(Non)local phase coexistence models

Enrico Valdinoci

University of Western Australia

enrico.valdinoci@uwa.edu.au

CAA Online Seminar – March 11, 2021

Overview

1. Setting of the problem
   - Allen-Cahn equation and perimeters
   - Anisotropic equations and interfaces
2. Main results
   - One-dimensional symmetry and rigidity theorems
3. Strategy of the proofs
   - Proof of fractional versions of De Giorgi Conjecture
   - Proof of asymptotical flatness results
4. Proof of general rigidity results

The nonlocal Allen-Cahn equation

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}}\, dy$$

$s \in (0,1)$

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u minimal solution, $u_\epsilon(x) := u\left(\frac{x}{\epsilon}\right)$ blow-down
u_ϵ converges to $\chi_{\mathbb{R}^n \setminus E} - \chi_E$, where E is a local minimizer for

- the classical perimeter, if $s \in \left[\frac{1}{2}, 1\right)$,
- the fractional perimeter, if $s \in \left(0, \frac{1}{2}\right)$.

(O. Savin, E. Valdinoci, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2012).

Fractional perimeter (L. Caffarelli, J.-M. Roquejoffre, O. Savin, Comm. Pure Appl. Math. 2010)

$$I_s(A,B) := \iint_{A\times B} \frac{dx\,dy}{|x-y|^{n+2s}},$$

$$A,B \subseteq \mathbb{R}^n, \qquad A\cap B = \varnothing, \qquad s \in \left(0,\frac{1}{2}\right),$$

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"counts" the interaction of E and E^c with respect to a reference domain Ω.

Critical points of Per_s possess "zero fractional mean curvature"

$$H^s_E(x) := \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+2s}}\,dy, \qquad \forall x \in \partial E.$$

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Associated geometric flow (motion by fractional mean curvature),

C. Imbert, Interfaces Free Bound. (2009),

L. Caffarelli, P. Souganidis, Arch. Ration. Mech. Anal. (2010),

A. Chambolle, M. Morini, M. Ponsiglione, Arch. Ration. Mech. Anal. (2015).

Anisotropic fractional operators

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \, \mu(y/|y|) \, dy.$$

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Main examples:

- fractional Laplacian $\mathcal{L} = (-\Delta)^s$. $\mu(\sigma) = 1$ for any $\sigma \in S^{n-1}$.

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Main examples:

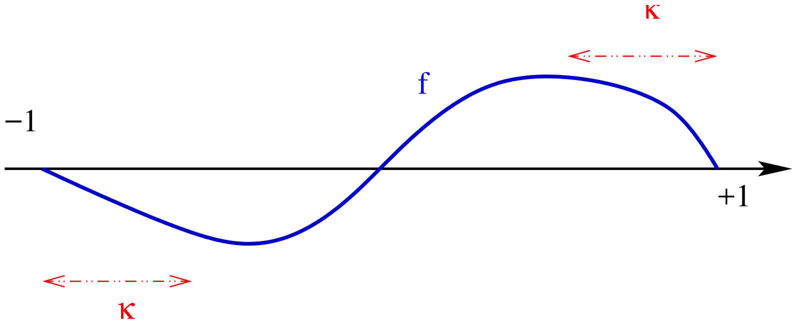
- fractional Laplacian $\mathcal{L} = (-\Delta)^s$. $\mu(\sigma) = 1$ for any $\sigma \in S^{n-1}$.
- anisotropic operators induced by a convex body K.

$$\|\sigma\|_K := \inf\{t > 0 \text{ s.t. } \sigma/t \notin K\},$$

$$\mu(\sigma) := \frac{|\sigma|^{n+2s}}{\|\sigma\|_K^{n+2s}}.$$

Bistable nonlinearities

$f(-1) = f(1) = 0$, f decreasing in $(-1, -1+\kappa) \cup (1-\kappa, 1)$.



