Pseudo-gaps of random hopping models (Analysis of critical energies)

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new contributions with:
Florian Dorsch, Joris De Moor, Christian Sadel

older papers with:
Maxim Drabkin, Lana Jitomirskaya, Werner Kirsch, Günther Stolz

papers on arXiv (one this week)

Overview

Random hopping models

- Pseudogaps and logarithmic singularities in DOS
- Formalism: transfer matrices and Prüfer variables
- Hyperbolic critical energies
- Renewal theory and optimal stopping theorem

Topological phase transitions in (generalized) SSH models

- Reduced transfer matrices
- Again hyperbolic critical energies

Elliptic critical energies

- Perturbation theory for Furstenberg measure at complex energies

Parabolic critical energies

- Band edges of Anderson or random Kronig-Penney model

Random hopping model ($1d$ discrete Schrödinger)

Random Hamiltonian on $\ell^2(\mathbb{Z})$ given by

$$(H\psi)_n \;=\; -t_{n+1}\,\psi_{n+1} \;-\; t_n\,\psi_{n-1}$$

where $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$

Hypothesis: $(t_n)_{n\in\mathbb{Z}}$ independent positive random variables

Technical simplification: support of distributions of t_n is compact

Two cases:

- $t_n \overset{\mathrm{d}}{=} t_{n+2}$ and $\mathbb{E}(\log t_{2n}) \neq \mathbb{E}(\log t_{2n+1})$ random hopping dimers
- t_n identically distributed, balanced random hopping

Bipartite symmetry:

$$J\,H\,J \;=\; -\,H \qquad , \qquad J\,|n\rangle \;=\; (-1)^n\,|n\rangle$$

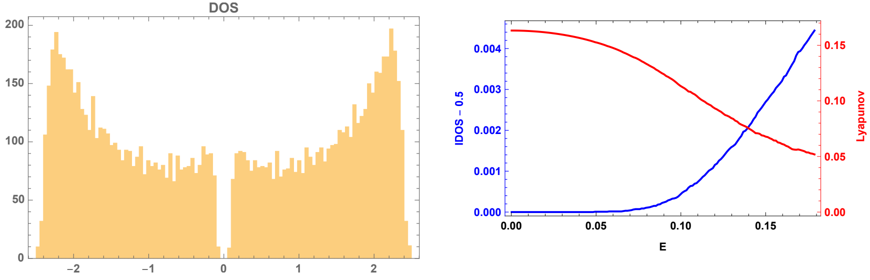
Hence center of band $E_c = 0$ is special

Integrated density of states (IDOS)

$$\mathcal{N}(E) = \lim_{N\to\infty} \frac{1}{N} \# \{\text{eigenvalues of } H_N = H\upharpoonright_{\ell^2(\{1,\ldots,N\})} \leqslant E\}$$

exists almost surely and Stieltjes functions giving DOS $\frac{d\mathcal{N}}{dE}(E)$

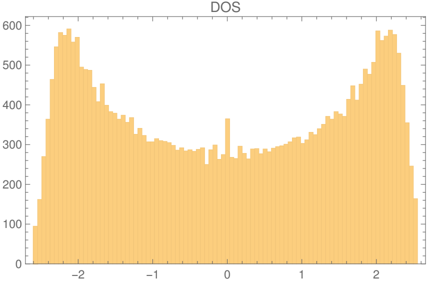
Numerics for random hopping dimers at $N = 9500$ and $\nu = 9.7$ (later)

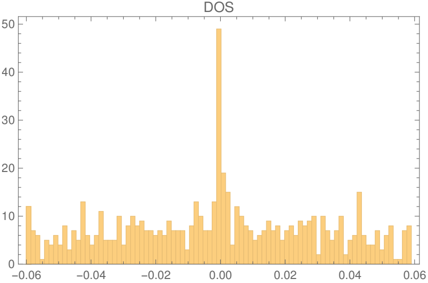


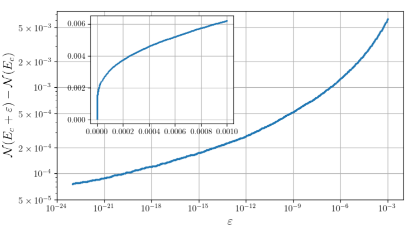
Pseudo-gap at $E_c = 0$ and Lyapunov exponent positive there

