Levin-Wen Models and Tensor Categories

Liang Kong

University of Erlangen-Nürnberg, Nov. 2011

Institute for Advanced Study in Tsinghua University, Beijing

a joint work with Alexei Kitaev
Goals:

- to provide a more rigorous and systematic study of Levin-Wen models;
- to enrich Levin-Wen models to include boundaries and defects of codimension 1,2,3;
- to show how the representation theory of tensor category enters the study of topological order at its full strength;
- to provide the physical meaning behind the so-called extended Turaev-Viro topological field theories;
1 Levin-Wen models

2 Kitaev’s Toric Code Model

3 Extended Topological Field Theories
Outline

1. Levin-Wen models
2. Kitaev’s Toric Code Model
3. Extended Topological Field Theories
Basics of unitary tensor category

unitary tensor category $\mathcal{C} = \text{unitary spherical fusion category}$

- semisimple: every object is a direct sum of simple objects;
- finite: there are only finite number of inequivalent simple objects, $i, j, k, l \in \mathcal{I}$, $|\mathcal{I}| < \infty$; $\dim \text{Hom}(A, B) < \infty$.
- monoidal: $(i \otimes j) \otimes k \cong i \otimes (j \otimes k)$; $1 \in \mathcal{I}$, $1 \otimes i \cong i \cong i \otimes 1$;
- the fusion rule: $\dim \text{Hom}(i \otimes j, k) = N_{ij}^k < \infty$;
- $\mathcal{C}$ is not assumed to be braided.

**Theorem** (Müger): The monoidal center $Z(\mathcal{C})$ of $\mathcal{C}$ is a modular tensor category.
Fusion matrices

The associator \((i \otimes j) \otimes k \xrightarrow{\alpha} i \otimes (j \otimes k)\) induces an isomorphism:

\[
\text{Hom}((i \otimes j) \otimes k, l) \xrightarrow{\cong} \text{Hom}(i \otimes (j \otimes k), l)
\]

Writing in basis, we obtain the fusion matrices:

\[
ij \quad k\quad m\quad l = \sum_{n} F_{ijk;lmn} ;
\]

(1)
Levin-Wen models

We fix a unitary tensor category $\mathcal{C}$ with simple objects $i, j, k, l, m, n \in \mathcal{I}$.

\[ \mathcal{H}_s = \mathcal{C}^{\mathcal{I}}, \quad \mathcal{H}_v = \bigoplus_{i,j,k} \text{Hom}_\mathcal{C}(i \otimes j, k). \]

\[ \mathcal{H} = \bigotimes_s \mathcal{H}_s \otimes_v \mathcal{H}_v. \]
**Hamiltonian**

Chose a basis of $\mathcal{H}$, $i, j, k \in I$ and $\alpha^{i'j';k'} \in \text{Hom}_C(i' \otimes j', k')$,

$$H = - \sum_v A_v - \sum_p B_p.$$

$$A_v |(i, j; k | \alpha^{i'j';k'})\rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'} |(i, j; k | \alpha^{i'j';k'})\rangle.$$

If the spin on $v$ is such that $A_v$ acts as 1, then it is called **stable**.
The definition of $B_p$ operator

$$B_p := \sum_{i \in I} \frac{d_i}{\sum_k d_k^2} B^i_p$$

- If there are unstable spins around the plaquette $p$, $B^i_p$ act on the plaquette as zero.
If all the spins at the corners are stable, then $B_p^k$ is defined as follow: suppressing all the spin labels,

$$B_p^k \left| i_1 \ i_2 \ j_1 \ i_3 \ j_2 \ j_3 \ i_4 \ j_4 \ i_5 \ j_5 \ i_6 \ j_6 \right\rangle = \left| i_1 \ i_2 \ j_1 \ i_3 \ j_2 \ j_3 \ i_4 \ j_4 \ i_5 \ j_5 \ i_6 \ j_6 \ k \right\rangle,$$

the right hand side of which is a sum of hexagons (without the $k$-loop) obtained by first fusing the $k$-loop with each $j$-edge then evaluating 6 triangles.