Example: $f(u) = u - u^3$ "Allen-Cahn type equation".

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$$\mathcal{L}u = f(u)$$

$$I(u) := \{|u| \le 1 - \kappa\}$$

will be called the interface of the solution u.

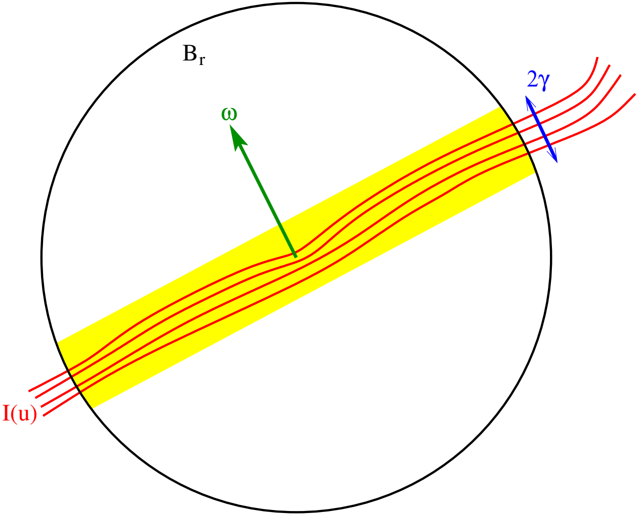
Trapping the interface

Given $r > 0$, $\gamma > 0$ and $\omega \in S^{n-1}$, we say that $I(u)$ is γ-trapped in B_r in direction ω if

$$\{\omega \cdot x \leq -\gamma\} \cap B_r \subseteq \{u \leq -1 + \kappa\} \cap B_r$$

and

$$\{u \leq 1 - \kappa\} \cap B_r \subseteq \{\omega \cdot x \leq \gamma\} \cap B_r .$$


B_r
ω
2γ
I(u)

1d symmetry

We say that $u : \mathbb{R}^n \to \mathbb{R}$ is 1d if there exist $u_o : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^{n-1}$ such that $u(x) = u_o(\omega \cdot x)$ for any $x \in \mathbb{R}^n$.

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Conjecture [De Giorgi 1978] If u (bounded and smooth) solves

$$-\Delta u = u - u^3$$

in $\mathbb{R}^n$, with

$$\frac{\partial u}{\partial x_n} > 0,$$

then u is 1d, at least if $n \leq 8$.

De Giorgi Conjecture

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- $4 \leq n \leq 8$ under the additional assumption

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- $n \geq 9$:

  M. del Pino, M. Kowalczyk, J. Wei, Ann. Math. (2011).

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  X. Cabré, J. Solà-Morales, Comm. Pure Appl. Math. (2005).

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- $n = 2,\ s \in (0, 1)$:

  Y. Sire, E. Valdinoci, J. Funct. Anal. (2009).

  X. Cabré, Y. Sire, Trans. Amer. Math. Soc. (2015).

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- $n = 3$, $s \in \left(0, \frac{1}{2}\right)$:
  S. Dipierro, A. Farina, E. Valdinoci, Calc. Var. Partial Differential Equations (2018).

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- $n = 3$, $s \in \left(0, \frac{1}{2}\right)$:

  S. Dipierro, A. Farina, E. Valdinoci, Calc. Var. Partial Differential Equations (2018).
- $n = 4$, $s = \frac{1}{2}$:

  A. Figalli, J. Serra, Invent. Math. 219 (2020).

Fractional De Giorgi Conjecture + limits at infinity

- Any $n \in \mathbb{N}$, $s \in (0,1)$, assuming that

$$\lim_{x_n \to \pm\infty} u(x_1, \dots, x_n) = \pm 1 \text{ uniformly in } (x_1, \dots, x_{n-1}):$$

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- $n \le 8$, $s \in \left(\frac{1}{2} - \varepsilon_0, \frac{1}{2}\right)$ and

