Low dimensional field theories with domain walls and related categorical structures

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Outline

- 2d TFT with domain walls
- bicategories from 2d field theories with domain walls
- brief discussion of generalisation in two directions (both incomplete): 3d TFT and 2d CFT
What are domain walls?

Describe domain wall by
- matching conditions on fields and/or additional terms in action localised on domain wall
- operators between the spaces of quantum states
2d topological field theory with defects

A 2d TFT is a symmetric monoidal functor from two-dimensional oriented bordisms to \( \mathbb{C} \)-vector spaces.
A 2d TFT with defects is a sym. mon. functor from two-dimensional oriented bordisms to $\mathbb{C}$-vector spaces with embedded or. 1-dim submf.

\[ \text{lin. map } H_1 \otimes H_2 \to H_3 \]
**2d topological field theory with defects**

Suszek, IR ’08, Davydov, Kong, IR ’11

A **2d TFT with defects** is a sym. mon. functor from two-dimensional oriented bordisms to $\mathbb{C}$-vector spaces with embedded or. 1-dim submf. and labelled components

Label sets $D_2, D_1$

2-dim domains $a, b, c \in D_2$

1-dim domain walls $w, x, y, z \in D_1$

Lin. map $H_1 \otimes H_2 \to H_3$
Example: 2d lattice TFT with defects

Lattice construction of 2d TFT:
Fix a Frobenius algebra with trace pairing $A$
(ie. an algebra $A$ such that $(a,b) \mapsto \text{tr}_A(L_a L_b)$ is non-deg.)

Bachas, Petropoulos '92
Fukuma, Hosono, Kawai '92
Example: 2d lattice TFT with defects

Lattice construction of 2d TFT:
Fix a Frobenius algebra with trace pairing $A$
(ie. an algebra $A$ such that $(a,b) \mapsto \text{tr}_A(L_a L_b)$ is non-deg.)

- $T : A \otimes A \otimes A \to \mathbb{C}$
  (from mult. and pairing)
- $P : \mathbb{C} \to A \otimes A$
  (copairing)

lin. map $Z(A) \otimes Z(A) \to Z(A)$
Example: 2d lattice TFT with defects

Now with defects:
Fix Frobenius algebras with trace pairing $A_a, a \in D_2$
finite dim. $A_t(x)-A_s(x)$-bimodules $X_x, x \in D_1$ $(s,t : D_1 \to D_2)$

Davydov, Kong, IR ’11

lin. map $H_1 \otimes H_2 \to H_3$
Example: 2d lattice TFT with defects

Now with defects:
Fix Frobenius algebras with trace pairing $A_a, a \in D_2$
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Davydov, Kong, IR ’11

$tft$+
lin. map $H_1 \otimes H_2 \to H_3$
Classification?

Known:
\[(\text{sym. mon. fun. 2-bord} \to \text{vect}) \cong (\text{comm. Frob alg.})\]

Unknown:
\[(\text{sym mon fun 2-bord} + \text{def.} \to \text{vect}) \cong (?)\]
A 2d TFT with defects gives a bicategory

Starting data:
- bulk phases $D_2$, domain walls $D_1$, maps $s,t : D_1 \to D_2$
- $\text{tft} : \text{sym. mon. functor } (\text{bord. with def}) \to \text{Vect}$

Bicategory
- objects $: D_2$
- 1-morph : lists of composable elements of $D_1$
- 2-morph : state spaces

Lattice example
- algebras $A_a$
- bimodules $X_x$

\[
\text{Hom}_{A_l A_r} \left( X_1 \otimes_{A_{12}} X_2 \otimes_{A_{23}} \cdots, \quad Y_1 \otimes_{B_{12}} Y_2 \otimes_{B_{23}} \cdots \right)
\]
**Remark: also true for 2d QFT**

**sym. mon. fun. Q from bord. with metric and defects to top. vector spaces**

- **topological domain wall:**
  - $Q$ depends only on isotopy class of 1-dim submanifold

- **scale and translation invariant state:**

Get bicategory

**objects**: $D_2$ `world sheet phases’

**1-morph**: lists of composable top. dom. walls

**2-morph**: scale and translation invariant states

\[ Q(\text{<diagram>}) = Q(\text{<diagram>}) \]
More dimensions or more dynamics

2d TFT (+ def.)
   \[ \xrightarrow{\text{add dimensions}} \]
   'categorify' (non-unique)
   \[ \xrightarrow{\text{compactify}} \]
   (evaluate on \( \Sigma \times S^1 \))

3d TFT (+ def.)
   \[ \xrightarrow{\text{add dimensions}} \]
   \[ \xrightarrow{\text{‘categorify’}} \]
   \[ \xrightarrow{\text{restrict to scale + transl. inv. states}} \]
   \( \xrightarrow{\text{apply } \text{Hom}_{C \boxtimes \overline{C}}(1, -)} \)

2d CFT (+ def.)
Some defects in 3d TFT

Framework:
Reshetikhin-Turaev 3d TFT from a modular category $C$
symmetric monoidal functor
(3-dim extended bordisms) $\rightarrow$ vector spaces
Example of an extended cobordism

\[ M : \text{extended bordism from } E \text{ to } E' \]

\[ U, V, W, R : \text{objects of } C \]

\[ f : U^* \to W \otimes V^* \]
Examples of 2-dimensional defects in 3d TFT

Given:

- embedded surface $\Sigma$ in 3-mf
- Frobenius algebra with trace pairing $A$ in $\mathbb{C}$

surface in ambient 3-mf

Fuchs, Schweigert, IR ’01
Kapustin, Saulina ’10
Examples of 2-dimensional defects in 3d TFT

Given:

- embedded surface $\Sigma$ in 3-mf
- Frobenius algebra with trace pairing $A$ in $C$

I) Pick a triangulation $\Sigma$

surface in ambient 3-mf

Fuchs, Schweigert, IR ’01
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Examples of 2-dimensional defects in 3d TFT

Given:
- embedded surface $\Sigma$ in 3-mf
- Frobenius algebra with trace pairing $A$ in $C$

1) Pick a triangulation $\Sigma$
2) pass to dual

surface in ambient 3-mf

Fuchs, Schweigert, IR ’01
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Examples of 2-dimensional defects in 3d TFT

Given:
- embedded surface $\Sigma$ in 3-mf
- Frobenius algebra with trace pairing $A$ in $C$

1) Pick a triangulation $\Sigma$
2) pass to dual
3) label vertices by morph.
   - $T : A \otimes A \otimes A \to I$
   - $P : I \to A \otimes A$

Result indep. of choice in 1)
From 2d TFT to 3d TFT:

Frobenius algebra over $\mathbb{C}$ is replaced by modular category $\mathcal{C}$

- Examples of codim. 1 domain walls from Frobenius algebra with trace pairing in $\mathcal{C}$

Note:
In context of Levin-Wen model for modular category $\mathcal{C}$, obtain codim. 1 domain walls from $\mathcal{C}$-$\mathcal{C}$-bimodule categories.
From 2d TFT to 2d CFT: enriching

- State spaces of 2d CFT are $V \otimes V$-representations
  ($V$ is a vertex operator algebra, e.g. $V_{\text{Vir}}$)

- Assume rational: $\mathcal{C} = \text{Rep} V$, a modular category

- State spaces are objects in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$
...from 2d TFT to 2d CFT: enriching

$D_2$ bulk phases; $D_1$ domain walls with $s, t : D_1 \rightarrow D_2$

Bicategory from 2d CFT via top. def + inv. states
obj: $D_2$, 1-morph: $D_1$, 2-morph $\in \mathbf{Vect}_{\text{fin}}$

enrich
“add dynamics”
(non-unique)

apply $\text{Hom}_{\mathbf{C} \boxtimes \overline{\mathbf{C}}}(1, -)$
“restrict to vacuum states”

Enriched bicategory from 2d CFT via top. def + all states
obj, 1-morph as above, 2-morph $\in \mathbf{C} \boxtimes \overline{\mathbf{C}}$
... from 2d TFT to 2d CFT: enriching

eg lattice TFT:
• Frob. alg. with trace pairing in $\text{Vect}_{\text{fin}}$,
• bimodules in $\text{Vect}_{\text{fin}}$,
• bimodule maps (Hom space in $\text{Vect}_{\text{fin}}$)

for CFT one takes:
• Frob. alg. with trace pairing in $\mathcal{C}$
• bimodules in $\mathcal{C}$
• bimodule maps (Hom space in $\text{Vect}_{\text{fin}}$)

Next:
- $\text{AB}$-bimodules in $\mathcal{C}$ form a $\mathcal{C} \boxtimes \overline{\mathcal{C}}$-module category
- internal action Hom is an object in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. 
...from 2d TFT to 2d CFT: enriching

Two $\mathcal{C}$-actions on $\text{AB-Bimod}$

$X \mapsto X \otimes U$, $U \in \mathcal{C}$ and $B$-action via $c_{U,B}$ resp. $c_{B,U^{-1}}$

Get $\mathcal{C} \boxtimes \overline{\mathcal{C}}$-action on $\text{AB-Bimod}$:

$X \mapsto X \cdot R$, $R \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$
... from 2d TFT to 2d CFT: enriching

Action allows to define *internal action hom*

\[
\text{Hom}_{AB}(X,Y) \in C \otimes \overline{C}
\]

via

\[
\text{Hom}_{C \otimes \overline{C}}(R, \text{Hom}_{AB}(X,Y)) \cong \text{Hom}_{AB}(X \cdot R, Y)
\]

for all \(R \in C \otimes \overline{C}\).

Physically, \(\text{Hom}_{AB}(X,Y)\) is an (infinite dimensional) \(V \otimes V\)-representation giving the full space of fields which can sit at the junction of an \(X\)- and a \(Y\)-type defect.
Knowing all top. defects means knowing all bulk sectors

**Theorem:**
Let $C$ : modular category,
\[ A : \text{Frobenius algebra with trace pairing in } C. \]
Then $\mathcal{Z}(AA\text{-Bimod}) \cong \mathcal{Z}(C).$
\[ \cong C \boxtimes \overline{C} \]  
(as $C$ modular, cf. Müger ’01)

I.e. in rational CFT:
If we know the tensor category of topological defects transparent to $V \otimes V$, then we know the braided monoidal category $\text{Rep} V \otimes V$. 

Schauenburg ’01
(under much weaker conditions)
Summary

• lattice 2d TFT with domain walls from Frobenius algebras with trace pairing and bimodules

• each 2d QFT with topological domain walls defines a bicategory (over Vect)

• rational 2d CFT : enrich over $\mathbb{C} \boxtimes \overline{\mathbb{C}}$

• knowing all ‘endo’-defects of a rational CFT amounts to knowing all $V \otimes V$-representations