Null-controllability of the heat equation on unbounded domains

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based on joint works with I. Veselić, A. Seelmann

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Statement of the problem

Let \( \Omega \subset \mathbb{R}^d \) be an open domain, \( \omega \subset \Omega \) be a measurable subset, \( \Delta \) be the Laplacian on \( \Omega \) with boundary conditions if \( \partial \Omega \neq \emptyset \), \( T > 0 \), \( u_0 \in L^2(\Omega) \), \( v \in L^2((0, T), L^2(\Omega)) \).

We consider the controlled heat equation:

\[
\frac{\partial}{\partial t} u(t) - \Delta u(t) = \chi_\omega v(t) \quad 0 < t \leq T
\]

\( u(0) = u_0 \)

where \( \chi_\omega \) is the characteristic function of \( \omega \).

Question 1

Null-controllability: Given \( u_0 \in L^2(\Omega) \) there exists a function \( v \in L^2((0, T), L^2(\Omega)) \) such that the (mild) solution of system (1) is zero at final time \( T \), i.e. \( u(T) \equiv 0 \)?

Question 2

Control cost's bound: If the answer to 1) is positive, can we found a “good” bound for \( C_T := \sup_{\|u_0\|_{L^2(\Omega)} = 1} \inf \{\|v\|_{L^2((0, T), L^2(\Omega))} : v \text{ makes the sol. of (1) zero at time } T \} \)?
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Our scope

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The general scope is to answer the initial question for unbounded domains in $\mathbb{R}^d$ obtaining an answer in dependence of geometric properties of the subset $\omega$. We will treat the case of $\Omega$ being $\mathbb{R}^d$, the infinite strip $(0, L)^{d-1} \times \mathbb{R}$, and unbounded domains with reflection symmetry.
Sufficient and necessary conditions for null-controllability

Theorem (J.-M. Coron '07, Tucsnack-Weiss '09): The controlled heat equation is null-controllable in time $T > 0$ if and only if the corresponding observability inequality is satisfied, i.e. $\exists C_{\text{obs}} > 0: \|e^{T \Delta} f\|_{L^2(\Omega)} \leq C_{\text{obs}} \int_0^T \|e^{t \Delta} f\|_{L^2(\omega)} \, dt \forall f \in L^2(\Omega)$.

Moreover, $C_T \leq \sqrt{C_{\text{obs}}}$.

Proposition (Beauchard, Pravda-Starov '17): Let $\Omega \subset \mathbb{R}^d$ and $\omega \subset \Omega$ as before, let $(\pi_E)_{E \in \mathbb{N}}$ be the family of spectral projections of $-\Delta$ up to value $E$. Assume $\exists c > 0: \|\pi_E f\|_{L^2(\Omega)} \leq e^{c \sqrt{E}} \|\pi_E f\|_{L^2(\omega)} \forall f \in L^2(\Omega) \forall E \in \mathbb{N}$ then $\exists C(c) > 1: \|e^{T \Delta} f\|_{L^2(\Omega)} \leq C e^{C/c} \int_0^T \|e^{t \Delta} f\|_{L^2(\omega)} \, dt \forall T > 0 \forall f \in L^2(\Omega)$. 

M. Egidi
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Let \( \cdot \mid \cdot \) denote the Lebesgue measure on \( \mathbb{R}^d \). Let \( S \subset \mathbb{R}^d \) be measurable. We say that \( S \) is thick if there exist \( \gamma \in (0, 1] \) and \( a = (a_1, \ldots, a_d) \in (0, \infty)^d \) such that
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|S \cap (x + [0, a_1] \times \cdots \times [0, a_d])| \geq \gamma d \prod_{j=1}^d a_j \quad \forall x \in \mathbb{R}^d.
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(2) We also refer to \( S \) as \( (\gamma, a) \)-thick. Example: a periodic arrangement of balls is a thick set.
Let $|\cdot|$ denote the Lebesgue measure on $\mathbb{R}^d$. Let $S \subset \mathbb{R}^d$ be measurable. We say that $S$ is thick if there exist $\gamma \in (0, 1]$ and $a = (a_1, \ldots, a_d) \in (0, \infty)^d$ such that $|S \cap (x + [0, a_1] \times \ldots \times [0, a_d])| \geq \gamma d \prod_{j=1}^d a_j \quad \forall x \in \mathbb{R}^d$.