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1d symmetry from flatness at infinity

Theorem (1)

S. Dipierro, J. Serra, E. Valdinoci

Let $\mathcal{L}u = f(u)$ *in* $\mathbb{R}^n$*. Assume that there exists* $a : (1, +\infty) \to (0, 1]$ *such that*

$$\lim_{r \to +\infty} a(r) = 0$$

and $I(u)$ *is* $a(r)\, r$*-trapped in* B_r *in some direction* ω_r*.*

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Then u *is* $1d$*.*

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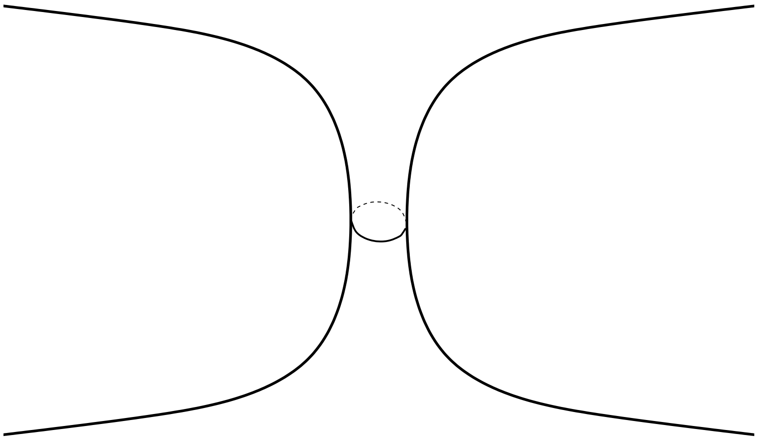
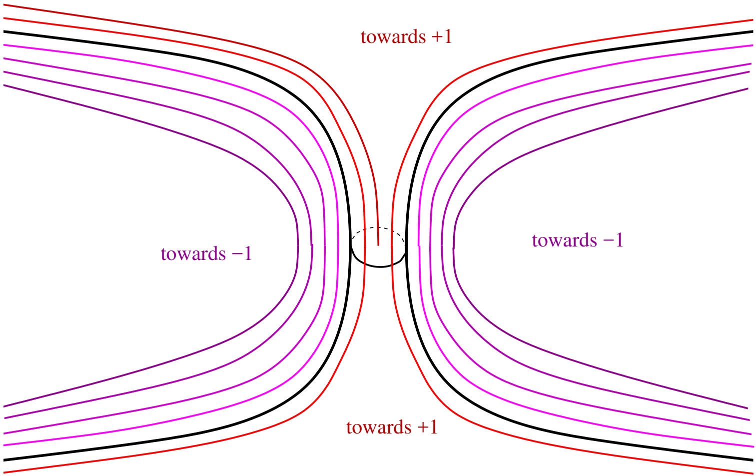
"Transition from a catenoid".

M. del Pino, M. Kowalczyk, J. Wei, J. Differential Geom. (2013).

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towards +1

Advantadges and disadvantadges of nonlocal structures.

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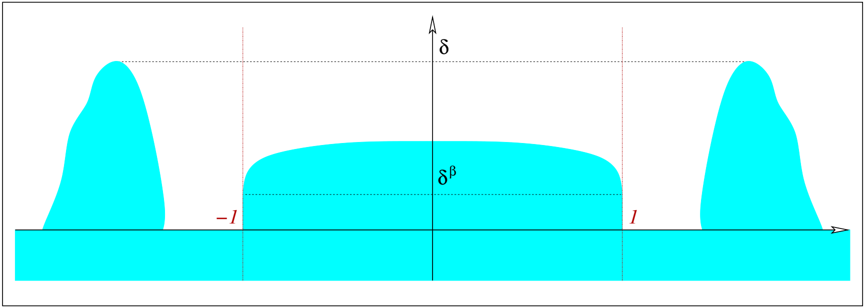
Nonlocal minimal surfaces are “rather different” from the classical ones and exhibit some “exotic behavior”.

