Minicourse: Multiscale behaviour in selection-mutation systems

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Part 5: Emergence and fixation on a infinite geographic space Ω_N in the $N \to \infty$ limit

1. Hierarchical mean-field limit and renormalization

2. The Basic Scenario

A Sequence of Space and Time Scales

Step 1 (*Times Scales*)

Define three types of time scales for the $j-$th step in the transition, i.e. for $j-$equilibration, $(j+1)-$fixation and $(j+1)-$equilibration by considering for each j :

$$T(N)N^j : T(N) \uparrow \infty, \text{ with } \frac{T(N)}{\log N} \to 0, \text{ as } N \to \infty \qquad (0.1)$$

$$S(N)N^j : S(N) \uparrow \infty, S(N)N^{-1} \to 0 \text{ as } N \to \infty, \text{ with} \qquad (0.2)$$

$$\liminf_{N\to\infty} \frac{S(N)}{\log N} \text{ sufficiently large,}$$

$$sN^{j+1} \text{ with } s \in (0, \infty). \qquad (0.3)$$

Step 2 (*Space scales and corresponding characteristic functionals*)

We introduce a sequence indexed by $k \in \mathbb{N}_0$, of *spatial rescalings* (with scaling parameter N) of the spatial system as follows.

Define now the block averages over $k-$balls of the process $X(t) = (x_\xi(t))_{\xi \in \Omega_N}$, as

$$x_{\xi,k}(t) = N^{-k} \sum_{d(\xi',\xi) \leq k} x_{\xi'}(t), \quad \xi \in \Omega_N. \tag{0.4}$$

$$x_{\xi,k}(t) \in \mathcal{P}(\cup_j E\ell) \tag{0.5}$$

Furthermore we define the level$-k$ empirical measure for the level-$(k-1)$ averages:

$$\Xi^N_{\xi,k}(t) = \frac{1}{N} \sum_{\xi' \in \Omega_N^{k-1}, d_{k-1}(\xi,\xi') \leq 1} \delta_{x_{\xi',k-1}(t)}, \quad \xi \in \Omega_N^{(k-1)}. \qquad (0.6)$$

$$\Xi_{\xi,k}(t) \in \mathcal{P}(\mathcal{P}(\cup_j E_\ell)) \qquad (0.7)$$

Step 3 *(Renormalization via multiple time-space scales)*

Definition (Renormalized system of index (j,k) and scaling parameter N)

Fix $N \geq 2$. Then we look at the following $\mathbb{N}_0^2-$indexed *collection of space-time rescaled systems* (we display now the dependence of $X(t)$ on N by writing $X^N(t) = ((x_\xi^N(t))_{\xi \in \Omega_N})$):

$$\left\{\left\{(x_{\xi,k}^N(U(N)N^j + tN^k))_{t \geq 0}, \quad \xi \in \Omega_N\right\}, \quad j,k \in \mathbb{N}\right\}, \tag{0.8}$$

with U being either S, T or sN from (0.1) - (0.3).

This collection induces for every (j,k) and U as above a random field (since the field is constant in $k-$balls for the original index set Ω_N)

$$\left\{(x_{\xi,k}^N(U(N)N^j + tN^k))_{t \geq 0}, \quad \xi \in \Omega_N^{(k)}\right\}. \tag{0.9}$$

This recipe defines in each of the *three sets of time scales* for every index $(j,k) \in \mathbb{N}_0^2$ and every value of the scaling parameter N a *new renormalized system* again indexed by locations ξ.

Step 4 *(Droplet description of fitter types)*

Next we have in order to understand the transitions between the different regimes to provide for the $j-$th step a precise description of the (very small) mass of the types in E_{j+1} in times of order N^j. This is referred to as droplet formation.

$$\widehat{x}_{1,j}(t) = \sum_{d_j(\xi,0)\leq 1} x_{\xi,j}(t), \quad \xi \in \Omega_N^{(j)}, \quad t \in [0, Const \cdot N^j \log N]. \quad (0.10)$$

Assign to every $j-$block within the $(j+1)-$block which we consider and which is characterized by $\ell \in \{0, 1, 2, \cdots, N\}$ the $j-$th digit in $\xi \in \Omega_N$ a label $a_j(\ell)$, so that for each j we get a collection

$$a_j(\ell) \quad , \quad \ell \in \mathbb{N}, \tag{0.11}$$

which are i.i.d. uniformly $[0, 1]-$distributed. Then consider for the $j-$th transition step the *atomic measure-valued process* defined as

$$\mathcal{J}^N_{j,t} = \sum_{\ell} \sum_{i \in E_{j+1}} x^N_{\xi(\ell),j}(tN^j)(\{i\})[\delta_{a_j(\ell)} \otimes \delta_{\{i\}}] \in \mathcal{P}([0,1] \times E_{j+1}), \tag{0.12}$$

where $\xi(\ell) = (0, \cdots, \ell, 0, \cdots), \quad \ell \in \{0, 1, 2, 3, \cdots\}$ with ℓ on the $j-$th position.

