K-theory in solid state physics

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Plan for the lectures

- What is a topological insulator?
- What are the main experimental facts?
- What are the main theoretical elements?
- Almost everything in a one-dimensional toy model (SSH model)
- Toy models for higher dimension
- Algebraic formalism (crossed product C^*-algebras)
- Measurable quantities as topological invariants
- Bulk-edge correspondence
- Index theorems for invariants
- Implementation of symmetries (periodic table of topological ins.)

Math tools: K-theory, index theory and non-commutative geometry

1 Experimental facts

What is a topological insulator?

- d-dimensional disordered system of independent Fermions with a combination of basic symmetries

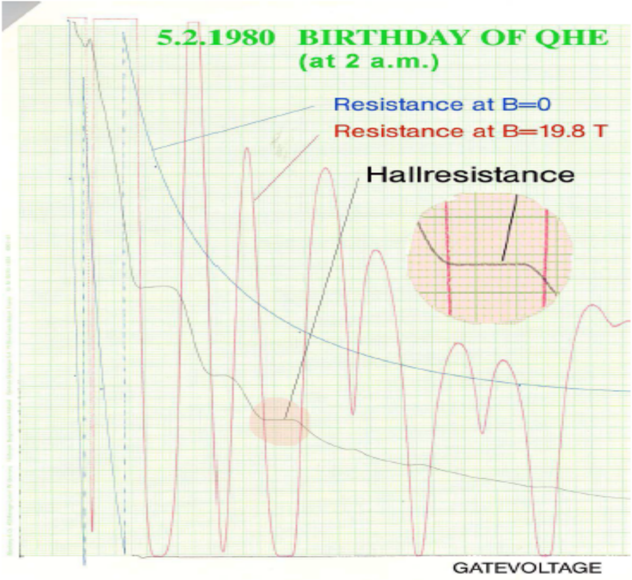
TRS, PHS, CHS = time reversal, particle hole, chiral symmetry

- Fermi level in a Gap or Anderson localization regime
- Topology of bulk (in Bloch bundles over Brillouin torus):

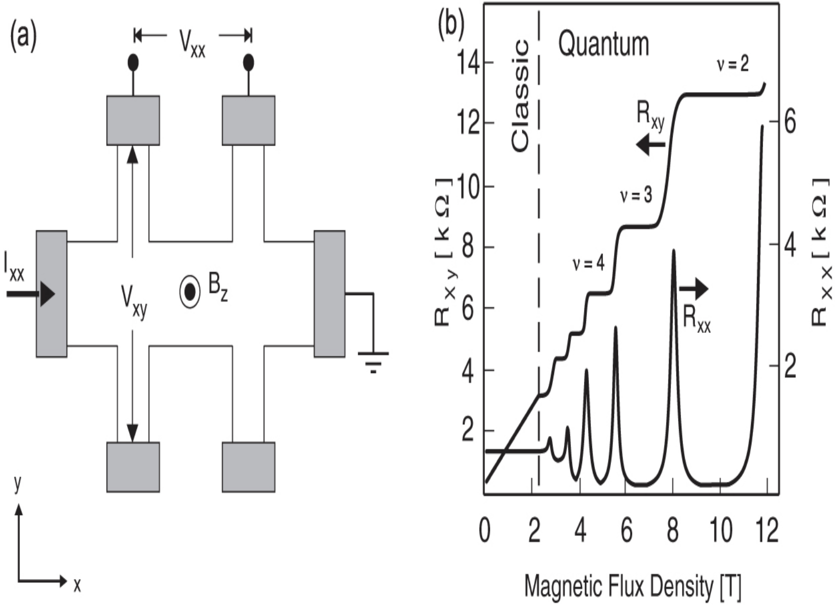
winding numbers, Chern numbers, $\mathbb{Z}_2$-invariants, higher invariants

- Delocalized edge modes with non-trivial topology
- Bulk-edge correspondence
- Topological bound states at defects (zero modes)
- Toy models: tight-binding Hamiltonians
- Wider notions include interactions, bosons, spins, photonic crys.

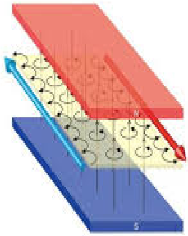
Quantum Hall Effect: first topological insulator

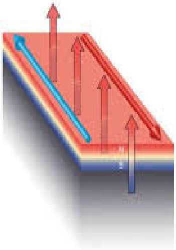


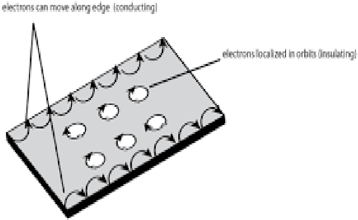
Schematic representation of IQHE



Most important facts for IQHE







Two-dimensional electron gas between two doted semiconductors

(Spot error in picture!) Measure of macroscopic (!) Hall tension

$$\sigma \;=\; \frac{I_{x,x}}{V_{x,y}} \;=\; n\,\frac{e^2}{h} \qquad \text{with } n \in \mathbb{N}$$

Integer quantization with relative error 10^{-8} with fundamental constant

Strong magnetic field and electron density can be modified

Anderson localizated states can be filled without changing conductivity

Prizes and further advances on the QHE

Nobel prizes:

- Klitzing (1985)
- Störmer-Tsui-Laughlin (1998) for fractional QHE
- Thouless (2016) explanation of integer QHE & Thouless-Kosterlitz
- Haldane (2016) anomalous QHE & Haldane spin chain

  NO exterior magnetic field, only magnetic material
- QHE in graphene at room temperature

  Novoselov, Geim et al 2007 (Nobel 2005)
- Anomalous QHE at room temperature in SnGe (Chinese group 2016)

  Review: Ren, Qiao, Niu 2016

Quantum spin Hall systems

Prior to 2005: no magnetic field $\Longrightarrow$ no topology

Kane-Mele (2005):

$\mathbb{Z}_2$-topology in two-dimensional systems with time-reversal symmetry

First erronous proposal: spin orbit coupling in graphene (too small)

Theoretical prediction by Bernevig and Zhang (2006): look into HgTe

Measurement by Molenkamp group in Würzburg

Complicated samples, inconsistencies with theory, so still disputed

Measurement in more conventional Si-semiconductor by Du group (Rice 2014) Surprise: stability w.r.t. magnetic field

Majorana zero modes

First proposal (Read-Green 2000):
attached to flux tubes in 2d $(p + ip)$-wave superconductors

Second proposal (Kitaev, Beenacker group, Alicea, *etc.*):
at ends of dirty superconductor wires placed on a semiconductor

Measurement in C. Marcus group (2014-2016 Bohr Inst., Kopenhagen)

Further measurements in Delft and Princeton groups

2017: `http://www.seethroughthe.cloud/2017/01/23/`

Headline is: Microsoft Steps Away From The Chalk Board
to Create Quantum Computer

Mysterious citation:
The magic recipe involves a combination of
semiconductors and superconductors

Higher dimensional topological insulators?

J. Phys. Soc. Jpn. **82** (2013) 102001 INVITED REVIEW PAPERS Y. ANDO

Table I. Summary of topological insulator materials that have bee experimentally addressed. The definition of (1;111) etc. is introduced in Sect. 3.7. (In this table, S.S., P.T., and SM stand for surface state, phase transition, and semimetal, respectively.)

Type	Material	Band gap	Bulk transport	Remark	Reference
2D, $\nu = 1$	CdTe/HgTe/CdTe	<10 meV	insulating	high mobility	31
2D, $\nu = 1$	AlSb/InAs/GaSb/AlSb	~4 meV	weakly insulating	gap is too small	73
3D (1;111)	$Bi_{1-x}Sb_x$	<30 meV	weakly insulating	complex S.S.	36, 40
3D (1;111)	Sb	semimetal	metallic	complex S.S.	39
3D (1;000)	Bi_2Se_3	0.3 eV	metallic	simple S.S.	94
3D (1;000)	Bi_2Te_3	0.17 eV	metallic	distorted S.S.	95, 96
3D (1;000)	Sb_2Te_3	0.3 eV	metallic	heavily p-type	97
3D (1;000)	Bi_2Te_2Se	~0.2 eV	reasonably insulating	ρ_{xx} up to 6 Ω cm	102, 103, 105
3D (1;000)	$(Bi,Sb)_2Te_3$	<0.2 eV	moderately insulating	mostly thin films	193
3D (1;000)	$Bi_{2-x}Sb_xTe_{3-y}Se_y$	<0.3 eV	reasonably insulating	Dirac-cone engineering	107, 108, 212
3D (1;000)	$Bi_2Te_{1.6}S_{1.4}$	0.2 eV	metallic	n-type	210
3D (1;000)	$Bi_{1.1}Sb_{0.9}Te_2S$	0.2 eV	moderately insulating	ρ_{xx} up to 0.1 Ω cm	210
3D (1;000)	Sb_2Te_2Se	?	metallic	heavily p-type	102
3D (1;000)	$Bi_2(Te,Se)_2(Se,S)$	0.3 eV	semi-metallic	natural Kawazulite	211
3D (1;000)	$TlBiSe_2$	~0.35 eV	metallic	simple S.S., large gap	110–112
3D (1;000)	$TlBiTe_2$	~0.2 eV	metallic	distorted S.S.	112
3D (1;000)	$TlBi(S,Se)_2$	<0.35 eV	metallic	topological P.T.	116, 117
3D (1;000)	$PbBi_2Te_4$	~0.2 eV	metallic	S.S. nearly parabolic	121, 124
3D (1;000)	$PbSb_2Te_4$	?	metallic	p-type	121
3D (1;000)	$GeBi_2Te_4$	0.18 eV	metallic	n-type	102, 119, 120
3D (1;000)	$PbBi_4Te_7$	0.2 eV	metallic	heavily n-type	125
3D (1;000)	$GeBi_{4-x}Sb_xTe_7$	0.1–0.2 eV	metallic	n (p) type at $x = 0$ (1)	126
3D (1;000)	$(PbSe)_5(Bi_2Se_3)_6$	0.5 eV	metallic	natural heterostructure	130
3D (1;000)	$(Bi_2)(Bi_2Se_{2.6}S_{0.4})$	semimetal	metallic	$(Bi_2)_n(Bi_2Se_3)_m$ series	127

2 Elements of basic theory

First for QHE in continuous physical space:

Landau-operator with disordered potential

$$H \;=\; \frac{1}{2m}(i\,\partial_{x_1} - eA_1)^2 \;+\; \frac{1}{2m}(i\,\partial_{x_2} - eA_2)^2 \;+\; \lambda\, V_{\text{dis}}$$

on Hilbert space $L^2(\mathbb{R}^2)$. Landau gauge $A_1 = 0$ and $A_2 = BX_1$

If there is no disorder $\lambda = 0$, Fourier transform in 2-direction works

$$\mathcal{F}_2 H \mathcal{F}_2^* \;=\; \int_{\mathbb{R}}^{\oplus} dk_2\, H(k_2)$$

with $H(k_2) = H(k_2)^*$ shifted one-dimensional harmonic oscillator

$\Longrightarrow$ infinitely degenerate so-called Landau bands.

Projection P on lowest band has integral kernel with Hall conductance

$$\begin{aligned} \mathrm{Ch}(P) \;&=\; 2\pi i\,\langle 0|P[i[X_1,P], i[X_2,P]]|0\rangle \\ &=\; \pi \int_{\mathbb{C}} dx \int_{\mathbb{C}} dy\; e^{-\frac{1}{2}(|x|^2+|y|^2-x\overline{y})}\,(x\overline{y} - y\overline{x}) \;=\; -1 \end{aligned}$$

Effect of disorder

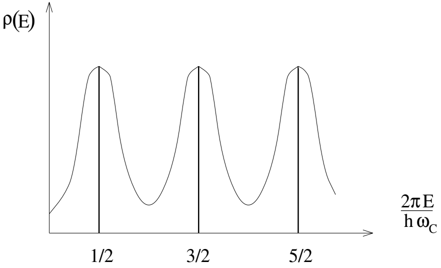
Typical model from i.i.d. $\omega_n \in [-1, 1]$ and $v \in C_K^\infty(B_1)$ with $\|v\|_\infty \leqslant 1$

$$V_{\text{dis}}(x) \;=\; \sum_{n \in \mathbb{Z}^2} \omega_n \, v(x-n)$$

Landau band widens by $\lambda \neq 0$. Gap closes at $\lambda \approx 1$

Expectation: all states Anderson localized, except at one energy

Proof at band edges by Barbaroux, Combes, Hislop 1997, others...



