).$$

The map defined by $\langle \eta, \cdot \rangle$ is called the Winding number form and we denote it by $\mathrm{Wind} : K_1(B) \to \mathbb{C}$, since it generalizes the classical form

$$u \in C^1(S^1) \mapsto i \int_{S^1} u\,du^* \in \mathbb{Z}.$$

Its interpretation as a homomorphism on $K_1(B)$ automatically gives us the usual homotopy and additivity properties.

6 Toeplitz extensions for one-parameter group actions

In this section we present some facts about the Toeplitz extension for a one-parameter group action. In the case of an $\mathbb{R}$-action this coincides with the smoothed Toeplitz extension introduced by [21], which we slightly generalize to periodic actions. This section has two purposes: one is to motivate the construction of the more general exact sequences of the next section, the other is to appeal to the similarities to the usual bulk-boundary exact sequences modeled after the Wiener-Hopf-extension or the discrete Toeplitz extension. As we will see in the applications, the advantage of the one-parameter-Toeplitz extension is that it can naturally describe a larger variety of boundary algebras (e.g. systems on a semi-infinite lattice with a boundary that is not axis-parallel).

As a general assumption for this section let A be a unital C^*-algebra and $A \subset \mathcal{B}(H)$ for some separable Hilbert space H. For $G = \mathbb{R}$ or $G = \mathbb{T}$ let α be a strongly continuous G-action on A.

Recall from Section 3 that the crossed product $A \rtimes_\alpha G$ is faithfully represented on $L^2(G, H)$ with the regular representation $\pi \times \lambda$. With the generator D of $t \mapsto \lambda(t)$ we had

$$\mathscr{C} := A \rtimes_\alpha G = C^* \left\{ \pi(a) f(D) : a \in A, f \in C_0(\hat{G}) \right\},$$

the smallest C^*-algebra in $\mathcal{B}(L^2(G,H))$ containing the set on the right hand side, i.e. algebraic closure followed by norm completion in $\mathcal{B}(L^2(G,H))$.

Definition 6.1. *We define the (smoothed) Toeplitz extension $\mathscr{T}$ associated to (A, G, α) as*

$$\mathscr{T} := C^* \left\{ \pi(a) f(D) : a \in A, f \in C_{0,*}(\hat{G}) \right\} \subset \mathcal{B}(L^2(G,H)),$$

where $C_{0,*}(\mathbb{R})$ respectively $C_{0,*}(\mathbb{Z})$ denote the continuous functions that vanish at $-\infty$ and admit a finite limit at $+\infty$.

The adjective "smoothed" is deserved only in the case $G = \mathbb{R}$ since this construction then has to be contrasted with the extension generated by the Toeplitz operators $\chi_{\mathbb{R}^+}(D)\pi(a)\chi_{\mathbb{R}^+}(D)$, which in general will not be contained in $\mathscr{C}$.

For a trivial action α the smoothed Toeplitz extension is not very interesting for our purposes. The relevant examples to keep in mind with respect to the applications are homogeneous observable algebras modeled as crossed products $A = B \rtimes \mathbb{Z}^d$ or $A = B \rtimes \mathbb{R}^d$ and to take for α a restriction of the dual action to a one-parameter subgroup. In special cases Takai duality then allows to rewrite e.g. $\mathscr{C} = A \rtimes_\alpha \mathbb{T} \simeq \mathbb{K} \otimes B \rtimes \mathbb{Z}^{d-1}$ such that one gets a "boundary" algebra modeled after a lower-dimensional space.

Let $h \in C(\hat{G})$ be a "switch" function, i.e. a smooth, monotonously increasing function that for some $\epsilon > 0$ satisfies

$$h(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > \epsilon. \end{cases}$$

We write $\mathscr{P} = h(D)$, which smoothly approximates the projection $\chi_{\mathbb{R}^+}(D)$ respectively can be chosen equal to the projection $\chi_{\mathbb{Z}^+}(D)$.

The following lemma is well-known in the case of $G = \mathbb{R}$ (see e.g. [21] or [25]).

Lemma 6.1. *Let $a \in A$. Then $[\mathscr{P}, \pi(a)] \in \mathscr{C}$ and hence $\mathscr{C}$ is a two-sided ideal in $\mathscr{T}$. Further, the element $\pi(a)\mathscr{P}$ is in $\mathscr{C}$ if and only if $a = 0$.*

Proof. For the first part it is enough to show that $[\mathscr{P}, \pi(a)] \in A \rtimes_\alpha G$ for $a \in C^\infty(A)$, the dense subset of smooth elements with respect to α. Let $h_n \to h$ be a sequence of smooth functions with compact support, that converges to h pointwise, i.e. $h_n(D) \to \mathscr{P}$ strongly. One computes using the multiplication law

$$[h_n(D), \pi(a)] = \int_G f_n(t) e^{i2\pi Dt} \mathrm{d}t$$

with

$$f_n(t) = (\alpha_t(a) - a)(\mathcal{F}h_n)(t).$$

Noting that

$$(\mathcal{F}h_n)(t) = \frac{(\mathcal{F}h_n')(t)}{it}$$

respectively

$$(\mathcal{F}h_n)(t) = \frac{(\mathcal{F}\Delta h_n)(t)}{1 + e^{i2\pi t}}$$

with the forward difference $\Delta h_n(x) = h_n(x+1) - h_n(x)$, we find that for $n \to \infty$ we have convergence to

$$[h(D), \pi(a)] = \int_G \frac{\alpha_t(a) - a}{it} (\mathcal{F}h')(t) e^{i2\pi Dt} \mathrm{d}t$$

respectively

$$[h(D), \pi(a)] = \int_G \frac{\alpha_t(a) - a}{1 + e^{it}} (\mathcal{F}\Delta h)(t) e^{i2\pi Dt} \mathrm{d}t,$$

which defines elements in $A \rtimes_\alpha G$ as they have smooth kernels that decay at infinity.

This shows that $\mathscr{C}$ is an ideal in $\mathscr{T}$, as for the generators we have e.g.

$$h(D)(\pi(a) f(D)) = [h(D), \pi(a)] f(D) + \pi(a)(h(D) f(D)) \in A \rtimes_\alpha G$$

for all $a \in A, f \in C_0(\hat{G}), h \in C_{0,*}(\hat{G})$.

The last part can be shown directly using Fourier analysis but, since we will later need a more general statement, we skip the proof for now. □

The statement about the commutators shows that we can write every element of $\mathscr{T}$ uniquely in the form $\pi(a)\mathscr{P} + c$ with $a \in A$ and $c \in A \rtimes_\alpha G$ and thus

Proposition 6.1. *The map $q : \mathscr{T} \to A$ defined through the equality*

$$q(\pi(a)\mathscr{P} + \mathscr{C}) = a$$

is a surjective homomorphism and hence there is the exact sequence

$$0 \to \mathscr{C} \hookrightarrow \mathscr{T} \xrightarrow{q} A \to 0. \tag{6.1}$$

We now turn our attention towards the connecting maps in K-theory. In the case $G = \mathbb{R}$ we write them as

$$\mathrm{Exp}_{\mathscr{T}} : K_0(A) \to K_1(A \rtimes_\alpha \mathbb{R})$$

and

$$\mathrm{Ind}_{\mathscr{T}} : K_0(A) \to K_1(A \rtimes_\alpha \mathbb{R}).$$

It is well-known that $K_j(A) \simeq K_{1-j}(A \rtimes_\alpha \mathbb{R})$ with natural isomorphisms given by the Connes-Thom-maps

$$\partial_j^\alpha : K_j(A) \to K_{1-j}(A \rtimes_\alpha \mathbb{R}),$$

which are the inverse maps to the connecting maps of the Wiener-Hopf-extension

$$0 \to C_0(\mathbb{R}, A) \rtimes_{\lambda\otimes\alpha} \mathbb{R} \to C_{0,*}(\mathbb{R}, A) \rtimes_{\lambda\otimes\alpha} \mathbb{R} \to A \rtimes_\alpha \mathbb{R} \to 0$$

with λ the left-translation on $C_{0,*}(\mathbb{R})$ [39].

In fact [21] proves that the smoothed Toeplitz extension can be considered the KK-inverse of the Wiener-Hopf-extension. As a consequence one has

Theorem 6.1. *Let $G = \mathbb{R}$. Then the connecting maps $\mathrm{Ind}_{\mathscr{T}}$ and $\mathrm{Exp}_{\mathscr{T}}$ of the smoothed Toeplitz extension are the inverses of the connecting maps of the Wiener-Hopf-extension corresponding to $(A, \alpha, \mathbb{R})$. This also means that they coincide with the Connes-Thom-isomorphisms $\partial_0^\alpha, \partial_1^\alpha$.*

There is no similar isomorphism in the case $G = \mathbb{T}$. However, the a dual action $\hat{\alpha}$ is a $\mathbb{Z}$-action for which one has the well-known Pimsner-Voiculescu sequence. Write $B = A \rtimes_\alpha \mathbb{Z}$, $\beta = \hat{\alpha}$ and consider the discrete Toeplitz extension for $\mathbb{Z}$-actions

$$0 \to C_0(\mathbb{Z}, B) \rtimes_{\lambda\otimes\beta} \mathbb{Z} \to C_{0,*}(\mathbb{Z}, B) \rtimes_{\lambda\otimes\beta} \mathbb{Z} \to B \rtimes_\beta \mathbb{Z} \to 0$$

with λ the left-translation action on $C_{0,*}(\mathbb{Z})$. The six-term exact sequence associated to this extension is called the Pimsner-Voiculescu sequence [5]. As the K-groups of the factors are isomorphic to either those of B or $B \rtimes_\beta \mathbb{Z}$, it facilitates the computation of $K_j(B \rtimes_\alpha \mathbb{Z})$ and is equivalent to

$$\begin{array}{ccccc}
K_0(B) & \xrightarrow{\mathrm{id}-\beta_*} & K_0(B) & \longrightarrow & K_0(B \rtimes_\beta \mathbb{Z}) \\
{\scriptstyle \mathrm{Ind}}\uparrow & & & & \downarrow{\scriptstyle \mathrm{Exp}} \\
K_1(B \rtimes_\beta \mathbb{Z}) & \longleftarrow & K_1(B) & \xleftarrow[\mathrm{id}-\beta_*]{} & K_1(B).
\end{array}$$

As a consequence of Takai duality we have

$$K_j(B \rtimes_\beta \mathbb{Z}) = K_j(A \rtimes_\alpha \mathbb{T} \rtimes_{\hat{\alpha}} \mathbb{Z}) \simeq K_j(A \otimes \mathbb{K}) = K_j(A)$$

and therefore the sequence may be written as

$$\begin{array}{ccccc}
K_0(A \rtimes_\alpha \mathbb{T}) & \xrightarrow{\mathrm{id}-\hat{\alpha}_*} & K_0(A \rtimes_\alpha \mathbb{T}) & \longrightarrow & K_0(A) \\
{\scriptstyle \mathrm{Ind}}\uparrow & & & & \downarrow{\scriptstyle \mathrm{Exp}} \\
K_1(A) & \longleftarrow & K_1(A \rtimes_\alpha \mathbb{T}) & \xleftarrow[\mathrm{id}-\hat{\alpha}_*]{} & K_1(A \rtimes_\alpha \mathbb{T}).
\end{array} \tag{6.2}$$

In fact the connecting maps are identical to those of the one-parameter-Toeplitz extension, which one might hope to prove by connecting the two sequences in a commutative diagram. However, the explicit forms of those embeddings would likely be complicated as they would involve Takai duality. We will therefore argue in a more indirect fashion.

First note that the $\mathbb{T}$-action α can be considered an $\mathbb{R}$-action with $\alpha_{t+1} := \alpha_t$. As we will see below, there is a surjective homomorphism $q : A \rtimes_\alpha \mathbb{R} \to A \rtimes_\alpha \mathbb{T}$ and as an intermediate result in the proof of the Pimsner-Voiculescu-sequence, [5] shows that the connecting maps of (6.2) can be expressed with the Connes-Thom-isomorphisms as

$$\mathrm{Exp} : K_0(A) \to K_1(A \rtimes_\alpha \mathbb{T}), \quad \mathrm{Exp} = q_* \circ \partial_0^\alpha$$

and

$$\mathrm{Ind} : K_1(A) \to K_0(A \rtimes_\alpha \mathbb{T}), \quad \mathrm{Ind} = q_* \circ \partial_1^\alpha.$$

We can consider the smoothed Toeplitz extension for α as either a $\mathbb{T}$- or $\mathbb{R}$-action and will denote them by $\mathscr{T}_\mathbb{T}$ and $\mathscr{T}_\mathbb{R}$, defined on $L^2(\mathbb{T}, H)$ and $L^2(\mathbb{R}, H)$ using the covariant representations $(\pi_\mathbb{T}, \lambda_\mathbb{T})$ and $(\pi_\mathbb{R}, \lambda_\mathbb{R})$ respectively.

Let $\psi \in L^2(\mathbb{R}, H)$. Then for $z \in \mathbb{Z}, t \in (0,1]$ the periodicity $\alpha_{t+1} = \alpha_t$ gives

$$(\pi_\mathbb{R}(a)\psi)(z+t) = \alpha_{z+t}^{-1}(a)\psi(z+t) = \alpha_t^{-1}(a)\psi(z+t)$$

and for $y \in \mathbb{Z}, r \in (0,1]$

$$(\lambda_\mathbb{R}(y+r)\psi)(z+t) = \psi(z-y+t-r).$$

With $p : L^2(\mathbb{R}, H) \to L^2(\mathbb{T}, H)$ the restriction to the interval $[0,1]$ one sees from the formulas above that p intertwines the generators of $\mathscr{C}_\mathbb{R}$ and $\mathscr{C}_\mathbb{T}$, i.e.

$$p \circ \pi_\mathbb{R}(a) = \pi_\mathbb{T}(a) \circ p$$
$$p \circ \lambda_\mathbb{R}(y+r) = \lambda_\mathbb{T}(r) \circ p.$$

The latter relation shows

$$p \circ f(D) = f(\tilde{D}) \circ p, \quad \forall f \in C_0(\mathbb{R}).$$

with D, $\tilde{D}$ the generators of the $\mathbb{R}$ respectively $\mathbb{T}$-actions.

Hence, p densely defines a morphism on the generators that implements the surjection $q : A \rtimes_\alpha \mathbb{R} \to A \rtimes_\alpha \mathbb{T}$. For a switch function $h \in C_{0,*}(\mathbb{R})$ as above there is a switch function $\tilde{h} \in C_{0,*}(\mathbb{Z})$ such that

$$p \circ h(D) = \tilde{h}(\tilde{D}) \circ p.$$

As the generators of $\mathscr{T}_\mathbb{R}$ can be written uniquely as $\pi_\mathbb{R}(a)(h(D) + g(D))$ with $g \in C_0(\mathbb{R})$ this implies that the spatial morphism defined through p extends to $q : \mathscr{T}_\mathbb{R} \to \mathscr{T}_\mathbb{T}$ and that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{C}_\mathbb{R} & \longrightarrow & \mathscr{T}_\mathbb{R} & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow q & & \downarrow q & & \| & & \\ 0 & \longrightarrow & \mathscr{C}_\mathbb{T} & \longrightarrow & \mathscr{T}_\mathbb{T} & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

commutes. The naturalness of the connecting maps then gives

$$\mathrm{Ind}_{\mathbb{T}} = q_* \circ \mathrm{Ind}_{\mathbb{R}}$$

and

$$\mathrm{Exp}_{\mathbb{T}} = q_* \circ \mathrm{Exp}_{\mathbb{R}}.$$

We conclude using Theorem 6.1:

Proposition 6.2. *Let $G = \mathbb{T}$. Denote the connecting maps of the one-parameter Toeplitz extension* (6.1) *by* Exp_G *and* Ind_G *and those of* (6.2) *by* Exp *and* Ind. *Then we have the commutative diagrams*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\mathrm{Exp}_{\mathbb{R}}} & K_1(A \rtimes_\alpha \mathbb{R}) \\ \big\downarrow{\scriptstyle i_*} & \searrow^{\mathrm{Exp}} & \big\downarrow{\scriptstyle q_*} \\ K_0(A) & \xrightarrow{\mathrm{Exp}_{\mathbb{T}}} & K_1(A \rtimes_\alpha \mathbb{T}) \end{array}$$

and

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\mathrm{Ind}_{\mathbb{R}}} & K_0(A \rtimes_\alpha \mathbb{R}) \\ \big\downarrow{\scriptstyle i_*} & \searrow^{\mathrm{Ind}} & \big\downarrow{\scriptstyle q_*} \\ K_1(A) & \xrightarrow{\mathrm{Ind}_{\mathbb{T}}} & K_0(A \rtimes_\alpha \mathbb{T}) \end{array}$$

where one identifies $K_j(A) \simeq K_j(A \rtimes_\alpha \mathbb{T} \rtimes_{\hat{\alpha}} \mathbb{Z}) \simeq K_j(A \otimes \mathbb{K})$.

These relations and Takai duality make it possible to prove for the (smoothed) one-parameter Toeplitz extension duality theorems for the Chern-cocycles similar to [36] by applying existing methods for the Toeplitz extension of $(A \rtimes_\alpha \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}$ or the Wiener-Hopf-extension of $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$ respectively. Even though there are numerous results in the literature along these lines, none seem to be applicable directly. However, we note that it seems possible to extend the proofs in [23] or [24] to also cover those cases. This would then complete the bulk-boundary correspondence for the smoothed Toeplitz extension. We will not pursue this further in this work, but rather study less regular extensions in the following, for which such a reduction is not possible.

7 Exact sequences for weakly continuous actions

In this section we use the general assumption that A is a unital C^*-algebra with a strongly continuous G-action α, $G = \mathbb{R}$ or $G = \mathbb{T}$, and an α-invariant faithful continuous finite trace τ. Let (π, λ) be the regular representation of A on a Hilbert space $L^2(G, H)$. As in section 4.1 we may construct the von Neumann algebras $L^\infty(A)$ and $L^\infty(A \rtimes_\alpha G)$ with induced weakly continuous actions α and invariant traces τ respectively $\hat{\tau}$. To shorten the notation we write

$$\mathcal{E} := A \rtimes_\alpha G$$

and denote the L^p-spaces as $L^p(A)$ and $L^p(\mathcal{E})$.