Definition (Renormalized droplets of index j)

The collection of renormalized droplets at time-space scale (j, j) is given by the processes

$$\{(\mathcal{J}^N_{j,t})_{t \geq 0}, \quad j \in \mathbb{N}\}. \tag{0.13}$$

Step 5 *(Quasi-equilibria)*

In the *hierarchical mean-field limit* of $N \to \infty$, the limiting object (certain nonlinear Markov processes) will allow us to describe asymptotically the various states (quasi-equilibria) through which the system passes on its way to the final equilibrium as time tends to infinity.

The *quasi-equilibrium* is defined as the equilibrium of a limiting dynamics and hence becomes a precise mathematical meaning.

Five Phases of Transition

Phase 0: *Approach to quasi-equilibrium of E_j-population; germs for $E_{j+1}-$droplets.*

(i) At the time $T(N)N^j$ a level$-j$ *quasi-equilibrium* concentrated on E_j-types arises in the limit $N \to \infty$ as equilibrium of a limiting dynamic derived from the selection-mutation-migration mechanism for the averages in $k-$blocks within a tagged $j-$block for $k \leq j$. The quasi-equilibrium of this tagged $j-$block is described by a *nonlinear McKean - Vlasov equation* with selection, mutation for the limiting empirical measure $\Xi^N_{\xi,j}$.

(ii) However on a sparse set in space, a subset of the tagged $(j+1)-$block and consisting of $j-$blocks but whose number is $o(N)$(as $N \to \infty$), these blocks are already dominated by fitter $E_{j+1}-$types which appeared through rare mutation and the subsequent selection-migration dynamic. These sparse subsets are called droplets which we described by $(\mathcal{J}_{j,t}^N)_{t\geq 0}$.

Phase 1: *Emergence of $E_{j+1}-$ types*

The *emergence* of level-$(j+1)$ types in the tagged $j-$block (i.e. a fixed threshold of type $E_{j+1}-$mass is reached with positive probability as $N \to \infty$) occurs at times

$$T_j N^j + C N^j \log N = N^j (T_j + C \log N) \tag{0.14}$$

for some *constant* C and a positive *random variable* T_j with values in $\mathbb{R}$. Both C and $\mathcal{L}[T_j]$ we determine via the dual process.

The time above has two distinct components.

1. The (large) part $CN^j \log N$ in (0.14) describes what occurs between time $T_j N^j$ and the time $CN^j \log N + T_j N^j$, the time interval at which time some fixed positive intensity ε of the new type can be established following a deterministic evolution starting from a large number of germs of the $E_{j+1}-$type which have developed by time $T(N)$.
2. The second part T_j arises if we wait in the *beginning*, i.e. times in $[0, T_j N^j]$, for a droplet of the rare mutants consisting of a finite random number of $j-$blocks. This initial randomness remains visible in the system after the much larger time $C \log N$ has passed.

Phase 2: *Fixation of $E_{j+1}-$ types*

The *fixation* (or *take-over*) of the tagged $j-$block or the $(j+1)-$blockaverage by $E_{j+1}-$types occurs rather fast compared to phase 1 and is captured considering the time scale

$$(T_j N^j + CN^j \log N) + tN^j \quad \text{for varying } t \in \mathbb{R}, (= N^j(T_j + C \log N + t)). \tag{0.15}$$

1. The limiting dynamic of the $(j+1)-$block empirical measure $\Xi^N_{0,j}$ is a nonlinear McKean - Vlasov process leading from a state-concentrated almost completely on E_j as $t \to \infty$ to fixation on E_{j+1}.
2. We find for a tagged $j-$blockaverage in the limit $N \to \infty$ an evolution, which is described by an entrance law of a Markov process in a random medium, the latter describing the independent feeding in of the rare mutants from somewhere in the $(j+1)-$block.
3. In order to track the genealogy and spatial spread of the mutant types in that phase we will use the *historical process* and corresponding to it as a dual object a *spatial coalescent*.

Phase 3: *Neutral evolution of $E_{j+1}-$ types*

After fixation a friendly *(neutral)* *coexistence* of all the $E_{j+1}-$types develops. Namely at the time scale $S(N)N^j$ the invasion and complete fixation of a region of radius $j+1$ on mutants of $E_{j+1}-$type has occurred. As time varies from $S(N)N^j$ to $S(N)N^j + t_N N^j$ with $t_N \uparrow +\infty$ and $0 \leq t_N = o(N)$ an *intermediate "neutral "quasi-equilibrium with immigration-emigration* on $E_{j+1}-$types develops in the tagged $j-$block.

Role of migration process in neutral quasi-equilibrium - the spatial distribution on the E_{j+1}-types within the j-block depends on the $\{c_\ell : \ell < j\}$.

$\{x^N_{\xi,k}(U(N)N^j)\}_{k=j,j-1,\dots,0}$ converges to a Markov chain with state space $\mathcal{P}(E_{j+1})$ [DGV95], which exhibits coexistence or clustering.

Phase 4: *Approach to quasi-equilibrium of E_{j+1} population*

Finally in times of order N^{j+1} , i.e. much later as we evolved in phase 3, the selection among the types of level $j+1$ starts to act and migration connects the N different $j-$blocks within distance $j+1$ and one observes in time scale tN^{j+1} a limiting dynamic for the $(j+1)-$blockaverages.

Later by time $T(N)N^{j+1}$ convergence to a mutation-selection *quasi-equilibrium concentrated now on level $(j+1)-$types* in the average over the $(j+1)-$block occurs.

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