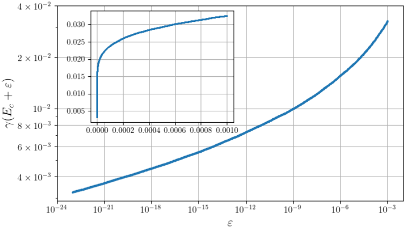
Second plot by Birkhoff telescoping of rotation number and Lyapunov

Balanced random hopping (DOS for $N = 16900$)









Results on DOS at $E_c = 0$ of random hopping

Theorem (Hölder continuity for hopping dimers)

There exists unique $\nu > 0$ obeying $\mathbb{E}\big((\frac{t_{2n+1}}{t_{2n}})^\nu\big) = 1$ if $\mathbb{E}(\log(\frac{t_{2n+1}}{t_{2n}})) > 0$

There are $C_- < C_+$ such that the IDS satisfies for ϵ sufficiently small

$$C_- \leqslant \frac{|\,\mathcal{N}(E_c + \epsilon) - \mathcal{N}(E_c)\,|}{|\epsilon|^\nu} \leqslant C_+$$

Deep pseudo-gap for large ν is possible (see numerics above)

Theorem (logarithmic divergence for balanced hopping)

There exists constant C such that

$$\Big|\,\mathcal{N}(E_c + \epsilon) - \mathcal{N}(E_c) - \tfrac{1}{2}\,\mathrm{Var}\big(\log\,(t_0)\big)\,\big(\log(\epsilon)\big)^{-2}\,\Big| \leqslant C\,|\log(\epsilon)|^{-3}$$

Dyson prediction (1953), weak form proved by Kotowski, Virág (2017)

Transfer matrices and critical energies

$$H\psi = E\psi \qquad \Longleftrightarrow \qquad \begin{pmatrix} t_{n+1}\psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} -E\frac{1}{t_n} & -t_n \\ \frac{1}{t_n} & 0 \end{pmatrix} \begin{pmatrix} t_n\psi_n \\ \psi_{n-1} \end{pmatrix}$$

Two-step $\mathrm{SL}(2,\mathbb{R})$ transfer matrices i.i.d. for random hopping dimer:

$$\begin{aligned} \mathcal{T}_n^E &= \begin{pmatrix} -E\frac{1}{t_{2n+1}} & -t_{2n+1} \\ \frac{1}{t_{2n+1}} & 0 \end{pmatrix} \begin{pmatrix} -E\frac{1}{t_{2n}} & -t_{2n} \\ \frac{1}{t_{2n}} & 0 \end{pmatrix} \\ &= -\begin{pmatrix} \frac{t_{2n+1}}{t_{2n}} & 0 \\ 0 & \frac{t_{2n}}{t_{2n+1}} \end{pmatrix} + E \begin{pmatrix} 0 & \frac{t_{2n}}{t_{2n+1}} \\ -\frac{1}{t_{2n}\,t_{2n+1}} & 0 \end{pmatrix} + E^2 \begin{pmatrix} \frac{1}{t_{2n}\,t_{2n+1}} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Definition

(i) E_c critical energy for random family $\mathcal{T}_\sigma^E \Longleftrightarrow$ all $\mathcal{T}_\sigma^{E_c}$ commute

(ii) Critical energy E_c is hyperbolic $\Longleftrightarrow \mathrm{spec}(\mathcal{T}_\sigma^{E_c}) \cap \mathbb{S}^1 = \emptyset$ for some σ

Random dynamics on Prüfer variables

$$e_{\theta_n^\epsilon} = \frac{\mathcal{T}_n^{E_c+\epsilon}\, e_{\theta_{n-1}^\epsilon}}{\|\mathcal{T}_n^{E_c+\epsilon}\, e_{\theta_{n-1}^\epsilon}\|} \quad , \quad e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

and the condition $\theta_{n+1}^\epsilon - \theta_n^\epsilon \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ fixing the branch

Markov process $(\theta_n^\epsilon)_{n\in\mathbb{N}}$ on $\mathbb{R}$ induced by $\mathrm{SL}(2,\mathbb{R})$-action

