Minicourse: Multiscale behaviour in selection-mutation systems

D. Dawson, A. Greven

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Part 5: Emergence and fixation on a infinite geographic space $\Omega_N$ in the $N \to \infty$ limit
Hierarchical mean-field limit and renormalization

The Basic Scenario

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A Sequence of Space and Time Scales

Step 1 (Times Scales)

Define three types of time scales for the $j$-th step in the transition, i.e. for $j$—equilibration, $(j + 1)$—fixation and $(j + 1)$—equilibration by considering for each $j$:

$$T(N)N^j : T(N) \uparrow \infty, \text{ with } \frac{T(N)}{\log N} \to 0, \text{ as } N \to \infty \quad (0.1)$$

$$S(N)N^j : S(N) \uparrow \infty, S(N)N^{-1} \to 0 \text{ as } N \to \infty, \text{ with } \liminf_{N \to \infty} \frac{S(N)}{\log N} \text{ sufficiently large,} \quad (0.2)$$

$$sN^{j+1} \text{ with } s \in (0, \infty). \quad (0.3)$$
Step 2 (*Space scales and corresponding characteristic functionals*)

We introduce a sequence indexed by $k \in \mathbb{N}_0$, of *spatial rescalings* (with scaling parameter $N$) of the spatial system as follows.

Define now the **block averages** over $k$–balls of the process $X(t) = (x_\xi(t))_{\xi \in \Omega_N}$, as

$$x_{\xi,k}(t) = N^{-k} \sum_{d(\xi',\xi) \leq k} x_{\xi'}(t), \quad \xi \in \Omega_N. \quad (0.4)$$

$$x_{\xi,k}(t) \in \mathcal{P}(\bigcup_j E_\ell) \quad (0.5)$$
Furthermore we define the **level–k empirical measure** for the level-(k − 1) averages:

\[
\Xi_{\xi, k}^N(t) = \frac{1}{N} \sum_{\xi' \in \Omega_{k-1}^N, d_{k-1}(\xi, \xi') \leq 1} \delta_{\xi, k-1}(t), \quad \xi \in \Omega_{N}^{(k-1)}. \tag{0.6}
\]

\[
\Xi_{\xi, k}(t) \in \mathcal{P}(\mathcal{P}(\cup_j E_\ell)) \tag{0.7}
\]
Step 3 (Renormalization via multiple time-space scales)

Definition (Renormalized system of index \((j, k)\) and scaling parameter \(N\))

Fix \(N \geq 2\). Then we look at the following \(\mathbb{N}_0^2\)-indexed collection of space-time rescaled systems (we display now the dependence of \(X(t)\) on \(N\) by writing \(X^N(t) = ((x^N_{\xi}(t))_{\xi \in \Omega_N})\):

\[
\left\{ (x^N_{\xi,k}(U(N)N^j + tN^k))_{t \geq 0}, \quad \xi \in \Omega_N \right\}, \quad j, k \in \mathbb{N} \right\}, \quad (0.8)
\]

with \(U\) being either \(S\), \(T\) or \(sN\) from (0.1) - (0.3).

This collection induces for every \((j, k)\) and \(U\) as above a random field (since the field is constant in \(k\)-balls for the original index set \(\Omega_N\))

\[
\left\{ (x^N_{\xi,k}(U(N)N^j + tN^k))_{t \geq 0}, \quad \xi \in \Omega_N^{(k)} \right\}. \quad (0.9)
\]

This recipe defines in each of the three sets of time scales for every index \((j, k) \in \mathbb{N}_0^2\) and every value of the scaling parameter \(N\) a new renormalized system again indexed by locations \(\xi\).
Step 4 (Droplet description of fitter types)

Next we have in order to understand the transitions between the different regimes to provide for the $j$–th step a precise description of the (very small) mass of the types in $E_{j+1}$ in times of order $N^j$. This is referred to as droplet formation.

$$\hat{x}_{1,j}(t) = \sum_{d_j(\xi,0) \leq 1} x_{\xi,j}(t), \quad \xi \in \Omega^{(j)}_N, \quad t \in [0, Const \cdot N^j \log N].$$ (0.10)
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Assign to every $j$—block within the $(j + 1)$—block which we consider and which is characterized by $\ell \in \{0, 1, 2, \cdots, N\}$ the $j$—th digit in $\xi \in \Omega_N$ a label $a_j(\ell)$, so that for each $j$ we get a collection

$$a_j(\ell), \quad \ell \in \mathbb{N},$$

(0.11)

which are i.i.d. uniformly $[0, 1]$—distributed. Then consider for the $j$—th transition step the \textit{atomic measure-valued process} defined as

$$\mathcal{J}^N_{j,t} = \sum_{\ell} \sum_{i \in E_{j+1}} x^N_{\xi(\ell),j}(tN^j)({i})[\delta_{a_j(\ell)} \otimes \delta_{\{i\}}] \in \mathcal{P}([0, 1] \times E_{j+1}),$$

(0.12)

where $\xi(\ell) = (0, \cdots, \ell, 0, \cdots)$, $\ell \in \{0, 1, 2, 3, \cdots\}$ with $\ell$ on the $j$—th position.

\textbf{Definition (Renormalized droplets of index $j$)}

The collection of renormalized droplets at time-space scale $(j,j)$ is given by the processes

$$\{(\mathcal{J}^N_{j,t})_{t \geq 0}, \quad j \in \mathbb{N}\}.$$  

(0.13)
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Step 5 (Quasi-equilibria)

In the hierarchical mean-field limit of $N \to \infty$, the limiting object (certain nonlinear Markov processes) will allow us to describe asymptotically the various states (quasi-equilibria) through which the system passes on its way to the final equilibrium as time tends to infinity.