In our applications A will be an algebra of observables describing solid state systems with an infinite spatial extent, while $\mathcal{E}$ will model semi-infinite systems that are located on the boundary of a system whose bulk is described by A.

For the classification of topological invariants in physics it will be useful to consider elements, that are not in the continuous subalgebra A but rather in the von Neumann algebra $L^\infty(A)$. The goal would be to relate projections and unitary elements of $L^\infty(A)$ to counterparts in $L^\infty(\mathcal{E})$ using the connecting maps of K-theory. However, the full von Neumann algebras are too large for this purpose since those have trivial K-groups. We will have to resort to using suitable subalgebras that ideally also allow us to densely define certain cocycles.

We note that a similar problem is well-studied in the case of the commutative algebras $A = C(\mathbb{T})$ respectively $L^\infty(A) = L^\infty(\mathbb{T})$. In this case there are index theorems for Toeplitz operators with symbols in $QC(\mathbb{T})$ (quasicontinuous functions on the torus), $PC(\mathbb{T})$ (piecewise-continuous functions) and other commutative algebras on the torus (see [7] for a review).

In order to take advantage of Breuer-Fredholm theories, it will be convenient to consider extensions of subalgebras of $L^\infty(A)$ with subalgebras of the $\hat{\tau}$-compact operators $\mathcal{K}_{\hat{\tau}} \subset L^\infty(\mathcal{E})$. Therefore, we define the quasicontinuous extension of A in analogy with the commutative case

$$QA := \{a \in L^\infty(A) : [\pi(a), P] \in \mathcal{K}_{\hat{\tau}}\},$$

where π is extended to the embedding $\pi : L^\infty(A) \to L^\infty(\mathcal{E})$ and $P = \chi_{(0,\infty)}(D) \in L^\infty(\mathcal{E})$ is the spectral projection of the generator D of $U(t)$. It is easy to see that QA is a $*$-algebra and closed since $\mathcal{K}_{\hat{\tau}}$ is a norm-closed ideal. We also note that in the classical case $A = C(\mathbb{T})$ the algebra QA allows an intrinsic characterization (bounded functions with vanishing mean oscillation), which can probably be generalized to the non-commuative case but won't be needed for this work.

Since the dual trace is α-invariant and π is α-covariant, the action restricts to QA but it usually will not be strongly continuous. In the classical case $A = C(\mathbb{T})$, for example, with α acting as translation, the orbit of an element $f \in L^\infty(\mathbb{T})$ is norm-continuous if and only if f is a continuous function in the usual sense. As there are quasicontinuous functions that are not continuous, this condition must fail for some elements of QA. Therefore, we also cannot use the (smoothed) Toeplitz extension to construct an exact sequence involving QA. However, we will show in this section that the sequence

$$0 \to \mathcal{K}_{\hat{\tau}} \to \pi(QA)P + \mathcal{K}_{\hat{\tau}} \xrightarrow{q} QA \to 0$$

is exact based on properties of the dual trace.

There is also the problem that QA is likely too large for the numerical bulk-boundary correspondence, since it is not clear whether $W_{1,1}(A)$ is dense in QA and that the generalized winding number defines a pairing with $K_1(QA)$. It will therefore be more convenient to work with smaller, separable algebras.

For any subalgebra $B \subset L^\infty(A)$ we may still define the Toeplitz algebra

$$\mathcal{T}(B) := C^*(\pi(B)P),$$

and the commutator ideal

$$\mathscr{C}(B) := \mathscr{T}(B) \cdot C^*([\pi(B), P]) \cdot \mathscr{T}(B),$$

where $C^*(\cdot)$ denotes the smallest C^*-subalgebra of $L^\infty(\mathcal{E})$ that contains the respective elements. In particular, $\mathscr{C}(B)$ is the smallest ideal in $\mathscr{T}(B)$ containing the commutators with P. As shown by [25] these algebras form an exact sequence for $B = A$, i.e. $\mathscr{T}(A)/\mathscr{C}(B) \simeq A$, but the arguments partly rely on the strong continuity of α (for the commutative case the same has been known since at least [14]). We now show that the sequence is also exact in the more general case if we restrict to $B \subseteq QA$ and later give constructive criteria for this condition.

The following lemma shows that the compact operators in some sense vanish at infinity:

Lemma 7.1. *Let $e \in \mathcal{K}_{\hat{\tau}}$ and $P_n := \chi_{(n,n+1]}(D)$. Then*

$$\lim_{n\to\infty} \hat{\tau}(P_n e^* e P_n) = 0.$$

Proof. First note that the expression makes sense, since e is bounded and P_n is $\hat{\tau}$-summable. For $e \in L^\infty(\mathcal{E}) \cap L^2(\mathcal{E})$ we have using Pythagoras

$$\|e\|_2^2 = \hat{\tau}(e^*e) = \hat{\tau}\left(\sum_{n\in\mathbb{Z}} P_n e^* e P_n\right) = \sum_{n\in\mathbb{Z}} \hat{\tau}(P_n e^* e P_n)$$

and therefore the sequence not only converges to zero but is also summable. Because this set is dense in $\mathcal{K}_{\hat{\tau}}$ we just have to show that the property is preserved by limits. Let $(e_k)_{k\in\mathbb{N}}$ be a sequence in $L^\infty(\mathcal{E}) \cap L^2(\mathcal{E})$ that converges to $e \in \mathcal{K}_{\hat{\tau}}$ in operator norm. Let $\delta > 0$ and choose k_0 such that

$$\|e - e_k\| < \delta$$

for all $k \geq k_0$ and n_0 such that

$$\hat{\tau}(P_n e_{k_0}^* e_{k_0} P_n) < \delta$$

for all $n \geq n_0$. Then

$$\begin{aligned}\hat{\tau}(P_n e^* e P_n) &= \hat{\tau}(P_n e_{k_0}^* e_{k_0} P_n) + \hat{\tau}(P_n e_{k_0}^*(e - e_{k_0})P_n) + \hat{\tau}(P_n(e - e_{k_0})^* e P_n)\\ &\leq \delta + \hat{\tau}(P_n)(\|e_{k_0}\|\,\|e - e_{k_0}\| + \|e\|\,\|e - e_{k_0}\|) \leq \delta(1 + \hat{\tau}(P_0)(\|e_{k_0}\| + \|e\|))\end{aligned}$$

since $\hat{\tau}(P_n) = \hat{\tau}(P_0)$.

As $\|e_k\|$ is uniformly bounded, choosing δ small enough shows convergence to zero. □

Corollary 7.1. *For $a \in L^\infty(A)$ we have $\pi(a)P \in \mathcal{K}_{\hat{\tau}}$ if and only if $a = 0$.*

Proof. We have

$$\hat{\tau}(P_n P \pi(a^*)\pi(a) P P_n) = \hat{\tau}(P_n \pi(a^*a) P_n) = \tau(a^*a)\hat{\tau}(P_0)$$

which is constant in n and since τ is faithful the expression vanishes if and only if $a = 0$. □

Proposition 7.1. *Let B be a unital C^*-subalgebra of QA and let C be a C^*-algebra with*

$$[\pi(B), P] \subseteq C \subseteq \mathcal{K}_{\hat{\mathfrak{t}}}.$$

Then we have for

$$\mathscr{T}(B, C) := C^*(\pi(B)P + C)$$

and

$$\mathscr{C}(B, C) := \mathscr{T}(B, C) \cdot C \cdot \mathscr{T}(B, C)$$

the exact sequence

$$0 \to \mathscr{C}(B, C) \to \mathscr{T}(B, C) \xrightarrow{q} B \to 0 \tag{7.1}$$

with the symbol map q densely defined by

$$q(\pi(b)P + c) = b, \quad \forall b \in B, c \in \mathscr{C}.$$

Proof. First note that $\mathscr{T} = B \oplus \mathscr{C}$ as a linear space since

$$(\pi(a_1)P + c_1)(\pi(a_2)P + c_2) \in \pi(a)\pi(b)P + \mathscr{C}$$

$$(\pi(a)P + c)^* \in \pi(a)^*P + \mathscr{C}$$

and because $\pi(a)P$ is never in $\mathcal{K}_{\hat{\mathfrak{t}}}$ and thus never in $\mathscr{C}$. Hence we can write any element of $\mathscr{T}(B, C)$ uniquely as $\pi(b)P + c$. Therefore, q is well-defined and as $\mathcal{K}_{\hat{\mathfrak{t}}}$ is an ideal, it is easy to see that it is a $*$-morphism and surjective with kernel $\mathscr{C}$. □

Since $P\mathscr{T}(B, C)P$ is a closed subalgebra, we also have

Corollary 7.2. *Let B,C be as above and define the genuine Toeplitz algebra*

$$\hat{\mathscr{T}}(B, C) := P\mathscr{T}(B, C)P$$

and write the restriction of q as

$$\hat{q} : \hat{\mathscr{T}}(B, C) \to B, \quad q(P\pi(b)P + c) = b.$$

Then

$$\hat{\mathscr{C}}(B, C) := ker\ \hat{q} = (ker\ q) \cap \hat{\mathscr{T}}(B, C) = P\mathscr{C}(B, C)P$$

is a C^*-algebra and the sequence

$$0 \to \hat{\mathscr{C}}(B, C) \to \hat{\mathscr{T}}(B, C) \xrightarrow{q} B \to 0 \tag{7.2}$$

is exact.

The genuine Toeplitz extension $\hat{\mathscr{T}}(B, C)$ is more closely aligned with the usual definition of Toeplitz algebras on half-spaces as in e.g. [14]. In contrast to $\mathscr{T}(B, C)$ it does not contain the commutators $[\pi(b), P]$ but only the semi-commutators of the form $P\pi(a)\pi(b)P - P\pi(a)P\pi(b)P$. However, it has the advantage that $\hat{\mathscr{T}}(B, C)$ is unital with the unit P, which simplifies some arguments.

As an application we have the exact sequences

$$0 \to \mathcal{K}_{\hat{\tau}} \to \pi(QA)P + \mathcal{K}_{\hat{\tau}} \to QA \to 0$$

and

$$0 \to P\mathcal{K}_{\hat{\tau}}P \to P\pi(QA)P + P\mathcal{K}_{\hat{\tau}}P \to QA \to 0$$

which show that we may consider QA a subalgebra of the Calkin-algebra.

To be precise, we have in general

Proposition 7.2. *The following diagram has exact rows and commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{\mathscr{C}}(B,C) & \longrightarrow & \hat{\mathscr{T}}(B,C) & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \mathcal{K}_{\hat{\tau}} & \longrightarrow & L^\infty(\mathcal{E}) & \longrightarrow & L^\infty(\mathcal{E})/\mathcal{K}_{\hat{\tau}} & \longrightarrow & 0 \end{array}$$

where

$$j : B \to L^\infty(\mathcal{E})/\mathcal{K}_{\hat{\tau}}, \quad a \mapsto P\pi(a)P + \mathcal{K}_{\hat{\tau}}$$

is an embedding of C^*-algebras.

Hence the diagram

$$\begin{array}{ccc} K_1(B) & \xrightarrow{\text{Ind}} & K_0(\hat{\mathscr{C}}(B,C)) \\ \downarrow{\scriptstyle j_*} & & \| \\ K_1(L^\infty(\mathcal{E})/\mathcal{K}_{\hat{\tau}}) & \xrightarrow{\text{Ind}} & K_0(\mathcal{K}_{\hat{\tau}}) \end{array}$$

commutes.

Proof. The diagram obviously commutes so we just have to check the claim about j. The map j is linear and a homomorphism since e.g.

$$P\pi(a)P\pi(b)P + \mathcal{K}_{\hat{\tau}} = P\pi(ab)P + P[\pi(a), P]\pi(b)P + \mathcal{K}_{\hat{\tau}} = P\pi(ab)P + \mathcal{K}_{\hat{\tau}}$$

by definition of QA. It is injective by Corollary 7.1. □

Hence, any extension encodes at least the Breuer-Fredholm index.

Another special case of Proposition 7.1 is the (smoothed) Toeplitz extension as $A \rtimes_\alpha G \subset \mathcal{K}_{\hat{\tau}}$. To see this, just note that for $a \in A$ and $f \in L^1(\hat{G})$ we have $\pi(a)f(D) \in L^\infty(A \rtimes_\alpha G) \cap L^1(A \rtimes_\alpha G)$ and those elements span a dense subalgebra of $A \rtimes_\alpha G$.

8 The Breuer-index of a Toeplitz operator

In this section we again use the general assumptions and notation of section 7 and additionally require that we have two algebras B,C that satisfy the conditions of Proposition 7.1.

To an element $a \in B$ we may associate the Toeplitz operator with symbol a

$$T_a = P\pi(a)P \in \hat{\mathcal{T}}(B, C).$$

We will show that the Breuer-Index of some Toeplitz operators can be computed in terms of their symbol but we first need a technical lemma:

Lemma 8.1. *Let $a \in L^1(\mathcal{E})$ and write $P_{k,r} := \chi_{(kr,(k+1)r]}(D)$ for $k \in \mathbb{Z}$ and $r > 0$.*

We have

$$a = \sum_{k\in\mathbb{Z}} P_{k,r} a$$

with the sum converging in $L^1(\mathcal{E})$ and thus

$$\hat{\tau}(a) = \sum_{k\in\mathbb{Z}} \hat{\tau}(P_{k,r} a).$$

Proof. Assume first $a \in L^\infty(\mathcal{E}) \cap L^1(\mathcal{E})$ and set $P^{(n)} = \sum_{k=-n}^{k=n} P_{k,r}$. Clearly we have $P^{(n)} \nearrow 1$ and

$$0 \le a^* P^{(n)} a \le a^* P^{(m)} a \le a^* a$$

for $n < m$. As the square root is operator monotone,

$$n \mapsto \sqrt{a^* P^{(n)} a}$$

forms a monotone sequence. This shows

$$\lim_n \|P^{(n)} a\|_1 = \sup_n \hat{\tau}\left(\sqrt{a^* P^{(n)} a}\right) \le \hat{\tau}(|a|) = \|a\|_1 \tag{8.1}$$

as the trace is normal. As the sequence is monotone and bounded, $P^{(n)} a$ is a Cauchy-sequence in $L^1(\mathcal{E})$ and Lemma 4.1 shows that it converges strongly to its L^1-limit, which must then coincide with $s\text{-}\lim_{n\to\infty} P^{(n)} a = a$.

For general $a \in L^1(\mathcal{E})$ approximate with $b \in L^\infty(\mathcal{E}) \cap L^1(\mathcal{E})$ such that $\|a - b\|_1 < \epsilon/3$ to get

$$\|P^{(n)} a - a\|_1 \le \|b - a\|_1 + \|P^{(n)} b - b\|_1 + \|P^{(n)} a - P^{(n)} b\|_1 < \epsilon/3 + \epsilon/3 + \epsilon/3$$

for n large enough.

Since the error term converges to zero, the second equality is obvious. □

The same argument holds for other approximate units.

Proposition 8.1. *Let $u \in L^\infty(A) \cap W_{1,1}(A)$ be unitary with $[\pi(u), P] \in L^1(\mathcal{E})$. Then*

$$\hat{\tau}\text{-Ind}(T_u) = \frac{i\mathcal{N}}{2\pi} \tau(u^* \delta u)$$

with the index taken with respect to the von Neumann algebra $PL^\infty(\mathcal{E})P$,

$$\mathcal{N} = \hat{\tau}(\chi_{(0,1]}(D))$$

and δ the derivation associated to α, where we regard a $\mathbb{T}$-action as periodic $\mathbb{R}$-action with $\alpha_{t+1} = \alpha_t$.

Proof. Note first that P is the unit of $PL^\infty(\mathcal{E})P$ and

$$P - T_u T_u^* = P[\pi(u), P][\pi(u^*), P] \in L^1(\mathcal{E}).$$

Therefore T_u is Fredholm by Proposition 4.3 with $n = 1$.

Writing $S = 1 - 2P =: \operatorname{sgn}(D)$, we get using equation (4.7) and the cyclicity of the trace

$$\begin{aligned}\hat\tau\text{-Ind}(T_u) &= \hat\tau(T_u T_u^* - T_u^* T_u)\\ &= -\frac{1}{4}\hat\tau(P[S,\pi(u)][S,\pi(u^*)] - P[S,\pi(u^*)][S,\pi(u)])\\ &= \frac{1}{8}\hat\tau(S[S,\pi(u)][S,\pi(u^*)] - S[S,\pi(u^*)][S,\pi(u)])\end{aligned}$$

where the right hand sides are trace class since the commutators are in $L^\infty \cap L^1(\mathcal{E})$.

It is enough to consider the first term by writing $a = \pi(u)$ and $b = \pi(u^*)$ and because of its trace class properties, we may introduce partitions of unity $1 = \sum_{k\in\mathbb{Z}} P_{k,r}$, with $P_{k,r} := \chi_{(kr,(k+1)r]}(D)$. The dual action $\hat\alpha$ leaves the trace invariant and we have

$$\hat\alpha_{rx}(a) = a, \quad \hat\alpha_{rx}(P_{k,r}) = P_{k-x,r}.$$

We rewrite the first term

$$\begin{aligned}&\hat\tau(\sum_{n\in\mathbb{Z}} SP_{n,r}[S,a][S,b]) = \sum_{n\in\mathbb{Z}} \hat\tau(SP_{n,r}[S,a][S,b]P_{n,r})\\ &= \sum_{n,m\in\mathbb{Z}} \hat\tau(SP_{n,r}[S,a]P_{m,r}[S,b]P_{n,r})\\ &= -\sum_{n,m\in\mathbb{Z}} \hat\tau(\operatorname{sgn}(n)(\operatorname{sgn}(n)-\operatorname{sgn}(m))^2 P_{n,r}aP_{m,r}bP_{n,r})\\ &= -\sum_{n,m\in\mathbb{Z}} \hat\tau(\operatorname{sgn}(n)(\operatorname{sgn}(n)-\operatorname{sgn}(m))^2 P_{0,r}aP_{m-n,r}bP_{0,r})\\ &= -4\sum_{k\in\mathbb{Z}} \hat\tau(kP_{0,r}aP_{k,r}bP_{0,r})\end{aligned}$$

where we used cyclicity to insert a further partition of unity, the $\hat\alpha$-invariance, then changed variables $k = m - n$ and computed the sum over the free index

$$\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)(\operatorname{sgn}(n) - \operatorname{sgn}(k+n))^2 = 4k$$

with $\operatorname{sgn}(0) := 1$ for this whole computation.