Spectrum of edge states

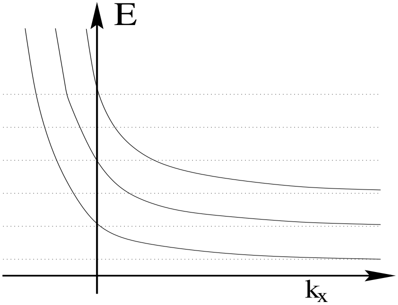
$\widehat{H}_L$ half-space restriction on $L^2(\mathbb{R}_{\geqslant 0} \times \mathbb{R})$ with Dirichlet

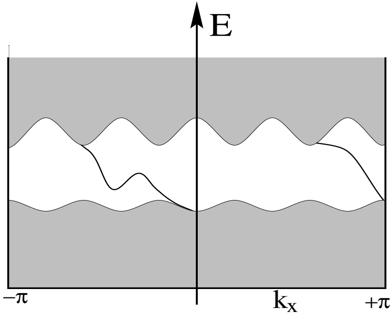
Still without disorder, Fourier transform works also for half-space:

$$\mathcal{F}_2 \widehat{H} \mathcal{F}_2^* \;=\; \int_{\mathbb{R}}^{\oplus} dk_2 \, \widehat{H}(k_2)$$

with $\widehat{H}(k_2) = \widehat{H}(k_2)^*$ cut off shifted harmonic oscillator on $L^2(\mathbb{R}_{\geqslant 0})$

Read off basic bulk-edge correspondence (right pic for generic gap)





Harper model

This is a lattice or tight-binding model on $\ell^2(\mathbb{Z}^2)$

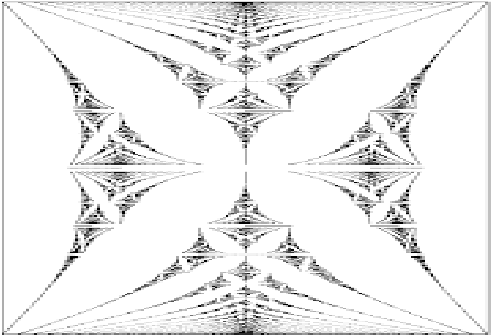
$$H \;=\; U_1 + U_1^* \;+\; U_2 + U_2^*$$

Here $U_1 = S_1$ shift in 1-direction, and $U_2 = e^{iBX_1} S_2$ (Landau gauge)

Plotted: spectrum as a function of B (Hofstadter's butterfly)

Spectrum fractal for irrational B. Most gaps close with V_{dis}

In each gap there are edge state bands (on $\ell^2(\mathbb{Z} \times \mathbb{N})$, Hatsugai 1993)



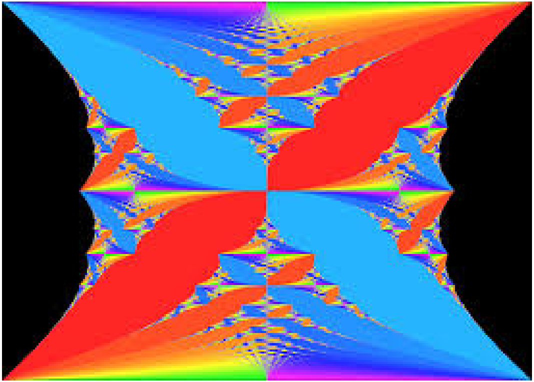
Coloured Hofstadter butterfly (Avron, Osadchy)

For each Fermi energy μ one has $P = \chi(H \leqslant \mu)$

If μ in gap, then Chern number well-defined

$$\mathrm{Ch}(P) \;=\; 2\pi i \,\langle 0|P[i[X_1, P], i[X_2, P]]|0\rangle \;\in\; \mathbb{Z}$$

Different values, different colours



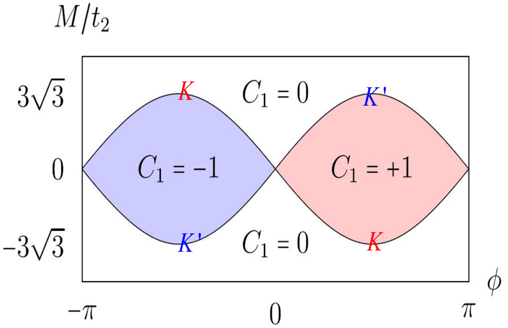
Haldane model for anomalous QHE (1988)

On honeycomb lattice = decorated triangular lattice, so on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$

$$H_{\mathrm{Hal}} = M \begin{pmatrix} 0 & S_1^* + S_2^* + 1 \\ S_1 + S_2 + 1 & 0 \end{pmatrix} + t_2 \sum_{j=1}^{3} \begin{pmatrix} e^{i\phi} S_j + (e^{i\phi} S_j)^* & 0 \\ 0 & e^{i\phi} S_j + (e^{i\phi} S_j)^* \end{pmatrix}$$

Here $S_3 = S_1 S_2$. Complex hopping, but only periodic magnetic field

Then central gap with $P = \chi(H \leqslant 0)$ and Chern number $C_1 = \mathrm{Ch}(P)$



Kane-Mele model for QSHE

On honeycomb lattice with spin $\frac{1}{2}$, so on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$

$$H_{\text{KM}} = \begin{pmatrix} H_{\text{Hal}} & 0 \\ 0 & \overline{H_{\text{Hal}}} \end{pmatrix} + H_{\text{Ras}}$$

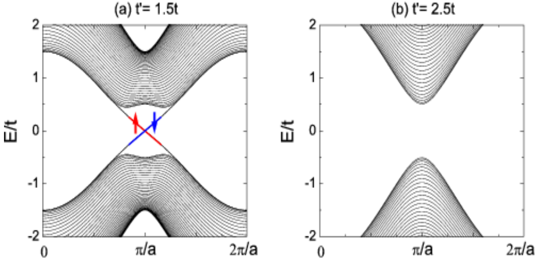
First term comes from spin-orbit coupling to next nearest neighbors

Second Rashba spin-orbit term is off-diagonal breaks chiral symmetry

If H_{Ras} small, central gap still open

Chern number vanishes (TRS), but non-trivial $\mathbb{Z}_2$-invariant

This leads to edge states



Discrete symmetries (invoking real structure)

Given commuting real, skew- or selfadjoint unitaries $J_{\rm ch}$, $S_{\rm tr}$, $S_{\rm ph}$

$$\begin{aligned} &\text{chiral symmetry (CHS)}: && J_{\rm ch}^* \, H \, J_{\rm ch} \;=\; -H \\ &\text{time reversal symmetry (TRS)}: && S_{\rm tr}^* \, \overline{H} \, S_{\rm tr} \;=\; H \\ &\text{particle-hole symmetry (PHS)}: && S_{\rm ph}^* \, \overline{H} \, S_{\rm ph} \;=\; -H \end{aligned}$$

$S_{\rm tr} = e^{i\pi s^y}$ orthogonal on $\mathbb{C}^{2s+1}$ with $S_{\rm tr}^2 = \pm \mathbf{1}$ even or odd

$S_{\rm ph}$ orthogonal on $\mathbb{C}^2_{\rm ph}$ with $S_{\rm ph}^2 = \pm \mathbf{1}$ even or odd

So typical Hamiltonian acts on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^{2s+1} \otimes \mathbb{C}^2_{\rm ph}$

Note: TRS + PHS $\implies$ CHS with $J_{\rm ch} = S_{\rm tr} S_{\rm ph}$

10 combinations of symmetries: none (1), one (5), three (4)

10 Cartan-Altland-Zirnbauer classes (CAZ): 2 complex, 8 real

Further distinction in each of the 10 classes: topological insulators

Periodic table of topological insulators

Schnyder-Ryu-Furusaki-Ludwig, Kitaev 2008: just strong invariants

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
1	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Complex (strong) invariants come from:

$$K_0(C_0(\mathbb{R}^d)) = \begin{cases} 0\,, & d \text{ odd} \\ \mathbb{Z}\,, & d \text{ even} \end{cases} \qquad K_1(C_0(\mathbb{R}^d)) = \begin{cases} \mathbb{Z}\,, & d \text{ odd} \\ 0\,, & d \text{ even} \end{cases}$$

Supplementary weak invariants from

$$K_0(C(\mathbb{T}^d)) \cong \mathbb{Z}^{2^{d-1}} \cong K_1(C(\mathbb{T}^d))$$

For strong Real structures: $\tau : \mathbb{T}^d \to \mathbb{T}^d$ inversion involution $\tau(k) = -k$

$$KR_j(C_0(\mathbb{R}^d_\tau)) = \pi_{j-1-d}(O)$$

where O stable orthogonal group with

j	0	1	2	3	4	5	6	7
$\pi_j(O)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\,\mathbb{Z}$	0	0	0	$\mathbb{Z}$

3 One-dimensional toy model (SSH, see [PS])

Su-Schrieffer-Heeger (1980, conducting polyacetelyn polymer)

$$H \;=\; \tfrac{1}{2}(\sigma_1 + i\sigma_2) \otimes S + \tfrac{1}{2}(\sigma_1 - i\sigma_2) \otimes S^* + m\,\sigma_2 \otimes \mathbf{1}$$

where S bilateral shift on $\ell^2(\mathbb{Z})$, $m \in \mathbb{R}$ mass and Pauli matrices

In their grading

$$H \;=\; \begin{pmatrix} 0 & S - im \\ S^* + im & 0 \end{pmatrix} \qquad \text{on } \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

Off-diagonal $\cong$ chiral symmetry $\sigma_3^* H \sigma_3 = -H$. In Fourier space:

$$H = \int_{[-\pi,\pi)}^{\oplus} dk\, H_k \qquad H_k = \begin{pmatrix} 0 & e^{-ik} - im \\ e^{ik} + im & 0 \end{pmatrix}$$

Topological invariant for $m \neq -1, 1$

$$\mathrm{Wind}(k \in [-\pi,\pi) \mapsto e^{ik} + im) \;=\; \delta(m \in (-1,1))$$

Chiral bound states

Half-space Hamiltonian

$$\widehat{H} \;=\; \begin{pmatrix} 0 & \widehat{S} - im \\ \widehat{S}^* + im & 0 \end{pmatrix} \qquad \text{on } \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$$

where $\widehat{S}$ unilateral right shift on $\ell^2(\mathbb{N})$

Still chiral symmetry $\sigma_3^* \widehat{H} \sigma_3 = -\widehat{H}$

If $m = 0$, simple bound state at $E = 0$ with eigenvector $\psi_0 = \binom{|0\rangle}{0}$.

Perturbations, *e.g.* in m, cannot move or lift this bound state ψ_m!