$B_p$ is a projector. $A_v$ and $B_p$ commute.
Ground states

$$A_v |0\rangle = |0\rangle, \quad B_p |0\rangle = |0\rangle.$$ 

If the model is defined on a surface $\Sigma$, then the space of ground states is exactly given by the $TV(\Sigma)$. It has been known for a long time. But only rigorously proved recently by Kirillov Jr. (2011)
Remark:

- Given a unitary tensor category $\mathcal{C}$, we obtain a lattice model.

- Conversely, Levin-Wen showed how the axioms of the unitary tensor category can be derived from the requirement to have a fix-point wave function of a string-net condensation state.
Edge theories

If we cut the lattice, we automatically obtain a lattice with a boundary with all boundary strings labeled by simple objects in $C$.

We will call such boundary as a $C$-boundary or $C$-edge.

**Question:** Are there any other possibilities?
\( \mathcal{M} \)-edge

It is possible to label the boundary strings by a different finite set \( \{\lambda, \sigma, \ldots\} \) which can be viewed as the set of inequivalent simples objects of another finite unitary semisimple category \( \mathcal{M} \).

The requirement of giving a fix-point wave function of string-net condensation state is equivalent to require that \( \mathcal{M} \) has a structure of \( C \)-module. We call such boundary an \( c\mathcal{M} \)-boundary or \( c\mathcal{M} \)-edge.
\( \mathcal{C}\)-module \( \mathcal{M} \):

For \( i \in \mathcal{C}, \gamma, \lambda \in \mathcal{M} \),

- \( i \otimes \gamma \) is an object in \( \mathcal{M} \) \((\otimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M})\)
- \( \text{dim Hom}_{\mathcal{M}}(i \otimes \gamma, \lambda) = N_{i, \gamma}^\lambda < \infty \);
- \( 1 \otimes \gamma \cong \gamma \);
- associator \((i \otimes j) \otimes \lambda \xrightarrow{\alpha} i \otimes (j \otimes \lambda)\);
- fusion matrices:

\[
\begin{array}{ccc}
& j & \\
\lambda & \sigma & i \\
& \gamma & \\
\end{array} = \sum_n F_{ijk;l}^{mni} \\
\begin{array}{ccc}
& j & \\
\lambda & \rho & i \\
& \gamma & \\
\end{array}
\]
Boundary excitations

For a given region $\bar{R}$ (with $n = 2$-external $C$-legs), $\bar{R} = \partial\bar{R} \cup R$, an excitation is given by a Hilbert subspace $\text{Im} P_{\bar{R}} \subset \mathcal{H}_{\bar{R}} = \mathcal{H}_{\partial\bar{R}} \otimes \mathcal{H}_R$ such that the projector $P_{\bar{R}}$ commutes with the action of $\otimes_{i=1}^3 B_{p_i}$ on the plaquettes immediately outside $\partial\bar{R}$.
The action of \( \bigotimes_{i=1}^{3} B_{p_i} \) on the plaquettes immediately outside \( \partial R \) can be written as \( \sum_r Q_{\text{ext}}^r \otimes Q_{\partial \bar{R}}^r \) where \( Q_{\text{ext}}^r \) acts on \( \mathcal{H}_{\text{ext}} \) and \( Q_{\partial \bar{R}}^r \) on \( \mathcal{H}_{\partial \bar{R}} \). The linear independence of \( Q_{\text{ext}}^r \) implies that \( Q_{\partial \bar{R}}^r \) commute with \( P_{\bar{R}} \).
\{ Q^r_{\partial R} \} \text{ generate an algebra } A^{(n)}_{\mathcal{M}\mathcal{M}} \text{ (n=2) spanned by:}

\[ \begin{array}{c}
\lambda \\
\alpha^* \\
\sigma \\
\vdots \\
\beta \\
\gamma \\
\rho \\
\end{array} \]

**Theorem:** A boundary excitation = a module over $A^{(n)}_{\mathcal{M}\mathcal{M}}$.

**Theorem:** $A^{(m)}_{\mathcal{M}\mathcal{M}}$ and $A^{(n)}_{\mathcal{M}\mathcal{M}}$ are Morita equivalent.
The case $n = 0$

We will construct the local operator algebra $A_{\mathcal{M}\mathcal{M}} = A_{\mathcal{M}\mathcal{M}}^{(0)}$.