Now we have

$$\hat\tau\left(\sum_k kP_{0,r}aP_{k,r}bP_{0,r}\right) = \frac{1}{r}\left(\hat\tau(P_{0,r}a[D,b]P_{0,r}) - \hat\tau(P_{0,r}af_r(D)bP_{0,r})\right)$$

with the function

$$f_r = \mathrm{id} - r\lfloor \mathrm{id}/r \rfloor$$

that expresses the difference between D and the approximation $\sum_k rkP_{k,r}$.

For $G = \mathbb{T}$ we have $\sigma(D) = \mathbb{Z}$, such that for $r = 1$ the error term becomes identically zero. In the case $G = \mathbb{R}$ one estimates

$$\left|\frac{1}{r}\hat{\tau}(P_{0,r} a f_r(D) b P_{0,r})\right| \leq \frac{1}{r}\hat{\tau}(P_{0,r}) \cdot \|a\| \, \|b\| \sup_{x\in(0,1]} |f_r(x)|$$

and since the first factor is equal to $\mathcal{N}$ and the supremum tends to 0 as $r \to 0$, the error term vanishes in the limit.

For the first term we get in either case from (4.3)

$$\frac{1}{r}\hat{\tau}(P_{0,r} a[D,b] P_{0,r}) = \frac{-i}{2\pi}\tau(u\delta u^*)\frac{1}{r}\hat{\tau}(P_{0,r}) = \frac{-i\mathcal{N}}{2\pi}\tau(u\delta u^*)$$

as the definitions of π and D imply

$$[D, \pi(u^*)] = \frac{-i}{2\pi}\pi(\delta(u^*)).$$

We therefore conclude

$$\hat{\tau}(S[S,a][S,b]) = 4\frac{i\mathcal{N}}{2\pi}\tau(u\delta u^*)$$

and the Leibniz formula combined with the α-invariance of the trace give us

$$\tau(u\delta u^*) = -\tau(u^*\delta u)$$

such that

$$\hat{\tau}(S[S,a][S,b]) - \hat{\tau}(S[S,b][S,a]) = 8\frac{i\mathcal{N}}{2\pi}\tau(u\delta u^*).$$

Substituting back gives the result. □

The multiplicative constant originates from the normalizations of D and the dual trace and may change in the applications.

We will now interpret this result as a pairing between cyclic cohomology and K-theory. Let $u \in \mathcal{U}_n(B)$ be a unitary element defining a class $[u]_1 \in K_1(B)$. We want to compute an image under the index map $\mathrm{Ind} : K_1(B) \to K_0(\hat{\mathscr{C}}(B,C))$ for the sequence (7.2).

Note that $\hat{u} := T_u \in M_n(\mathscr{T}(\hat{B},C))$ is a lift of u to a contraction $\|T_u\| \leq 1$. Hence we may compute the index as in Proposition 2.3:

As P is the unit of $\hat{\mathscr{T}}(B,C)$ we have the unitary lift

$$v = \begin{pmatrix} \hat{u} & (P - \hat{u}\hat{u}^*)^{1/2} \\ -(P - \hat{u}^*\hat{u})^{1/2} & \hat{u}^* \end{pmatrix}$$

such that

$$\mathrm{Ind}[u]_1 = \left[v\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}v^*\right]_0 - [P]_0 = \left[\begin{pmatrix} \hat{u}\hat{u}^* & \hat{u}(P - \hat{u}^*\hat{u})^{1/2} \\ (P - \hat{u}^*\hat{u})^{1/2}\hat{u}^* & P - \hat{u}^*\hat{u} \end{pmatrix}\right]_0 - [P]_0, \quad (8.2)$$

where we identify the unit of $\hat{\mathscr{C}}(B,C)^+$ with P.

We want to pair $K_0(\hat{\mathscr{C}}(B,C))$ with the dual trace $\hat{\tau}$, however, this can only be done if every class in $K_0(\hat{\mathscr{C}})$ can be represented by projections in the unitization of $L^1(\mathcal{E})$. Hence we need to assume from now on that $\hat{\mathscr{C}}(B,C)\cap L^1(\mathcal{E})$ is dense in $\hat{\mathscr{C}}(B,C)$ with respect to the C^*-norm. As remarked under Proposition 5.3, this shows that $K_0(\hat{\mathscr{C}}(B,C)\cap L^1(\mathcal{E})) \simeq K_0(\hat{\mathscr{C}}(B,C))$ and with the isomorphism induced by the embedding map.

The dual trace $\hat{\tau}$ obviously defines a 0-cycle on $M_n(\hat{\mathscr{C}}\cap L^1(\mathcal{E}))$ and therefore the pairing with a projection $p \in M_n((\hat{\mathscr{C}} \cap L^1(\mathcal{E}))^+)$ is given by the extension

$$\langle \hat{\tau}, p\rangle = \hat{\tau}_n(p - s(p)),$$

where $\hat{\tau}_n = \hat{\tau} \otimes \mathrm{Tr}_n$ and $s(p)$ is the scalar part of p, i.e. $p - s(p) \in M_n(L^1(\mathcal{E}))$.

Proposition 8.2. *Assume that $\hat{\mathscr{C}}(B,C)\cap L^1(\mathcal{E})$ is dense in $\hat{\mathscr{C}}(B,C)$.*
Let $u \in L^\infty(A)$ be unitary and $[\pi(u),P] \in L^1(A \rtimes_\alpha G)$. We then have

$$\hat{\tau}_*(\mathrm{Ind}[u]_1) = \hat{\tau}\text{-}\mathrm{Ind}(T_u)$$

with the index taken with respect to the von Neumann algebra $PL^\infty(\mathcal{E})P$.

Proof. We may write $P - \hat{u}^*\hat{u} = c^*c$ with $c = P[\pi(u),P] \in L^1(\mathcal{E}) \cap L^\infty(\mathcal{E})$ which shows that

$$P - \hat{u}^*\hat{u},\ P - \hat{u}\hat{u}^*, (P - \hat{u}^*\hat{u})^{\frac{1}{2}} \in \hat{\mathscr{C}}(B,C) \cap L^1(\mathcal{E}).$$

Considering $P - vv^*$ from the image of the index map (8.2), it is then easy to see that

$$\begin{aligned} s(P - vv^*) &= s\left(\begin{pmatrix} \hat{u}\hat{u}^* & \hat{u}(P-\hat{u}^*\hat{u})^{\frac{1}{2}} \\ (P-\hat{u}^*\hat{u})^{\frac{1}{2}}\hat{u}^* & P - \hat{u}^*\hat{u}\end{pmatrix}\right) \\ &= s\left(\begin{pmatrix} \hat{u}\hat{u}^* - P & \hat{u}(P-\hat{u}^*\hat{u})^{\frac{1}{2}} \\ (P-\hat{u}^*\hat{u})^{\frac{1}{2}}\hat{u}^* & P - \hat{u}^*\hat{u}\end{pmatrix} + \begin{pmatrix} P & 0 \\ 0 & 0\end{pmatrix}\right) = \begin{pmatrix} P & 0 \\ 0 & 0\end{pmatrix} \end{aligned}$$

since the scalar part is just the part proportional to P and $\hat{u}^*\hat{u} \in P + \hat{\mathscr{C}}(B,C)$.

Therefore

$$\langle \hat{\tau}, \mathrm{Ind}[u]_1\rangle = \hat{\tau}_2\left(\begin{pmatrix} \hat{u}\hat{u}^* - P & \hat{u}(P-\hat{u}^*\hat{u})^{\frac{1}{2}} \\ (P-\hat{u}^*\hat{u})^{\frac{1}{2}}\hat{u}^* & P - \hat{u}^*\hat{u}\end{pmatrix}\right) = \hat{\tau}(P - \hat{u}^*\hat{u}) - \hat{\tau}(P - \hat{u}\hat{u}^*)$$

and the last expression coincides with the index by Proposition 4.3. □

Since $M_n(A) \rtimes G = M_n(A \rtimes G)$, the result also holds if one takes a general $u \in \mathcal{U}_n(A)$ with the index taken with respect to $M_n(PL^\infty(\mathcal{E})P)$.

We may therefore write up the results of this section as

Theorem 8.1. *Let B,C be algebras satisfying the conditions of Proposition 7.1 and assume that $\hat{\mathscr{C}}(B,C) \cap L^1(\mathcal{E})$ is dense.*

If $[u]_1 \in K_1(B)$ is represented by a unitary $u \in \mathcal{U}_n(B)$ with $[\pi(u), P] \in M_n(L^1(\mathcal{E}))$ and $u \in W_{1,1}(A)$ then

$$\langle \hat{\tau}, \mathrm{Ind}[u]_1 \rangle = \frac{i\mathcal{N}}{2\pi} \tau(u\delta u^*) = \hat{\tau}\text{-}\mathrm{Ind}(T_u).$$

If we further assume that $W_{1,1}(A) \cap B$ is dense, we can write this as

$$\langle \hat{\tau}, \mathrm{Ind}([u]_1) \rangle = \frac{\mathcal{N}}{2\pi} Wind([u]_1),$$

with the Winding number form associated to δ.

Similar results have been obtained in the past for various different conditions and settings. The most prominent version is the Gohberg-Krein theorem for Toeplitz operators with symbols in $C(\mathbb{T})$. A similar formula for the non-commutative case was shown by Connes [15] for crossed products $A \rtimes_\alpha \mathbb{R}$ using the Connes-Thom-isomorphism. This was later interpreted in terms of the smoothed Toeplitz extension and Breuer-Fredholm index in [25] and extended to the semifinite case in [32]. All these approaches require smooth elements, i.e. $u \in C^2(A)$ or better (and possibly additional summability conditions). In this sense, our index theorem is a strict generalization of the one by Lesch. Newer works such as [12], [44], [2] extend these results and interpret them in terms of semifinite spectral triples and spectral flow. However, those approaches using unbounded spectral triples usually require that δu defines a bounded operator, such that there is also a spectral triple over the bulk algebra. This condition fails in our envisioned applications where $\delta u \in L^1(\mathcal{E})$ is genuinely unbounded (though these problems can probably be solved through some kind of regularization). The cyclic cohomology approach works more easily with the relaxed regularity, as e.g. the winding number form defines a cyclic cocycle irrespective of whether the derivative δu is bounded or not. In contrast, unbounded spectral triples require the manipulation of (in our case unbounded) commutators like $[D, u]$, which may induce the need to regularize the Dirac operator D to apply standard results like the local index formula. Therefore, the Winding number $\mathrm{Wind}([u]_1)$ can not necessarily be interpreted as an index but always as the pairing between a cyclic cocycle and $K_1(B)$.

9 Besov spaces for flows

Given the results of the previous section, we want to know a sufficient condition for

$$[\pi(a), P] \in L^1(A \rtimes_\alpha G)$$

given an element $a \in L^\infty(A)$ that is computable purely in terms of the algebra $L^\infty(A)$.

Commutators of this form are closely related to Hankel operators and it is known that in the classical case a Hankel operator with symbol in $L^\infty(\mathbb{T})$ is in the Schatten-ideal $\mathcal{S}^p$

if and only if the symbol is in the classical Besov space $B^{1/p}_{p,p}(\mathbb{T})$ [31]. In this section we partially extend this result to a non-commutative generalization of the Besov spaces.

We will first vaguely recall the definition of the classical vector-valued Besov spaces over $\mathbb{R}$ [30] (the spaces on the torus are defined analogously) and fill in the details later when discussing the non-commutative case. Let E be a Banach space and consider the Banach space $L^p(\mathbb{R}, E)$ with the usual p-norm. A function $f \in L^2 \cap L^p(\mathbb{R}, E)$ has a Fourier transform $\mathcal{F}f \in L^2(\mathbb{R}, E)$, which allows the definition of so-called Fourier-multipliers $\phi : \mathbb{R} \to \mathbb{C}$ through

$$\phi * f := \mathcal{F}^* \mathrm{Mult}(\phi)\mathcal{F}f$$

and which extends to a map $\phi *\cdot : L^p(\mathbb{R}, E) \to L^p(\mathbb{R}, E)$ if the function ϕ is well-behaved.

There are smooth partitions of unity $(W_k)_{k\in\mathbb{Z}}$, $W_k : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\sum_{k\in\mathbb{Z}} W_k(x) = x, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

and certain other properties such that they form a so-called dyadic decomposition of $\mathbb{R}$. The vector-valued Besov space $B^s_{p,q}(\mathbb{R}, E)$ with $0 < s \leq 1$, $1 \leq p, q \leq \infty$ is then defined as

$$B^s_{p,q}(\mathbb{R}, E) := \{a \in L^p(\mathbb{R}, E) : \sum_{k\in\mathbb{Z}} 2^{qsk} \|W_k * a\|_p^q < \infty\}.$$

The Besov spaces are similar to the fractional Sobolev spaces with regard to smoothness but are better behaved with respect to Fourier multipliers and approximation with analytic functions.

As the classical Besov spaces are defined primarily in terms of Fourier multipliers with certain properties, we first have to generalize this concept to the non-commutative case. In this section let M be a von Neumann algebra with a weakly continuous action α of a one-parameter-group $G = \mathbb{R}$ or $G = \mathbb{T}$ and let τ be a normal and finite α-invariant trace on M.

For a $\mathbb{T}$-action α we may define for an element $a \in L^p(M)$ the Fourier coefficients

$$\phi_x(a) := \int_{\mathbb{T}} \alpha_t(a) e^{i2\pi xt} \mathrm{d}t, \quad x \in \mathbb{Z},$$

which are also in $L^p(M)$ because α is isometric. Many analogues of results for vector-valued Fourier series also hold for this definition of Fourier coefficients, including convergence in the L^p-norm for $1 < p < \infty$ (the ideas of [13] developed for the non-commutative torus mostly carry over to this more general situation).

However, it is not as simple to define Fourier coefficients with respect to an $\mathbb{R}$-action, since the above integrals cannot exist due to isometry. A possible approach would be through approximations using the spectral subspaces introduced by Arveson [3] for abelian automorphism groups. However, since we will deal with analytic properties, it is more convenient to use the spectral decomposition on the Hilbert space $L^2(M)$ instead.

Recall that the induced action of α on the Hilbert space $L^2(M)$ is strongly continuous and therefore generated by an unbounded self-adjoint operator T with

$$\alpha_t(a) = e^{i2\pi T t}a, \quad a \in L^2(M).$$

Indeed, up to a factor T coincides with the derivation $\delta = -i[D, \cdot]$ induced by α on M but acts on a different space. The spectral decomposition of T gives a direct integral decomposition

$$L^2(M) = \int_{\sigma(T)}^{\oplus} H_\lambda \mu(\mathrm{d}\lambda) \tag{9.1}$$

with Hilbert spaces H_λ and a Borel-measure μ on $\sigma(T)$. As we assume that τ is a finite trace, we have $M \subset L^2(M)$ and hence every $a \in M$ has a "Fourier"- decomposition

$$a = \int_{\sigma(T)}^{\oplus} a_\lambda \mu(\mathrm{d}\lambda),$$

with a_λ a section of the field of Hilbert spaces $(H_\lambda)_{\lambda\in\sigma(T)}$. Note that this is a decomposition of a as a vector in the Hilbert space $L^2(M)$ and not as a linear operator, i.e. the decomposition is not compatible with multiplication. Furthermore, it is only defined uniquely up to null sets with respect to μ and we will call the μ-a.s. support of a_λ the Fourier spectrum of a (which coincides with the Arveson spectrum of a as introduced in [3]).

Definition 9.1. *We will call a Fourier multiplier any μ-measurable function $m : \mathbb{R} \to \mathbb{C}$. The action of m on an element $a \in L^2(M)$ is defined by*

$$m * a := \int_{\sigma(T)}^{\oplus} m(\lambda) a_\lambda \mu(\mathrm{d}\lambda),$$

whenever the right hand side defines an element of $L^2(M)$. A multiplier m is called L^p-bounded if this linear map extends to a bounded map $L^p(M) \to L^p(M)$.

By definition we have $m * a = m(T)a$ by functional calculus if this is well-defined and

$$\alpha_t(a) = \int_{\sigma(T)}^{\oplus} e^{i2\pi\lambda t} a_\lambda \mu(\mathrm{d}\lambda).$$

In general it is difficult to decide whether a Fourier multiplier is bounded, however, there is the following useful criterion that also holds in the non-commutative version:

Proposition 9.1. *Let m be the Fourier transform of a bounded function $\widehat{m} \in L^1(\mathbb{R})$. Then m is an L^p-bounded Fourier multiplier for any $1 \le p \le \infty$.*

Proof. For $a \in L^2(M)$ we have

$$m * a = \int_{\sigma(T)}^{\oplus} m(\lambda) a_\lambda \mu(\mathrm{d}\lambda) = \int_{\sigma(T)}^{\oplus} \left(\int_{\mathbb{R}} \widehat{m}(t) e^{i2\pi t\lambda} \mathrm{d}t \right) a_\lambda \mu(\mathrm{d}\lambda) = \int_{\mathbb{R}} \widehat{m}(t) \alpha_t(a) \mathrm{d}t \tag{9.2}$$

Hence, for any $a \in L^2(M) \cap L^p(M)$ we find that

$$\|m * a\|_p \leq \|\widehat{m}\|_1 \|a\|_p.$$

□

The Besov norm is defined in terms of a dyadic decomposition with smooth Fourier multipliers, for which we now give one out of several possible definitions.

Let ϕ be a Schwartz function on $\mathbb{R}$ that satisfies the following properties:

$$\begin{aligned} &\mathrm{supp}(\phi) \subset [-2, -2^{-1}] \cup [2^{-1}, 2] \\ &\phi > 0 \text{ on } (-2, -2^{-1}) \cup (2^{-1}, 2) \\ &\sum_{k \in \mathbb{Z}} \phi(2^{-k}x) = 1, \quad \forall x \in \mathbb{R} \setminus \{0\} \end{aligned} \tag{9.3}$$

If ψ is any smooth positive function with support exactly in $(-2, -2^{-1}) \cup (2^{-1}, 2)$, we can rescale it to also fulfill the third condition by setting

$$\phi(x) := \left(\sum_{k \in \mathbb{Z}} \psi(2^{-k}x) \right)^{-1} \psi(x).$$

Hence there are many possible ϕ that satisfy these conditions.