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Thick sets

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Previously known: Let $\omega \subset \mathbb{R}^d$ be an open subset.

(Miller '04) A necessary condition for null-controllability is $\sup_{x \in \mathbb{R}^d}(x, \omega) < \infty$.

(Miller '05) A sufficient condition for null controllability is that $\omega$ satisfies $\exists r, \delta > 0 \text{ s.t. } \forall y \in \mathbb{R}^n \exists x \in \omega: B(x, r) \subset \omega,$ and $\|y - x\| < \delta$.

Theorem 1 (E.-Veselic '18) Let $\Omega = \mathbb{R}^d$ and $T > 0$. The following statements are equivalent:

(i) The set $\omega$ is thick;

(ii) The controlled heat equation is null-controllable in any time $T > 0$.

In particular, if $\omega$ is $(\gamma, a)$-thick, the control cost satisfies $C_T \leq C_1 / 2 \exp(C_1/2T)$, $C_1 = (K_d \gamma K(d + \|a\|_1))$, where $K > 0$ is a universal constant.
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In particular, if $\omega$ is $(\gamma, a)$-thick, the control cost satisfies

$$C_T \leq C_1^{1/2} \exp \left( \frac{C_1}{2T} \right), \quad C_1 = \left( \frac{K^d}{\gamma} \right)^{K(d + \|a\|_1)},$$

where $K > 0$ is a universal constant.
Proof of "thickness ⇒ null-controllability"

Recall: we need to prove \( \| \pi_E f \|_{L^2(\mathbb{R}^d)} \leq e^{c\sqrt{E}} \| \pi_E f \|_{L^2(\omega)} \forall E \in \mathbb{N}, \forall f \in L^2(\mathbb{R}^d). \)
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Recall: we need to prove $\|\pi_E f\|_{L^2(\mathbb{R}^d)} \leq e^{c\sqrt{E}} \|\pi_E f\|_{L^2(\omega)}$ $\forall E \in \mathbb{N}$, $\forall f \in L^2(\mathbb{R}^d)$. This follows from

**Proposition 1 (Kovrijkine ’01)**

Let $g \in L^2(\mathbb{R}^d)$ such that $\text{supp} \hat{g} \subset J$, for $J$ a parallelepiped in $\mathbb{R}^d$ with sides parallel to coordinate axes and of length $b_1, \ldots, b_d$. Set $b = (b_1, \ldots, b_d)$. Let $\omega$ be a $(\gamma, a)$-thick set, then

$$\|g\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{K^d}{\gamma}\right)^{K(a \cdot b + d)} \|g\|_{L^2(\omega)}$$

where $K > 0$ is a universal constant where $a \cdot b = \sum_{j=1}^d a_j b_j$. 
Proof of "thickness $\Rightarrow$ null-controllability"

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Indeed: for any $E \in \mathbb{N}$, $\pi_E : L^2(\mathbb{R}^d) \to \{ f \in L^2(\mathbb{R}^d) : \text{supp} \, \hat{f} \subset B(0, \sqrt{E})\}$. 
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Recall: we need to prove \( \| \pi_E f \|_{L^2(\mathbb{R}^d)} \leq e^{c\sqrt{E}} \| \pi_E f \|_{L^2(\omega)} \) \( \forall E \in \mathbb{N}, \forall f \in L^2(\mathbb{R}^d) \).

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\| g \|_{L^2(\mathbb{R}^d)} \leq \left( \frac{K^d}{\gamma} \right)^{K(a \cdot b + d)} \| g \|_{L^2(\omega)}
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Hence for any \( f \in L^2(\mathbb{R}^d), \text{supp} \pi_E f \subset [-\sqrt{E}, \sqrt{E}]^d \) and Prop 1 applies to \( g = \pi_E f \).
Proof of "null-controllability $\Rightarrow$ thickness"

Proof by contradiction: we assume that $\omega$ is not thick and we construct a sequence $(g_k)_{k \in \mathbb{N}}$ of functions in $L^2(\mathbb{R}^d)$ for which the observability inequality

$$\|e^{T \Delta} g_k\|_{L^2(\mathbb{R}^d)} \leq C_{\text{obs}} \int_0^T \|e^{t \Delta} g_k\|_{L^2(\omega)}^2 dt$$

does not hold.