The quasi-equilibrium is defined as the equilibrium of a limiting dynamics and hence becomes a precise mathematical meaning.
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Part 5: Emergence and fixation on an infinite geographic space $\Omega_N$ in the $N \rightarrow \infty$ limit
**Phase 0:** Approach to quasi-equilibrium of $E_j$–population; germs for $E_{j+1}$–droplets.

(i) At the time $T(N)N^j$ a level–$j$ quasi-equilibrium concentrated on $E_j$–types arises in the limit $N \to \infty$ as equilibrium of a limiting dynamic derived from the selection-mutation-migration mechanism for the averages in $k$–blocks within a tagged $j$–block for $k \leq j$. The quasi-equilibrium of this tagged $j$–block is described by a nonlinear McKean - Vlasov equation with selection, mutation for the limiting empirical measure $\Xi^N_{\xi,j}$. 
(ii) However on a **sparse set in space**, a subset of the tagged $(j+1)$–block and consisting of $j$–blocks but whose number is $o(N)$ (as $N \to \infty$), these blocks are already dominated by fitter $E_{j+1}$–types which appeared through rare mutation and the subsequent selection-migration dynamic. These sparse subsets are called **droplets** which we described by $(\mathcal{J}_j^N, t)_{t \geq 0}$. 
Phase 1:  *Emergence of* $E_{j+1}$ *types*

The *emergence* of level-$(j + 1)$ types in the tagged $j$–block (i.e. a fixed threshold of type $E_{j+1}$–mass is reached with positive probability as $N \to \infty$) occurs at times

$$T_j N^j + CN^j \log N = N^j (T_j + C \log N) \quad (0.14)$$

for some *constant* $C$ and a positive *random variable* $T_j$ with values in $\mathbb{R}$. Both $C$ and $\mathcal{L}[T_j]$ we determine via the dual process.
The time above has two distinct components.

1. The (large) part $CN^j \log N$ in (0.14) describes what occurs between time $T_j N^j$ and the time $CN^j \log N + T_j N^j$, the time interval at which time some fixed positive intensity $\varepsilon$ of the new type can be established following a deterministic evolution starting from a large number of germs of the $E_{j+1}$—type which have developed by time $T(N)$.

2. The second part $T_j$ arises if we wait in the beginning, i.e. times in $[0, T_j N^j]$, for a droplet of the rare mutants consisting of a finite random number of $j$—blocks. This initial randomness remains visible in the system after the much larger time $C \log N$ has passed.
**Phase 2:** *Fixation of $E_{j+1}$ types*

The *fixation* (or *take-over*) of the tagged $j$--block or the $(j + 1)$--block average by $E_{j+1}$--types occurs rather fast compared to phase 1 and is captured considering the time scale

$$ (T_j N^j + C N^j \log N) + t N^j \quad \text{for varying } t \in \mathbb{R}, \quad (\equiv N^j (T_j + C \log N + t)). $$

(0.15)
The limiting dynamic of the \((j + 1)\)-block empirical measure \(\Xi_{0,j}^N\) is a nonlinear McKean-Vlasov process leading from a state-concentrated almost completely on \(E_j\) as \(t \to \infty\) to fixation on \(E_{j+1}\).

We find for a tagged \(j\)-blockaverage in the limit \(N \to \infty\) an evolution, which is described by an entrance law of a Markov process in a random medium, the latter describing the independent feeding in of the rare mutants from somewhere in the \((j + 1)\)-block.

In order to track the genealogy and spatial spread of the mutant types in that phase we will use the historical process and corresponding to it as a dual object a spatial coalescent.
Phase 3: *Neutral evolution of* $E_{j+1}$-types

After fixation a friendly (neutral) coexistence of all the $E_{j+1}$-types develops. Namely at the time scale $S(N)N^j$ the invasion and complete fixation of a region of radius $j + 1$ on mutants of $E_{j+1}$-type has occurred. As time varies from $S(N)N^j$ to $S(N)N^j + t_N N^j$ with $t_N \uparrow +\infty$ and $0 \leq t_N = o(N)$ an intermediate ”neutral” quasi-equilibrium with immigration-emigration on $E_{j+1}$-types develops in the tagged $j$-block.

**Role of migration process in neutral quasi-equilibrium** - the spatial distribution on the $E_{j+1}$-types within the $j$-block depends on the \{c_\ell : \ell < j\}.

\{x_{\xi,k}^N(U(N)N^j)\}_{k=j,j−1,...,0} converges to a Markov chain with state space $\mathcal{P}(E_{j+1})$ [DGV95], which exhibits coexistence or clustering.
Phase 4: Approach to quasi-equilibrium of $E_{j+1}$ population

Finally in times of order $N^{j+1}$, i.e. much later as we evolved in phase 3, the selection among the types of level $j + 1$ starts to act and migration connects the $N$ different $j$–blocks within distance $j + 1$ and one observes in time scale $tN^{j+1}$ a limiting dynamic for the $(j + 1)$–blockaverages.

Later by time $T(N)N^{j+1}$ convergence to a mutation-selection quasi-equilibrium concentrated now on level $(j + 1)$–types in the average over the $(j + 1)$–block occurs.