For any such ϕ we have a so-called dyadic decomposition $(W_k)_{k \in \mathbb{Z}}$, $W_k := \phi(2^{-k} \cdot)$, which by Proposition 9.1 consists of L^p-bounded multipliers for all $p \geq 1$.

We can now define the Besov spaces as in the classical case [43] by replacing the usual notion of Fourier multipliers with our non-commutative version. Furthermore, this construction can be seen as a generalization of [13], which studies Fourier multipliers and Besov spaces on the non-commutative torus.

Definition 9.2. *Let ϕ, W_k be as above. The Besov space $B^s_{p,q}$ with $0 < s \leq 1$, $1 \leq p, q \leq \infty$ is defined as*

$$B^s_{p,q} := \{a \in L^p(M) : \sum_{k \in \mathbb{Z}} 2^{qsk} \|W_k * a\|_p^q < \infty\}$$

One can argue that, as in the classical case, this definition does not depend on the choice of ϕ, but for this work we just assume that some admissible function has been fixed once and for all.

For the von Neumann algebras $L^\infty(\mathbb{T})$ and $L^\infty(\mathbb{R})$ with the integral trace and α acting by translation, the definitions of Fourier multipliers and Besov spaces can be seen to be equivalent to the classical notions (for the latter we have to drop the assumption that the trace is finite which is irrelevant for the definition).

We also note that the main difference to the classical vector-valued Besov spaces is not really the lack of commutativity (since multiplication is not required at all) but rather that the non-commutative versions do not admit evaluations at points, i.e. they are no function spaces in general.

Proposition 9.2. *The Besov space $B^s_{p,q}$ is a Banach space with the norm*

$$\|a\|_{B^s_{p,q}} := \|a\|_p + \left(\sum_{k\in\mathbb{Z}} 2^{qsk}\|W_k * a\|_p^q\right)^{1/q}.$$

Proof. We will just show completeness. If $(a_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $B^s_{p,q}$-norm, then we have in p-norm $a_n \to a$ for some $a \in L^p(M)$ and the second term

$$\left(2^{sk}W_k * a\right)_{k\in\mathbb{N}}$$

can be seen to converge in the sequence space $\ell^q(\mathbb{Z}; L^p(M))$. We therefore also have pointwise convergence, such that for any $k \in \mathbb{Z}$

$$W_k * a_n \overset{n\to\infty}{\to} b_k$$

in p-norm for some b_k in $L^p(M)$. As the application of a bounded multiplier is continuous, we conclude $W_k * a = b_k$ for all $k \in \mathbb{Z}$ and therefore $a \in B^s_{p,q}$. □

We will now give sufficient conditions for the commutator $[P, \pi(a)] \in M \rtimes_\alpha G$ to be in $L^1(M \rtimes_\alpha G)$ with respect to the dual trace, where π is the canonical embedding of M in $M \rtimes_\alpha G$ and $P = \chi_{(0,\infty]}(D)$.

For convenience we won't work with the commutator directly:

Definition 9.3. *Denote by*

$$H_a := P\pi(a)(1-P) \in L^\infty(M \rtimes_\alpha G)$$

the Hankel operator with symbol $a \in L^\infty(M)$.

Since

$$[P, \pi(a)] = H_a - (H_{a^*})^*,$$

it is enough to show trace class properties for Hankel operators.

The proof presented here for $p = 1$ is based on a variation of the one in [31] for the classical vector-valued case. The main technical difference is that we cannot use the singular value decomposition for trace class operators in a general von Neumann algebra and therefore use the spectral decomposition for self-adjoint elements instead.

Lemma 9.1. *Let $a \in L^1(M) \cap M$ and assume that a is supported only on the spectral subspace of T corresponding to the interval $(-n, n) \cap \sigma(T)$. Then*

$$\|H_a\|_1 \le C(n+1)\|a\|_1$$

with a constant C that does not depend on a and n.

Proof. Note that a^* also fulfills the spectral condition as the Fourier spectrum is just mirrored by the involution (this can be seen by applying formula (9.2) to functions m with compact support). By splitting into $a = 2^{-1}(a + a^*) + 2^{-1}(a - a^*)$ it is therefore enough to prove the statement for $a = a^*$ self-adjoint.

Expanding the element $h(D)\pi(a)g(D)$ with $h, g \in L^1(\hat{G})$ in terms of the multiplication law (3.1) one finds that

$$h(D)\pi(a)g(D) = \int_G f(t)e^{i2\pi Dt}$$

with the convolution kernel $f : G \to M$ given by

$$\begin{aligned} f(s) &= \int_G (\mathcal{F}h)(t)(\mathcal{F}g)(s-t)\alpha_t(a)\mathrm{d}t \\ &= \int_G (\mathcal{F}h)(t)(\mathcal{F}g)(s-t)\int_{\sigma(T)}^{\oplus} e^{i2\pi\lambda t}a_\lambda \mu(\mathrm{d}\lambda)\mathrm{d}t \\ &= N\int_{\hat{G}}\left(\int_{\sigma(T)}^{\oplus} h(\lambda - z)g(z)a_\lambda\mu(\mathrm{d}\lambda)\right)e^{-i2\pi zs}\mathrm{d}z \end{aligned}$$

where N is some constant introduced by the convolution theorem.

Denoting $P_I := \chi_I(D)$ it therefore follows that

$$P_{(0,n]}\pi(a)P_{[-n,0)} = P_{(0,m]}\pi(a)P_{[-m,0]}$$

for all $m > n$ due to the spectral condition. As $P_{(0,m]}$ converges to P strongly by functional calculus and is uniformly bounded, we conclude that

$$H_a = P_{(0,n]}\pi(a)P_{[-n,0)}.$$

Since a is self-adjoint, it has a spectral decomposition and we may approximate it with commuting spectral projections $E_j \in L^\infty(M)$ given by

$$E_j = \chi\left((2j-1)\frac{\|a\|}{2k+1} \leq a < (2j+1)\frac{\|a\|}{2k+1}\right)$$

such that with

$$\nu_j = j\frac{\|a\|}{k}$$

we have

$$a = \sum_{i=-k}^{k} \nu_i E_i + R_k$$

with a remainder R_k that can be made arbitrarily small in both operator- and L^1-norm (since we assume that τ is finite).

For any $b \in L^2(M)$ we have using the formula (4.3) for the dual trace

$$\begin{aligned} \|P_{(0,n]}\pi(b)\|_2^2 &= \hat{\tau}(P_{(0,n]}\pi(b^*b)P_{[0,n]}) \\ &= \hat{\tau}(P_{(0,n]}\pi(b^*b)) = \hat{\tau}(P_{(0,n]}) \cdot \tau(b^*b) \\ &\leq C(n+1)\|b\|_2^2 \end{aligned}$$

with a constant that depends only on the normalization of the dual measure.

It follows using the Cauchy-Schwarz inequality

$$\begin{aligned} \|P_{(0,n]}\pi(a)P_{[-n,0)}\|_1 &\leq \|P_{(0,n]}\sum_{i=1}^{k} \nu_i \pi(E_i)P_{[-n,0)}\|_1 + \|R_k\| \, \|P_{(0,n]}\|_1 \\ &\leq \sum_{i=1}^{k} |\nu_i| \, \|P_{(0,n]}\pi(E_i)P_{[-n,0)}\|_1 + C\|R_k\|(n+1) \\ &\leq \sum_{i=1}^{k} |\nu_i| \, \|P_{(0,n]}\pi(E_i)\|_2 \|\pi(E_i)P_{[-n,0)}\|_2 + C\|R_k\|(n+1) \\ &\leq C(n+1)\sum_{i=1}^{k} |\nu_i| \, \|E_i\|_2^2 + C\|R_k\|(n+1) \\ &= C(n+1)\sum_{i=1}^{k} |\nu_i| \, \tau(E_i) + C\|R_k\|(n+1) \\ &= C(n+1)\tau\left(\sum_{i=1}^{k} |\nu_i| \, E_i\right) + C\|R_k\|(n+1) \end{aligned}$$

and the last expression converges to $C(n+1)\|a\|_1$ as the approximation is refined. □

Proposition 9.3. *For all $a \in M \cap B^1_{1,1}$*

$$H_a \in L^1(M \rtimes_\alpha G)$$

Proof. The function $\widetilde{W} := 1 - \sum_{k\in\mathbb{N}} W_k$ with the pointwise sum is smooth and its support is contained in $[-2,2]$, i.e. it is also a bounded Fourier multiplier. As $\widetilde{W} + \sum_{k\in\mathbb{N}} W_k = 1$ pointwise,

$$a = \widetilde{W} * a + \sum_{k\in\mathbb{N}} W_k * a$$

converges strongly by functional calculus with T. Then

$$H_a = H_{\widetilde{W}*a} + \sum_{k\in\mathbb{N}} H_{W_k * a}$$

also converges strongly, as π is a normal representation.

Noting the support conditions of the W_k and therefore of the Fourier spectra of the $W_k * a$, Lemma 9.1 implies

$$\sum_{k\in\mathbb{N}} \|H_{W_k * a}\|_1 \leq 2C \sum_{k\in\mathbb{N}} 2^k \|W_k * a\|_1 \leq 2C\|a\|_{B^1_{1,1}}.$$

Hence the sum $H_{\widetilde{W}*a} + \sum_{k\in\mathbb{N}} H_{W_k*a}$ converges in $L^1(M \rtimes_\alpha G)$ and its limit must coincide with its strong limit H_a by Lemma 4.1. □

Using the fact that for a self-adjoint we have

$$[P, \pi(a)] = H_a - (H_a)^*$$

and otherwise again splitting into real and imaginary parts we conclude

Corollary 9.1. *Under the same conditions as the Proposition we have*

$$[P, \pi(a)] \in L^1(M \rtimes_\alpha G).$$

Until now we have not shown that the Besov spaces are non-empty. The following proposition gives easy sufficient conditions:

Proposition 9.4. *If $a \in M$ has Fourier spectrum contained in a set of the form $[-r, -\epsilon) \cup \{0\} \cup (\epsilon, r]$ with $r, \epsilon > 0$, then $a \in B^s_{p,q}$ for all $0 < s \leq 1, 1 \leq p, q < \infty$.*

Proof. The series defining the Besov norm has only finitely many non-vanishing terms, which themselves are finite.

□

10 Applications

10.1 The disordered non-commutative torus

In this section we apply the index theorem to algebras of operators on the so-called disordered non-commutative torus. More information and proofs concerning the spaces introduced in this section can be found in [36] and [34].

Many solid state systems can be described by periodic operators (i.e. convolutions) or almost periodic operators (for systems in an irrational magnetic field) on a lattice $\mathbb{Z}^d$. Both cases can be accommodated algebraically with the (non-)commutative torus.

Definition 10.1. *Let $\theta = (\theta_{ij}) \in \mathbb{R}^{d\times d}$ be an anti-symmetric matrix.*

The non-commutative torus $\mathbb{T}^d_\theta$ is defined as the universal C^*-algebra generated by d unitary generators $u_1, ..., u_d$ that satisfy the commutation relations

$$u_i u_j = e^{i\theta_{i,j}} u_j u_i.$$

Alternatively one may construct $\mathbb{T}^d_\theta$ as an iterated crossed product

$$\mathbb{T}^d_\theta = (\mathbb{C} \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z} ... \rtimes_{\alpha_d} \mathbb{Z}$$

with the strongly continuous $\mathbb{Z}$-actions

$$\alpha_k : \mathbb{Z} \times \mathbb{T}^k_{\theta_{(k)}} \to \mathbb{T}^k_{\theta_{(k)}}$$

defined on the generators as

$$\alpha_k(x, u_i) = (u_k)^x u_i (u_k^*)^x, \quad x \in \mathbb{Z}.$$

The matrix θ consists of the magnetic field strengths and for $\theta = 0$ the action is trivial such that

$$\mathbb{T}^d_{\theta=0} = \mathbb{C} \rtimes \mathbb{Z}^d \simeq C(\mathbb{T}^d).$$

We also want to describe systems with random disorder, i.e. the convolution kernels may depend on the location on the lattice but only through ergodic random variables, such that the system is homogeneous at large scales. For simplicity we consider only product measures.

Definition 10.2. *Let $(\Omega_0, \mathbb{P}_0)$ be a probability space with the Borel-σ-algebra, with Ω_0 a compact topological space and $\mathbb{P}_0$ a probability measure with full support on Ω_0 . The total disorder space is the product*

$$\Omega = \Omega_0^{\mathbb{Z}^d}$$

with the product measure $\mathbb{P}$.

On the disorder space Ω there is the ergodic $\mathbb{P}$-invariant action

$$T : \mathbb{Z}^d \times \Omega \to \Omega, \quad (T_x(\omega))_y = \omega_{y-x}, \quad x, y \in \mathbb{Z}.$$

Definition 10.3. *Let $B = (B_{ij}) \in \mathbb{R}^{d \times d}$ be an anti-symmetric matrix and $(\Omega, \mathbb{P})$.*

The disordered non-commutative torus $\mathbb{T}^d_{\theta,\Omega}$ is defined as the universal C^*-algebra generated by d unitary generators $u_1, ..., u_d$ and the continuous functions $C(\Omega)$ together with the commutation relations

$$u_i u_j = e^{iB_{i,j}} u_j u_i$$

$$f u_j = u_j (f \circ T_{e_j}) \quad \forall f \in C(\Omega).$$

Denoting for $x \in \mathbb{Z}^d$ the monomial

$$u^x = u_1^{x_1} u_2^{x_2} ... u_d^{x_d}$$

the (disordered) non-commutative torus may be considered the completion of the algebra spanned by the formal Fourier series

$$a = \sum_{x \in \mathbb{Z}^d} a_x u^x$$

with only finitely many non-vanishing coefficients $a_x \in \mathbb{C}$ or $a_x \in C(\Omega)$.

On $\mathbb{T}^d_{\theta,\Omega}$ we have the strongly continuous $\mathbb{T}^d$-action

$$\rho_k(a) := \sum_{x\in\mathbb{Z}^d} e^{i2\pi k\cdot x} a_x u^x$$

which is dual to the twisted $\mathbb{Z}^d$ action and allows us to make explicit the Fourier coefficients

$$\psi_x(a) := \int_{\mathbb{T}^d} \rho_k(a(u^x)^*)\mathrm{d}k$$

where one integrates with respect to the normalized Haar-measure of the torus. The Fourier coefficients lie in the fixed-point-algebra, which is isomorphic to $\mathbb{C}$ respectively $C(\Omega)$. The coefficient maps allow us to write any element as a formal sum

$$a = \sum_{x\in\mathbb{Z}^d} \psi_x(a)u^x$$

even if this expression does not necessarily converge in the C^*-norm (it does however converge in an average sense).

Using the Fourier coefficients we define the finite continuous faithful trace

$$\tau : \mathbb{T}^d_{\theta,\Omega} \to \mathbb{C}, \quad \tau(a) = \int_\Omega \mathbb{P}(\mathrm{d}\omega)\psi_0(a,\omega).$$

As ψ_0 is invariant under ρ, the trace is also ρ-invariant.

We consider Hilbert space representations of $\mathbb{T}^d_{\theta,\Omega}$ [36]:

Proposition 10.1. *Denote the standard basis of $L^2(\mathbb{Z}^d)$ in Dirac notation by*

$$|x\rangle, \quad x\in\mathbb{Z}^d,$$

for $y\in\mathbb{Z}^d$ define the shift operator as

$$S_y|x\rangle = |x+y\rangle$$

and define the unbounded position operators $X = (X_1, ..., X_d)$ by

$$X_j|x\rangle = x_j|x\rangle.$$

For θ_+ the lower triangular part of θ we have a family $(\pi_\omega)_{\omega\in\Omega}$ of $*$-representations of $\mathbb{T}^d_{\theta,\Omega}$ defined on the generators by

$$\pi_\omega(u_j) = e^{i\langle e_j|\theta_+|X\rangle} S_j$$

and

$$\pi_\omega(f) = \sum_{x\in\mathbb{Z}^d} f(T_x\omega)|x\rangle\langle x|.$$

The representations are non-degenerate and faithful for $\mathbb{P}$-almost all $\omega \in \Omega$ and are covariant with respect to ρ with the strongly continuous unitary representation U of $\mathbb{T}^d$ defined by

$$U(k) := e^{2\pi i k \cdot X}$$

such that

$$U(k)\pi_\omega(a)U(k)^* = \pi_\omega(\rho_k(a)).$$

For fixed ω the representation π_ω is almost surely faithful as we assume that the probability measure is ergodic with respect to translations. A family of operators $(A_\omega)_{\omega\in\Omega}$ with $A_\omega = \pi_\omega(a)$ for some element $a \in \mathbb{T}^d_{\theta,\Omega}$ is usually called an ergodic family of operators.

The ergodicity implies a self-averaging property

$$\tau(a) = \int_\Omega \mathbb{P}(\mathrm{d}\omega)\langle 0|\pi_\omega(a)|0\rangle = \lim_{L\to\infty} \frac{1}{(2L)^d} \sum_{\substack{x\in\mathbb{Z}^d \\ \|x\|_\infty < L}} \langle x|\pi_\omega(a)|x\rangle$$

for almost every $\omega \in \Omega$, i.e. τ is the trace per unit volume.

The Hilbert space $L^2(\mathbb{T}^d_{\theta,\Omega})$ will be the completion of $\mathbb{T}^d_{\theta,\Omega}$ under the L^2-norm

$$\tau(a^* b) = \int_\Omega \mathbb{P}(\mathrm{d}\omega) \sum_{x\in\mathbb{Z}^d} a_x^*(\omega) b_x(\omega)$$

and is isometrically isomorphic to $L^2(\mathbb{Z}^d \times \Omega)$ with the isomorphism given by

$$a \in \mathbb{T}^d_{\theta,\Omega} \mapsto \int_\Omega^\oplus \mathbb{P}(\mathrm{d}\omega)\pi_\omega(a)|0\rangle \in L^2(\mathbb{Z}^d \times \Omega),$$

i.e. the GNS-representation constructed from τ is equivalent to

$$\pi_\tau = \int_\Omega^\oplus \mathbb{P}(\mathrm{d}\omega)\pi_\omega.$$

Hence we can define $L^\infty(\mathbb{T}^d_{\theta,\Omega})$ using this representation, set

$$L^\infty(\mathbb{T}^d_{\theta,\Omega}) = (\pi_\tau(\mathbb{T}^d_{\theta,\Omega}))''$$

as in the constructions of Proposition 4.1 and ρ extends to a weakly continuous action. The Fourier coefficient maps ψ_x extend to $L^\infty(\mathbb{T}^d_{\theta,\Omega})$ such that we can still write elements of $L^\infty(\mathbb{T}^d_{\theta,\Omega})$ as formal trigonometric series. Since the Fourier coefficients of a can be recovered from $\pi(a)$ by using the matrix elements

$$(\psi_x(a))(\omega) = e^{\phi(\theta,x)}\langle 0|\pi_\omega(a)|x\rangle$$

with $\phi(\theta, x)$ some phase factor, the Fourier series at least exists in the weak topology. The trace τ extends to a ρ-invariant finite normal trace that is again given by the expectation value of ψ_0.