Positive chirality conserved: $\sigma_3 \psi_m = \psi_m$

Theorem 3.1 (Basic bulk-boundary correspondence)

If $\widehat{P}$ projection on bound states of $\widehat{H}$, then

$$\text{Wind}(k \mapsto e^{ik} + im) \;=\; \text{Tr}(\widehat{P}\sigma_3)$$

Disordered model

Add uniformly bounded i.i.d. random mass term $\omega = (m_n)_{n\in\mathbb{Z}}$:

$$H_\omega \;=\; H \;+\; \sum_{n\in\mathbb{Z}} m_n\,\sigma_2 \otimes |n\rangle\langle n|$$

Still chiral symmetry $\sigma_3^* H_\omega \sigma_3 = -H_\omega$ so

$$H_\omega \;=\; \begin{pmatrix} 0 & A_\omega^* \\ A_\omega & 0 \end{pmatrix}$$

Bulk gap at $E = 0 \Longrightarrow A_\omega$ invertible

Non-commutative winding number, also called first Chern number:

$$\mathrm{Wind}(A) \;=\; \mathrm{Ch}_1(A) \;=\; i\,\mathbf{E}_\omega\,\mathrm{Tr}\,\langle 0|A_\omega^{-1}\,i[X, A_\omega]|0\rangle$$

where $\mathbf{E}_\omega$ is average over probability measure $\mathbb{P}$ on i.i.d. masses

Index theorem and bulk-boundary correspondence

Theorem 3.2 (Disordered Noether-Gohberg-Krein Theorem)

If Π is Hardy projection on positive half-space, then $\mathbb{P}$-almost surely

$$\mathrm{Wind}(A) \;=\; \mathrm{Ch}_1(A) \;=\; -\,\mathrm{Ind}(\Pi A_\omega \Pi)$$

For periodic model as above, $A_\omega =$ Mult. by $e^{ik} \in C(\mathbb{S}^1)$

In this case, Fredholm operator is standard Toeplitz operator

Theorem 3.3 (Disoreded bulk-boundary correspondence)

If $\widehat{P}_\omega$ projection on bound states of $\widehat{H}_\omega$, then

$$\mathrm{Wind}(A) \;=\; \mathrm{Ch}_1(A) \;=\; \mathrm{Ch}_0(\widehat{P}_\omega) \;=\; \mathrm{Tr}(\widehat{P}_\omega \sigma_3)$$

Structural robust result:

holds for chiral Hamiltonians with larger fiber, other disorder, etc.

Index in linear algebra

Rank theorem for $T \in \mathrm{Mat}(N \times M, \mathbb{C})$

$$\begin{aligned} M &= \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Ran}(T)) \\ &= \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Ker}(T^*)^\perp) \\ &= \dim(\mathrm{Ker}(T)) + \big(N - \dim(\mathrm{Ker}(T^*))\big) \end{aligned}$$

Hence stability of index defined by

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*))) = M - N$$

Homotopy invariance: under continuous perturbation $t \in \mathbb{R} \mapsto T_t$

$$t \in \mathbb{R} \mapsto \mathrm{Ind}(T_t) \text{ konstant}$$

For quadratic matrices, *i.e.* $N = M$, always $\mathrm{Ind}(T) = 0$

Index in infinite dimension

Definition 3.4

$T \in \mathcal{B}(\mathcal{H})$ continuous Fredholm operator on $\mathcal{H}$

$\iff T\mathcal{H}$ closed, $\dim(\mathrm{Ker}(T)) < \infty$, $\dim(\mathrm{Ker}(T^*)) < \infty$

Then: $\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*))$

Theorem 3.5 (Dieudonné, Krein)

Ind *is a compactly stable homotopy invariant:*

$$\mathrm{Ind}(T) \;=\; \mathrm{Ind}(T+K) \;=\; \mathrm{Ind}(T_t)$$

Example: shift $\widehat{S} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $\widehat{S}\psi = (\psi_{n-1})_{n\in\mathbb{N}}$ on $\psi = (\psi_n)_{n\in\mathbb{N}}$

$$\mathrm{Ker}(\widehat{S}) \;=\; \mathrm{span}\{(1,0,0,\dots)\} \qquad , \qquad \mathrm{Ker}(\widehat{S}) \;=\; \{0\}$$

Thus $\mathrm{Ind}(\widehat{S}) = 1$

Index theorems connect index to a topological invariant

Structure: Toeplitz extension (no disorder)

S bilateral shift on $\ell^2(\mathbb{Z})$, then $\mathrm{C}^*(S) \cong C(\mathbb{S}^1)$

$\widehat{S}$ unilateral shift on $\ell^2(\mathbb{N})$, only partial isometry with a defect:

$$\widehat{S}^*\widehat{S} = \mathbf{1} \qquad \widehat{S}\,\widehat{S}^* = \mathbf{1} - |0\rangle\langle 0|$$

Then $\mathrm{C}^*(\widehat{S}) = \mathcal{T}$ Toeplitz algebra with exact sequence:

$$0 \;\to\; \mathcal{K} \;\overset{i}{\hookrightarrow}\; \mathcal{T} \;\overset{\pi}{\to}\; C(\mathbb{S}^1) \;\to\; 0$$

K-groups for C^*-algebra $\mathcal{A}$ with unitization $\mathcal{A}^+$:

$$\begin{aligned} K_0(\mathcal{A}) &= \{[P]-[s(P)] \;:\; \text{projections in some } M_n(\mathcal{A}^+)\} \\ K_1(\mathcal{A}) &= \{[U] \;:\; \text{unitary in some } M_n(\mathcal{A}^+)\} \end{aligned}$$

Abelian group operation: Whitney sum

Example: $K_0(\mathbb{C}) = \mathbb{Z} = K_0(\mathcal{K})$ with invariant $\dim(P)$

Example: $K_1(C(\mathbb{S}^1)) = \mathbb{Z}$ with invariant given by winding number

6-term exact sequence for Toeplitz extension

C*-algebra short exact sequence $\Longrightarrow$ K-theory 6-term sequence

$$\begin{array}{ccccc} K_0(\mathcal{K}) = \mathbb{Z} & \xrightarrow{i_*} & K_0(\mathcal{T}) = \mathbb{Z} & \xrightarrow{\pi_*} & K_0(C(\mathbb{S}^1)) = \mathbb{Z} \\ \uparrow \mathrm{Ind} & & & & \downarrow \mathrm{Exp} \\ K_1(C(\mathbb{S}^1)) = \mathbb{Z} & \xleftarrow{\pi_*} & K_1(\mathcal{T}) = 0 & \xleftarrow{i_*} & K_1(\mathcal{K}) = 0 \end{array}$$

Here: $[A]_1 \in K_1(C(\mathbb{S}^1))$ and $[\widehat{P}\sigma_3]_0 = [\widehat{P}_+]_0 - [\widehat{P}_-]_0 \in K_0(\mathcal{K})$

$$\mathrm{Ind}([A]_1) \;=\; [\widehat{P}_+]_0 - [\widehat{P}_-]_0 \qquad \text{(bulk-boundary for } K\text{-theory)}$$

$$\mathrm{Ch}_0(\mathrm{Ind}(A)) \;=\; \mathrm{Ch}_1(A) \qquad \text{(bulk-boundary for invariants)}$$

Disordered case: analogous

4 K-theory krash kourse [RLL, WO, CMR]

K-theory developed to classify vector bundles over topological space X

Swan-Serre Theorem: $\{\text{vector bundles}\} \cong \{\text{projections in } M_n(C(X))\}$

Replace $C(X)$ by non-commutative C*-algebra $\mathcal{A}$ (no Real structures)

Definition 4.1

$(\mathcal{A}, +, \cdot, \|\,.\,\|)$ Banach algebra over $\mathbb{C}$ if $\|AB\| \leqslant \|A\|\,\|B\|$, etc.

Then: $\mathcal{A}$ is C*-algebra $\iff \|A^*A\| = \|A\|^2$

Gelfand: commutative C* algebras are $\mathcal{A} = C_0(X)$ with spectrum X

GNS: For any state on $\mathcal{A}$ $\exists$ Hilbert $\mathcal{H}$ and representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$

Example 1: $\mathcal{A} = \mathbb{C}$ or $\mathcal{A} = M_n(\mathbb{C})$

Example 2: Calkin's exact sequence over a Hilbert space $\mathcal{H}$:

$$0 \to \mathcal{K}(\mathcal{H}) \overset{i}{\hookrightarrow} \mathcal{B}(\mathcal{H}) \overset{\pi}{\to} \mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to 0$$

Definition of $K_0(\mathcal{A})$

Unitization $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ of C*-algebra $\mathcal{A}$ by

$$(A,t)(B,s) \;=\; (AB + As + Bt, ts) \qquad , \qquad (A,t)^* = (A^*, \bar{t})$$

There is natural C*-norm $\|(A,t)\|$. Unit $\mathbf{1} = (0,1) \in \mathcal{A}^+$

Exact sequence of C*-algebras $0 \to \mathcal{A} \overset{i}{\hookrightarrow} \mathcal{A}^+ \overset{\rho}{\to} \mathbb{C} \to 0$

ρ has right inverse $i'(t) = (0,t)$, then $s = i' \circ \rho : \mathcal{A}^+ \to \mathcal{A}^+$ scalar part

$$\mathcal{V}_0(\mathcal{A}) \;=\; \left\{ V \in \bigcup_{n \geqslant 1} M_{2n}(\mathcal{A}^+) \;:\; V^* \;=\; V \;,\;\; V^2 \;=\; \mathbf{1} \;,\;\; s(V) \sim_0 E_{2n} \right\}$$

where $s(V) \sim_0 E_{2n}$ means homotopic to $E_{2n} = E_2^{\oplus n}$ with $E_2 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Equivalence relation $\sim_0$ on $\mathcal{V}_0(\mathcal{A})$ by homotopy and $V \sim_0 \begin{pmatrix} V & 0 \\ 0 & E_2 \end{pmatrix}$

Then $K_0(\mathcal{A}) = \mathcal{V}_0(\mathcal{A})/\sim_0$ abelian group via $[V]_0 + [V']_0 = [\begin{pmatrix} V & 0 \\ 0 & V' \end{pmatrix}]_0$

Definition of $K_0(\mathcal{A})$ is equivalent to standard one via $V = 2P - \mathbf{1}$:

$$K_0(\mathcal{A}) \;\cong\; \widehat{K}_0(\mathcal{A}) \;=\; \{[P] - [s(P)] \;:\; \text{projections in some } M_n(\mathcal{A}^+)\}$$

Theorem 4.2 (Stability of K_0)

$K_0(\mathcal{A}) = K_0(M_n(\mathcal{A})) = K_0(\mathcal{A} \otimes \mathcal{K})$

Example 1: $K_0(\mathbb{C}) = K_0(\mathcal{K}) = \mathbb{Z}$, invariant $\dim(P) = \dim(\mathrm{Ker}(V - \mathbf{1}))$
Example 2: $K_0(\mathcal{B}(\mathcal{H})) = 0$ for every separable $\mathcal{H}$ by [RLL] 3.3.3
Example 3: $K_0(C(\mathbb{S}^1)) = \mathbb{Z}$ and $K_0(\mathcal{T}) = \mathbb{Z}$ for Toeplitz (also dim)

Dimensions are examples of invariants, *e.g.* used for gap-labelling:

Theorem 4.3 (0-cocyles paired with $K_0(\mathcal{A})$)

If $\mathcal{T}$ tracial state on all $\mathcal{A}$, then class map $\mathcal{T} : K_0(\mathcal{A}) \to \mathbb{R}$ defined by

$$\mathcal{T}[V]_0 \;=\; \mathcal{T}(P) \;=\; \tfrac{1}{2}\,\mathcal{T}(V + \mathbf{1})$$

Definition of $K_1(\mathcal{A})$

For definition of $K_1(\mathcal{A})$ set

$$\mathcal{V}_1(\mathcal{A}) \;=\; \left\{ U \in \bigcup_{n\geqslant 1} M_n(\mathcal{A}^+) \;:\; U^{-1} \;=\; U^* \right\}$$

Equivalence relation $\sim_1$ by homotopy and $U \sim_1 \begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}$

Then $K_1(\mathcal{A}) = \mathcal{V}_1(\mathcal{A})/\sim_1$ with addition $[U]_1 + [U']_1 = [U \oplus U']_1$

If $\mathcal{A}$ unital, one can work with $M_n(\mathcal{A})$ instead of $M_n(\mathcal{A}^+)$ in $\mathcal{V}_1(\mathcal{A})$

Alternative: even chiral symmetry $K_{2n} = K_2^{\oplus^n}$ with $K_2 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ extended diagonally $K = \oplus_{n\geqslant 1} K_{2n}$ to $\cup_{n\geqslant 1} M_{2n}(\mathcal{A}^+)$. Then

$$V_1(\mathcal{A}) \;=\; \{ V \in V_0(\mathcal{A}) \;:\; K^* \, V \, K \;=\; -\, V \}$$
$$=\; \left\{ V \in \bigcup_{n\geqslant 1} M_{2n}(\mathcal{A}^+) \;:\; V^* \;=\; V \,,\; V^2 \;=\; \mathbf{1} \,,\; K^* \, V \, K \;=\; -\, V \right\}$$