Stickiness phenomenon (S. Dipierro, O. Savin, E. Valdinoci):



Genuinely nonlocal fractional versions of De Giorgi Conjecture

Theorem (2)

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Let $(-\Delta)^s u = u - u^3$ *in* $\mathbb{R}^n$, *with*

$$\frac{\partial u}{\partial x_n} > 0$$

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$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Assume that $n \leq 8$ *and* s *is sufficiently close to* $1/2$ *(i.e.* $s \in (s_n,\, 1/2)$ *for a suitable* $s_n \in [0,\, 1/2)$*).*

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A general trapping result

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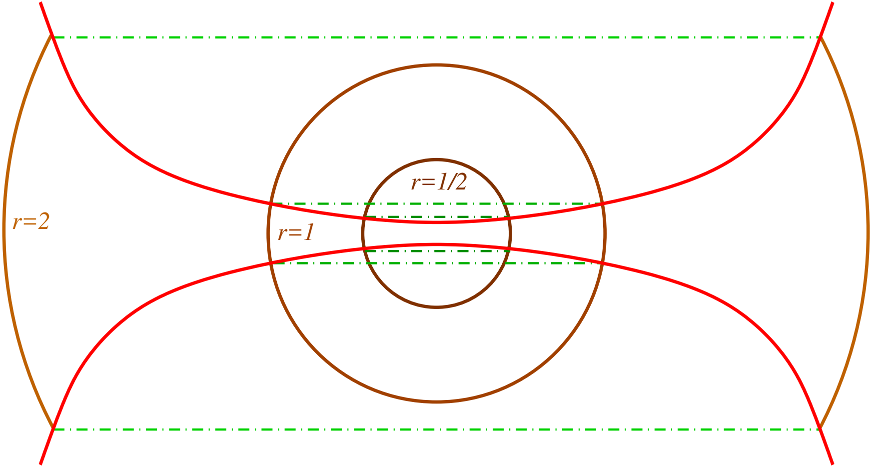
Then Trap(j) holds for $j = -1$.


r=1/2
r=2
r=1

Theorem (0) (General trapping)

$\Downarrow$

Theorem (1) (Flatness from infinity)

$\Downarrow$

Theorems (2) and (3) (De Giorgi type)

Proof of Theorem (3)

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$$E_v(x, z) := P * v(x, z) = \int_{\mathbb{R}^n} P(y, z)\, v(x - y)\, dy.$$

Proof of Theorem (3)

Step 2: Renormalized energy methods.

$$\mathcal{F}_R(v) := \iint_{\mathbb{R}^{2n}\setminus(B_R^c)^2} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{B_R} F(v(x))\,dx$$

$$\mathcal{E}_R(V) := \int_{\mathcal{B}_R^+} z^{1-2s}\,|\nabla V(x,z)|^2\,dx\,dz + \int_{B_R} F(V(x,0))\,dx.$$

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$$\inf_{\Phi\in C_0^\infty(\mathbb{R}^{n+1})} \mathcal{E}_R(E_v+\Phi) - \mathcal{E}_R(E_v) = \inf_{\varphi\in C_0^\infty(\mathbb{R}^n)} \mathcal{F}_R(v+\varphi) - \mathcal{F}_R(v).$$

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Step 2: Renormalized energy methods.

$$\mathcal{F}_R(v) := \iint_{\mathbb{R}^{2n}\setminus(B_R^c)^2} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{B_R} F(v(x))\,dx$$

$$\mathcal{E}_R(V) := \int_{\mathcal{B}_R^+} z^{1-2s}\,|\nabla V(x,z)|^2\,dx\,dz + \int_{B_R} F(V(x,0))\,dx.$$

$$\inf_{\Phi\in C_0^\infty(\mathbb{R}^{n+1})} \mathcal{E}_R(E_v+\Phi) - \mathcal{E}_R(E_v) = \inf_{\varphi\in C_0^\infty(\mathbb{R}^n)} \mathcal{F}_R(v+\varphi) - \mathcal{F}_R(v).$$

E_v is an extended local minimizer if and only if v is a local minimizer.