In the following we usually identify $\mathbb{T}^d_{\theta,\Omega}$ and $L^\infty(\mathbb{T}^d_{\theta,\Omega})$ with their respective images under the faithful representation π_τ.

10.2 Half-spaces

Given a unit vector $\xi \in S^{d-1}$ we may define the $\mathbb{R}$-action

$$\alpha : \mathbb{R} \times \mathbb{T}^d_{\theta,\Omega} \to \mathbb{T}^d_{\theta,\Omega}, \quad \alpha_t(a) := \rho_{\xi \cdot t}(a).$$

By construction the action defines a strongly continuous group of automorphisms that extends to a weakly continuous group of automorphisms of $L^\infty(\mathbb{T}^d_{\theta,\Omega})$ and the generator of α in $L^2(\mathbb{T}^d \times \Omega)$ is given by

$$D_\xi = \xi \cdot X$$

with X the position operator.

Denote by $\Gamma_\xi := \xi \cdot \mathbb{Z}^d$ the discrete subgroup of $\mathbb{R}$ that coincides with the point spectrum of $\xi \cdot X$. The components of ξ are rationally dependent (i.e. ξ is a scalar multiple of a vector in $\mathbb{Q}^d$) if and only if the closure $\overline{\Gamma_\xi}$ is discrete in $\mathbb{R}$, otherwise Γ_ξ is dense in $\mathbb{R}$.

In the rationally dependent case we have $\Gamma_\xi = \Lambda_\xi \mathbb{Z}$, with Λ_ξ the smallest positive element of Γ_ξ. The action has the minimal period Λ_ξ^{-1} and therefore restricts to a $\mathbb{T}$-action. In either case, since the trace τ is ρ-invariant, it is also α-invariant and we can proceed in constructing the crossed products as above:

The crossed product $\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G$ with $G = \mathbb{R}$ (or $G = \mathbb{T}$ where applicable) is defined by its regular representation $\pi \times \lambda$ on the Hilbert space $L^2(G, L^2(\mathbb{Z}^d \times \Omega))$ and is given by the C^*-algebraic span of the products

$$\pi(a) f(D)$$

with the representations on $L^2(\mathbb{Z}^d \times \Omega)$ defined by

$$\pi(a) f(D) \mapsto \pi_\tau(a) f(D_\xi), \quad \forall a \in \mathbb{T}^d_{\theta,\Omega}, f \in C_0(\hat{G}).$$

An element $\hat{a} \in \mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G$ can still be considered a covariant family $\hat{a} = (\hat{a}_\omega)_{\omega \in \Omega}$ with $\hat{a}_\omega = (\pi_\omega \times U)(\hat{a})$ in the integrated representation as the covariant representation factors $\pi_\tau \times U = \int_\Omega^\oplus \mathbb{P}(\mathrm{d}\omega) \pi_\omega \times U$.

Note that $f(D_\xi) = f(\xi \cdot X)$ is a multiplication operator that only depends on the displacement in the direction ξ relative to some arbitrary reference point. In particular $\Pi_\xi := \chi_{\mathbb{R}^+}(\xi \cdot X)$ is the restriction to the half-space of all $x \in \mathbb{Z}^d$ with $\xi \cdot x > 0$. Hence the crossed product can be used to describe restrictions of elements of $\mathbb{T}^d_{\theta,\Omega}$ to half-spaces, i.e. physical systems with a boundary.

However, the representation of $\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha \mathbb{R}$ on $L^2(\mathbb{Z}^d \times \Omega)$ can only be faithful if Γ_ξ is dense or if Γ_ξ is discrete and $G = \mathbb{T}$. For if $\sigma(D_\xi) = \overline{\Gamma_\xi}$ is discrete, all elements $f(D)$ with $\mathrm{supp}(f) \cap \Gamma_\xi = \emptyset$ will be mapped to zero. This is the main reason why we covered both cases $G = \mathbb{R}$ and $G = \mathbb{T}$ in this work.

We also note that for rationally independent ξ it would be in some ways more natural to consider crossed products with the dual group of Γ_ξ with the discrete topology instead of $\mathbb{R}$. However, this would eventually lead to the same von Neumann algebras as for the $\mathbb{R}$-action, since $L^2(\mathbb{R}) \simeq L^2(\hat{\Gamma}_\xi)$ and the generator D would have $\sigma(D) = \mathbb{R}$ anyways, so we may work with the continuous versions instead.

The construction of section 4.1 then gives us the crossed products $L^\infty(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$ with a dual trace that is densely defined by

$$\hat{\tau}(\pi(a)f(D)) = \tau(a) \int_{\hat{G}} f(x)\mu(\mathrm{d}x)$$

for some Haar measure μ on $\hat{G}$. We fix the normalization such that for $G = \mathbb{R}$ we have

$$\hat{\tau}(f(D)) = \int_{\mathbb{R}} f(x)\mathrm{d}x$$

and for $G = \mathbb{T}$

$$\hat{\tau}(f(D)) = \Lambda_\xi \sum_{k \in \mathbb{Z}} f(\Lambda_\xi k).$$

Recall that up to now it was understood that the action is rescaled to be 1-periodic. We now use a different convention and indicate the changes when necessary. For the physical interpretation it is more convenient to choose $\sigma(D) = \Lambda_\xi \mathbb{Z}$ such that the spectrum of D labels the orthogonal distances of lattice points from the hyperplane $\xi \cdot \mathbb{R}^d = 0$. The idea is that $\hat{\tau}(f(D_\xi))$ for a slowly varying function $f \in C_0(\mathbb{R})$ should give approximately the same value for all $\xi \in S^{d-1}$.

Another motivation for the normalization is

Proposition 10.2. *Assume ξ rationally dependent with Λ_ξ as above. Define the cubes*

$$C_n = \{x \in \mathbb{Z}^d : -n\Lambda_\xi \leq \xi \cdot x \leq n\Lambda_\xi, \|x - (\xi \cdot x)\xi\|_\infty \leq n\}$$

and the trace per unit area

$$\hat{\mathcal{T}}(\hat{a}_\omega) = \lim_{n\to\infty} \frac{1}{(2n)^{d-1}} \sum_{x \in C_n} \langle x|\hat{a}_\omega|x\rangle$$

for $\hat{a} \in \mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha \mathbb{T}$.

Then for almost all $\omega \in \Omega$ we have

$$\hat{\tau}(\hat{a}) = \hat{\mathcal{T}}(\hat{a}_\omega).$$

Proof. We check the formula on the generators $\pi(a)f(D)$:

$$\hat{\mathcal{T}}(\pi_\omega(a)f(\xi \cdot X)) = \lim_{n\to\infty} \frac{1}{(2n)^{d-1}} \sum_{x \in C_n} f(\xi \cdot x)\langle x|\pi_\omega(a)|x\rangle$$

We rewrite the sum with the slices $S_k := \{x \in \mathbb{Z}^d | \, \xi \cdot x = k\Lambda_\xi\}$:

$$\sum_{x \in C_n} f(\xi \cdot x)\langle x|\pi_\omega(a)|x\rangle = \sum_{k=-n}^{n} \sum_{x \in S_k \cap C_n} f(k\Lambda_\xi)\langle x|\pi_\omega(a)|x\rangle.$$

Asymptotically in n each $S_k \cap C_n$ contains approximately the same number of elements independent of k and thus

$$|S_k \cap C_n| \sim \frac{|C|_n}{2n+1} \sim \frac{\Lambda_\xi (2n+1)^d}{2n+1} = \Lambda_\xi (2n)^{d-1}.$$

The self-averaging property applied to each S_k separately then gives $\mathbb{P}$-almost surely

$$\hat{\mathcal{T}}(\pi_\omega(a) f(\xi \cdot X)) = \sum_{k \in \mathbb{Z}} \Lambda_\xi f(k\Lambda_\xi)\tau(a).$$

□

One would expect that a similar formula also holds for irrational ξ but the combinatorics seem fairly complicated.

In the following we will always denote the angular parameter as ξ and keep the dependence of α and $\hat{\tau}$ on ξ implicit. We set $G = \mathbb{T}$ if ξ is rationally dependent and $G = \mathbb{R}$ if otherwise.

10.3 Summability

In order to check the summability criteria we have to consider the Fourier decomposition with respect to α. The action of α on some $a = \sum_{x \in \mathbb{Z}^d} a_x u^x$ is given by

$$\alpha_t(a) = \sum_{x \in \mathbb{Z}^d} e^{i2\pi t(\xi \cdot x)} a_x u^x$$

and therefore its generator on $L^2(\mathbb{Z}^d \times \Omega) \simeq L^2(\mathbb{T}^d_{\theta,\Omega})$ is given by the multiplication operator $T = \xi \cdot X$ or more explicitly

$$Ta = \sum_{x \in \mathbb{Z}^d} (\xi \cdot x) a_x u^x$$

which generates the Fourier decomposition as in (9.1). Due to our choice of representation T happens to coincide with D_ξ but we prefer to keep the notation consistent with Section 9. The position basis provides an orthonormal basis for T and we define

$$E_\lambda := \{x \in \mathbb{Z}^d \,|\, \xi \cdot x = \lambda\}$$

and denote $V_\lambda = L^2(E_\lambda \times \Omega)$. The spectral decomposition is thus given by

$$L^2(\mathbb{T}^d_{\theta,\Omega}) = \bigoplus_{\lambda \in \sigma(T)} V_\lambda$$

with only countably many non-vanishing summands.

The action of a Fourier multiplier $m : \mathbb{R} \to \mathbb{C}$ is therefore given by

$$m * a = \sum_{\lambda \in \sigma(T)} \sum_{x \in E_\lambda} m(\lambda) a_x u^x = \sum_{x \in \mathbb{Z}^d} m(\xi \cdot x) a_x u^x.$$

The Fourier spectrum is hence given by all $\lambda \in \sigma(T)$ for which there is an $x \in E_\lambda$ with $a_x \neq 0$. By Proposition 9.4 any element with only finitely many Fourier coefficients (sometimes called finite hopping range) is contained in any Besov space $B^s_{p,q}(\mathbb{T}^d_{\theta,\Omega})$.

Writing out the Besov norm gives

$$\|a\|_{B^s_{p,q}} = \|a\|_p + \left(\sum_{k\in\mathbb{Z}} \left\| \sum_{x\in\mathbb{Z}^d} 2^{qsk} W_k(\xi \cdot x) a_x u^x \right\|_p^q \right)^{1/q}$$

and we now establish sufficient conditions for elements with rapidly decaying Fourier coefficients:

Proposition 10.3. *For $a = \sum_{x\in\mathbb{Z}^d} a_x u^x \in L^\infty(\mathbb{T}^d_{\theta,\Omega})$ one has*

$$\|a\|_{B^1_{1,1}} \leq \sum_{x\in\mathbb{Z}^d} (2 + 6|x|) \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)|.$$

Proof. By [36, 3.3.5] we have for any $a \in L^\infty(\mathbb{T}^d_{\theta,\Omega})$ the estimate

$$\|a\|_1 \leq 2 \sum_{x\in\mathbb{Z}^d} \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)|. \tag{10.1}$$

We apply this to the Besov norm to get

$$\begin{aligned}\|W_k * a\|_1 &\leq 2 \sum_{\pm\lambda\in(2^{k-1},2^{k+1})} \sum_{x\in E_\lambda} |W_k(\lambda)| \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)| \\ &\leq 2 \sum_{\pm\lambda\in(2^{k-1},2^{k+1})} \sum_{x\in E_\lambda} \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)|\end{aligned}$$

where the sum again only has countably many non-vanishing terms. Since any $\lambda \in \mathbb{R}\setminus\{0\}$ is contained in the support of at most two (consecutive) of the W_k, we estimate

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^k \|W_k * a\|_1 &\leq 2 \sum_{k\in\mathbb{Z}} 2^k \sum_{\pm\lambda\in(2^{k-1},2^{k+1})} \sum_{x\in E_\lambda} \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)| \\ &= 2 \sum_{\lambda\in\sigma(T)\setminus\{0\}} \sum_{x\in E_\lambda} (2^{\lceil \log_2(|\lambda|)\rceil - 1} + 2^{\lceil \log_2(|\lambda|)\rceil}) \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)| \\ &\leq 2 \sum_{\lambda\in\sigma(T)\setminus\{0\}} \sum_{x\in E_\lambda} (|\lambda| + 2|\lambda|) \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)| \\ &= \sum_{\lambda\in\sigma(T)\setminus\{0\}} \sum_{x\in E_\lambda} 6|\xi \cdot x| \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)| \\ &\leq 6 \sum_{x\in\mathbb{Z}^d} |x| \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)|\end{aligned}$$

where we just changed the order of summation and hence

$$\|a\|_{B^1_{1,1}} = \|a\|_1 + \sum_{k\in\mathbb{Z}} 2^k \|W_k * a\|_1 \leq \sum_{x\in\mathbb{Z}^d} (2+6|x|) \int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)|.$$

□

Corollary 10.1. *Let $a = \sum_{x\in\mathbb{Z}^d} a_x u^x \in L^\infty(\mathbb{T}^d_{\theta,\Omega})$ and assume that a decays exponentially, i.e. there exist $A, \gamma > 0$ such that*

$$\int_\Omega \mathbb{P}(\mathrm{d}\omega)|a_x(\omega)| < Ae^{-\gamma|x|}.$$

Then we have $a \in B^1_{1,1}$.

For periodic systems we can give a much better characterization that shows that the Besov norm can be finite even if the Fourier coefficients only have a slow polynomial decay. For the rest of this section we therefore assume $\theta = 0$, $\Omega = \{*\}$. We then have $\mathbb{T}^d_\theta = C(\mathbb{T}^d)$ and $L^\infty(\mathbb{T}^d_\theta) = L^\infty(\mathbb{T}^d)$ as well as $L^p(\mathbb{T}^d_\theta) = L^p(\mathbb{T}^d)$, where all isomorphisms are induced by the Fourier transform

$$a = \sum_{x\in\mathbb{Z}^d} a_x u^x \mapsto a \in L^p(\mathbb{T}^d), \quad a(k) = \sum_{x\in\mathbb{Z}^d} a_x e^{i2\pi x\cdot k}.$$

That this is an isomorphism is clear for $C(\mathbb{T}^d)$ because of the density of trigonometric polynomials and it extends to an isomorphism of the von Neumann algebras. Since the map preserves the trace, i.e.

$$\tau(a) = \psi_0(a) = \int_{\mathbb{T}^d} \rho_k(a)\mathrm{d}k = \int_{\mathbb{T}^d} a(k)\mathrm{d}k,$$

it defines an isometric isomorphism of the L^p-spaces. The actions ρ and α become translations on $\mathbb{T}^d$

$$\rho_q(a)(k) = a(k-q), \quad \alpha_t(a)(k) = a(k-\xi t)$$

and the Fourier coefficients coincide with the classical ones. For any $a \in L^p(\mathbb{T}^d)$ consider the function

$$f_a : \mathbb{R} \to L^p(\mathbb{T}^d), \quad \phi_a(t) := \alpha_t(a) = a(\cdot - \xi t).$$

If α is periodic (i.e. ξ is rationally dependent) f_a is periodic with the period given by Λ_ξ^{-1}. The key insight is that we may then write f_a as a function of $\mathbb{T}$ and the Fourier coefficients with respect to α coincide with the $L^p(\mathbb{T}^d)$-valued Fourier coefficients of f_a. Therefore also the Fourier multipliers become classical Fourier multipliers on the torus and the Besov-norms coincide with those of the classical vector-valued Besov spaces.

Unfortunately, the analogous construction fails for the non-periodic case because $f_a(t)$ then is almost-periodic and therefore not integrable with respect to t. The following proposition must therefore restrict to the periodic case.

Proposition 10.4. *If Γ_ξ is discrete, the Besov spaces with respect to α isometrically embed into the vector-valued Besov spaces $B^s_{p,q}(\mathbb{T}, L^p(\mathbb{T}^d))$.*

Hence we have so-called characterization by differences:

The norm $\|a\|_{B^s_{p,q}}$ equivalent to the norm

$$\|a\|_p + \left(\int_{[0,1]} \frac{\|\alpha_t(a) + \alpha_{-t}(a) - 2a\|_p^q}{t^{1+sq}} \mathrm{d}t \right)^{1/q}.$$

Proof. The embedding as defined above is given by

$$a \mapsto f : \mathbb{T} \to L^p(\mathbb{T}^d), \quad f(t) := \alpha_t(a)$$

and we will check that it is an isometry. As $\sigma(T) = \Lambda_\xi \mathbb{Z}$ we write

$$f(t) = \sum_{n \in \mathbb{Z}} f_n e^{i2\pi n \Lambda_\xi t}$$

with

$$f_n = \sum_{x \in E_{n\Lambda_\xi}} a_x u^x.$$

Now compute the action of W_k on f

$$\begin{aligned}
(W_k * f)(t) &= \sum_{n \in \mathbb{Z}} W_k(n\Lambda_\xi) f_n e^{i2\pi n \Lambda_\xi t} = \sum_{n \in \mathbb{Z}} W_k(n\Lambda_\xi) \sum_{x \in E_{n\Lambda_\xi}} a_x u^x e^{i2\pi n \Lambda_\xi t} \\
&= \sum_{n \in \mathbb{Z}} W_k(n\Lambda_\xi) \alpha_t \left(\sum_{x \in E_{n\Lambda_\xi}} a_x u^x \right) = \alpha_t \left(\sum_{n \in \mathbb{Z}} W_k(n\Lambda_\xi) \sum_{x \in E_{n\Lambda_\xi}} a_x u^x \right) \\
&= \alpha_t(W_k * a)
\end{aligned}$$

and hence

$$\|W_k * f\|_{L^p(\mathbb{T}, L^p(\mathbb{T}^d))} = \left(\int_{\mathbb{T}} \|\alpha_t(W_k * a)\|_p^p \mathrm{d}t \right)^{1/p} = \|W_k * a\|_p$$

since α is an isometry and we choose the torus to have unit volume. This clearly shows that the mapping between the Besov spaces is isometric.