Oscillation theory and rotation number calculation for IDOS

$$\mathcal{N}(E_c+\epsilon) - \mathcal{N}(E_c) = \lim_{N\to\infty} \frac{1}{N}\frac{1}{2\pi}\mathbb{E}(\theta_N^\epsilon) = \lim_{N\to\infty} \frac{1}{N}\frac{1}{2\pi}\sum_{n=1}^{N}\mathbb{E}(\theta_n^\epsilon - \theta_{n-1}^\epsilon)$$

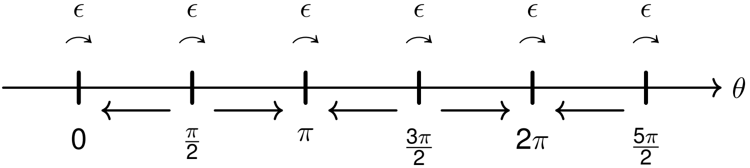
Lyapunov exponent

$$\gamma(E_c+\epsilon) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\Big(\log(\|\mathcal{T}_{n+1}^{E_c+\epsilon} e_{\theta_n^\epsilon}\|)\Big)$$

Hyperoblic action and perturbation by energy

$$\mathcal{T}_n^{E_c} = -\begin{pmatrix} \frac{t_{2n+1}}{t_{2n}} & 0 \\ 0 & \frac{t_{2n}}{t_{2n+1}} \end{pmatrix} = -\begin{pmatrix} \kappa_n & 0 \\ 0 & \frac{1}{\kappa_n} \end{pmatrix} \quad , \qquad \kappa_n = \frac{t_{2n+1}}{t_{2n}}$$

Fixed points $e_{k\pi} = \binom{\pm 1}{0}$ and $e_{\frac{\pi}{2}+k\pi} = \binom{0}{\pm 1}$ and invariant intervals



Energy dependent perturbation generates random shift of order ϵ:

$$\mathcal{T}_n^{E_c+\epsilon} = -\begin{pmatrix} \kappa_n & 0 \\ 0 & \frac{1}{\kappa_n} \end{pmatrix} \left[\mathbf{1} + \epsilon \begin{pmatrix} 0 & -1 \\ \frac{1}{t_{2n+1}} & 0 \end{pmatrix} + \mathcal{O}(\epsilon^2) \right]$$

For $\epsilon > 0$ to right $\Longrightarrow$ fixed points semipermeable $\Longrightarrow$ renewal theory

Probabilistic analysis

Dyson-Schmidt variables $x_n^\epsilon = -\cot(\theta_n^\epsilon) \in \mathbb{R}$ have fixed points 0 and ∞

Focus on half-lines $(0,\infty)$ There set $y_n^\epsilon = \log(x_n^\epsilon)$. Then for $\epsilon = 0$

$$y_{n+1}^0 = y_n^0 + 2\log(\kappa_n)$$

Balanced random hopping: $\mathbb{E}(\log\kappa_n) = 0$ random walk y_n^ϵ on $\mathbb{R}$

- effectively hard wall at $-\log(\frac{1}{\epsilon})$, sufficient to arrive at $\log(\frac{1}{\epsilon})$
- by CLT roughly $\log(\epsilon)^2$ time steps needed (see Theorem)
- construction of comparison martingales (upper and lower bound)
- control of error terms depending on ϵ
- optional stopping theorem for estimated interarrival times
- elementary renewal theorem connects to rotation number

Random hopping dimer: $\mathbb{E}(\log\kappa_n) > 0$ drift to left/right

- $e_{k\pi}$ attractive, $e_{k\pi+\frac{\pi}{2}}$ repulsive (on average), so passage unlikely
- large deviation regime for crossing $(-\log(\frac{1}{\epsilon}), \log(\frac{1}{\epsilon}))$

Generalized Su-Schrieffer-Heeger (SSH) model

H on $\ell^2(\mathbb{Z},\mathbb{C}^{2L})$ satisfying $JHJ = -H$ with $J|n\rangle = (-1)^n \begin{pmatrix} \mathbf{1}_L & 0 \\ 0 & -\mathbf{1}_L \end{pmatrix} |n\rangle$

$$(H\Phi)_n \;=\; T_{n+1}\Phi_{n+1} \;+\; V_n\Phi_n \;+\; T_n^*\Phi_{n-1} \qquad , \qquad \Phi_n \in \mathbb{C}^{2L}$$