Proof of Theorem (3)

Step 3: Classification of the profiles at infinity ($n = 3$).

$$\underline{u}(x_1, x_2) := \lim_{x_3 \to -\infty} u(x_1, x_2, x_3),$$
$$\overline{u}(x_1, x_2) := \lim_{x_3 \to +\infty} u(x_1, x_2, x_3).$$

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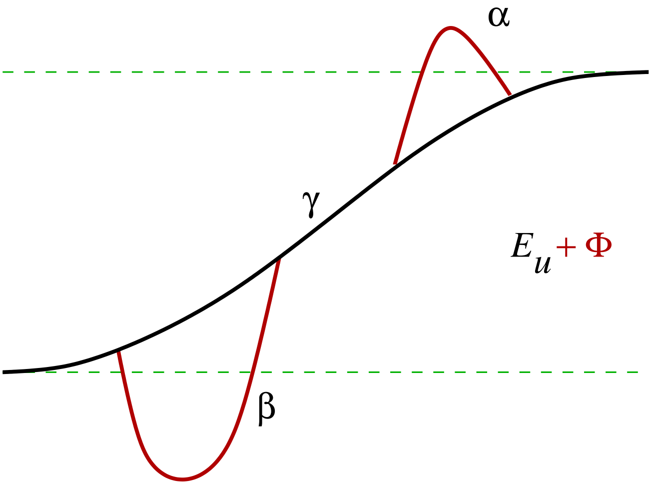
They are stable solutions in $\mathbb{R}^2$ and thus 1d
(X. Cabré and Y. Sire. Trans. Amer. Math. Soc. 2015, Y. Sire and E. Valdinoci, J. Funct. Anal., 2009).

Proof of Theorem (3)

Step 4: Inheriting minimality from that at infinity.
If $\underline{u}$ and $\overline{u}$ are local minimizers (in $\mathbb{R}^{n-1}$), then E_u is an extended local minimizer (in $\mathbb{R}^{n+1}_+$) and u is a local minimizer (in $\mathbb{R}^n$).

D. Jerison, R. Monneau, Ann. Mat. Pura Appl., 2004.

Proof of Theorem (3)



Proof of Theorem (3)

$$\mathcal{E}_{\{E_u+\Phi>E_{\overline{u}}\}}(\alpha) \geq \mathcal{E}_{\{E_u+\Phi>E_{\overline{u}}\}}(E_{\overline{u}}), \qquad \text{by the minimality of } \overline{u},$$
$$\mathcal{E}_{\{E_u+\Phi<E_{\underline{u}}\}}(\beta) \geq \mathcal{E}_{\{E_u+\Phi<E_{\underline{u}}\}}(E_{\underline{u}}), \qquad \text{by the minimality of } \underline{u},$$
$$\mathcal{E}_{\{E_{\underline{u}}<E_u+\Phi<E_{\overline{u}}\}}(\gamma) \geq \mathcal{E}_{\{E_{\underline{u}}<E_u+\Phi<E_{\overline{u}}\}}(E_u), \qquad \text{by sliding method.}$$