Some examples of K_1-groups

Example 1: $K_1(\mathbb{C}) = K_1(\mathcal{K}) = 0$

Example 2: $K_1(C(\mathbb{S}^1)) = \mathbb{Z}$ with invariant "winding number"

Example 3: $K_1(\mathcal{A}^+) = K_1(\mathcal{A})$

Example 4: $K_1(\mathcal{B}(\mathcal{H})) = 0$ by Kuipers' theorem (holds for all W*'s)

Example 5: For Calkin $K_1(\mathcal{Q}(\mathcal{H})) = \mathbb{Z}$ with invariant $=$ Noether index

Suspension and Bott map

Definition 4.4

Suspension of a C*-algebra $\mathcal{A}$ is the C*-algebra $S\mathcal{A} = C_0(\mathbb{R}) \otimes \mathcal{A}$

Alternatively upon rescaling: $S\mathcal{A} \cong C_0((0,1),\mathcal{A})$

Theorem 4.5 (Suspension)

One has an isomorphism $\Theta : K_1(\mathcal{A}) \to K_0(S\mathcal{A})$, *described below*

Theorem 4.6 (Bott map)

One has isomorphism $\beta : K_0(\mathcal{A}) \cong \widehat{K}_0(\mathcal{A}) \to K_1(S\mathcal{A})$ *given by*

$$\beta([P]_0 - [s(P)]_0) \;=\; \big[t \in (0,1) \mapsto (\mathbf{1} - P) + e^{2\pi i t} P\big]_1$$

Note that r.h.s. indeed a unitary in $(S\mathcal{A})^+$

Corollary 4.7 (Bott periodicity)

$K_0(SS\mathcal{A}) = K_0(\mathcal{A})$

Standard construction of $\Theta : K_1(\mathcal{A}) \to K_0(S\mathcal{A})$ [WO, RLL]

Given $U \in M_n(\mathcal{A})$, $\mathrm{diag}(U, U^*)$ is homotop to $\mathbf{1}_{2n}$ in $M_{2n}(\mathcal{A})$

Let $t \in [0,1] \mapsto W_t$ be the connecting path

Then

$$\Theta[U]_1 \;=\; [W_1\,\mathrm{diag}(\mathbf{1},0)\,W_1^*]_0 - [\mathrm{diag}(\mathbf{1},0)]_0 \;\in\; K_0(S\mathcal{A})$$

Possible choice:

$$W_t \;=\; R_t\,\mathrm{diag}(U^*,\mathbf{1})\,R_t^*\,\mathrm{diag}(U,\mathbf{1})$$

with

$$R_t \;=\; \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & \sin\left(\frac{\pi t}{2}\right) \\ -\sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix}$$

Construction of $\Theta^{-1} : K_0(S\mathcal{A}) \to K_1(\mathcal{A})$ with adiabatic evolution:

$$0 \longrightarrow S\mathcal{A} \xrightarrow{\ i\ } C(\mathbb{S}^1,\mathcal{A}) \xrightarrow{\ \mathrm{ev}\ } \mathcal{A} \longrightarrow 0$$

After rescaling is given a loop $t \in [0,2\pi) \mapsto P_t = \frac{1}{2}(V_t + \mathbf{1}) \in M_N(\mathcal{A})$

With P_0 viewed as constant loop, $[P]_0 - [P_0]_0 \in K_0(S\mathcal{A})$

Indeed $\mathrm{ev}([P]_0 - [P_0]_0) = 0$ so identified with element in $K_0(S\mathcal{A})$

Aim: find preimage under Θ in $K_1(\mathcal{A})$

For $H_t = H_t^* \in M_N(\mathcal{A})$ satisfying $[H_t, P_t] = 0$ unitary solution $U_t \in \mathcal{A}^+$ of

$$i\,\partial_t U_t \;=\; \big(H_t + i[\partial_t P_t, P_t]\big)\,U_t\;, \qquad U_0 = \mathbf{1}_N$$

Then $P_t = U_t P_0 U_t^*$ and $U_{2\pi}\,P_0\,U_{2\pi}^* = P_0$

$$\Theta^{-1}\big([P]_0 - [P_0]_0\big) \;=\; [P_0 U_{2\pi} P_0 \;+\; \mathbf{1}_N \;-\; P_0]_1$$

R.h.s. is unitary! Choice of H_t determines lift. Details in [PS] □

Natural push-forwards maps in K-theory

Associated to an exact sequence of C*-algebras

$$0 \;\to\; \mathcal{K} \;\stackrel{i}{\hookrightarrow}\; \mathcal{A} \;\stackrel{\pi}{\to}\; \mathcal{Q} \;\to\; 0$$

there are natural push-forward maps:

$$i_* \;:\; K_j(\mathcal{K}) \to K_j(\mathcal{A}) \qquad , \qquad \pi_* \;:\; K_j(\mathcal{A}) \to K_j(\mathcal{Q})$$

given $i_*[V]_0 = [i(V)]_0$, $\pi_*[V]_0 = [\pi(V)]_0$, etc.

$\mathrm{Ker}(\pi_*) = \mathrm{Ran}(i_*)$, so short exact sequences of abelian groups:

$$K_0(\mathcal{K}) \;\stackrel{i_*}{\to}\; K_0(\mathcal{A}) \;\stackrel{\pi_*}{\to}\; K_0(\mathcal{Q})$$

and

$$K_1(\mathcal{Q}) \;\stackrel{\pi_*}{\leftarrow}\; K_1(\mathcal{A}) \;\stackrel{i_*}{\leftarrow}\; K_1(\mathcal{K})$$

Connecting maps close diagram to a cyclic 6-term diagram

Connecting maps from $K_j(\mathcal{Q})$ to $K_{j+1}(\mathcal{K})$

Definition 4.8 (Exponential map: $K_0(\mathcal{Q}) \to K_1(\mathcal{K})$)

Let $B = B^* \in M_n(\mathcal{A}^+)$ be contraction lift of unitary $V = V^* \in M_n(\mathcal{Q}^+)$

$$\begin{aligned}\mathrm{Exp}[V]_0 &= \left[\exp\left(2\pi i (\tfrac{1}{2}(B + \mathbf{1}))\right)\right]_1 \\ &= [-\cos(\pi B) - i\sin(\pi B)]_1 \\ &= [2B\sqrt{\mathbf{1} - B^2} + i(\mathbf{1} - 2B^2)]_1\end{aligned}$$

Definition 4.9 (Index map: $K_1(\mathcal{Q}) \to K_0(\mathcal{K})$)

Let $B \in M_n(\mathcal{A}^+)$ be contraction lift of unitary $U \in M_n(\mathcal{Q}^+)$, namely $\pi^+(B) = U$ and $\|B\| \leqslant 1$. Then define

$$\mathrm{Ind}[U]_1 = \left[\begin{pmatrix} 2BB^* - \mathbf{1} & 2B\sqrt{\mathbf{1} - B^*B} \\ 2B^*\sqrt{\mathbf{1} - BB^*} & \mathbf{1} - 2B^*B \end{pmatrix}\right]_0$$

Index map versus index of Fredholm operator

B unitary up to compact on $\mathcal{H} \iff \mathbf{1} - B^*B\,,\ \mathbf{1} - BB^* \in \mathcal{K}(\mathcal{H})$

$\implies$ B Fredholm operator and $U = \pi(B) \in \mathcal{Q}(\mathcal{H})$ unitary

Fedosov formula if $\mathbf{1} - B^*B$ and $\mathbf{1} - BB^*$ are traceclass:

$$\begin{aligned}
\mathrm{Ind}(B) &= \dim(\mathrm{Ker}(B)) - \dim(\mathrm{Ker}(B^*)) \\
&= \mathrm{Tr}(\mathbf{1} - B^*B) - \mathrm{Tr}(\mathbf{1} - BB^*) \\
&= \mathrm{Tr}\begin{pmatrix} BB^* - \mathbf{1} & B(\mathbf{1} - B^*B)^{\frac{1}{2}} \\ (\mathbf{1} - B^*B)^{\frac{1}{2}}B^* & \mathbf{1} - B^*B \end{pmatrix} \\
&= \tfrac{1}{2}\,\mathrm{Tr}\big(V - E_2\big) \qquad \text{with } V \text{ as above} \\
&= \tfrac{1}{2}\,\mathrm{Tr}\big(\mathrm{Ind}[U]_1 - E_2\big)
\end{aligned}$$

Hence there is a connection...

6-term exact sequence

Theorem 4.10

For every $0 \to \mathcal{K} \overset{i}{\hookrightarrow} \mathcal{A} \overset{\pi}{\to} \mathcal{Q} \to 0$, *above definitions lead to*

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \xrightarrow{i_*} & K_0(\mathcal{A}) & \xrightarrow{\pi_*} & K_0(\mathcal{Q}) \\ \uparrow \scriptstyle{\mathrm{Ind}} & & & & \downarrow \scriptstyle{\mathrm{Exp}} \\ K_1(\mathcal{Q}) & \xleftarrow{\pi_*} & K_1(\mathcal{A}) & \xleftarrow{i_*} & K_1(\mathcal{K}) \end{array}$$

Proof in the books...

Example 4.11

Toeplitz extension $0 \to \mathcal{K}(\ell^2(\mathbb{N})) \overset{i}{\hookrightarrow} \mathcal{T} \overset{\pi}{\to} C(\mathbb{S}^1) \to 0$

Bilateral shift $S \in C(\mathbb{S}^1)$ gives class $[S]_1 \in K_1(C(\mathbb{S}^1))$

Contraction lift is unilateral shift $\widehat{S} \in \mathcal{T} \subset \mathcal{B}(\ell^2(\mathbb{N}))$ with $\widehat{S}\widehat{S}^* = \mathbf{1} - P_0$

From definition $\mathrm{Ind}[S]_1 = [\mathrm{diag}(\mathbf{1} - 2P_0, -\mathbf{1})]_0$

Exact sequence of the sphere

$$\mathbb{D}^{d+1} \subset \overline{\mathbb{D}^{d+1}} \qquad , \qquad \partial\, \overline{\mathbb{D}^{d+1}} = \mathbb{S}^d$$

leads to an exact sequence of C*-algebras

$$0 \to C_0(\mathbb{D}^{d+1}) \cong C_0(\mathbb{R}^{d+1}) \stackrel{i}{\hookrightarrow} C(\overline{\mathbb{D}^{d+1}}) \stackrel{\pi}{\to} C(\mathbb{S}^d) \to 0$$

All K-groups are well-known [WO]. For for $d = 2n+1$ odd

$$\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z} & \xrightarrow{\pi_*} & \mathbb{Z} = K_0(C(\mathbb{S}^d)) \\
\uparrow \text{Ind} & & & & \downarrow \text{Exp} \\
\mathbb{Z} & \xleftarrow{\pi_*} & 0 & \xleftarrow{i_*} & 0
\end{array}$$

while for $d = 2n$ even

$$\begin{array}{ccccc}
0 & \xrightarrow{i_*} & \mathbb{Z} & \xrightarrow{\pi_*} & \mathbb{Z}^2 = K_0(C(\mathbb{S}^d)) \\
\uparrow \text{Ind} & & & & \downarrow \text{Exp} \\
0 & \xleftarrow{\pi_*} & 0 & \xleftarrow{i_*} & \mathbb{Z}
\end{array}$$

Aim: analyze one of the connecting maps, say Ind for d odd

Bott element

Let us write out $\mathrm{Ind} : K_1(C(\mathbb{S}^{2n-1})) = \mathbb{Z} \to K_0(C_0(\mathbb{D}^{2n})) = \mathbb{Z}$

For $n = 1$, generator is function $z : \mathbb{S}^1 \to \mathbb{S}^1$ with unit winding number

Lift is $z : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ which is *not* invertible, but a contraction

Bott element is "the" non-trivial self-adjoint unitary on $\mathbb{D}^2$:

$$\mathrm{Ind}([z]_1) \;=\; \left[\begin{pmatrix} 2|z|^2 - 1 & 2z\sqrt{1-|z|^2} \\ 2\overline{z}\sqrt{1-|z|^2} & 1-2|z|^2 \end{pmatrix}\right]_0 \;\in\; K_0(C(\mathbb{D}^2))$$