Hypothesis: $\mathrm{rank}(T_n) = 1$, so say $T_n = t_n\,|1\rangle\langle 2L|$ and set

$$\begin{pmatrix} \langle 1|(E-V_n)^{-1}|1\rangle & \langle 2L|(E-V_n)^{-1}|1\rangle \\ \langle 1|(E-V_n)^{-1}|2L\rangle & \langle 2L|(E-V_n)^{-1}|2L\rangle \end{pmatrix} \;=\; \begin{pmatrix} G_n^{E,-,-} & G_n^{E,-,+} \\ G_n^{E,+,-} & G_n^{E,+,+} \end{pmatrix}$$

Reduced transfer matrices from $\mathrm{SL}(2,\mathbb{R})$:

$$\mathcal{T}_n^E = \begin{pmatrix} (G_n^{E,-,+})^{-1} & -(G_n^{E,-,+})^{-1}G_n^{E,-,-} \\ G_n^{E,+,+}(G_n^{E,-,+})^{-1} & G_n^{E,+,-} - G_n^{E,+,+}(G_n^{E,-,+})^{-1}G_n^{E,-,-} \end{pmatrix} \begin{pmatrix} \frac{1}{t_n} & 0 \\ 0 & \overline{t_n} \end{pmatrix}$$

$$= \begin{pmatrix} \kappa_n & 0 \\ 0 & \frac{1}{\kappa_n} \end{pmatrix} \;+\; \mathcal{O}(E)$$

Again $E_c = 0$ hyperbolic critical energy, now with $\kappa_n = \frac{1}{G_n^{0,-,+}\,t_n}$

Some background facts on the random SSH model

Prototype of a $1d$ chiral topological insulator (not designed for)

Randomness: $t_n = 1 + \lambda\omega_n$ and $V_n = \frac{1}{2}(m \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} + \mu\omega'_n)$

Closes gap at Fermi level $E_c = 0$ for λ, μ sufficiently large

Noncommutative winding number w.r.t. position operator X on $\ell^2(\mathbb{Z})$

$$\mathrm{Wind}(H) \;=\; \tfrac{1}{2}\,\mathbb{E}\,\mathrm{Tr}\,\big(\langle 0|J\,H^{-1}\,[H,X]|0\rangle\big)\;\in\;\mathbb{Z}$$

Topological phases pending on parameters λ and μ (for $m > 0$ fixed)

Mondragon-Shem *et al.* (2014) plotted phase diagram using $\mathrm{Wind}(H)$

Anderson localization away from E_c (proofs by J. Shapiro 2021)

Bulk-boundary correspondence

Half-space restrictions have chiral surface states (similar to Majorana)

Persists in localization regime with closed gap (Graf-Shapiro 2018)

Divergence of DOS at topological phase transition

By definition, balanced points make up phase boundary:

$$\mathcal{P} \;=\; \big\{(\lambda,\mu)\in\mathbb{R}^2 \;:\; \gamma(E_c=0) = |\mathbb{E}\,\log(\kappa_n)| = 0\big\}$$

Theorem

Off $\mathcal{P}$: pseudo-gap at $E_c = 0$

On $\mathcal{P}$: logarithmic divergence of DOS

With Prodan (2016) and Stoiber (2022):

change of topological invariant (here winding number)

$\Longrightarrow$ no dynamical Anderson localization at E_c on $\mathcal{P}$

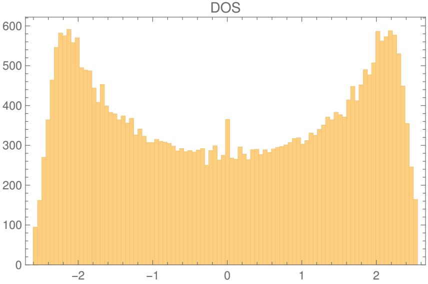
Open questions: quantitative lower bound on transport

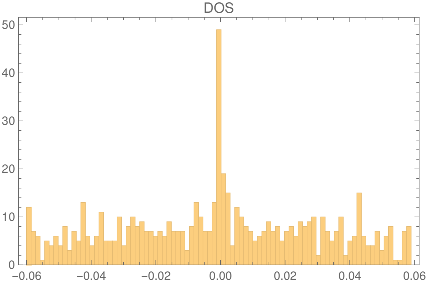
(many states, but $\gamma(E_c+\epsilon)$ rapidly increasing in ϵ)