The classical Besov norm for functions $f : \mathbb{T} \to L^p(\mathbb{T}^d)$ is equivalent to (see [43] for the scalar case and [30] for comments on the vector-valued case)

$$\|f\|_{L^p(\mathbb{T}, L^p(\mathbb{T}^d))} + \left(\int_{[0,1]} \frac{\|f(\cdot + t) + f(\cdot - t) - 2f\|_p^q}{t^{1+sq}} \mathrm{d}t \right)^{1/q}$$

and substituting $f = \alpha_\cdot(a)$ the fact that α is an isometry gives the result. □

It seems plausible that the equivalence of norms also holds for ξ rationally independent since the formula does not depend directly on ξ. In fact it may be possible to show it directly for the non-commutative Besov spaces without a detour to the classical versions, since the characterization by differences can be proven using only the properties of certain Fourier multipliers (as is done in [13] for the non-commutative torus, though in a somewhat different setting).

We now have a convenient criterion that only requires us to estimate the integral of a function on $\mathbb{T}^d$. In particular this shows that being in $B^1_{1,1}$ is not much stronger than Sobolev-$W_{1,1}(\mathbb{T}^d)$-differentiability and therefore does not require rapid decay of the Fourier coefficients.

Let us now quickly comment on the non-commutative Sobolev-spaces. On the non-commutative torus it is usual to consider the Sobolev-spaces with respect to ρ, i.e we choose d independent generators of ρ respectively derivations $\partial_1, ..., \partial_d$ such that formally

$$\partial_j \sum_{x\in\mathbb{Z}^d} a_x u^x = i \sum_{x\in\mathbb{Z}^d} x_j a_x u^x$$

with the corresponding Sobolev spaces given by

$$\begin{aligned} W_{s,k}(\mathbb{T}^d_{\theta,\Omega}) &= \{a \in L^s(\mathbb{T}^d_{\theta,\Omega}) : \sum_{1\leq|\beta|\leq k} \|\partial_1^{\beta_1}..\partial_d^{\beta_d} a\|_s\} \\ &= \{a \in L^s(\mathbb{T}^d_{\theta,\Omega}) : \sum_{1\leq|\beta|\leq k} \left\| \sum_{x\in\mathbb{Z}^d} x_1^{\beta_1}..x_d^{\beta_d} a_x u^x \right\|_s < \infty\}. \end{aligned}$$

We define the derivation associated to α as

$$\delta(a) = \lim_{t\to 0} \frac{\alpha_t(a) - a}{t} = i2\pi \sum_{x\in\mathbb{Z}^d} (\xi \cdot x) a_x u^x = 2\pi \sum_j \xi_j \partial_j a.$$

Applying this to an element $a = \sum_x a_x u^x$, one finds that the Sobolev norm with respect to α is equivalent to

$$\|a\|_{s,k} \sim \sum_{0\leq n\leq k} \|\delta^n a\|_s = \sum_{0\leq n\leq k} \left\| \sum_{x\in\mathbb{Z}^d} (\xi \cdot x)^n a_x u^x \right\|_s$$

Ostensibly, being in $W_{s,k}(\mathbb{T}^d_{\theta,\Omega})$ implies membership of the Sobolev space for α independent of ξ. Furthermore we have for all $a \in W_{1,1}(\mathbb{T}^d_{\theta,\Omega})$:

$$\mathrm{Wind}_\xi([u]_1) := i\tau(u\delta u^*) = 2\pi i \sum \xi_j \tau(u\partial_j u^*) =: 2\pi \sum \xi_j \mathrm{Wind}_j([u]_1)$$

with Wind_j the winding number form for the derivation δ_j. Exponential decay of the Fourier coefficients clearly implies Sobolev for any combination of parameters. For periodic models we can use the Fourier Transformation to reduce to the usual notions of differentiability on the torus, i.e. ∂_j becomes an ordinary partial derivative.

We further note that all results above generalize to elements with a matrix fiber if we take the trace $\tau \otimes \mathrm{Tr}_n$ on $M_n(\mathbb{T}^d_{\theta,\Omega})$ and the respective Schatten norms on the L^p-spaces.

10.4 The index map for chiral Hamiltonians

We consider a system described by a self-adjoint Hamilton operator $h \in M_{2N}(\mathbb{T}^d_{\theta,\Omega})$, $h = h^*$ with a chiral symmetry, i.e.

$$JhJ = -h, \quad J = \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}$$

which is equivalent to

$$h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

with some $a \in M_N(\mathbb{T}^d_{\theta,\Omega})$. A chiral symmetry in solid state physics usually arises from a sublattice symmetry (e.g. on a bipartite lattice). We also note that the bulk-boundary correspondence for such Hamiltonians has been treated in [36] under the assumptions of a spectral gap and axis-parallel boundaries.

An important consequence of the chiral symmetry is

$$f(h) = Jf(-h)J \tag{10.2}$$

for all bounded Borel-functions f. This further simplifies if f is symmetric or anti-symmetric.

We now consider the polar decomposition $h = |h|\mathrm{sgn}(h)$. Since sgn is an odd function, the chiral symmetry allows us to conclude that $\mathrm{sgn}(h)$ is also chiral and therefore off-diagonal

$$\mathrm{sgn}(h) = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \in M_{2N}(L^\infty(\mathbb{T}^d_{\theta,\Omega}))$$

with some $u \in M_N(L^\infty(\mathbb{T}^d_{\theta,\Omega}))$. Since $\mathrm{sgn}(x)^2 = 1 - \chi_{\{0\}}(x)$ we have

$$u^*u \oplus uu^* = 1 - P_{\ker h},$$

which shows that u is a partial isometry whose initial and final projection form an orthogonal decomposition of the range of h.

To simplify the notation we assume in the following that the matrix fiber has dimension $N = 1$, though it is understood that nothing depends on this. Furthermore, we usually will not distinguish between spaces X and $M_2(X)$ if the dimension is either irrelevant or clear from the context.

For the rest of this section we will make the following regularity assumptions:

Assumption 10.1. *1. $h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ with $a \in \mathbb{T}^d_{\theta,\Omega}$ that has only finitely many non-vanishing Fourier coefficients (i.e. finite hopping range).*

2. 0 is not an eigenvalue of $\pi_\tau(h)$ (but possibly $0 \in \sigma(h)$), i.e. 0 is almost surely not an eigenvalue of $\pi_\omega(h)$.

3. *By item 2 the partial isometry $u \in L^\infty(\mathbb{T}^d_{\theta,\Omega})$ of the polar decomposition of a is unitary and we assume it is in $B^1_{1,1}(\mathbb{T}^d_{\theta,\Omega})$ and $W_{1,1}(\mathbb{T}^d_{\theta,\Omega})$*

Consider the C^*-algebra

$$B := C^*(1, a, u) \subset L^\infty(\mathbb{T}^d_{\theta,\Omega}),$$

i.e. the smallest C^*-algebra containing the finite algebraic combinations of a,u and the unit 1. Under the natural inclusion one has $h \in M_2(B)$ and $\mathrm{sgn}(h) \in M_2(B)$.

Since both a and u are in $B^1_{1,1}$, their commutators with $P = \chi_{\mathbb{R}^+}(D)$ lie in $L^1(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$ by Corollary 9.1. Because $L^\infty \cap L^1(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$ is an ideal, the Leibniz-rule of the commutator shows that

$$[\pi(b), P] \in L^1(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$$

for any element $b \in B$ in the finite algebraic span of a and u. Hence, any element in the norm closure B has a compact commutator, that is $B \subset Q\mathbb{T}^d_{\theta,\Omega}$, the quasicontinuous extension of $\mathbb{T}^d_{\theta,\Omega}$. By Proposition 7.1 we therefore have at least the two commuting exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{\mathscr{C}}(B,C) & \longrightarrow & \hat{\mathscr{T}}(B,C) & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \| & & \\
0 & \longrightarrow & \hat{\mathcal{K}}_{\hat{\tau}} & \longrightarrow & \hat{\mathscr{T}}(B,\mathcal{K}_{\hat{\tau}}) & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

where

$$C := C^*([\pi(B), P]) = C^*([\pi(a), P], [\pi(u), P]) \subset L^\infty(\mathbb{T}^d_{\theta,\Omega}) \rtimes_\alpha G$$

is the smallest C^*-algebra containing the commutators of P with $\pi(B)$, $\hat{\mathcal{K}}_{\hat{\tau}} := P\mathcal{K}_{\hat{\tau}}P$ are the half-sided compact elements and $\hat{\mathscr{C}}$, $\hat{\mathscr{T}}$ are defined as in Corollary 7.2.

The index map allows us to relate the class $[u]_1 \in K_1(B)$ to some $[p]_0 - [q]_0 \in K_0(\hat{\mathscr{C}}(B,C))$ or $[p]_0 - [q]_0 \in K_0(\hat{\mathcal{K}}_{\hat{\tau}})$ and since the conditions of Theorem 8.1 are fulfilled, we have in either case the numerical equality

$$\langle \hat{\tau}, \mathrm{Ind}([u]_1) \rangle = \frac{1}{2\pi} \mathrm{Wind}([u]_1) = \sum_j \xi_j \mathrm{Wind}_\xi([u]_1).$$

The normalization is clear for $G = \mathbb{R}$ but we changed the convention for $G = \mathbb{T}$: Since the action is now Λ_ξ^{-1}-periodic instead of 1-periodic, the relation between the generator D of the crossed product and δ incurs an additional factor of Λ_ξ^{-1} that is then canceled by our choice of normalization for the dual trace.

Consider the restriction of h to the half-space

$$\hat{h} := P\pi(h)P \in M_2(\hat{\mathscr{T}}(B,C)),$$

which clearly also satisfies $\hat{h}^* = \hat{h}$ and $J\hat{h}J = -h$. It follows as above that the partial isometry of the polar decomposition of $\hat{h}$ is given by

$$\mathrm{sgn}(\hat{h}) = \begin{pmatrix} 0 & \hat{u} \\ \hat{u}^* & 0 \end{pmatrix} \in M_2\left(L^\infty(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)\right)$$

with a partial isometry $\hat{u} \in L^\infty(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$ satisfying

$$\hat{u}^*\hat{u} \oplus \hat{u}\hat{u}^* = 1 - P_{\ker \hat{h}} = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}.$$

Moreover, since $\hat{h}^2$ is diagonal and $\hat{a} := P\pi(a)P = |\hat{a}|\,\hat{u}$ is a polar decomposition we have

$$\begin{aligned} p_+ &= P_{\mathrm{Ker}\,(\hat{a}\hat{a}^*)} = P_{\mathrm{Ker}\,|\hat{a}|} = 1 - \hat{u}^*\hat{u} \\ p_- &= P_{\mathrm{Ker}\,(\hat{a}^*\hat{a})} = P_{\mathrm{Ker}\,|\hat{a}^*|} = 1 - \hat{u}\hat{u}^*. \end{aligned} \tag{10.3}$$

Contrary to h, the restriction $\hat{h}$ can have a non-trivial kernel and we would like to show

$$\hat{\tau}(JP_{\ker \hat{h}}) = \hat{\tau}_N(p_+ - p_-) = \sum_j \xi_j \mathrm{Wind}_\xi([u]_1).$$

Given the above, a sufficient condition for this is that $p_+, p_- \in \mathcal{K}_{\hat{\tau}}$, such that $[p_+]_0 - [p_-]_0$ defines a class in $K_0(\mathcal{K}_{\hat{\tau}})$ and coincides with the image of the index map.

We will first check this for the case where the bulk Hamiltonian h is invertible. The idea is to lift the unitary u to the partial isometry of a polar decomposition and then use Proposition 2.2 to compute the index map. Since polar decomposition is not available in a general C^*-algebra we remind that one can embed B into the Calkin-algebra using Proposition 7.2:

Proposition 10.5. *Let h, u be as above, but assume further that $0 \notin \sigma(h)$, i.e. h is invertible, and let $\hat{h} = P\pi(h)P + k$ with $k \in M_2\left(\hat{\mathcal{K}}_{\hat{\tau}}\right)$ chirally symmetric $JkJ = -k$. Then $\hat{h}$ is Breuer-Fredholm and*

$$\mathrm{Ind}([u]_1) = [p_+]_0 - [p_-]_0.$$

Proof. We have

$$\hat{h} + \mathcal{K}_{\hat{\tau}} = P\pi(h)P + \mathcal{K}_{\hat{\tau}} = j(h)$$

with $j : B \to M_2\left(L^\infty(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)/\mathcal{K}_{\hat{\tau}}\right)r$ the embedding from Proposition 7.2 and therefore $\hat{h}$ is invertible in the Calkin-algebra. Since $\hat{h}$ is Breuer-Fredholm and chirally symmetric, the partial isometry of its polar decomposition is given by

$$\mathrm{sgn}(\hat{h}) = \begin{pmatrix} 0 & \hat{u} \\ \hat{u}^* & 0 \end{pmatrix}$$

with $\hat{u}$ a partial isometry whose image in the Calkin-algebra is unitary. As $\hat{u}$ is a lift of u we have using (10.3)

$$\mathrm{Ind}([u]_1) = [P - \hat{u}^*\hat{u}]_0 - [P - \hat{u}\hat{u}^*]_0 = [P_{\ker(\hat{a})}]_0 - [P_{\ker(\hat{a}^*)}]_0 = [p_+]_0 - [p_-]_0$$

and, as remarked above, the index map does not depend on the exact sequence that is used. □

If h is not invertible, the problem of relating the kernel of $\hat{h}$ to the index map becomes much harder. Without any further assumptions we can at least state

Proposition 10.6. *Let h, u be as above. There is a norm-continuous path of chiral Hamiltonians $(\hat{h}_t)_{t\in\mathbb{R}^+} \in M_2(\hat{\mathscr{T}}(B,C))$ such that $\hat{h}_0 = P\pi(h)P$ and $\hat{h}_t$ is Breuer-Fredholm for $t > 0$. We therefore have*

$$\mathrm{Ind}([u]_1) = [p_{t,+}]_0 - [p_{t,-}]_0$$

with $p_{t,+}$, $p_{t,-}$ the decomposition of $P_{ker\,(\hat{h}_t)}$ as above.

Proof. Write $S = \mathrm{sgn}(\pi(h))$ and set

$$h_t := h + tS = (|h| + t)S = |h + tS|S$$

with the last equality holding because

$$(h + tS)^2 = h^2 + 2hS + t^2 = |h|^2 + 2t|h| + t^2 = (|h| + t)^2.$$

Clearly h_t is invertible for $t > 0$ and hence

$$\hat{h}_t := P\pi(h_t)P$$

has the required properties. □

This shows that if $\mathrm{Wind}_\xi([u]_1) \neq 0$, the half-space Hamiltonian $\hat{h}$ is arbitrarily close to a gapped, chirally symmetric Hamiltonian that has a non-trivial kernel. That property is manifestly stable under chirally symmetric boundary terms (i.e. compact perturbations). Unfortunately, it is not clear if this allows us to conclude anything about the unmodified Hamiltonian.

We will therefore need an additional analytical property:

Definition 10.4. *A self-adjoint operator $h \in L^\infty(\mathbb{T}^d_{\theta,\Omega})$ defines an integrated density of states (IDOS)-measure*

$$\mu_T(I) := \tau(\chi_I(T)),$$

for all intervals $I \subset \mathbb{R}$ such that $\tau(f(|T|)) = \int_{\sigma(h)} f(E)\mu_T(\mathrm{d}E)$ for all positive measurable functions f.

We say that h has small density of states at zero if there is some $\delta > 0$ such that μ_T is absolutely continuous on the interval $(-\delta, \delta)$ with a density function ρ and constants $A, s > 0$ such that

$$\rho(|\epsilon|) \leq A|\epsilon|^s.$$

Much is known about the IDOS of ergodic Schrödinger operators, e.g. in the absence of a magnetic field the absolute continuity of the density of states measure follows for all periodic Hamiltonians or for disordered models under mild conditions on the random convolution kernels.

If h is chirally symmetric and has an absolutely continuous density of states measure, then the same holds for h^2. For any symmetric function $f(E) = f(-E)$ we have $f(h) = f(|h|)$ and hence for $0 \leq a \leq b$ we get

$$\chi_{(-b,-a)}(h) + \chi_{(a,b)}(h) = \chi_{(a^2,b^2)}(h^2).$$

The chiral symmetry gives us $J\chi_{(-b,-a)}(h)J = \chi_{(a,b)}(h)$ (see (10.2)) and hence

$$\tau(\chi_{(-b,-a)}(h)) = \tau(\chi_{(a,b)}(h)).$$

Therefore the transformation formula allows us to recover the density of states from that of h^2

$$\rho_h(E) = |E|\,\rho_{h^2}(E^2)$$

with ρ_{h^2} the density of μ_{h^2}. Thus a chirally symmetric Hamiltonian will usually have a small density of states at zero, since ρ_{h^2} can have at most an integrable singularity.

The property shows that h is invertible in some sense:

Proposition 10.7. *Assume that h has small density of states at zero. Since* 0 *is not an eigenvalue of h, the inverse h^{-1} at least exists as an unbounded, densely defined operator affiliated to $L^\infty(\mathbb{T}^d_{\theta,\Omega})$.*

We have $h^{-1} \in L^1(\mathbb{T}^d_{\theta,\Omega})$ and

$$\left\| \frac{1}{h} - \frac{1}{h+z} \right\|_1 \leq C_\gamma |\mathrm{Im}\, z|^s + O(|\mathrm{Im}\, z|)$$

uniformly for $z \in \mathbb{C} \setminus \sigma(h)$ small enough with $|\mathrm{Re}\, z| < \gamma |\mathrm{Im}\, z|$ for any $\gamma > 0$.