$$A_{\mathcal{M}\mathcal{M}} := \bigoplus i, \lambda_1, \lambda_2, \gamma_1, \gamma_2 \text{Hom}_{\mathcal{M}}(i \otimes \lambda_2, \lambda_1) \otimes \text{Hom}_{\mathcal{M}}(\gamma_1, i \otimes \gamma_2).$$

For $\xi \in \text{Hom}_{\mathcal{M}}(i \otimes \lambda_2, \lambda_1)$ and $\zeta \in \text{Hom}_{\mathcal{M}}(\gamma_1, i \otimes \gamma_2)$, the element $\xi \otimes \zeta \in A_{\mathcal{M}\mathcal{M}}$ can be expressed by the following graph:

For $i \in \mathcal{C}$ and $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathcal{M}$. 

\[
\begin{array}{c}
\lambda_1 \\
\Rightarrow \xi \\
\lambda_2 \\
\downarrow i \\
\gamma_1 \\
\leftarrow \zeta \\
\gamma_2
\end{array}
\]
The multiplication $A_{\mathcal{M}\mathcal{M}} \otimes A_{\mathcal{M}\mathcal{M}} \rightarrow A_{\mathcal{M}\mathcal{M}}$ is defined by

\[
\begin{array}{c}
\lambda_1 & \xi & \lambda_2 \\
\gamma_1 & \zeta & \gamma_2 \\
\end{array}
\begin{array}{c}
\lambda'_1 & \xi' & \lambda'_2 \\
\gamma'_1 & \zeta' & \gamma'_2 \\
\end{array}
\end{array}

\cdot

\begin{array}{c}
\lambda_1 & \xi & \lambda_2 & \xi' & \lambda'_3 \\
\gamma_1 & \zeta & \gamma_2 & \zeta' & \gamma'_3 \\
\end{array}

= \delta_{\lambda_2} \lambda'_2 \delta_{\gamma_2} \gamma'_2

where the last graph is a linear span of graphs in $A_{\mathcal{M}\mathcal{M}}$ by applying F-moves twice and removing bubbles.
Figure: Two elements of local operator algebra $A_{M,M}$ act on an edge excitation (up to an ambiguity of the excited region).
$A_{\mathcal{M}\mathcal{M}}$ is bialgebra with above comultiplication.
With some small modifications, one can turn $A_{\mathcal{M},\mathcal{M}}$ into a weak $C^*$-Hopf algebra so that the boundary excitations form a finite unitary fusion category (Hayashi99, Szlachanyi00; Ostrik01, etc., see also Hendryk Pfeiffer’s talk).

**Theorem** [Ostrik]:

The category of $A_{\mathcal{M},\mathcal{M}}$-modules $\cong \text{Fun}_C(\mathcal{M},\mathcal{M})$.

**A physical proof**: use the set-up to show that excitations are classified by closed string operators that commute with the Hamiltonian (Levin-Wen). It is fairly straightforward to show that the latter objects are equivalent to $C$-module functors.
Close the boundary to a circle, a closed string operator on it is nothing but a systematic reassignment of boundary string labels and spin labels:

\[
\gamma \mapsto F(\gamma) \in \mathcal{M},
\]

\[
\text{Hom}_\mathcal{M}(i \otimes \gamma, \lambda) \mapsto \text{Hom}_\mathcal{M}(i \otimes F(\gamma), F(\lambda))
\]

This assignment is essentially the same data forming a functor from $\mathcal{M}$ to $\mathcal{M}$. Physical requirements (Levin-Wen) add certain consistency conditions which turn it into a $\mathcal{C}$-module functor.

**Theorem:** Excitations on a $\mathcal{C}\mathcal{M}$-edge are given by simple objects in the category $\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{M})$ of $\mathcal{C}$-module functors.
a defect line or a domain wall

\[ i, j, k, l \in \mathcal{C}, \lambda_1, \ldots, \lambda_9 \in \mathcal{M}, i', j', k', l' \in \mathcal{D}. \mathcal{C} \text{ and } \mathcal{D} \text{ are unitary tensor categories and } \mathcal{M} \text{ is a } \mathcal{C}\mathcal{D}\text{-bimodule. We call such defect } \mathcal{C}\mathcal{M}\mathcal{D}\text{-defect line or } \mathcal{C}\mathcal{M}\mathcal{D}\text{-wall.} \]
A $\mathcal{M}$-edge can be viewed as $\mathcal{CM}_{\text{Hilb}}$-wall.