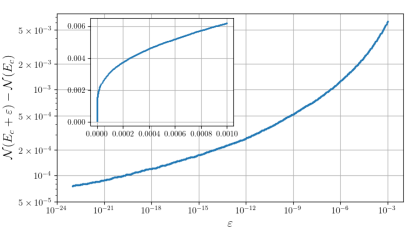
Lyapunov exponents, Furstenberg measure

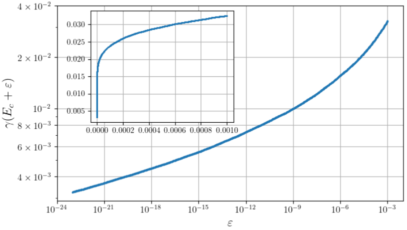
level statistics, enhenced area law (as Müller, Pastur, Schulte 2020)

Balanced random hopping (DOS for $N = 16900$)









Elliptic and parabolic critical energies

Definition

Random family $E \in \mathbb{R} \mapsto \mathcal{T}_\sigma^E$ in $\mathrm{SL}(2,\mathbb{R})$

(i) E_c critical energy $\Longleftrightarrow$ all $\mathcal{T}_\sigma^{E_c}$ commute

(ii) E_c hyperbolic critical energy $\Longleftrightarrow \mathrm{spec}(\mathcal{T}_\sigma^{E_c}) \cap \mathbb{S}^1 = \emptyset$ for some σ

(iii) E_c elliptic critical energy $\Longleftrightarrow \mathrm{spec}(\mathcal{T}_\sigma^{E_c}) \subset \mathbb{S}^1$, no Jordan, for all σ

(iv) E_c parabolic point $\Longleftrightarrow \mathrm{spec}(\mathcal{T}_\sigma^{E_c}) \subset \mathbb{S}^1$ with Jordan blocks

At elliptic E_c there exists M with $M\,\mathcal{T}_\sigma^{E_c}\,M^{-1} = R_{\eta_\sigma}$ random rotation

Appears in random polymer model

Also: 1d Anderson model $H = \Delta + \lambda V_{\mathrm{dis}}$, but in parameter λ with $\lambda_c = 0$

So-called anomalies if $\eta_\sigma \in \{0, \frac{\pi}{2}\}$ for all σ (center of band of Anderson)

Parabolic E_c at band edges of Anderson and random Kronig-Penney

Furstenberg measure

Projective space in $\mathbb{R}^2$ via Prüfer phases $e^{i\theta_n^\epsilon}$

Furstenberg measure μ^ϵ is (unique) invariant measure on $\mathbb{S}^1$ with

$$\int_{\mathbb{S}^1} \mu^\epsilon(d\theta)\, f(e^{i\theta}) \;=\; \mathbb{E} \int_{\mathbb{S}^1} \mu^\epsilon(d\theta)\, f(\mathcal{T}_\sigma^{E_c+\epsilon} \cdot e^{i\theta}) \qquad , \qquad f \in C(\mathbb{S}^1)$$

At elliptic E_c Furstenberg is weakly Lebesgue, up to errors:

Theorem (Random phase approximation, Pastur-Figotin)

At elliptic critical energy E_c and away from anomaly,

$$\int_{\mathbb{S}^1} \mu^\epsilon(d\theta)\, f(e^{i\theta}) \;=\; \int_{\mathbb{S}^1} \frac{d\theta}{2\pi}\, f(e^{i\theta}) \;+\; \mathcal{O}(\epsilon^2)$$

and for some computable $\mathcal{D} > 0$

$$\gamma(E_c + \epsilon) \;=\; \mathcal{D}\,\epsilon^2 + \mathcal{O}(\epsilon^3)$$

Imaginary energy

For quantum transport, add imaginary part $i\delta$ with $\delta = \frac{1}{T} > 0$ to energy

Thus two-parameter family $(\epsilon, \delta) \mapsto \mathcal{T}_\sigma^{\epsilon,\delta} \in \mathrm{SL}(2, \mathbb{C})$, *e.g.* $= \mathcal{T}_\sigma^{E_c+\epsilon+i\delta}$

Dynamics $e \mapsto \frac{\mathcal{T} e}{\|\mathcal{T} e\|}$ now on unit vectors in $\mathbb{C}^2$, namely $\mathbb{C}P(1)$

Due to sign of δ, stereographic projection remains in closed unit disc $\overline{\mathbb{D}}$

$$z = \Pi(e) = \Pi\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \frac{a - ib}{a + ib} \in \overline{\mathbb{D}}$$