Since $\omega$ is not thick, for all $k \in \mathbb{N}$ there exists a point $\xi_k \in \mathbb{R}^d$ such that

$$|\omega \cap (\xi_k + [0, 2k])| < \frac{1}{k}.$$ Then there exist points $x_k \in \mathbb{R}^d$ such that

$$|\omega \cap B(x_k, k)| < \frac{1}{k}.$$

Set $g_k(x) := \exp\left(-\frac{\|x-x_k\|^2}{2}\right)$.

Show that

$$\|e^{T \Delta} g_k\|_{L^2(\mathbb{R}^d)} \geq C > 0$$

for all $k$, but

$$\int_0^T \|e^{t \Delta} g_k\|_{L^2(\omega)}^2 dt \to k \to \infty 0.$$
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- Set $g_k(x) := \exp(-\frac{\|x-x_k\|^2}{2})$. 

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- Since $\omega$ is not thick, for all $k \in \mathbb{N}$ there exists a point $\xi_k \in \mathbb{R}^d$ such that $|\omega \cap (\xi_k + [0, 2k]^d)| < \frac{1}{k}$. Then there exist points $x_k \in \mathbb{R}^d$ such that $|\omega \cap B(x_k, k)| < \frac{1}{k}$.
- Set $g_k(x) := \exp(-\frac{|x-x_k|^2}{2})$.
- Show that $\| e^{t\Delta} g_k \|_{L^2(\mathbb{R}^d)} \geq C > 0$ for all $k$, but $\int_0^T \| e^{t\Delta} g_k \|_{L^2(\omega)}^2 \, dt \to_{k \to \infty} 0$. 
The case of an infinite strip

Let \( L > 0 \), \( d \geq 2 \), \( \Omega = (0, L)^d - 1 \times \mathbb{R}^d \) and impose Dirichlet or Neumann boundary conditions on \( \partial \Omega \). The following statements are equivalent:

(i) The measurable set \( \omega \subset \Omega \) satisfies the condition: there exist \( \gamma \in (0, 1] \) and \( a \in \mathbb{R}^d \) such that \( |\omega \cap P| \geq \gamma |P| \) for all \( P \subset \Omega \) parallelepipeds with sides of length \( a_1, \ldots, a_d \) and parallel to coordinate axes.

(ii) The controlled heat equation on \( \Omega \) is null-controllable in any time \( T > 0 \).

If \( \omega \) satisfies (i) with parameters \( \gamma \) and \( a \), then the control cost satisfies the bound

\[
C_T \leq C_1 / 2^{2} \exp (C_2^2 T),
\]

where \( C_2 = (2 K^d \gamma) \frac{1}{2} \sqrt{2 K (d + \|a\|_1)} \), for \( K > 0 \) a universal constant.

The control cost is independent of the width of the strip.
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Theorem 2 (E. ’18)

Let $L > 0$, $d \geq 2$, $\Omega = (0, L)^{d-1} \times \mathbb{R}$ and impose Dirichlet or Neumann boundary conditions on $\partial \Omega$. The following statements are equivalent:

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If $\omega$ satisfies (i) with parameters $\gamma$ and $a$, then the control cost satisfies the bound $C_T \leq C_1/2^2 \exp(C_2^2T)$, $C_2 = (2K^d)\gamma^{1/2} \sqrt{2K(d + \|a\|_1)}$, for $K > 0$ a universal constant.

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If $\omega$ satisfies (i) with parameters $\gamma$ and $a$, then the control cost satisfies the bound

$$C_T \leq C_2^{1/2} \exp \left( \frac{C_2}{2T} \right), \quad C_2 = \left( \frac{(2K)^d}{\gamma} \right)^{12\sqrt{2}K(d+\|a\|_1)},$$

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Sketch of proof of Theorem 2

(i) $\Rightarrow$ (ii): Using the cartesian structure of $\Omega$ we develop a Kovrijkine-type inequality for functions $f \in L^2(\Omega)$ such that $f(x_1, \cdot)$ has Fourier coefficients contained in a compact set and $f(\cdot, x_2)$ has compactly supported Fourier transform. From here we derive the desired spectral inequality using the tensor structure of the Laplacian on $\Omega$.