For higher odd d, irrep $\gamma_1, \ldots, \gamma_d$ of Clifford $\mathbb{C}_d$. Generator of $K_1(\mathbb{S}^d)$

$$U \;=\; \sum_{j=1,\ldots,d} x_j\,\gamma_j \;+\; i\,x_{d+1} \qquad , \qquad x = (x_1, \ldots, x_{d+1}) \in \mathbb{S}^d$$

Lift $B \in C(\overline{\mathbb{D}^{d+1}})$ same formula with $x \in \overline{\mathbb{D}^{d+1}}$. Then with $r = \|x\|$

$$\mathrm{Ind}[U]_1 \;=\; \left[\begin{pmatrix} 2r^2 - 1 & 2(1-r^2)^{\frac{1}{2}}B \\ 2B^*(1-r^2)^{\frac{1}{2}} & -(2r^2-1) \end{pmatrix}\right]_0$$

Another connecting map (for Floquet systems)

Theorem 4.12 ([SS])

$$0 \to \mathcal{K} \overset{\imath}{\hookrightarrow} \mathcal{A} \overset{\pi}{\to} \mathcal{Q} \to 0$$

Recall $\mathrm{Ind} : K_1(S\mathcal{Q}) \to K_0(S\mathcal{K})$ *and* $\Theta^{-1} : K_0(S\mathcal{K}) \to K_1(\mathcal{K})$*, so*

$$\Theta^{-1} \circ \mathrm{Ind} : K_1(S\mathcal{Q}) \to K_1(\mathcal{K})$$

Given smooth path $(0, 2\pi) \mapsto U(t) \in \mathcal{Q}$ *specifying class* $K_1(S\mathcal{Q})$

$$\Theta^{-1}(\mathrm{Ind}([(0, 2\pi) \mapsto U(t)]_1)) \;=\; [\widehat{U}(2\pi)]_1$$

where $\widehat{U}(2\pi) - \mathbf{1} \in \mathcal{K}$ *is end point of initial value problem in* $\mathcal{A}$

$$i\,\partial_t \widehat{U}(t) \;=\; \widehat{H}(t)\,\widehat{U}(t) \qquad\qquad \widehat{U}(0) \;=\; \mathbf{1}$$

associated to self-adjoint lift $\widehat{H}(t) \in \mathcal{A}$ *of* $H(t) = -i\,U(t)\partial_t U(t)^* \in \mathcal{Q}$

5 Observable algebra for tight-binding models

One-particle Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L$

Fiber $\mathbb{C}^L = \mathbb{C}^{2s+1} \otimes \mathbb{C}^r$ with spin s and r internal degrees

e.g. $\mathbb{C}^r = \mathbb{C}^2_{\mathrm{ph}} \otimes \mathbb{C}^2_{\mathrm{sl}}$ particle-hole space and sublattice space

Typical Hamiltonian

$$H_\omega \;=\; \Delta^B \,+\, W_\omega \;=\; \sum_{i=1}^{d} (t_i^* S_i^B + t_i (S_i^B)^*) \,+\, W_\omega$$

Magnetic translations $S_j^B S_i^B = e^{iB_{i,j}} S_i^B S_j^B$ in Laudau gauge:

$$S_1^B = S_1 \qquad S_2^B = e^{iB_{1,2}X_1} S_2 \qquad S_3^B = e^{iB_{1,3}X_1 + iB_{2,3}X_2} S_3$$

t_i matrices $L \times L$, e.g. spin orbit coupling, (anti)particle creation

matrix potential $W_\omega = W_\omega^* = \sum_{n\in\mathbb{Z}^d} |n\rangle \omega_n \langle n|$ with i.i.d. matrices ω_n

Configurations $\omega = (\omega_n)_{n\in\mathbb{Z}^d} \in \Omega$ compact probability space $(\Omega, \mathbb{P})$

$\mathbb{P}$ invariant and ergodic w.r.t. $T : \mathbb{Z}^d \times \Omega \to \Omega$

Covariant operators (generalizes periodicity)

Covariance w.r.t. to dual magnetic translations $V_a = S_j^B V_a (S_j^B)^*$

$$V_a H_\omega V_a^* \;=\; H_{T_a\omega} \qquad , \qquad a \in \mathbb{Z}^d$$

$\|A\| = \sup_{\omega\in\Omega} \|A_\omega\|$ is C*-norm on

$$\begin{aligned} \mathcal{A}_d \;&=\; \mathrm{C}^* \left\{ A = (A_\omega)_{\omega\in\Omega} \text{ finite range covariant operators} \right\} \\ &\cong\; \text{twisted crossed product } C(\Omega) \rtimes_B \mathbb{Z}^d \end{aligned}$$

Fact: Suppose Ω contractible (say ω_n from matrix ball)

$\Longrightarrow$ rotation algebra $\mathrm{C}^*(S_1^B, \ldots, S_d^B)$ is deformation retract of $\mathcal{A}_d$

In particular: K-groups of $\mathrm{C}^*(S_1^B, \ldots, S_d^B)$ and $\mathcal{A}_d$ coincide

Theorem 5.1 (Pimsner-Voiculescu 1980)

$K_0(\mathcal{A}_d) = \mathbb{Z}^{2^{d-1}}$ and $K_1(\mathcal{A}_d) = \mathbb{Z}^{2^{d-1}}$

More precisely, from $0 \to \mathcal{A}_{d-1} \otimes \mathcal{K} \to \mathcal{T}(\mathcal{A}_d) \to \mathcal{A}_d \to 0$ one has

$$\begin{array}{ccccc}
K_0(\mathcal{A}_{d-1}) & \xrightarrow{i_*} & K_0(\mathcal{T}(\mathcal{A}_d)) & \xrightarrow{\pi_*} & K_0(\mathcal{A}_d) \\
\uparrow{\scriptstyle \text{Ind}} & & & & \downarrow{\scriptstyle \text{Exp}} \\
K_1(\mathcal{A}_d) & \xleftarrow{\pi_*} & K_1(\mathcal{T}(\mathcal{A}_d)) & \xleftarrow{i_*} & K_1(\mathcal{A}_{d-1})
\end{array}$$

But Pimsner-Voiculescu also show $K(\mathcal{T}(\mathcal{A}_d)) \cong K(\mathcal{A}_{d-1})$

Under this isomorphism, one then has (note that i_* moved!)

$$\begin{array}{ccccc}
K_0(\mathcal{A}_{d-1}) & \xrightarrow{0} & K_0(\mathcal{A}_{d-1}) & \xrightarrow{i_*} & K_0(\mathcal{A}_d) \\
\uparrow{\scriptstyle \text{Ind}} & & & & \downarrow{\scriptstyle \text{Exp}} \\
K_1(\mathcal{A}_d) & \xleftarrow{i_*} & K_1(\mathcal{A}_{d-1}) & \xleftarrow{0} & K_1(\mathcal{A}_{d-1})
\end{array}$$

Hence there are two short exact sequences of K-groups

Generators of $K_j(\mathcal{A}_d)$ from PV's Toeplitz extension

From the above:

$$0 \;\to\; K_0(\mathcal{A}_{d-1}) \;\overset{i_*}{\to}\; K_0(\mathcal{A}_d) \;\overset{\mathrm{Exp}}{\to}\; K_1(\mathcal{A}_{d-1}) \;\to\; 0$$

$$0 \;\to\; K_1(\mathcal{A}_{d-1}) \;\overset{i_*}{\to}\; K_1(\mathcal{A}_d) \;\overset{\mathrm{Ind}}{\to}\; K_0(\mathcal{A}_{d-1}) \;\to\; 0$$

No torsion $\implies K_j(\mathcal{A}_d) = K_0(\mathcal{A}_{d-1}) \oplus K_1(\mathcal{A}_{d-1}) = \mathbb{Z}^{2^{d-2}} \oplus \mathbb{Z}^{2^{d-2}}$

Iterative construction of generators using inverse of Ind and Exp

Explicit generators $[G_I]$ of K-groups labelled by subsets $I \subset \{1, \ldots, d\}$

Top generator $I = \{1, \ldots, d\}$ identified with Bott in $K_j(C(\mathbb{S}^d))$

Example $G_{\{1,2\}}$ Powers-Rieffel projection in $C^*(S_1^B, S_2^B)$

In general, any projection $P \in M_n(\mathcal{A}_d)$ can be decomposed as

$$[P]_0 \;=\; \sum_{I \subset \{1,\ldots,d\}} n_I \,[G_I]_0 \qquad\qquad n_I \in \mathbb{Z},\; |I| \text{ even}$$

Questions: calculate $n_I = c_I \,\mathrm{Ch}_I(P)$ and give physical significance

K-group elements of physical interest

Fermi level $\mu \in \mathbb{R}$ in spectral gap of H_ω

$$P_\omega \;=\; \chi(H_\omega \leqslant \mu) \qquad \text{covariant Fermi projection}$$

Hence: $P = (P_\omega)_{\omega\in\Omega} \in \mathcal{A}_d$ fixes element in $[P]_0 \in K_0(\mathcal{A}_d)$

If chiral symmetry present: Fermi unitary $U = A|A|^{-1}$ from

$$H_\omega \;=\; -J_{\rm ch}^* H_\omega J_{\rm ch} \;=\; \begin{pmatrix} 0 & A_\omega \\ A_\omega^* & 0 \end{pmatrix} \qquad , \qquad J_{\rm ch} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

If $\mu = 0$ in gap, $A = (A_\omega)_{\omega\in\Omega} \in \mathcal{A}_d$ invertible and $[U]_1 = [A]_1 \in K_1(\mathcal{A}_d)$

Remark Sufficient to have an approximate chiral symmetry

$$H_\omega \;=\; \begin{pmatrix} B_\omega & A_\omega \\ A_\omega^* & C_\omega \end{pmatrix} \qquad \text{with invertible } A_\omega$$

Strong and weak invariants in *K*-theory terms

Fermi level $\mu \implies$ Fermi projection P or Fermi unitary A

Decompositions

$$[P]_0 = \sum_{I\subset\{1,\ldots,d\}} n_I \, [G_I]_0 \qquad , \qquad [A]_1 = \sum_{I\subset\{1,\ldots,d\}} n_I \, [G_I]_1$$

Invariants n_I, top invariant $n_{\{1,\ldots,d\}} \in \mathbb{Z}$ called *strong* , others weak

A systems with $n_{\{1,\ldots,d\}} \neq 0$ is called a strong topological insulator

If $n_{\{1,\ldots,d\}} = 0$, but some other $n_I \neq 0$, weak topological insulator

For Class A (no symmetry) and Class AIII (chiral symmetry):

	dimension d	1	2	3	4	5	6	7	8
A	strong invariant	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	strong invariant	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

$\mathbb{Z}$-entries are parts of the K-groups. Calculation of number next

Non-commutative analysis tools [BES, PS]

Definition 5.2 (Non-commutative integration and derivatives)

Tracial state $\mathcal{T}$ on $\mathcal{A}_d$ given by

$$\mathcal{T}(A) \;=\; \mathbf{E}_{\mathbb{P}}\, \mathrm{Tr}_L \, \langle 0|A_\omega|0\rangle$$

Derivations $\nabla = (\nabla_1, \ldots, \nabla_d)$ densely defined by

$$\nabla_j A_\omega \;=\; i[X_j, A_\omega]$$

Then define $C^k(\mathcal{A})$, $C^\infty(\mathcal{A})$, *etc.*

Usual rules: $\mathcal{T}(AB) = \mathcal{T}(BA)$, $\nabla(AB) = \nabla(A)B + A\nabla(B)$, *etc.*

Also: $\mathcal{T}(\nabla(A)) = 0$, so partial integration $\mathcal{T}(\nabla(A)B) = -\mathcal{T}(A\nabla(B))$

Proposition 5.3 (Birkhoff theorem for translation group)

$\mathcal{T}$ is $\mathbb{P}$-almost surely the trace per unit volume

$$\mathcal{T}(A) \;=\; \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{n\in\Lambda} \mathrm{Tr}_L \, \langle n|A_\omega|n\rangle$$