Action becomes Möbius transformation. Furstenberg measure on $\overline{\mathbb{D}}$:

$$\int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(dz)\, f(z) = \mathbb{E} \int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(dz)\, f(\mathcal{T}_\sigma^{\epsilon,\delta} \cdot z) \quad , \quad f \in C(\overline{\mathbb{D}})$$

Interest in regime of small ϵ and δ

Crossover regime for $\epsilon^2 \approx \delta$

Numerics for $1d$ Anderson model

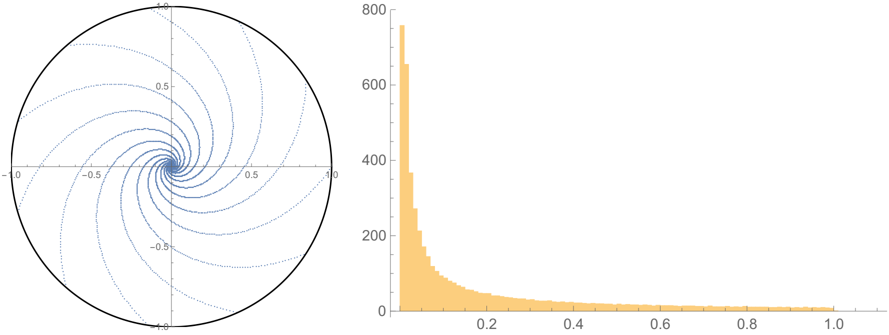
Orbit $(z_n)_{n=1,\ldots,N}$ in $\overline{\mathbb{D}}$ with $N = 5 \cdot 10^3$ computed from Anderson model

Parameters $\epsilon = 10^{-4}$ (disorder strength) and $\delta = 10^{-3}$ so that $\epsilon^2 \ll \delta$

$E = 0.52$ and initial condition $z_0 = 1$. Histogram of radii $|z_n|^2$

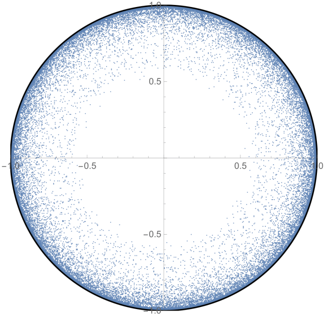
Tail of distribution merely due to first points

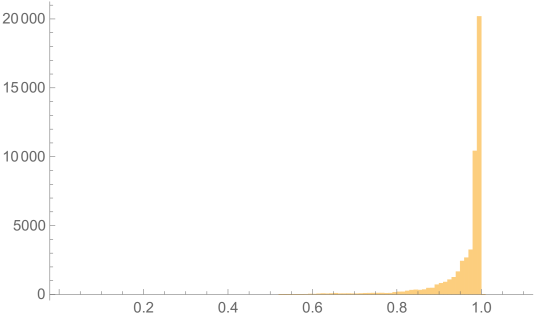
Furstenberg supported in central ball of size $\mathcal{O}(\delta)$



More numerics for $1d$ Anderson model

All as above, except $\epsilon = 0.1$ and $\delta = 10^{-5}$ so that $\epsilon^2 \gg \delta$





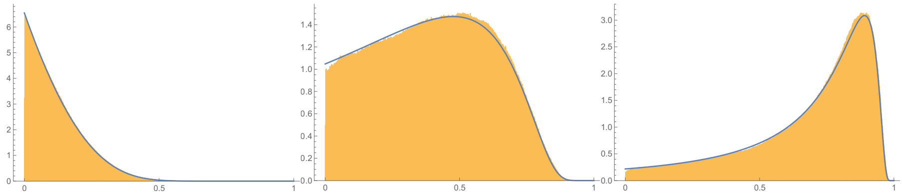
Main mass close in δ-ring of boundary $\mathbb{S}^1 = \partial\overline{\mathbb{D}}$

Further note: rotational symmetry (so away from anomaly)

There *are* orbits arriving at $z = 0$, but drift to outside due to $\gamma(E) > 0$

Yet more numerics for $1d$ Anderson model

$(\epsilon, \delta) = (0.05, 7.5 \cdot 10^{-4})$ $(0.05, 1.2 \cdot 10^{-4})$ $(0.05, 2.5 \cdot 10^{-5})$