Conversely, if we fold the system along the $\mathcal{CM}_D$-wall, we obtain a doubled bulk system determined by $\mathcal{C} \boxtimes \mathcal{D}^{\text{op}}$ with a single boundary determined by $\mathcal{M}$ which is viewed as a $\mathcal{C} \boxtimes \mathcal{D}^{\text{op}}$-module.

$$a \, \mathcal{CM}_D \text{-wall} = a \, \mathcal{C} \boxtimes \mathcal{D}^{\text{op}} \mathcal{M} \text{-edge}$$
Therefore, we have:

\[ C \mathcal{M}_D \text{-wall excitations} = C \boxtimes D^{\text{op}} \mathcal{M} \text{-edge excitations} = \text{Fun}_{C \boxtimes D^{\text{op}}}(\mathcal{M}, \mathcal{M}) = \text{Fun}_{C|D}(\mathcal{M}, \mathcal{M}) \]

\( \text{Fun}_{C|D}(\mathcal{M}, \mathcal{M}) := \text{the category of } C-D\text{-bimodule functors.} \)
As a special case, \(i, j, k, l, \lambda_1, \ldots, \lambda_9, i', j', k', l' \in C = \mathcal{M} = \mathcal{D}\).

\[
\begin{align*}
\text{a line in } C\text{-bulk} & \quad = \quad \text{a } cC_c\text{-wall} \\
C\text{-bulk excitations} & \quad = \quad cC_c\text{-wall excitations} \\
& \quad = \quad \text{Fun}_{C|C}(C, C) = Z(C)
\end{align*}
\]
A $\mathcal{C}\mathcal{M}_D$-wall can fuse with a $\mathcal{D}\mathcal{N}_E$-wall into a $\mathcal{C}(\mathcal{M} \boxtimes_D \mathcal{N})_E$-wall.

$\mathcal{C}\mathcal{M}_D$-wall (or $\mathcal{D}\mathcal{N}_E$-wall) excitations can fuse into $\mathcal{C}(\mathcal{M} \boxtimes_D \mathcal{N})_E$-wall as follow:

$$(\mathcal{M} \xrightarrow{F} \mathcal{M}) \mapsto (\mathcal{M} \boxtimes_D \mathcal{N} \xrightarrow{F \boxtimes_D id_N} \mathcal{M} \boxtimes_D \mathcal{N})$$

$$(\mathcal{N} \xrightarrow{G} \mathcal{N}) \mapsto (\mathcal{M} \boxtimes_D \mathcal{N} \xrightarrow{id_M \boxtimes_D G} \mathcal{M} \boxtimes_D \mathcal{N})$$
As a special case $\mathcal{M} = \mathcal{D}$: we obtain

the fusion of bulk excitations into wall excitations

as a monoidal functor:

$$(\mathcal{D} \xrightarrow{F} \mathcal{D}) \mapsto (\mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F \boxtimes_{\mathcal{D}} \text{id}_{\mathcal{N}}} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N})$$
a cospan: \[ Z(C) \xrightarrow{L_{\mathcal{M}}} \text{Fun}_{C|D}(\mathcal{M}, \mathcal{M}) \xleftarrow{R_{\mathcal{M}}} Z(D) \]

\[ L_{\mathcal{M}} : (C \xrightarrow{\mathcal{F}} C) \quad \rightarrow \quad (\mathcal{M} \cong C \boxtimes C \mathcal{M} \xrightarrow{\mathcal{F} \boxtimes C \text{id}_{\mathcal{M}}} C \boxtimes C \mathcal{M} \cong \mathcal{M}) \]

\[ R_{\mathcal{M}} : (D \xrightarrow{\mathcal{G}} D) \quad \rightarrow \quad (\mathcal{M} \cong \mathcal{M} \boxtimes D \mathcal{D} \xrightarrow{\text{id}_{\mathcal{M}} \boxtimes D \mathcal{G}} \mathcal{M} \boxtimes D \mathcal{D} \cong \mathcal{M}) \]
Definition: If $\mathcal{M} \boxtimes_D \mathcal{N} \cong \mathcal{C}$ and $\mathcal{N} \boxtimes_C \mathcal{M} \cong \mathcal{D}$, then $\mathcal{M}$ and $\mathcal{N}$ are called invertible; $\mathcal{C}$ and $\mathcal{D}$ are called Morita equivalent.