Periodic systems

For simplicity 1-periodic in all directions and no magnetic field

Then $\mathcal{A}_d = C(\mathbb{T}^d) \otimes \mathbb{C}^{L\times L}$ commutative up to matrix degree

non-commutative	A	$\nabla_j(A)$	$\mathcal{T}$
commutative	$k \mapsto A(k)$	$\partial_{k_j} A$	$\int_{\mathbb{T}^d} dk\, \mathrm{Tr}$

With dictionary: rewrite many formulas from solid state literature

Example: Kubo formula for conductivity at relaxation time τ

$$\int dk \sum_{n,m} \mathrm{Tr}\left(\partial_{k_i}(f_{\beta,\mu}(E_n(k))P_n(k))\big(E_n(k) - E_m(k) + \tfrac{1}{\tau}\big)^{-1}\partial_{k_j}(E_m(k)P_m(k))\right)$$

$$= \mathcal{T}\left(\nabla_i(f_{\beta,\mu}(H))\big(\mathcal{L}_H + \tfrac{1}{\tau}\big)^{-1}(\nabla_j(H))\right)$$

where $\mathcal{L}_H = i[H,\,.\,]$ Liouville operator

6 Topological invariants in solid state systems

$A \in \mathcal{A}_d$ invertible and $|I|$ odd with $\rho : \{1, \ldots, |I|\} \to I$ and $\mathrm{sig}(\rho) = (-1)^\rho$:

$$\mathrm{Ch}_I(A) \;=\; \frac{i(i\pi)^{\frac{|I|-1}{2}}}{|I|!!} \sum_{\rho \in S_I} (-1)^\rho \, \mathcal{T}\left(\prod_{j=1}^{|I|} A^{-1} \nabla_{\rho_j} A \right) \;\in\; \mathbb{R}$$

where $\quad \mathcal{T}(A) = \mathbf{E}_{\mathbb{P}} \, \mathrm{Tr}_L \, \langle 0 | A_\omega | 0 \rangle \quad$ and $\quad \nabla_j A_\omega = i[X_j, A_\omega]$

For even $|I|$ and projection $P \in \mathcal{A}_d$:

$$\mathrm{Ch}_I(P) \;=\; \frac{(2i\pi)^{\frac{|I|}{2}}}{\frac{|I|}{2}!} \sum_{\rho \in S_I} (-1)^\rho \, \mathcal{T}\left(P \prod_{j=1}^{|I|} \nabla_{\rho_j} P \right) \;\in\; \mathbb{R}$$

Theorem 6.1 (Connes 1985, [Con])

$\mathrm{Ch}_I(A)$ *and* $\mathrm{Ch}_I(P)$ *homotopy invariants; pairings with* $K(\mathcal{A}_d)$

Rewriting

Let d be even and $\mathbb{C}_d$ complex Clifford generated by $\gamma_1, \ldots, \gamma_d$

Extend $\mathcal{A}_d$ to $\mathcal{A}_d \otimes \mathbb{C}_d$ so that degree of form can be counted

Exterior derivatives are $dA \otimes v = \sum_{j=1}^{d} \nabla_j A \otimes \gamma_j v$

Finally let $\mathrm{ev}(\gamma_1 \cdots \gamma_j) = \delta_{j,d}$

Then

$$\mathrm{Ch}_{\{1,\ldots,d\}}(P) \;=\; \frac{(2i\pi)^{\frac{|I|}{2}}}{\frac{|I|}{2}!}\,\mathcal{T} \circ \mathrm{ev}\,(PdP \cdots dP)$$

Special case $d = 2$ gives ”first” Chern number:

$$\begin{aligned}
\mathrm{Ch}_{\{1,2\}}(P) &= 2\pi i\, \mathcal{T} \circ \mathrm{ev}\,(PdPdP) \\
&= 2\pi i\, \mathcal{T}\,(P[\nabla_1 P, \nabla_2 P]) \\
&= 2\pi i \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2}\, \mathrm{Tr}\big(P(k)[\partial_1 P(k), \partial_2 P(k)]\big)
\end{aligned}$$

where $P = \int_{\mathbb{T}^2}^{\oplus} dk\, P(k)$

Link to Volovik-Essin-Gurarie invariants

Express the invariants in terms of Green function/resolvent

Consider path $z : [0,1] \to \mathbb{C}\backslash\sigma(H)$ encircling $(-\infty,\mu] \cap \sigma(H)$

Set

$$G(t) \;=\; (H - z(t))^{-1}$$

Theorem 6.2 ([PS])

For $|I|$ even and with $\nabla_0 = \partial_t$,

$$\mathrm{Ch}_I(P_\mu) \;=\; \frac{(i\pi)^{\frac{|I|}{2}}}{i(|I|-1)!!} \sum_{\rho \in S_{I \cup \{0\}}} (-1)^\rho \int_0^1 dt\, \mathcal{T}\left(\prod_{j=0}^{|I|} G(t)^{-1} \nabla_{\rho_j} G(t) \right)$$

Isomorphism via Bott map $\beta : K_0(\mathcal{A}_d) \to K_1(S\mathcal{A}_d)$ leads to

$$\beta[P_\mu]_0 \;=\; \big[t \in [0,1] \mapsto G(t)\big]_1$$

Combine with suspension result on cyclic cohomology side

Similar results for odd pairings

Dimensional reduction for d even

Theorem 6.3 ([STo])

$H \in \mathcal{A}_d$ only nearest neighbor hopping with fibers $\mathbb{C}^{L\times L}$

Also: $H = H_0 + \lambda H_1$ with H_0 periodic in $d-1$ directions along boundary

Let $\delta > 0$ and λ sufficiently small, $P = \chi(H \leqslant \mu) \in \mathcal{A}_d$ Fermi projection

$$\mathrm{Exp}[P]_0 \;=\; -[(\widehat{G}^{\mu+\imath\delta} - \imath\,\mathbf{1}_L)(\widehat{G}^{\mu+\imath\delta} + \imath\,\mathbf{1}_L)^{-1}]_1$$

where, with Π_1 restriction to boundary Hilbert space $\ell^2(\mathbb{Z}^{d-1}\times\{1\},\mathbb{C}^L)$,

$$\widehat{G}^z \;=\; \Pi_1(\widehat{H}-z)^{-1}\Pi_1^*$$

Effective chiral Hamiltonian $h_{\rm eff} \in \mathcal{A}_{d-1}$

$$h_{\rm eff} \;=\; \begin{pmatrix} 0 & (\widehat{V}^z)^* \\ \widehat{V}^z & 0 \end{pmatrix} \qquad , \qquad \widehat{V}^z \;=\; (\widehat{G}^z - \imath\,\mathbf{1})(\widehat{G}^z + \imath\,\mathbf{1})^{-1}$$

Open question: dimensional reduction in odd dimension

Generalized Streda formulæ

In QHE: integrated density of states grows linearly in magnetic field

integrated density of states: $\mathbf{E}\,\langle 0|P|0\rangle \;=\; \mathrm{Ch}_{\emptyset}(P)$

$$\partial_{B_{1,2}}\,\mathrm{Ch}_{\emptyset}(P) \;=\; \frac{1}{2\pi}\,\mathrm{Ch}_{\{1,2\}}(P)$$

Theorem 6.4 (Elliott 1984, [PS])

$$\partial_{B_{i,j}}\,\mathrm{Ch}_I(P) \;=\; \frac{1}{2\pi}\,\mathrm{Ch}_{I\cup\{i,j\}}(P) \qquad |I| \text{ even},\ i,j\notin I$$

$$\partial_{B_{i,j}}\,\mathrm{Ch}_I(A) \;=\; \frac{1}{2\pi}\,\mathrm{Ch}_{I\cup\{i,j\}}(A) \qquad |I| \text{ odd},\ i,j\notin I$$

Application: magneto-electric effects in $d=3$

Time is 4th direction needed for calculation of polarization

Non-linear response is derivative w.r.t. B given by $\mathrm{Ch}_{\{1,2,3,4\}}(P)$

Index theorem for strong invariants and odd d

$\gamma_1, \ldots, \gamma_d$ irrep of Clifford C_d on $\mathbb{C}^{2^{(d-1)/2}}$

$$D \;=\; \sum_{j=1}^{d} X_j \otimes \mathbf{1} \otimes \gamma_j \qquad \text{Dirac operator on } \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L \otimes \mathbb{C}^{2^{(d-1)/2}}$$

Dirac phase $F = \frac{D}{|D|}$ provides odd Fredholm module on $\mathcal{A}_d$:

$$F^2 = \mathbf{1} \qquad [F, A_\omega] \text{ compact and in } \mathcal{L}^{d+\epsilon} \text{ für } A = (A_\omega)_{\omega\in\Omega} \in \mathcal{A}_d$$

Theorem 6.5 (Local index = generalizes Noether-Gohberg-Krein)

Let $\Pi = \frac{1}{2}(F + \mathbf{1})$ *be Hardy projection for* F*. For invertible* A_ω

$$\mathrm{Ch}_{\{1,\ldots,d\}}(A) \;=\; \mathrm{Ind}(\Pi\, A_\omega \Pi)$$

The index is $\mathbb{P}$*-almost surely constant.*

Proof based on key geometric identities

Let $d = 2k + 1$

Given $x_1, \ldots, x_{2k+2} \in \mathbb{R}^{2k+1}$ with x_{2k+2} fixed at the origin

$\gamma_1, \ldots, \gamma_{2k+1}$ irrep on $\mathbb{C}^{2^k}$ of complex Clifford Cl_{2k+1}

$$\int_{\mathbb{R}^{2k+1}} dx \ \mathrm{tr}\Big(\prod_{j=1}^{2k+1} \big(\mathrm{sgn}\langle \gamma, x_j + x\rangle - \mathrm{sgn}\langle \gamma, x_{j+1} + x\rangle\big)\Big)$$
$$= -\frac{2^{2k+1}(i\pi)^k}{(2k+1)!!} \sum_{\rho \in \mathcal{S}_{2k+1}} (-1)^\rho \prod_{j=1}^{2k+1} x_{j,\rho_j}$$

For $d = 1$: standard element in proof of Noether-Gohberg-Krein

Analog for $d = 2$: Connes' triangle equality

Extension: index theory for weak invariants (Prodan-SB)

Alternative proof: semifinite index theory (Bourne-SB)

Local index theorem for even dimension d

As above $\gamma_1, \ldots, \gamma_d$ Clifford, grading $\Gamma = -i^{-d/2}\gamma_1 \cdots \gamma_d$

Dirac $D = -\Gamma D \Gamma = |D| \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}$ even Fredholm module

Theorem 6.6 (Connes $d = 2$, Prodan, Leung, Bellissard 2013)

Almost sure index $\mathrm{Ind}(P_\omega F P_\omega)$ *equal to* $\mathrm{Ch}_{\{1,\ldots,d\}}(P)$

Special case $d = 2$: $F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$ and

$$\mathrm{Ind}(P_\omega F P_\omega) \;=\; 2\pi i\, \mathcal{T}(P[[X_1, P], [X_2, P]])$$

Proof: again geometric identity of high-dimensional simplexes

Advantages: phase label also for dynamical localized regime

implementation of discrete symmetries (CPT)

Numerical technique for strong invariants

H chiral with Fermi unitary A. For tuning parameter $\kappa > 0$ introduce:

$$L_\kappa \;=\; H \,+\, \kappa \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \;=\; \begin{pmatrix} \kappa\, D & A \\ A^* & -\kappa\, D \end{pmatrix}$$ **spectral localizer**

A_ρ restriction of A (Dirichlet b.c.) to range of $\chi(|D| \leqslant \rho)$

$$L_{\kappa,\rho} \;=\; \begin{pmatrix} \kappa\, D_\rho & A_\rho \\ A_\rho^* & -\kappa\, D_\rho \end{pmatrix}$$

Clearly selfadjoint matrix:

$$(L_{\kappa,\rho})^* \;=\; L_{\kappa,\rho}$$

Fact 1: $L_{\kappa,\rho}$ is gapped, namely $0 \notin L_{\kappa,\rho}$

Fact 2: $L_{\kappa,\rho}$ has spectral asymmetry measured by signature

Fact 3: signature linked to topological invariant

Theorem 6.7 ([LS2])

Given $D = D^*$ with compact resolvent and invertible A with invertibility gap $g = \|A^{-1}\|^{-1}$. Provided that

$$\|[D, A]\| \leqslant \frac{g^3}{12\,\|A\|\,\kappa} \qquad (*)$$

and

$$\frac{2\,g}{\kappa} \leqslant \rho \qquad (**)$$

the matrix $L_{\kappa,\rho}$ is invertible and with $\Pi = \chi(D \geqslant 0)$

$$\tfrac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho}) = \mathrm{Ind}\big(\Pi A \Pi + (\mathbf{1} - \Pi)\big)$$

How to use: form (*) infer κ, then ρ from (**)

If A unitary, $g = \|A\| = 1$ and $\kappa = (12\|[D, A]\|)^{-1}$ and $\rho = \frac{2}{\kappa}$

Hence **small** matrix of size $\leqslant$ 100 sufficient! Great for numerics!