Proof. Write

$$h = \int_{\sigma(h)} \lambda \mathrm{d}E_\lambda$$

such that

$$h^{-1} = \int_{\sigma(h)} \lambda^{-1} \mathrm{d}E_\lambda$$

with its usual domain given by unbounded functional calculus. By the density of states condition we have

$$\tau(|h^{-1}|) \leq 2\frac{\|h\| - \delta}{\delta} + \int_{(-\delta,\delta)} \frac{1}{|\lambda|}\rho(\lambda)\mathrm{d}\lambda < \infty$$

and therefore h^{-1} exists as an L^1-convergent Riemann-Stieltjes integral.

For the second assertion we consider

$$\tau\left(\left|\frac{1}{h} - \frac{1}{h+z}\right|\right) = \int_{\sigma(h)\setminus(-\delta,\delta)} \left|\frac{1}{\lambda} - \frac{1}{\lambda+z}\right| \tau(\mathrm{d}E_\lambda) + \int_{(-\delta,\delta)} \left|\frac{1}{\lambda} - \frac{1}{\lambda+z}\right| \rho(\lambda)\mathrm{d}\lambda$$

The first term is analytic in z and can be bounded $O(|z|)$ using the first derivative, so we focus on the second one.

We first treat the case $z = ix$ where we write

$$\frac{1}{\lambda} - \frac{1}{\lambda + z} = \frac{x^2}{\lambda(x^2+\lambda^2)} + \frac{ix}{x^2+\lambda^2}$$

such that

$$\int_{(-\delta,\delta)} \mathrm{d}\lambda \left|\frac{1}{\lambda} - \frac{1}{\lambda+z}\right| \rho(\lambda) \leq \int_{(-\delta,\delta)} \mathrm{d}\lambda A \left|\lambda^{s-1} \frac{x^2}{x^2+\lambda^2}\right| + A\left|\lambda^s \frac{x}{x^2+\lambda^2}\right|.$$

Splitting into positive and negative parts to get rid of the modulus and making the substitution $t = \lambda^2/\delta^2$, this can be written in terms of the hypergeometric function

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b-c)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t$$

such that

$$\int_{(-\delta,\delta)} \mathrm{d}\lambda \left|\lambda^s \frac{\alpha}{\beta^2+\lambda^2}\right| = \text{const}\ \frac{|\alpha|}{\beta^2} F\left(1, \frac{1+s}{2}, 1+\frac{1+s}{2}; -\frac{\delta^2}{\beta^2}\right)$$

with a constant that only depends on s and δ.

For $x \to 0$ one has the asymptotics

$$F\left(a,b,c;-\frac{1}{x}\right) \sim C_1 x^a + C_2 x^b + O(x^{1+a} + x^{1+b})$$

which can be derived by using a transformation formula that maps the singularity to a neighborhood of zero [1, 15.3.7] and the series expansion of $F(a,b,c;z)$ around $z = 0$.

We therefore have

$$\int_{(-\delta,\delta)} \mathrm{d}\lambda \left|\lambda^s \frac{\alpha}{\beta^2+\lambda^2}\right| \sim \tilde{C}_1 |\alpha| + \tilde{C}_2 |\alpha| |\beta|^{s-1} + O(|\beta|^{\frac{3+s}{2}} + |\beta|^2)$$

and substituting the respective values for α,β and s shows the claim in the special case. For general $z = y + ix$ we take

$$\left|\frac{1}{\lambda} - \frac{1}{\lambda+z}\right| \leq \left|\frac{1}{\lambda} - \frac{1}{\lambda+ix}\right| + \left|\frac{1}{\lambda+ix} - \frac{1}{\lambda+z}\right|$$

and bound the latter term

$$\begin{aligned}\left|\frac{1}{\lambda+ix} - \frac{1}{\lambda+z}\right| &= \left|-\frac{yx^2+y^2\lambda+y\lambda^2}{(x^2+\lambda^2)(x^2+(y+\lambda)^2)} + i\left(-\frac{x}{x^2+\lambda^2} + \frac{x}{x^2+(y+\lambda)^2}\right)\right| \\ &\leq \left|\frac{y}{x^2+\lambda^2}\right| + \left|\frac{y^2}{\lambda(y+\lambda)^2}\right| + \left|\frac{y}{(y+\lambda)^2}\right| + \left|\frac{x}{x^2+\lambda^2}\right| + \left|\frac{x}{(y+\lambda)^2}\right|\end{aligned}$$

where the first equality is just an identical rewriting. The first and fourth terms can be integrated as above to get asymptotics in the form $y^m x^{s-m+k}$ which can be made uniform in x by posing a condition $|y| \leq \gamma\,|x|$.

The integrals of the remaining terms can be rewritten analogously

$$\int_{(-\delta,\delta)} \mathrm{d}\lambda \left|\lambda^s \frac{\alpha}{(\beta+\lambda)^2}\right| = \text{const}\,\frac{|\alpha|}{\beta^2} F\left(2, 1+s, 2+s; -\frac{\delta}{\beta}\right) \sim \tilde{C}_1\,|\alpha| + \tilde{C}_2\,|\alpha|\,|\beta|^{s-1} + O(|\beta|).$$

We conclude

$$\tau\left(\left|\frac{1}{h} - \frac{1}{h+z}\right|\right) \leq C_\beta\,|\mathrm{Im}\,z|^s + O(|\mathrm{Im}\,z|).$$

□

This bound lets us control the divergence of the resolvent $\frac{1}{h+z}$ at zero well enough to use functional calculus with functions that are not continuous at zero.

Proposition 10.8. *Let h be as in the general assumptions 10.1 and with small density of states at zero.*

Set

$$\hat{h} = P\pi(h)P + k$$

with $k \in \hat{\mathcal{K}}_{\hat{\tau}}$ a finite hopping range boundary term, i.e. there is some $P_{[-n,n]} = \chi_{[-n,n]}(D)$ with $kP_{[-n,n]} = k$. Then

$$\mathrm{sgn}(\hat{h}) = P\pi(\mathrm{sgn}(h))P + c$$

with $c \in \hat{\mathcal{K}}_{\hat{\tau}} \cap L^2(\mathbb{T}^d_{\theta,\Omega}) \rtimes_\alpha G$.

Proof. We begin with the following resolvent identity

$$\frac{P}{\hat{h}+Pz} = P\pi\left(\frac{1}{h+z}\right)P + \frac{P}{\hat{h}+Pz}(P\pi(h)(1-P)+k)\pi\left(\frac{1}{h+z}\right)P$$

where $\frac{P}{\hat{h}+Pz}$ denotes the inverse in $PL^\infty P$ (this identity can be seen by writing $\frac{P}{\hat{h}+Pz} - P\pi\left(\frac{1}{h+z}\right) =: R$ and multiplying out). On the right hand side we recognize the Hankel operator $H_h = P\pi(h)(1-P)$, which also has a finite range and hence we introduce

$$V = (P\pi(h)(1-P)+k)$$

for which there is some $m \in \mathbb{N}$ with $VP_{[-m,m]} = V$. In total we have

$$\frac{P}{\hat{h}+Pz} - P\pi\left(\frac{1}{h+z}\right)P = \frac{P}{\hat{h}+Pz} VP_{[-m,m]}\pi\left(\frac{1}{h+z}\right)P$$

with a $\hat{\tau}$-trace class element on the right-hand side

$$\left\|P_{[-m,m]}\pi\left(\frac{1}{h+z}\right)\right\|_1 = \left\|P_{[-m,m]}\right\|_1 \left\|\frac{1}{h+z}\right\|_1.$$

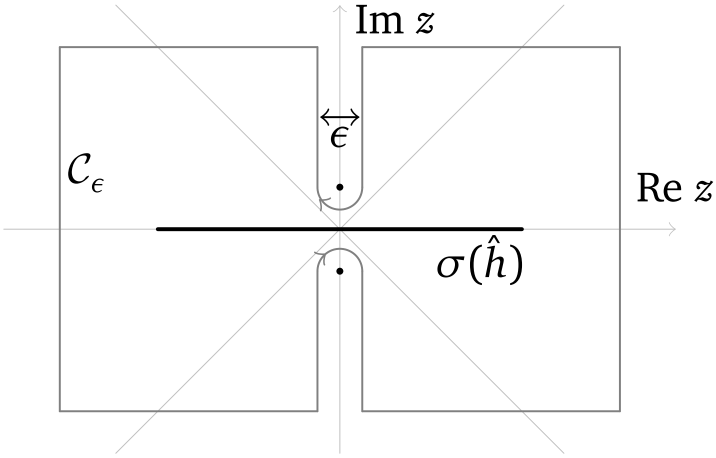


Figure 1: The contour for integration. It scales in such a way that the semicircles stay inside their respective quarter-planes as the two singular points move closer to the spectrum.

The strategy (inspired by the proof of [34, Proposition 3.31]) is now to approximate the sign function with holomorphic functional calculus such that we can apply our resolvent estimate. Note that pointwise $\operatorname{sgn}_\epsilon(x) := \tanh(x/\epsilon) \to \operatorname{sgn}(x)$ and therefore $\operatorname{sgn}_\epsilon(\hat{h}) \to \operatorname{sgn}(\hat{h})$ strongly.

We choose for $\tanh(x/\epsilon)$ a contour $\mathcal{C}_\epsilon$ that surrounds $\sigma(\hat{h})$ at a distance larger $\delta > 0$ except for two keyholes close to 0 such as to avoid the poles of $\tanh(x/\epsilon)$ at $\pm i\epsilon\frac{\pi}{2}$.

Then we have

$$c_\epsilon := \operatorname{sgn}_\epsilon(\hat{h}) - P\pi(\operatorname{sgn}_\epsilon(h))P = \frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \mathrm{d}z \tanh(z/\epsilon)\frac{P}{Pz-\hat{h}}VP_{[-m,m]}\pi\left(\frac{1}{z-h}\right)P.$$

The integrand is analytic in the Banach algebra $L^1 \cap L^\infty(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$ at any point $z \in \mathcal{C}_\epsilon$ and hence we may consider the integral equivalently in the strong L^1-sense, i.e. as an L^1-convergent Riemann integral.

Write $f(z) := \frac{1}{z-h} - \frac{1}{h}$ and note that f is analytic in the Banach space $L^1(\mathbb{T}^d_{\theta,\Omega})$ at any $z \notin \sigma(\hat{h})$. The map $a \mapsto \frac{1}{\|P_{[-m,m]}\|_1}P_{[-m,m]}\pi(a)$ extends to an isometry $L^1(\mathbb{T}^d_{\theta,\Omega}) \to L^1(\mathbb{T}^d_{\theta,\Omega}) \rtimes_\alpha G$, hence $P_{[-m,m]}\pi(f(z))$ is analytic in $L^1(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$.

Therefore we may write

$$\begin{aligned} c_\epsilon &= \frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \mathrm{d}z \tanh(z/\epsilon)\frac{P}{Pz-\hat{h}}VP_{[-m,m]}\left(\pi\left(\frac{1}{h}\right) + f(z)\right)P \\ &= \operatorname{sgn}_\epsilon(\hat{h})VP_{[-m,m]}\pi\left(\frac{1}{h}\right)P + \frac{1}{i2\pi}\int_{\mathcal{C}_\epsilon} \mathrm{d}z \tanh(z/\epsilon)\frac{P}{Pz-\hat{h}}VP_{[-m,m]}f(z)P. \end{aligned}$$

The first term is bounded in L^1-norm by

$$\left\| \operatorname{sgn}_\epsilon(\hat{h}) V P_{[-m,m]} \pi\left(\frac{1}{h}\right) P \right\|_1 \leq \|V\| \left\| P_{[-m,m]} \right\|_1 \left\| \frac{1}{h} \right\|_1 .$$

We now consider the second term. The keyholes are included in the cone $|\mathrm{Re}\, z| \leq |\mathrm{Im}\, z|$ and hence by Proposition 10.8 we may choose δ so small that

$$\left\| P_{[-m,m]} f(z) \right\|_1 = \left\| P_{[-m,m]} \right\|_1 \|f(z)\|_1 \leq C \left| \mathrm{Im}\, z \right|^s + O(|\mathrm{Im}\, z|) \tag{10.4}$$

for some $s > 0$ and all $z \in \mathcal{C}_\epsilon \cap B_\delta(0)$ uniformly in ϵ. Away from the spectrum the integral can be bounded with the standard resolvent estimates

$$\left\| \frac{P}{\hat{h} - Pz} \right\| \leq \frac{1}{\mathrm{dist}\,(z, \sigma(\hat{h})} \leq \frac{1}{|\mathrm{Im}\, z|}$$

and the length of the curve:

$$\begin{aligned} &\left\| \int_{\mathcal{C}_\epsilon \setminus (\mathcal{C}_\epsilon \cap B_\delta(0))} \mathrm{d}z \tanh(z/\epsilon) \frac{P}{Pz - \hat{h}} V P_{[-m,m]} f(z) P \right\|_1 \\ &\leq |\mathcal{C}_\epsilon \setminus (\mathcal{C}_\epsilon \cap B_\delta(0))| \sup_{z \in \mathcal{C}_\epsilon} |\tanh(z/\epsilon)| \frac{1}{\delta} \|V\| \left\| P_{[-m,m]} \right\|_1 \left(\left\| \frac{1}{h} \right\|_1 + \frac{1}{\delta} \right). \end{aligned}$$

For the keyholes we use the estimate (10.4)

$$\begin{aligned} &\left\| \int_{\mathcal{C}_\epsilon \cap B_\delta(0)} \mathrm{d}z \tanh(z/\epsilon) \frac{P}{Pz - \hat{h}} V P_{[-m,m]} f(z) P \right\|_1 \\ &\leq \int_{\mathcal{C}_\epsilon \cap B_\delta(0)} |\mathrm{d}z| \, |\tanh(z/\epsilon)| \, \|V\| \left\| P_{[-m,m]} \right\|_1 \left(\frac{1}{|\mathrm{Im}\, z|^{1-s}} + O(1) \right). \end{aligned}$$

As the singularity is integrable even for $\epsilon \to 0$ and $\tanh(z/\epsilon) \to \mathrm{sgn}(\mathrm{Re}\, z)$ pointwise, the integral is uniformly bounded in ϵ.

Therefore the sequence of norms $\|c_\epsilon\|_1$ is bounded by some constant $\tilde{C}$ but we have not shown that c_ϵ it is a Cauchy-sequence. Instead we conclude a weaker but sufficient result:

For a fixed $x \in L^1(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$ the map $a \in L^\infty(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G) \mapsto \hat{\tau}(ax)$ is σ-weakly continuous and hence strongly continuous on the unit ball [18]. Consider the functionals on $L^1(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$ defined by

$$\hat{\phi} := \hat{\tau}(\mathrm{sgn}(\hat{h}) \cdot)$$

and

$$\phi := \hat{\tau}(P \pi(\mathrm{sgn}(h)) P \cdot).$$

For the difference we have

$$(\hat{\phi} - \phi)(x) = \lim_{\epsilon \to 0} \hat{\tau}(c_\epsilon x) \leq \sqrt{2\tilde{C}} \, \|x\|_2$$

for all $x \in L^1(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G) \cap L^2(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$, where we used strong convergence, the Cauchy-Schwarz inequality and bounded $\|c_\epsilon\|_2^2 \leq \|c_\epsilon\| \, \|c_\epsilon\|_1$. Hence, the difference defines a bounded functional on L^2 and by duality we have

$$(\hat{\phi} - \phi)(x) = \hat{\tau}(ba)$$

with some $b \in L^2(\mathbb{T}^d_{\theta,\Omega} \rtimes_\alpha G)$. Clearly, $b = \operatorname{sgn}(\hat{h}) - P\pi(\operatorname{sgn}(h))P$ since the functionals coincide on a dense subset of L^2. The conclusion $\operatorname{sgn}(\hat{h}) - P\pi(\operatorname{sgn}(h))P \in \mathcal{K}_{\hat{\tau}}$ then follows from

$$\mathcal{K}_{\hat{\tau}} = \overline{L^\infty \cap L^2}.$$

□

Corollary 10.2. *Assume the same conditions as the Proposition and additionally that the boundary term k is chirally symmetric, i.e.* $\hat{h} = \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^* & 0 \end{pmatrix}$ *for some* $a \in \hat{\mathscr{T}}(B, \mathcal{K}_{\hat{\tau}})$.

We then have

$$\operatorname{Ind}([u]_1) = [p_+]_0 - [p_-]_0$$

with

$$P_{\operatorname{Ker}(\hat{h})} =: \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}.$$

By the index theorem we conclude

$$\hat{\tau}(JP_{\operatorname{Ker}\hat{h}}) = \operatorname{Wind}_\xi([u]_1).$$

Proof. The proposition shows that the partial isometry $\operatorname{sgn}(\hat{h})$ from the polar decomposition of $\hat{h}$ is in $M_2(\hat{\mathscr{T}}(B, \mathcal{K}_{\hat{\tau}})))$. By the chiral symmetry we have

$$\operatorname{sgn}(\hat{h}) = \begin{pmatrix} 0 & \hat{u} \\ \hat{u}^* & 0 \end{pmatrix}$$

with some partial isometry $\hat{u} \in \hat{\mathscr{T}}(B, \mathcal{K}_{\hat{\tau}})$. The Proposition shows

$$q(\operatorname{sgn}(\hat{h})) = \operatorname{sgn}(h)$$

and hence $\hat{u}$ lifts u to a partial isometry. As seen by (10.3), we then have

$$\operatorname{Ind}([u]_1) = [P - \hat{u}^*\hat{u}]_0 - [P - \hat{u}\hat{u}^*]_0 = [p_+]_0 - [p_-]_0$$

□

We finally remark that K-theory was apparently not necessary to prove the main result of this section; Breuer-Fredholm theory would have been enough since the $\hat{\tau}$-index is invariant under compact perturbations. This is simply a result of the coincidence that our boundary cocycle of interest, $\hat{\tau}$, is densely defined on $\mathcal{K}_{\hat{\tau}}$ such that we can take this largest possible extension. In general, however, we would want to extend the formalism to bulk-boundary correspondence of higher Chern numbers which one needs e.g. to discuss edge states for the Quantum Hall systems. It is then the case that the boundary cocycles, also involve the derivations $\partial_1, ..., \partial_d$ which are not densely defined on $\mathcal{K}_{\hat{\tau}}$. We then cannot argue using arbitrary compact perturbations of Breuer-Fredholm operators anymore but using the index/exponential map will still be a viable approach.