Here $\epsilon^2 \approx \delta$. More precisely, set with $\mathcal{D}$ as in random phase approx.

$$\lambda = \frac{2}{\mathcal{D}} \frac{\delta}{\epsilon^2}$$

Above: three non-trivial radial distributions given by

$$\varrho_\lambda(|z|^2) = \frac{\lambda}{(1-|z|^2)^2} \exp\left[-\frac{\lambda |z|^2}{1-|z|^2}\right]$$

Theorem

Suppose that E_c not an anomaly and randomness "non-trivial"

For $\delta \gg \epsilon^2$, the distribution is centered around $0 \in \mathbb{D}$

$$\int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(\mathrm{d}z)\, |z|^2 \;=\; \mathcal{O}(\delta, \epsilon, \epsilon^2\delta^{-1})$$

For $\delta \ll \epsilon^2$, the distribution is centered close to $\partial\mathbb{D} = \mathbb{S}^1$

$$\int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(\mathrm{d}z)\, |z|^2 \;=\; 1 \;+\; \mathcal{O}(\epsilon^{\frac{1}{2}}, \delta^{\frac{1}{2}}\epsilon^{-1})$$

For $\delta \approx \epsilon^2$ and λ and ϱ_λ as above, then for $h \in C^2([0,1])$

$$\int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(\mathrm{d}z)\, h(|z|^2) \;=\; \int_0^1 \mathrm{d}s\, \varrho_\lambda(s)\, h(s) \;+\; \mathcal{O}(\epsilon, \epsilon^{-1}\delta)$$

Application: lower bound on dynamics for random polymer model (avoiding large deviation estimate in work with Jitomirskaya, Stolz)

Rough idea of proof for the last claim

Due to invariance of $\mu^{\epsilon,\delta}$ sufficient to control Birkhoff sums:

$$\int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(dz)\, g(|z|^2) \;=\; \int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(dz)\;\; \mathbb{E}\, g\big(|\mathcal{T}_\sigma^{\epsilon,\delta}\cdot z|^2\big)$$

$$= \int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(dz)\, \mathbb{E}\, \frac{1}{N}\sum_{n=0}^{N-1}\, g\big(|z_n|^2\big)$$

Go back in history once and expand for smooth g (to higher order!):

$$\mathbb{E}\frac{1}{N}\sum_{n=0}^{N-1} g(|z_{n+1}|^2) = \mathbb{E}\frac{1}{N}\sum_{n=0}^{N-1} g(|z_n|^2) - \mathbb{E}\frac{1}{N}\sum_{n=0}^{N-1} 4\delta|z_n|^2 g'(|z_n|^2) + \mathcal{O}(\epsilon,\delta\epsilon)$$

In limit $N\to\infty$ oscillatory terms disappear and after a lot of algebra:

$$\int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(dz)\,(\mathcal{L}_\lambda g)(|z|^2) = \mathcal{O}(\epsilon, \tfrac{\delta}{\epsilon}) \quad \text{with} \quad \mathcal{L}_\lambda \;=\; \big(\lambda \,-\, (1-s)^2\partial_s\big)s\partial_s$$

Make Ansatz:

$$\int_{\overline{\mathbb{D}}} \mu^{\epsilon,\delta}(dz)\; g(|z|^2) \;=\; \int_0^1 ds\, \varrho_\lambda(s)\, g(s)$$

Then for all smooth g

$$\mathcal{O}(\epsilon, \delta\epsilon) \;=\; \int_0^1 ds\, \varrho_\lambda(s)\, (\mathcal{L}_\lambda g)(s) \;=\; \int_0^1 ds\, (\mathcal{L}_\lambda^* \varrho_\lambda)(s)\, g(s) \;+\; \text{b.t.}$$

Now boundary terms vanish, and $\mathcal{L}_\lambda$ and $\mathcal{L}_\lambda^*$ singular elliptic

Thus $\mathrm{Ker}(\mathcal{L}_\lambda^*) = \mathrm{span}\{\rho_\lambda\}$ and $\mathrm{Ker}(\mathcal{L}_\lambda) = \mathrm{span}\{1\}$ fundamental

So claim follows from

$$h(s) = \mathcal{L}_\lambda g + \Big(\int_0^1 ds\, \rho_\lambda(s)\, h(s) \Big)\, 1$$

Resumé = Overview

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