Theorem (Müger, Etingof-Nikshych-Ostrik, Kitaev) $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent iff $Z(\mathcal{C})$ is equivalent to $Z(\mathcal{D})$ as braided tensor categories.

- Invertible $\mathcal{C}$-$\mathcal{C}$-defects form a group called Picard group $\text{Pic}(\mathcal{C})$.
- We denote the auto-equivalence of $Z(\mathcal{C})$ as $\text{Aut}(Z(\mathcal{C}))$.

Theorem (Etingof-Nikshych-Ostrik09, Kitaev-K.09): 

$$\text{Aut}(Z(\mathcal{C})) \cong \text{Pic}(\mathcal{C}).$$
Defects of codimension 2

- A defect of codimension 2 is a junction between a $\mathcal{C}\mathcal{M}_D$-wall and a $\mathcal{C}\mathcal{N}_D$-wall. It corresponds to a module functor $\mathcal{F} \in \text{Fun}_{\mathcal{C}\mathcal{M}_D}(\mathcal{M}, \mathcal{N})$.

- Any excitation can be viewed as a defect of codimension 2.

- Any defect of codimension 2 is an excitation in the sense that it can be realized as a super-selection sector of a local operator algebra $A_{\mathcal{M}\mathcal{N}}$. 

Action of $A_{M,N}$ on defects of codimension 2

$$\lambda_1, \lambda_2, \lambda_3 \in M, \gamma_1, \gamma_2, \gamma_3 \in N$$
1. The category of $A_{\mathcal{M},\mathcal{N}}$-modules $= \text{Fun}_C(\mathcal{M}, \mathcal{N})$.

2. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{C}$-$\mathcal{D}$-walls, then we have the following commutative diagram:
Defects of codimension 3 (instantons)

If one takes into account the time direction, one can define a defect of codimension 3 by a natural transformation $\phi$ between module functors.

The Hamiltonian:

$$H \rightarrow H + H_t.$$ 

where $H_t$ is a local operator defined using $\phi$ (an instanton).
### Dictionary 1:

<table>
<thead>
<tr>
<th>Ingredients in LW-model</th>
<th>Tensor-categorical notions</th>
</tr>
</thead>
<tbody>
<tr>
<td>a bulk lattice</td>
<td>a unitary tensor category $\mathcal{C}$</td>
</tr>
<tr>
<td>string labels in a bulk</td>
<td>simple objects in a unitary tensor category $\mathcal{C}$</td>
</tr>
<tr>
<td>excitations in a bulk</td>
<td>simple objects in $Z(\mathcal{C})$ the monoidal center of $\mathcal{C}$</td>
</tr>
<tr>
<td>an edge</td>
<td>a $\mathcal{C}$-module $\mathcal{M}$</td>
</tr>
<tr>
<td>string labels on an edge</td>
<td>simple objects in a $\mathcal{C}$-module $\mathcal{M}$</td>
</tr>
<tr>
<td>excitations on a $\mathcal{M}$-edge</td>
<td>$\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{M})$: the category of $\mathcal{C}$-module functors</td>
</tr>
<tr>
<td>bulk-excitations fuse into an $\mathcal{M}$-edge</td>
<td>$Z(\mathcal{C}) = \text{Fun}_{\mathcal{C}</td>
</tr>
</tbody>
</table>

\[
(C \xrightarrow{\mathcal{F}} C) \mapsto (C \boxtimes C \mathcal{M} \xrightarrow{\mathcal{F}\text{id}_\mathcal{M}} C \boxtimes C \mathcal{M}).
\]
### Dictionary 2:

<table>
<thead>
<tr>
<th>Ingredients in LW-model</th>
<th>Tensor-categorical notions</th>
</tr>
</thead>
<tbody>
<tr>
<td>a domain wall</td>
<td>a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{N}$</td>
</tr>
<tr>
<td>string labels on a $\mathcal{N}$-wall</td>
<td>simple objects in a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{C}\mathcal{N}_\mathcal{D}$</td>
</tr>
<tr>
<td>excitations on a $\mathcal{N}$-wall</td>
<td>$\text{Fun}_{\mathcal{C}</td>
</tr>
<tr>
<td>fusion of two walls</td>
<td>$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$</td>
</tr>
<tr>
<td>an invertible $\mathcal{C}\mathcal{N}_\mathcal{D}$-wall</td>
<td>$\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent, i.e. $\mathcal{N} \otimes_{\mathcal{D}} \mathcal{N}^{\text{op}} \cong \mathcal{C}$, $\mathcal{N}^{\text{op}} \otimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{D}$.</td>
</tr>
<tr>
<td>bulk-excitation fuse into a $\mathcal{C}\mathcal{N}_\mathcal{D}$-wall</td>
<td>$Z(\mathcal{C}) = \text{Fun}_{\mathcal{C}</td>
</tr>
<tr>
<td>defects of codimension 2: a $\mathcal{M}$-$\mathcal{N}$-excitation</td>
<td>simple objects $\mathcal{F}, \mathcal{G} \in \text{Fun}_{\mathcal{C}</td>
</tr>
<tr>
<td>a defect of codimension 3 or an instanton</td>
<td>a natural transformation $\phi : \mathcal{F} \to \mathcal{G}$</td>
</tr>
</tbody>
</table>
Outline

1. Levin-Wen models
2. Kitaev’s Toric Code Model
3. Extended Topological Field Theories
Kitaev’s Toric Code Model

- Kitaev’s Toric Code Model is equivalent to Levin-Wen model associated to the category $\text{Rep}_{\mathbb{Z}_2}$ of representations of $\mathbb{Z}_2$.

- It is the simplest example that can illustrate the general features of Levin-Wen models.
Kitaev’s Toric Code Model

\[ \mathcal{H} = \bigotimes_{e \in \text{all edges}} \mathcal{H}_e; \quad \mathcal{H}_e = \mathbb{C}^2. \]

\[ H = - \sum_v A_v - \sum_p B_p. \]

\[ A_v = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4; \quad B_p = \sigma_z^5 \sigma_z^6 \sigma_z^7 \sigma_z^8. \]
Vacuum properties of toric code model:

A vacuum state $|0\rangle$ is a state satisfying $A_v |0\rangle = |0\rangle$, $B_p |0\rangle = |0\rangle$ for all $v$ and $p$.

- If surface topology is trivial (a sphere, an infinite plane), the vacuum is unique.

- Vacuum is given by the condensation of closed strings, i.e.

$$|0\rangle = \sum_{c \in \text{all closed string configurations}} |c\rangle.$$
Excitations

- The “set” of excitations determines the topological phase.

- An excitation is defined to be super-selection sectors (irreducible modules) of a local operator algebra.

- There are four types of excitations: $1, e, m, \epsilon$. We denote the ground states of these sectors as $|0\rangle, |e\rangle, |m\rangle, |\epsilon\rangle$. We have

$$
\exists v_0, \quad A_{v_0} |e\rangle = -|e\rangle,
\exists p_0, \quad B_{p_0} |m\rangle = -|m\rangle,
\exists v_1, p_1, \quad A_{v_1} |\epsilon\rangle = -|\epsilon\rangle, \quad B_{p_1} |\epsilon\rangle = -|\epsilon\rangle.
$$
1 = e \otimes e \sim \sigma_z^1 \sigma_z^2 \sigma_z^3 \sigma_z^4 \sigma_z^5 |0\rangle,

1 = m \otimes m \sim \sigma_x^6 \sigma_x^7 \sigma_x^8 |0\rangle,

\quad e \otimes m = \epsilon.

\footnote{1, e, m, \epsilon are simple objects of a \textbf{braided tensor category} \( Z(\text{Rep}_{\mathbb{Z}_2}) \) which is the \textbf{monoidal center} of \( \text{Rep}_{\mathbb{Z}_2} \).}
A smooth edge