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$$\begin{aligned}
& \mathcal{E}(E_u+\Phi) \\
= \; & \mathcal{E}_{\{E_u+\Phi>E_{\overline{u}}\}}(\alpha) + \mathcal{E}_{\{E_{\underline{u}}<E_u+\Phi<E_{\overline{u}}\}}(\gamma) + \mathcal{E}_{\{E_u+\Phi<E_{\underline{u}}\}}(\beta) \\
\geq \; & \mathcal{E}_{\{E_u+\Phi>E_{\overline{u}}\}}(E_{\overline{u}}) + \mathcal{E}_{\{E_{\underline{u}}<E_u+\Phi<E_{\overline{u}}\}}(E_u) + \mathcal{E}_{\{E_u+\Phi<E_{\underline{u}}\}}(E_{\underline{u}}) \\
= \; & \mathcal{E}_{\{E_u+\Phi>E_{\overline{u}}\}}(\gamma) + \mathcal{E}_{\{E_{\underline{u}}<E_u+\Phi<E_{\overline{u}}\}}(\gamma) + \mathcal{E}_{\{E_u+\Phi<E_{\underline{u}}\}}(\gamma) \\
= \; & \mathcal{E}(\gamma) \\
\geq \; & \mathcal{E}(E_u).
\end{aligned}$$

Proof of Theorem (3)

Step 5: Conclusions.

$$u_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right).$$

Since u is minimal, we have that $u_\varepsilon \to \chi_E - \chi_{E^c}$, and the level sets of u_ε approach ∂E with E fractional minimal surface in $\mathbb{R}^3$ (O. Savin, E. Valdinoci, J. Math. Pures Appl., 2014).

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A "Bernstein type result" gives that E is a halfplane.
Then, Theorem (1) gives that u is 1d. $\square$

Proof of Theorem (2)

u is a minimal solution, with limit interface a graph
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So we go back to the asymptotical flatness setting of Theorem (1) and we obtain that u is 1d. □

Proof of Theorem (1)

Iteration of Theorem (0):

- linear iteration: trapping in smaller balls,
- nonlinear iteration: trapping with α_0-decay.

This gives a control of the interface all the way to infinity: for any $j \in \mathbb{N}$ and $z \in I(u)$,

$$\{x_n \leq z_n - C2^{j(1-\delta)}\} \cap B_{2^j}(z) \subseteq \{u \leq -1 + \kappa\} \cap B_{2^j}(z)$$

and

$$\{u \leq 1 - \kappa\} \cap B_{2^j}(z) \subseteq \{x_n \leq z_n + C2^{j(1-\delta)}\} \cap B_{2^j}(z),$$

with $C > 0$ and $\delta \in (0, 1)$.

Proof of Theorem (1)

And thus a global Lipschitz control of the interface

$$\{x_n \le G(x') - C\} \subseteq \{u \le -1 + \kappa\}$$

and $$\{u \le 1 - \kappa\} \subseteq \{x_n \le G(x') + C\},$$

with G Lipschitz and sublinear at infinity.

Proof of Theorem (1)

This enables us to start the sliding method in a tilted direction:
Fix $e' \in \mathbb{R}^{n-1}$ with $|e'| = 1$, and $\varepsilon > 0$. Look at

$$e_\varepsilon := \frac{(e', \varepsilon)}{\sqrt{1+\varepsilon^2}} \in S^{n-1}.$$

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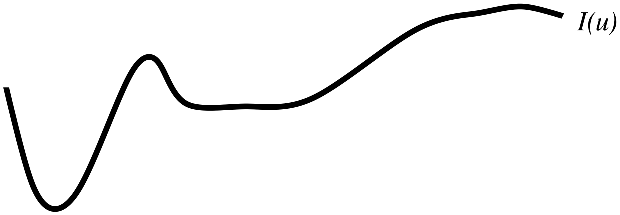
$$u^t(x) := u(x - e_\varepsilon\, t).$$

Proof of Theorem (1)



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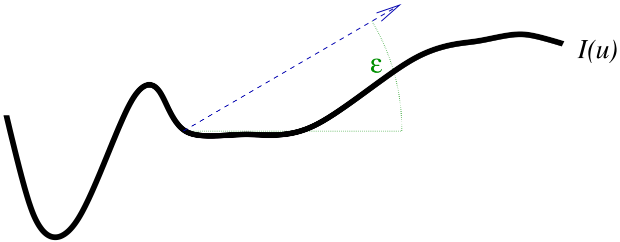