Why it can work:

Proposition 6.8

If (*) and (**) hold,

$$L^2_{\kappa,\rho} \geqslant \frac{g^2}{2}$$

Proof:

$$L^2_{\kappa,\rho} = \begin{pmatrix} A_\rho A_\rho^* & 0 \\ 0 & A_\rho^* A_\rho \end{pmatrix} + \kappa^2 \begin{pmatrix} D_\rho^2 & 0 \\ 0 & D_\rho^2 \end{pmatrix} + \kappa \begin{pmatrix} 0 & [D_\rho, A_\rho] \\ [D_\rho, A_\rho]^* & 0 \end{pmatrix}$$

Last term is a perturbation controlled by (*)

First two terms positive (indeed: close to origin and away from it)

Now $A^*A \geqslant g^2$, but $(A^*A)_\rho \neq A_\rho^* A_\rho$

This issue can be dealt with by tapering argument:

Proposition 6.9 (Bratelli-Robinson)

For $f : \mathbb{R} \to \mathbb{R}$ with Fourier transform defined without $\sqrt{2\pi}$,

$$\|[f(D), A]\| \;\leqslant\; \|\widehat{f'}\|_1 \; \|[D, A]\|$$

Lemma 6.1

$\exists$ even function $f_\rho : \mathbb{R} \to [0, 1]$ with $f_\rho(x) = 0$ for $|x| \geqslant \rho$
and $f_\rho(x) = 1$ for $|x| \leqslant \frac{\rho}{2}$ such that $\|\widehat{f'_\rho}\|_1 = \frac{8}{\rho}$

With this, $f = f_\rho(D) = f_\rho(|D|)$ and $\mathbf{1}_\rho = \chi(|D| \leqslant \rho)$:

$$\begin{aligned} A_\rho^* A_\rho \; &= \; \mathbf{1}_\rho A^* \mathbf{1}_\rho A \mathbf{1}_\rho \; \geqslant \; \mathbf{1}_\rho A^* f^2 A \mathbf{1}_\rho \\ &= \; \mathbf{1}_\rho f A^* A f \mathbf{1}_\rho \; + \; \mathbf{1}_\rho \big([A^*, f] f A + f A^* [f, A] \big) \mathbf{1}_\rho \\ &\geqslant \; g^2 \, f^2 \; + \; \mathbf{1}_\rho \big([A^*, f] f A + f A^* [f, A] \big) \mathbf{1}_\rho \end{aligned}$$

So indeed $A_\rho^* A_\rho$ positive close to origin

Then one can conclude... but a bit tedious □

Proof by spectral flow

Use Phillips' result for phase $U = A|A|^{-1}$ and properties of SF:

$$
\begin{aligned}
\mathrm{Ind}(\Pi A \Pi + \mathbf{1} - \Pi) &= \mathrm{SF}(U^* D U, D) \\
&= \mathrm{SF}(\kappa\, U^* D U, \kappa\, D) \\
&= \mathrm{SF}\left(\begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}^* \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&= \mathrm{SF}\left(\begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}^* \begin{pmatrix} \kappa\, D & \mathbf{1} \\ \mathbf{1} & -\kappa\, D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&= \mathrm{SF}\left(\begin{pmatrix} \kappa\, U^* D U & U \\ U^* & -\kappa\, D \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&= \mathrm{SF}\left(\begin{pmatrix} \kappa\, D & U \\ U^* & -\kappa\, D \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right)
\end{aligned}
$$

Now localize and use $\mathrm{SF} = \frac{1}{2}\,\mathrm{Sig}$ on paths of selfadjoint matrices □

Even pairings (in even dimension)

Consider gapped Hamiltonian H on $\mathcal{H}$ specifying $P = \chi(H \leqslant 0)$

Dirac operator D on $\mathcal{H} \oplus \mathcal{H}$ is odd w.r.t. grading $\Gamma = \left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{smallmatrix}\right)$

Thus $D = -\Gamma D \Gamma = \left(\begin{smallmatrix} 0 & D' \\ (D')^* & 0 \end{smallmatrix}\right)$ and Dirac phase $F = D'|D'|^{-1}$

Fredholm operator $PFP + (\mathbf{1} - P)$ has index = Chern number

Spectral localizer

$$L_\kappa = \begin{pmatrix} H & \kappa D' \\ \kappa (D')^* & -H \end{pmatrix} = H \otimes \Gamma + \kappa D$$

Theorem 6.10 ([LS3])

Suppose $\|[H, D']\| < \infty$ *and* D' *normal, and* κ, ρ *with* (*) *and* (**)

$$\mathrm{Ind}\big(PFP + (\mathbf{1} - P)\big) = \tfrac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho})$$

Elements of proof

Definition 6.11

A fuzzy sphere (X_1, X_2, X_3) of width $\delta < 1$ in C*-algebra $\mathcal{K}$ is a collection of three self-adjoints in $\mathcal{K}^+$ with spectrum in $[-1, 1]$ and

$$\left\| \mathbf{1} - (X_1^2 + X_2^2 + X_3^2) \right\| < \delta \qquad \qquad \| [X_j, X_i] \| < \delta$$

Proposition 6.12

If $\delta \leqslant \frac{1}{4}$, one gets class $[L]_0 \in K_0(\mathcal{K})$ by self-adjoint invertible

$$L = \sum_{j=1,2,3} X_j \otimes \sigma_j \in M_2(\mathcal{K}^+)$$

Reason: L invertible and thus has positive spectral projection

Remark: odd-dimensional spheres give elements in $K_1(\mathcal{K})$

Proposition 6.13

$L_{\kappa,\rho}$ homotopic to $L = \sum_{j=1,2,3} X_j \otimes \sigma_j$ in invertibles

Construction of that particular fuzzy sphere:

Smooth tapering $f_\rho : \mathbb{R} \to [0,1]$ with $\text{supp}(f_\rho) \subset [-\rho, \rho]$ as above

Define $F_\rho : \mathbb{R} \to [0,1]$ by

$$F_\rho(x)^4 \;+\; f_\rho(x)^4 \;=\; 1$$

If $D' = D_1 + iD_2$ with $D_j^* = D_j$, and $R = |D|$, set

$$\begin{aligned} X_1 &= F_\rho(R)\, R^{-\frac{1}{2}}\, D_{1,\rho}\, R^{-\frac{1}{2}}\, F_\rho(R) \\ X_2 &= F_\rho(R)\, R^{-\frac{1}{2}}\, D_{2,\rho}\, R^{-\frac{1}{2}}\, F_\rho(R) \\ X_3 &= f_\rho(R)\, H_\rho\, f_\rho(R) \end{aligned}$$

Theorem 6.14

$$\mathrm{Ind}\,[\pi(P\,F\,P + \mathbf{1} - P)]_1 \;=\; [L_{\kappa,\rho}]_0$$

Proof:

General tool:

Image of K-theoretic index map can be written as fuzzy sphere

$$\mathrm{Ind}[\pi(A)]_1 \;=\; \Big[\sum_{j=1,2,3} Y_j \otimes \sigma_j\Big]_0$$

(by choosing an almost unitary lift A)

Formulas for Y_1, Y_2, Y_3 are explicit (but long)

General tool for $P\,F\,P\,+\mathbf{1}-P$ provides fuzzy sphere (Y_1, Y_2, Y_3)

Final step: find classical degree 1 map $M:\mathbb{S}^2\to\mathbb{S}^2$ such that

$$M(Y_1, Y_2, Y_3)\;\sim\;(X_1, X_2, X_3)$$

Numerics for toy model: $p + ip$ superconductor

Hamiltonian on $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ depending on μ and δ

$$H = \begin{pmatrix} S_1 + S_1^* + S_2 + S_2^* - \mu & \delta\big(S_1 - S_1^* + \imath(S_2 - S_2^*)\big) \\ \delta\big(S_1 - S_1^* + \imath(S_2 - S_2^*)\big)^* & -(S_1 + S_1^* + S_2 + S_2^* - \mu) \end{pmatrix} + \lambda\, V_{\text{dis}}$$

and disorder strength λ and i.i.d. uniformly distributed entries in

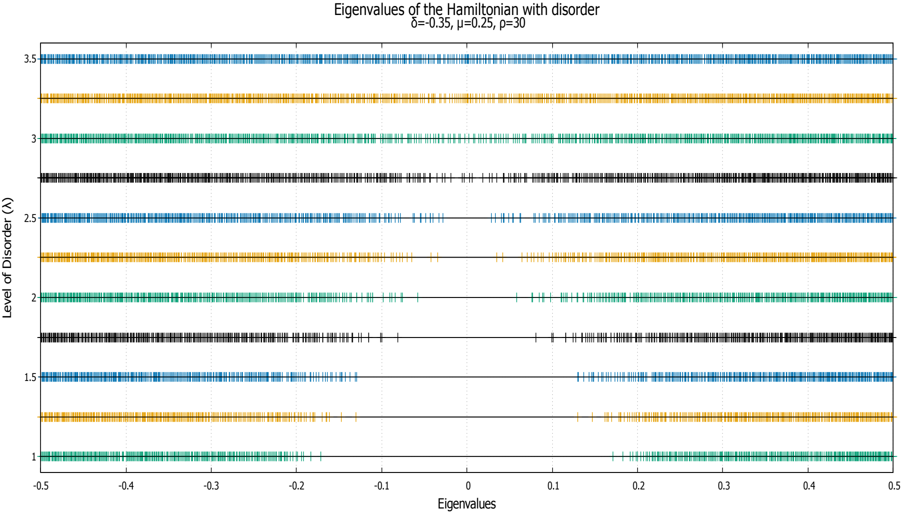
$$V_{\text{dis}} = \sum_{n \in \mathbb{Z}^2} \begin{pmatrix} v_{n,0} & 0 \\ 0 & -v_{n,1} \end{pmatrix} |n\rangle\langle n|$$

Build even spectral localizer from $D = X_1\sigma_1 + X_2\sigma_2 = -\sigma_3 D \sigma_3$:

$$L_{\kappa,\rho} = \begin{pmatrix} H_\rho & \kappa\,(X_1 + iX_2)_\rho \\ \kappa\,(X_1 - iX_2)_\rho & -H_\rho \end{pmatrix}$$

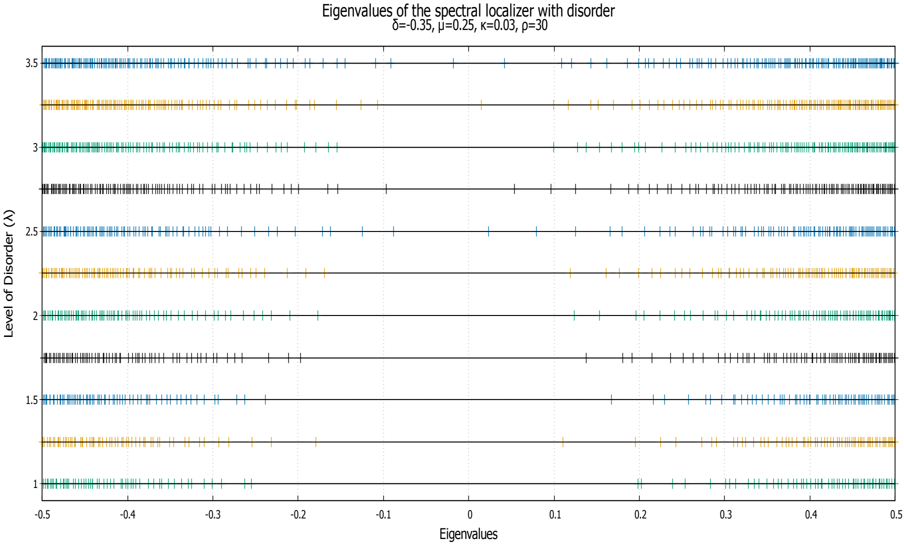
Calculation of signature by block Chualesky algorithm