This assignment actually gives a monoidal functor

\[ Z(\text{Rep}_{\mathbb{Z}_2}) \rightarrow \text{Rep}_{\mathbb{Z}_2} = \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}}(\text{Rep}_{\mathbb{Z}_2}, \text{Rep}_{\mathbb{Z}_2}). \]
This assignment gives another monoidal functor
\[ Z(\text{Rep}_{\mathbb{Z}_2}) \rightarrow \text{Rep}_{\mathbb{Z}_2} = \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}}(\text{Hilb}, \text{Hilb}). \]
defects of codimension 1, 2

\[ B_{p_1} = \sigma_7^x \sigma_3^x \sigma_2^x \sigma_5^x; \quad B_{p_2} = \sigma_3^x \sigma_7^x \sigma_8^x \sigma_9^x; \]

\[ B_Q = \sigma_6^x \sigma_1^y \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z \sigma_5^z \sigma_6^z \sigma_7^z \sigma_8^z \sigma_9^z \sigma_{10}^z \sigma_{11}^z \sigma_{12}^z \sigma_{13}^z \sigma_{14}^z \sigma_{15}^z \sigma_{16}^z \sigma_{17}^z \sigma_{18}^z \sigma_{19}^z \sigma_{20}^z. \]
defects of codimension 1

\[ 1 \mapsto 1 \mapsto 1, \quad e \xrightarrow{\sigma_3^3} \text{Ext}^{\text{defect}}_{3|7,8,9} \xrightarrow{\sigma_5^8} m, \]

\[ m \mapsto \text{Ext}^{\text{defect}}_{7|3,2,5} \mapsto e, \quad \epsilon \mapsto \text{Ext}^{\text{defect}}_{2,5,7,8,9,3} \mapsto \epsilon. \]

This assignment gives an invertible monoidal functor

\[ Z(\text{Rep}_{\mathbb{Z}_2}) \rightarrow \text{Fun}_{\text{Rep}_{\mathbb{Z}_2} \mid \text{Rep}_{\mathbb{Z}_2}}(\text{Hilb}, \text{Hilb}) \rightarrow Z(\text{Rep}_{\mathbb{Z}_2}). \]
Two eigenstates of $B_Q$ correspond to two simple $\text{Rep}_{\mathbb{Z}_2} \text{-Rep}_{\mathbb{Z}_2}$-bimodule functors $\text{Hilb} \to \text{Rep}_{\mathbb{Z}_2}$. 
Outline

1 Levin-Wen models
2 Kitaev’s Toric Code Model
3 Extended Topological Field Theories
Levin-Wen models enriched by defects of codimension 1, 2, 3 provides a physical meaning behind the so-called extended Turaev-Viro topological field theories.

Algebraic structures appeared in extended Turaev-Viro TQFT can be summarized as a conjectured boundary-to-bulk (or holography) functor between two tri-categories as we will discuss.
The building blocks of the lattice models:

which 0-1-2-3 cells of a tri-category, or “equivalently”,

Levin-Wen Models and Tensor Categories

Liang Kong University of Erlangen-Nürnberg, Nov. 2011
Excitations (topological phases):

\[ Z(\mathcal{M}) \]

\[ Z(\mathcal{C}) \rightarrow Z(\mathcal{M}, \mathcal{N})_{\mathcal{F}} \rightarrow Z(\mathcal{M}, \mathcal{N})_{\mathcal{G}} \rightarrow Z(\mathcal{D}) \]

\[ Z(\mathcal{M}) := \text{Fun}_{C|D}(\mathcal{M}, \mathcal{M}), \ Z(\mathcal{N}) := \text{Fun}_{C|D}(\mathcal{N}, \mathcal{N}), \]
\[ \mathcal{F}, \mathcal{G} \in Z(\mathcal{M}, \mathcal{N}) := \text{Fun}_{C|D}(\mathcal{M}, \mathcal{N}). \]
**Conjecture (Functoriality of Holography):** The assignment $Z$ is a functor between two tricategories.

**Remark:** It also says that the notion of monoidal center is functorial.
**General philosophy:** for $n+1$-dim extended TQFT,

$$\text{pt} \mapsto n\text{-category of boundary conditions}.$$ 

**Extended Turaev-Viro (2+1) TQFT:** the bicategory of boundary conditions of LW-models $= \mathcal{C}\text{-Mod},$

$$\text{pt}_{+,-} \mapsto \mathcal{C}, \mathcal{D} \text{ or } (\mathcal{C}\text{-Mod} \cong \mathcal{D}\text{-Mod}),$$

an interval $\mapsto \mathcal{M}_\mathcal{D}, \mathcal{N}_\mathcal{C}$ (invertible)

$$S^1 \mapsto Tr(\mathcal{C}) = Z(\mathcal{C}),$$
Thank you!