(ii) $\Rightarrow$ (i) follows by contradiction with the same strategy as before. This time, the negation of the condition on $\omega$ gives points $x_k \in \Omega$ such that $|\omega \cap Q_k| < L^{d-1}/k$, for $Q_k$ parallelepipeds centred at $x_k$ with sides of length $(L, \ldots, L, k)$. Then we choose $g_k(t) = e^{t \Delta K_\Omega(1, \cdot, x_k)}$, where $K_\Omega$ denotes the heat kernel on $\Omega$, and show that the $L^2(\Omega)$-norm of $g_k(T)$ remains bounded away from zero, while the $L^2((0, T), L^2(\omega))$-norm of $g_k$ tends to zero as $k$ grows.
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The case of (unbounded) domains with a reflection symmetry

Let $\Omega \subset (0, \infty) \times \mathbb{R}^{d-1}$ for which there exists an open set $\tilde{\Omega} \subset \mathbb{R}^d$ such that $\Gamma := \tilde{\Omega} \cap \{0\} \times \mathbb{R}^{d-1} \neq \emptyset$, $\Omega = \tilde{\Omega} \cap ((0, \infty) \times \mathbb{R}^{d-1})$, $\tilde{\Omega}$ is symmetric with respect to the reflection $M$ with respect to the first coordinate. Then $\tilde{\Omega} = \Omega \cup \tilde{M}(\Omega) \cup \Gamma$.

Theorem 3 (E.-Seelmann '19)
If the controlled heat equation on $\tilde{\Omega}$ is null-controllable in time $T > 0$ from the set $\tilde{\omega}$ with control cost bounded by $C > 0$, then also the controlled heat equation on $\Omega$ is null-controllable in time $T > 0$ from $\omega$, and its control cost does not exceed $C$.

$\Omega$ and $\tilde{\Omega}$ do not need to be unbounded. It is possible to consider domains symmetric with respect to any hyperplane. The theorem is also valid for divergence-type operators plus potentials, as long as the matrix coefficients and the potential are compatible with the symmetry.
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- The theorem is also valid for divergence-type operators plus potentials, as long as the matrix coefficients and the potential are compatible with the symmetry.
Let $f \in L^2(\Omega)$ and set $X_f = f \oplus (-f)$ for Dirichlet boundary conditions and $X_f = f \oplus f$ for Neumann boundary conditions.

To prove the theorem (in this simple case) it is enough to notice that $f \in L^2(\Omega)$, then $X_f \in L^2(\tilde{\Omega}) = L^2(\Omega) \oplus L^2(M(\Omega))$, $f \in D(-\Delta_\Omega)$ then $X_f \in D(-\Delta_{\tilde{\Omega}})$, $X(-\Delta_{\Omega} f) = -\Delta_{\tilde{\Omega}} (X_f)$ and $e^{t\Delta_{\tilde{\Omega}}} X_f = X e^{t\Delta_{\Omega}} f$.

Therefore, since the observability estimate holds for the larger systems, we have

$$\|e^{t\Delta_{\Omega}} f\|_{L^2(\Omega)}^2 = \frac{1}{2} \|X e^{t\Delta_{\Omega}} f\|_{L^2(\tilde{\Omega})}^2 = \frac{1}{2} \|X e^{t\Delta_{\tilde{\Omega}}} f\|_{L^2(\tilde{\Omega})}^2 \leq \frac{1}{2} C_{\text{obs}} \int_0^T \|e^{t\Delta_{\tilde{\Omega}}} X_f\|_{L^2(\tilde{\Omega})}^2 dt = \frac{1}{2} C_{\text{obs}} \int_0^T \|X e^{t\Delta_{\Omega}} f\|_{L^2(\tilde{\Omega})}^2 dt,$$

which proves the claim.

Corollary 4 (E.-Seelmann '19)

Let $\Omega$ be the half-space, the positive orthant or a sector of angle $\pi/2^n$, $n \geq 2$, and let $S \subset \mathbb{R}^d$ be a $(\gamma, a)$-thick set. Set $\omega = S \cap \Omega$, then the controlled heat equation on $\Omega$ is null-controllable from $\omega$ in any time $T > 0$. 

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\[
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\[
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THANK YOU FOR YOUR ATTENTION!