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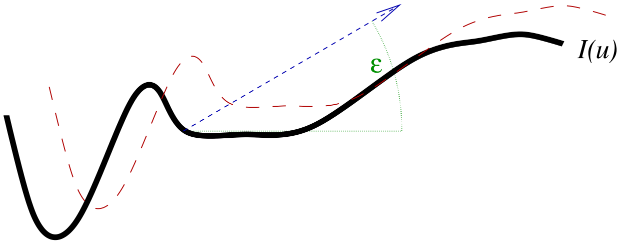

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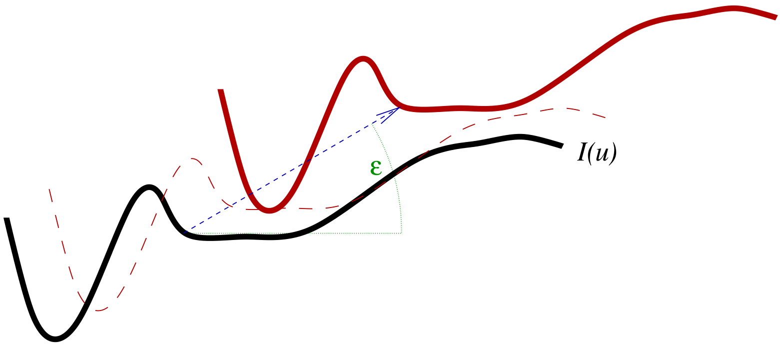

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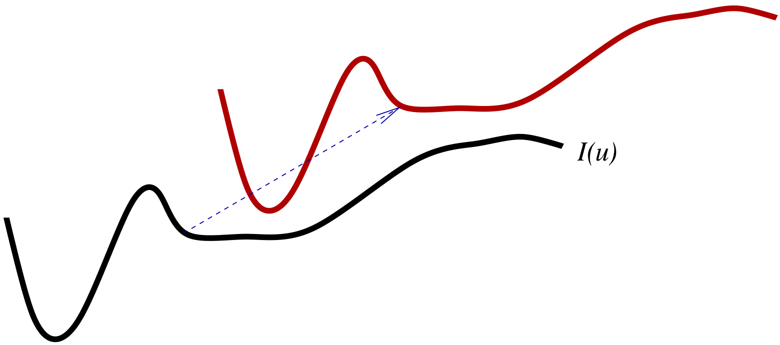

Proof of Theorem (1)

For large $t \geq T_\varepsilon$, we have that

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For $t \geq T_\varepsilon$, we have that if $x \in \{u^t \geq u\}$
then $u(x) \in (-1,-1+\kappa] \cup [1-\kappa,1)$
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That is, $\{u^t \geq u\}$ lies in the monotonicity regime for the nonlinearity and comparison principle can be applied.

Proof of Theorem (1)

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Now take $\varepsilon \searrow 0$:

$$u(x' - e' t, x_n) \leq u(x). \qquad \square$$

Proof of Theorem (0)

Modeling solutions via the distance function.

- ϕ 1d solution.
- $\xi(x') := (1 + |x'|^2)^{\frac{1+\alpha}{2}} - 1$ superlinear barrier for the level set.
- For small $b > 0$, one takes d_b as the distance from $\{x_n \geq b\xi(x')\}$.
- Modeled 1d solution: $\phi_b(x) := \phi(d_b(x))$.