10.5 Examples

Gapped systems

A topological insulator with a chiral symmetry is described by a chirally symmetric Hamiltonian

$$h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \in M_{2N}(\mathbb{T}^d_{\theta,\Omega})$$

with $a \in M_N(\mathbb{T}^d_{\theta,\Omega})$ having a finite hopping range. We assume that h has a spectral gap at zero, i.e. there is some $\delta > 0$ with $\sigma(h) \cap (-\delta, \delta) = \emptyset$. The well-known Combes-Thomas-estimate implies that the Fermi unitary u is exponentially localized (see e.g. [36])

$$\int_\Omega \mathbb{P}(\mathrm{d}\omega)|\phi_x(u)| \leq Ae^{-\beta|x|}$$

which implies $u \in B^1_{1,1}(\mathbb{T}^d_{\theta,\Omega}) \cap W_{1,1}(\mathbb{T}^d_{\theta,\Omega})$ and therefore all requirements of Assumption 10.1 are satisfied.

Furthermore the half-space operator

$$\hat{h} = P\pi(h)P$$

is Breuer-Fredholm with the pseudo-inverse $P\pi(h^{-1})P$.

Therefore Proposition 10.5 implies

Proposition 10.9. *Let h, u satisfy the Assumption (10.1) and assume further that h is invertible. For arbitrary $\xi \in S^{d-1}$ let*

$$\hat{h} = P\pi(h)P + k \in M_{2N}\left(L^\infty(\mathbb{T}_{\theta,\Omega}) \rtimes_\alpha G\right)$$

with a chirally symmetric compact boundary term $k \in M_{2N}\left(\hat{\mathcal{K}}_{\hat{\tau}}\right)$.

Then we have

$$\hat{\tau}(JP_{\mathrm{Ker}\,\hat{h}}) = \mathrm{Wind}_\xi([u]_1) = \sum_k \xi_k \mathrm{Wind}_k([u]_1).$$

In particular $\hat{h}$ has boundary states at zero energy whose density is at least $|\mathrm{Wind}_\xi([u]_1)|$.

This result is stable under perturbations in two senses: For one, we may add an arbitrary chirally symmetric compact perturbation to the half-space Hamiltonian, since this does not change its Breuer-Fredholm-index and the pair of kernel projections still is an image of the index map. It is also stable under a norm-continuous homotopy of the bulk Hamiltonian $t \mapsto h_t$, $h_0 = h$ as long as all the conditions (10.1) are satisfied for h_t, u_t, the gap remains open and the path $t \mapsto u_t$ is norm-continuous (i.e. the K_1-class of u is unchanged by the homotopy).

We also note that this result slightly sharpens the one obtained by [36], which uses a continuous approximation of the Kernel projection and therefore does not directly imply non-triviality of the kernel of $\hat{h}$.

Mobility gap

In the mobility gap regime the bulk Hamiltonian h does not have a spectral gap at zero but still satisfies an exponential localization condition:

Definition 10.5. *h is called exponentially localized in the spectral interval $(-\delta,\delta)$, if for any $s\in(0,1)$ there are $A_s,\beta_s>0$ such that*

$$\int_\Omega \mathbb{P}(\mathrm{d}\omega)\sum_{x\in\mathbb{Z}^d}|\langle 0|(h_\omega - E \pm i\epsilon)^{-1}|x\rangle|^s \le A_s e^{-\beta_s|x|} \tag{10.5}$$

holds uniformly for all $E\in(-\delta,\delta)$ and $\epsilon\to 0$.

Such a phase would be expected to arise from a periodic Hamiltonian with a spectral gap when subjected to strong chiral disorder terms that cause the gap to close.

Exponential localization implies that u is exponentially summable (see [36]) and therefore in $W_{1,1}$ and $B^1_{1,1}$ and therefore we can apply Corollary 10.2 to get

Proposition 10.10. *Let h,u satisfy the Assumption (10.1) and assume that h is exponentially localized for some $\delta>0$ and has a small density of states at 0. For arbitrary $\xi\in S^{d-1}$ let*

$$\hat{h} = P\pi(h)P + k \in M_{2N}\left(L^\infty(\mathbb{T}_{\theta,\Omega})\rtimes_\alpha G\right)$$

with a chirally symmetric, finite range boundary term $k\in M_{2N}\left(\hat{\mathcal{K}}_{\hat{\tau}}\right)$.

Then we have

$$\hat{\tau}(JP_{\mathrm{Ker}\,\hat{h}}) = \mathrm{Wind}_\xi([u]_1) = \sum_j \xi_j \mathrm{Wind}_j([u]_1).$$

In particular $\hat{h}$ has boundary states at zero energy whose density is at least $|\mathrm{Wind}_\xi([u]_1)|$.

This result is, however, of a rather tentative nature, since bounds such as (10.5) have not been established for any concrete examples. We note that a similar bulk-boundary correspondence [20] and also localization bounds [41] have recently been obtained for one-dimensional chiral chains through the use of hard analysis.

Pseudogap state

One application that in many ways motivated this work are periodic chiral systems in a pseudogap state, i.e. systems in which 0 is in the bulk spectrum but is only a band-touching point and therefore the density of states vanishes polynomially. As a concrete example we consider the discrete laplacian on the honeycomb lattice, which is often used as a first order approximation to graphene. After fixing two triangular sublattices of the honeycomb lattice and mapping them to $\mathbb{Z}^2$, a possible parametrisation is

$$h := \begin{pmatrix} 0 & 1+u_1+u_1^*u_2 \\ 1+u_1^*+u_2^*u_1 & 0 \end{pmatrix}$$

where the off-diagonals are the three nearest-neighbor-hopping terms. For vanishing magnetic field we Fourier transform to get

$$h : \mathbb{T}^2 \to M_2(\mathbb{C}), \quad h_k = \begin{pmatrix} 0 & 1 + e^{i2\pi k_1} + e^{i2\pi(k_2-k_1)} \\ 1 + e^{-i2\pi k_1} + e^{-i2\pi(k_2-k_1)} & 0 \end{pmatrix} \tag{10.6}$$

and indeed the matrix vanishes at the Dirac points $k_\pm = (\frac{3\pm 1}{6}, 0)$. At momenta close to these Dirac points the physical dispersion relation takes the form

$$E(k_\pm + q) \approx |l_\pm \cdot q|$$

and therefore vanishes linearly. This condition will be strong enough for our form of bulk-boundary correspondence.

The Fourier transform of the Fermi unitary is given by

$$u(k) = \frac{1 + e^{i2\pi k_1} + e^{i2\pi k_2}}{|1 + e^{i2\pi k_1} + e^{i2\pi k_2}|}$$

and is manifestly not continuous at the Dirac points. This is directly related to the topology of the Hamiltonian since indeed $u(k)$ has winding numbers ± 1 around the Dirac points, which could not be possible if it were a smooth function.

We now establish regularity for this type of periodic Hamilton operator. For simplicity we restrict to the two-dimensional case.

Proposition 10.11. *Let $a : \mathbb{T}^2 \to \mathbb{C}$ a trigonometric polynomial that has finitely many zeroes $k^{(1)}, \ldots, k^{(n)}$ such that*

$$a(k^{(p)} + q) = l_1^{(p)} q_1 + i l_2^{(p)} q_2 + O(q^2) \tag{10.7}$$

in the vicinity of the zeroes. Then $u := \frac{a}{|a|} \in B^1_{1,1}(\mathbb{T}^2) \cap W_{1,1}(\mathbb{T}^2)$. Furthermore the Hamiltonian

$$h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

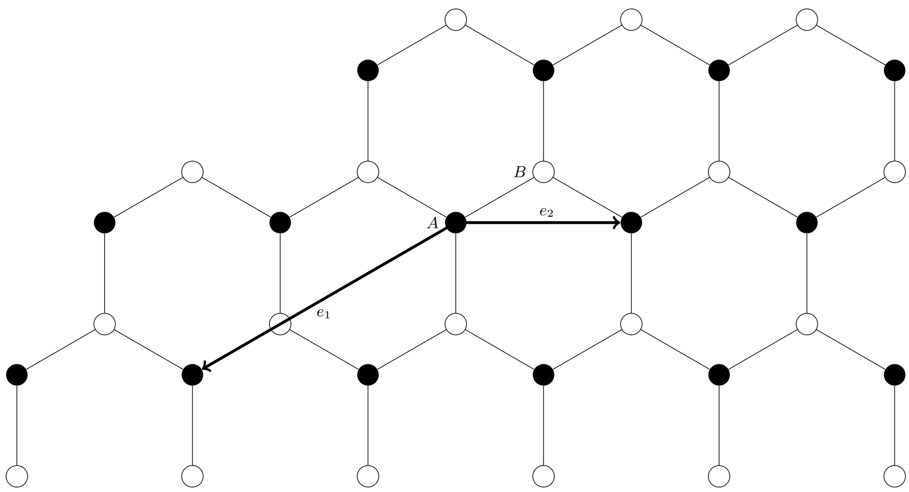
has a small density of states at 0.

Proof. For the Besov norm we have to bound

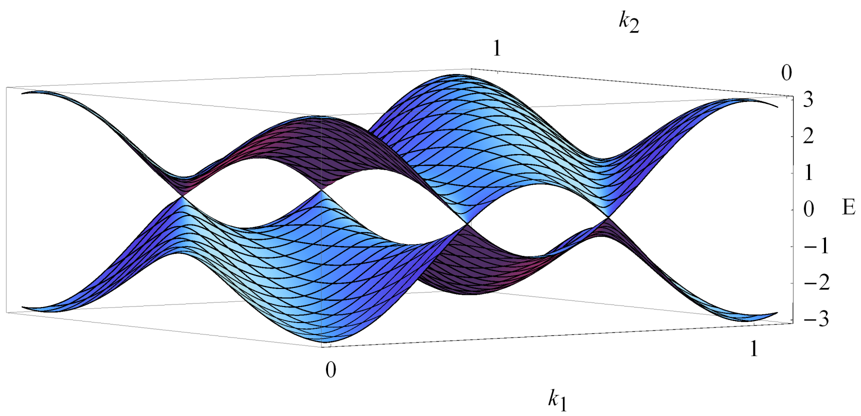
$$\int_{[0,1]^3} \frac{|u(k+\xi t) + u(k-\xi t) - 2u(k)|}{t^2} \mathrm{d}^2 k \mathrm{d}t$$

and since the region of integration is compact it is enough to show that the integrand is locally integrable everywhere. For $t > 0$ and k not a zero of a this is clear since then u is continuous and bounded at (t, k) with

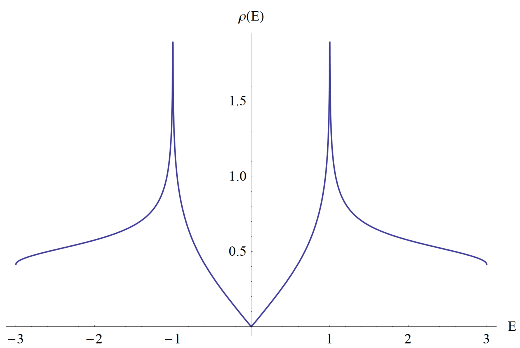
$$\frac{|u(k+\xi t) + u(k-\xi t) - 2u(k)|}{t^2} \leq \frac{4}{t^2}.$$



(a)



(b)



(c)

Figure 2: a) The choice of basis vectors for the honeycomb lattice. Terminating the lattice parallel to e_1 gives so-called armchair boundary conditions and parallel to e_2-direction gives zigzag or bearded edges. b) The band structure of the Hamiltonian (10.6), i.e. the spectrum of $h(k)$ as a function of the momenta. c) The density of states vanishes linearly at the band-touching points.

For $t = 0$ and k not a zero, u is smooth therefore the directional derivative

$$\lim_{t\to 0} \frac{u(k+\xi t)+u(k-\xi t)-2u(k)}{t^2} = u''(k)$$

is locally bounded and continuous.

Hence, we only have to consider the case $t = 0$ and $k = k^{(p)}$ a zero of a. In this situation we change to spherical coordinates around $k^{(p)}$:

$$\begin{aligned} k_1 - k_1^{(p)} &= r\sin(\theta)\sin(\phi) =: x_1(r,\theta,\phi) \\ k_2 - k_2^{(p)} &= r\sin(\theta)\cos(\phi) =: x_2(r,\theta,\phi) \\ t &= r\cos(\theta) \end{aligned}$$

On a neighborhood of $(t, k^{(p)})$ this transforms the integral to

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\epsilon} \frac{1}{\cos^2(\theta)} |u(x+r\cos(\theta)\xi)+u(x-r\cos(\theta)\xi)-2u(x)| \sin(\theta) \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

The integrand is bounded away from $\theta_\pm = \pm\frac{\pi}{2}$, so we just have to check for potential singularities on those two lines. Expanding to order $O(\theta^2)$ we get

$$\begin{aligned} &\lim_{\theta\to\theta_\pm} \frac{1}{\cos^2(\theta)} (u(x+r\cos(\theta)\xi)+u(x-r\cos(\theta)\xi)-2u(x)) \\ &\qquad = u(r\sin(\phi), r\cos(\phi)) + r^2 u''(r\sin(\phi), r\cos(\phi)) \end{aligned}$$

with u'' the directional derivative in the direction ξ.

Using the asymptotics (10.7) we have for $r \to 0$

$$\begin{aligned} u(r\sin(\phi), r\cos(\phi)) &\sim l_1^{(p)}\sin(\phi) + i l_2^{(p)}\cos(\phi) \\ u'(r\sin(\phi), r\cos(\phi)) &\sim \frac{G(\phi)}{r} \\ u''(r\sin(\phi), r\cos(\phi)) &\sim \frac{H(\phi)}{r^2} \end{aligned}$$

with smooth and bounded functions H, G. Hence, the transformed integrand is bounded on a neighborhood of 0 and therefore the original integrand is locally integrable. The same asymptotics also give us $u \in W_{1,1}(\mathbb{T}^2)$ (in fact one has the inclusion $B^1_{1,1} \subset W_{1,1}$ for the classical function spaces).

For the claim about the density of states note that only the parts of h close to the zeroes $k^{(1)}, ..., k^{(n)}$ contribute to the density of states at zero and h^2 has the form

$$h^2(k^{(p)}+k) = \mathrm{id}_2 \otimes \left(\left(l_1^{(p)}\right)^2 k_1^2 + \left(l_2^{(p)}\right)^2 k_2^2 + O(k^3)\right).$$

Hence $|h|$ is approximately linear and therefore the density of states tends to a linear function close to 0. □

It is reasonable to expect that the conclusion of the Proposition also holds in higher dimension since Dirac type singularities then have even better integrability.

Proposition 10.12. *Assume a vanishing magnetic field and let* $h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ *with* a *as in the proposition above.*
For $\xi \in S^1$ *rationally dependent let*

$$\hat{h} = P\pi(h)P + k \in M_{2N}\left(L^\infty(\mathbb{T}_{0,\Omega}) \rtimes_\alpha \mathbb{T}\right)$$

with a chirally symmetric, finite range boundary term $k \in M_{2N}\left(\hat{\mathcal{K}}_{\hat{\tau}}\right)$.
Then we have

$$\hat{\tau}(JP_{\operatorname{Ker}\hat{h}}) = \operatorname{Wind}_\xi([u]_1) = \sum_k \xi_k \operatorname{Wind}_k([u]_1).$$

In particular $\hat{h}$ *has boundary states at zero energy whose density is at least* $|\operatorname{Wind}_\xi([u]_1)|$.

As an example we revisit the Hamiltonian of the honeycomb-lattice

$$h(k) = \begin{pmatrix} 0 & a(k) \\ \overline{a(k)} & 0 \end{pmatrix}$$

with the off-diagonal term

$$a(k) = 1 + e^{i2\pi k_1} + e^{i2\pi(k_2 - k_1)}$$

giving the Fermi unitary

$$u(k) = \frac{1 + e^{i2\pi k_1} + e^{i2\pi(k_2-k_1)}}{|1 + e^{i2\pi k_1} + e^{i2\pi(k_2-k_1)}|}$$

with Dirac points at $(k_1, k_2) = (\frac{3\pm 1}{6}, 0)$. We compute the winding numbers

$$\tau(u\partial_{k_1}u^*) = \int_{[0,1]} \mathrm{d}k_2 \int_{[0,1]} \mathrm{d}k_1 u(k)\partial_{k_1}u^*(k).$$

Indeed, the integral for fixed k_2 is just the winding number along one slice and the homotopy invariance implies that it must be locally constant in k_2, i.e. the value of the inner integral can only change at the discontinuities of u. For $k_2 = \frac{1}{2}$ we get $a(k_1, k_2) = 1 + 2i\sin(2\pi k_1)$ which has no winding as all points lie on a complex line. Hence the winding number is identically zero for all k_2

$$\tau(u\partial_{k_1}u^*) = 0.$$

For the other direction we have two regions between the Dirac points, one is composed of $\frac{1}{3} < k_1 < \frac{2}{3}$ and the other contains the rest. For the latter region the winding number again vanishes as can be seen by substituting $k_1 = 0$. The other region has winding number 1 since $u(\frac{1}{2}, k_2) = -e^{i2\pi k_2}$. Hence

$$\tau(u\partial_{k_2}u^*) = \int_{[0,1]} \mathrm{d}k_1 \chi_{(\frac{1}{3},\frac{2}{3})}(k_1) = \frac{1}{3}.$$

We consider restrictions to half-spaces of the form $\xi_1 x_1 + \xi_2 x_2 \geq 0$ for $x \in \mathbb{Z}^2$. The components ξ_1 and ξ_2 are rationally dependent if and only if the separating line contains more than one lattice point of the same sublattice. The extreme cases $\xi = e_1$ and $\xi = e_2$ are called armchair and zigzag boundaries respectively (see Figure 10.5). It is well-known that the half-space restriction exhibits a zero energy edge state for zigzag but not for armchair boundary conditions [27].

Our index theorem for a chiral half-space restriction $\hat{h}$ allows the easy computation for all angles simultaneously

$$\hat{\tau}\left(JP_{\mathrm{Ker}\,\hat{h}}\right) = \frac{1}{3}\xi_2,$$

i.e. unless we cut the honeycomb-lattice exactly through the line $x_2 = 0$ we necessarily get a non-vanishing edge-state at zero energy. Notably this result also holds for rough boundaries, i.e. with random finite range boundary terms, as long as the chiral symmetry is preserved.

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Erklärung

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Erlangen, den 19. September, 2018