Proof of Theorem (0)

Layer cake representation of the operator, i.e., if $v(0) = w(0)$,

$$w(y) - v(y) = \int_{\mathbb{R}} \chi_{[v(y),\, w(y)]}(\theta) - \chi_{[w(y),\, v(y)]}(\theta)\, d\theta,$$

and so

$$\begin{aligned} & \mathcal{L}v(0) - \mathcal{L}w(0) \\ = & \iint_{\mathbb{R}\times\mathbb{R}^n} \Big(\chi_{[v(y),\, w(y)]}(\theta) - \chi_{[w(y),\, v(y)]}(\theta)\Big) \frac{\mu(y/|y|)}{|y|^{n+2s}}\, d\theta\, dy. \end{aligned}$$

We find that

$$\mathcal{L}\phi_b = f(\phi_b) + \text{“small errors”}.$$

Proof of Theorem (0)

For instance, if the level sets $\{v(y) \leq \theta \leq w(y)\}$ are a small quadratic deviation from a hyperplane, of the form $\{0 \leq y_n \leq b|y'|\}$, the local contribution has the form

$$\int_{|y'|\leq 1} \int_{0\leq y_n \leq b|y'|^2} \frac{dy}{|y|^{n+2s}} \sim b \int_{|y'|\leq 1} \frac{dy}{|y|^{n+2s-2}}$$

$$\sim b \int_0^1 \frac{r^{n-2}\,dr}{r^{n+2s-2}} \sim b.$$

Proof of Theorem (0)

General improvement of oscillation Lemma

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$$k_a := \left\lfloor \frac{\log_2(1/a)}{\alpha} \right\rfloor - m_0.$$

Suppose that

$$\int_{B_2} u \geq 0$$

$$\text{and} \qquad \{u \leq 1-\kappa\} \subseteq \{x_n \leq a2^{j(1+\alpha)}\}$$

in $\{|x'| \leq 2^j\} \times (-2^{k_a}, 2^{k_a})$ for all $j \in \{0, \ldots, k_a\}$.

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$$\{u \leq 1-\kappa\} \subseteq \{x_n \leq a(1-\eta_0)\}$$

in $\{|x'| \leq 1/2\} \times (-2^{k_a}, 2^{k_a})$, for some $\eta_0 \in (0,1)$.

Proof of Theorem (0)

The proof aims at a comparison between u and $\phi^b\left(\frac{x-te_n}{\varepsilon}\right)$ (sliding from $t=+\infty$ to some $t_0 \le a(1-2\eta_0)$). The goal is to prove

$$u(x) \ge \phi^b\left(\frac{x-t_0 e_n}{\varepsilon}\right).$$

To check this, first one checks that for t large the two functions are ordered, then takes the smallest possible t and proves by contradiction that such t lies below $a(1-2\eta_0)$.

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To check this, first one checks that for t large the two functions are ordered, then takes the smallest possible t and proves by contradiction that such t lies below $a(1-2\eta_0)$.
Then, we obtain that

$$u(x) \le 1-\kappa \Rightarrow \phi^b\left(\frac{x-t_0 e_n}{\varepsilon}\right) \le 1-\kappa$$
$$\Rightarrow x_n - t_0 \le O(\varepsilon) \Rightarrow x_n \le t_0 + O(\varepsilon) \le a(1-\eta_0).$$

Proof of Theorem (0)

With the oscillation result in hand, one performs vertical rescalings in the last coordinate:

$$\{x_n \le a g_a(x')\} \subseteq \{u \le -1+\kappa\}$$
$$\text{and} \qquad \{u \le 1-\kappa\} \subseteq \{x_n \le a g^a(x')\},$$

with g_a and g^a approaching some g, with $g(0)=0$, which is s-harmonic in $\mathbb{R}^{n-1}$.

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with g_a and g^a approaching some g, with $g(0)=0$, which is s-harmonic in $\mathbb{R}^{n-1}$.

Hence g must be linear. So ag_a and ag^a must be locally comparable to $ag + O(a^{1+\sigma}) = a\omega' \cdot x' + O(a^{1+\sigma})$. Up to a rotation, $\omega' = 0$, so the interface is locally trapped in $|x_n| \le O(a^{1+\sigma})$. $\square$

Thank you very much!