Low-lying spectrum of Hamiltonian

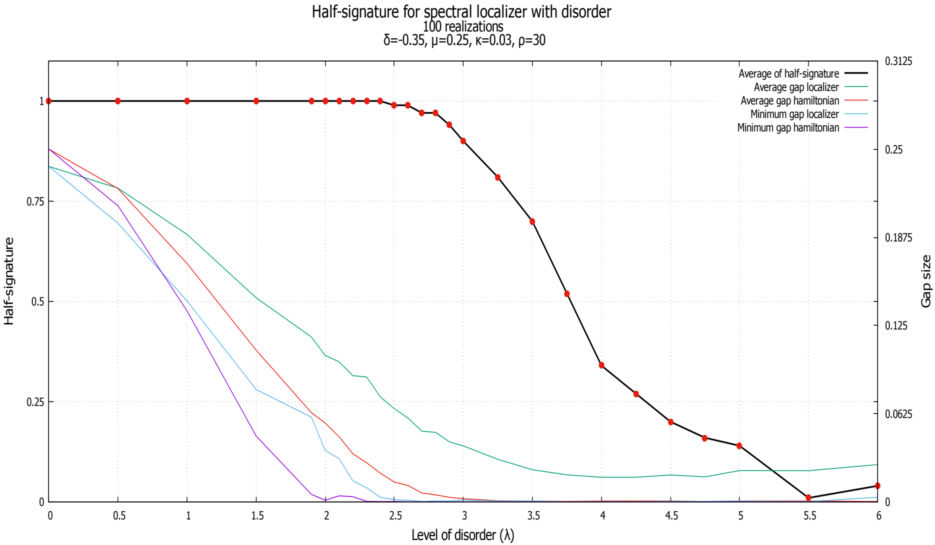


Gap of localizer open in dynamical localization regime with no gap of H

Low-lying spectrum of spectral localizer [LSS]



Half-signature and gaps for $p + ip$ superconductor



7 Invariants as response coefficients

- Hall conductance via Kubo formula: $\mathrm{Ch}_{\{i,j\}}$ with $i \neq j$
- polarization for periodically driven systems: $\mathrm{Ch}_{\{0,j\}}$ with 0 time
- orbital magnetization at zero temperature
- magneto-electric effect: $\mathrm{Ch}_{\{0,1,2,3\}}$ with 0 time
- chiral polarization: $\mathrm{Ch}_{\{j\}}$

Current operator $J = (J_1, \ldots, J_d)$ in d dimension:

$$J \;=\; \dot{X} \;=\; i\,[H, X] \;=\; \nabla H$$

Current density at equilibrium expressed by Fermi-Dirac state:

$$j_{\beta,\mu} \;=\; \mathcal{T}(f_{\beta,\mu}(H)\, J) \qquad , \qquad f_{\beta,\mu}(H) \;=\; (\mathbf{1} + e^{\beta(H-\mu)})^{-1}$$

Proposition 7.1 ([BES])

If $H = H^* \in C^1(\mathcal{A})$ and $f \in C_0(\mathbb{R})$, then $\mathcal{T}(f(H)\nabla H) = 0$

Proof: Leibniz implies $0 = \mathcal{T}(\nabla H^n) = n\mathcal{T}(H^{n-1}\nabla H)$ for all $n \geqslant 1$ □

Hence no current at equilibrium! Add external electric field $\mathcal{E} \in \mathbb{R}^d$

$$H_{\mathcal{E}} \;=\; H + \mathcal{E} \cdot X$$

Then $H_{\mathcal{E}}$ neither bounded nor homogeneous and thus not in $\mathcal{A}$

Nevertheless associated time evolution remains in the algebra $\mathcal{A}$

In the Schrödinger picture it is governed by the Liouville equation:

$$\partial_t \, \rho \;=\; -\,i\,[H_{\mathcal{E}}, \rho] \;=\; -\,i\,[H + \mathcal{E} \cdot X, \rho] \;=\; -\,\mathcal{L}_H(\rho) \;+\; \mathcal{E} \cdot \nabla(\rho)$$

Now Dyson series with Liouville $\mathcal{L}_H$ as perturbation is iteration of

$$e^{t\mathcal{L}_{H_{\mathcal{E}}}} \;=\; e^{t\mathcal{E}\cdot\nabla} \;+\; \int_0^t ds \; e^{(t-s)\mathcal{E}\cdot\nabla} \mathcal{L}_H e^{s\mathcal{L}_{H_{\mathcal{E}}}}$$

This shows:

Proposition 7.2

$\pm\mathcal{L}_H + \mathcal{E} \cdot \nabla$ are generators of automorphism groups in $\mathcal{A}$

Next time-averaged current under the dynamics with $\mathcal{E}$:

$$j_{\beta,\mu,\mathcal{E}} \;=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \mathcal{T}\big(f_{\beta,\mu}(H)\; e^{t\mathcal{L}_{H_{\mathcal{E}}}}(J)\big)$$

As trace $\mathcal{T}$ invariant under both ∇ and $\mathcal{L}_H$,

$$j_{\beta,\mu,\mathcal{E}} \;=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \mathcal{T}\big(J\, e^{-t\mathcal{L}_{H_{\mathcal{E}}}}(f_{\beta,\mu}(H))\,\big)$$

(Schrödinger picture $\Longleftrightarrow$ Heisenberg picture). Now

Proposition 7.3 (Bloch Oscillations)

Time-averaged current $j_{\beta,\mu,\mathcal{E}}$ along direction of $\mathcal{E}$ vanishes

Proof.

$$\mathcal{E}\cdot J(t) \;=\; e^{t\mathcal{L}_{H_{\mathcal{E}}}}(\mathcal{E}\cdot\nabla(H)) \;=\; e^{t\mathcal{L}_{H_{\mathcal{E}}}}(\mathcal{L}_{H_{\mathcal{E}}}(H)) \;=\; \frac{dH(t)}{dt}$$

Taking the time average gives us

$$\frac{1}{T}\int_0^T dt\,\mathcal{E}\cdot J(t) = \frac{H(T)-H}{T}$$

Since H bounded and $\|H(t)\| = \|H\|$, r.h.s. vanishes as $T\to\infty$ □

Modify dynamics by bounded linear collision term (like Boltzmann eq.):

$$\partial_t \rho + \mathcal{L}_H(\rho) - \mathcal{E} \cdot \nabla(\rho) \;=\; -\Gamma(\rho)$$

Main property is invariance of equilibrium: $\Gamma(f_{\beta,\mu}(H)) = 0$

Again Dyson series shows existence of dynamics:

$$\rho(t) \;=\; e^{-t(\mathcal{L}_H - \mathcal{E}\cdot\nabla + \Gamma)}(\rho(0))$$

Initial state chosen to be $\rho(0) = f_{\beta,\mu}(H)$

Exponential time-averaged current density shows:

$$\begin{aligned} j_{\beta,\mu,\mathcal{E}} \;&=\; \lim_{\delta\to 0}\; \delta \int_0^\infty dt\; e^{-\delta t}\; \mathcal{T}(J\rho(t)) \\ &=\; \lim_{\delta\to 0}\; \delta\, \mathcal{T}\left(J\; \frac{1}{\delta + \Gamma + \mathcal{L}_H - \mathcal{E}\cdot\nabla}(f_{\beta,\mu}(H))\right) \end{aligned}$$

By Proposition 7.1 and $(\mathcal{L}_H + \Gamma)(f_{\beta,\mu}(H)) = 0$ no current at equilibrium:

$$0 \;=\; \delta\,\mathcal{T}\left(J\,\frac{1}{\delta}\; f_{\beta,\mu}(H)\right) \;=\; \delta\,\mathcal{T}\left(J\,\frac{1}{\delta + \mathcal{L}_H + \Gamma}\,(f_{\beta,\mu}(H))\right)$$

Subtract this from $j_{\beta,\mu,\mathcal{E}}$ and use resolvent identity

$$j_{\beta,\mu,\mathcal{E}} \;=\; \lim_{\delta\to 0} \mathcal{T}\left(J\,\frac{1}{\delta+\Gamma+\mathcal{L}_H-\mathcal{E}\cdot\nabla}\,\mathcal{E}\cdot\nabla\,\frac{\delta}{\delta+\Gamma+\mathcal{L}_H}(f_{\beta,\mu}(H))\right)$$

Now, again $(\mathcal{L}_H+\Gamma)(f_{\beta,\mu}(H))=0$,

$$j_{\beta,\mu,\mathcal{E}} \;=\; \lim_{\delta\to 0} \sum_{j=1}^{d}\mathcal{E}_j\;\mathcal{T}\left(J\,\frac{1}{\delta+\Gamma+\mathcal{L}_H-\mathcal{E}\cdot\nabla}(\nabla_j f_{\beta,\mu}(H))\right)$$

This contains all non-linear terms in the electric field

Limit $\delta\to 0$ can be taken, if inverse exists

Linear coefficients of $j_{\beta,\mu,\mathcal{E}}$ in $\mathcal{E}$ give conductivity tensor

In **relaxation time approximation** (RTA) on replaces Γ by $\frac{1}{\tau}>0$

Theorem 7.4 (Kubo formula in RTA [BES])

$$\sigma_{i,j}(\beta,\mu,\tau) \;=\; \mathcal{T}\left(\nabla_i H\,\frac{1}{\frac{1}{\tau}+\mathcal{L}_H}(\nabla_j f_{\beta,\mu}(H))\right)$$

Hall conductance $i \neq j$ at zero temperature $\beta = \infty$ and $\tau = \infty$ exists

$$\sigma_{i,j}(\beta = \infty, \mu, \tau = \infty) \;=\; \mathcal{T}\Big((\mathcal{L}_H)^{-1} (\nabla_i H)\, \nabla_j P \Big)$$

where $P = \chi(H \leqslant \mu)$. As

$$\nabla_j P \;=\; P \nabla_j P (\mathbf{1} - P) \;+\; (\mathbf{1} - P) \nabla_j P P$$

and

$$(\mathcal{L}_H)^{-1} (P \,\nabla_j H\, (\mathbf{1} - P)) \;=\; -i\, P\, \nabla_j P\, (\mathbf{1} - P)$$

$$(\mathcal{L}_H)^{-1} ((\mathbf{1} - P)\, \nabla_j H\, P) \;=\; i\, (\mathbf{1} - P)\, \nabla_j P\, P$$

Hence

$$\sigma_{i,j}(\beta = \infty, \mu, \tau = \infty) \;=\; i\, \mathcal{T}\left(P [\nabla_i P, \nabla_j P] \right) \;=\; \frac{1}{2\pi}\, \mathrm{Ch}_{\{i,j\}}(P)$$

R.h.s. is integer-valued in dimension $d = 2$ and $d = 3$ (3D QHE)

This result holds also in a mobility gap regime [BES]

Electric polarization

$t \in [0, 2\pi) \cong \mathbb{S}^1 \mapsto H(t)$ periodic gapped Hamiltonian (changes dyn.)

Change ΔP in polarization is integrated induced current density:

$$\Delta P \;=\; \int_0^{2\pi} dt\, \mathcal{T}(\rho(t)\, J(t)) \qquad , \qquad \rho(0) \;=\; P_0 = \chi(H \leqslant \mu)$$

with $J(t) = i[H(t), X]$. Algebraic reformulation:

$$\Delta P \;=\; \int_0^{2\pi} dt\, \mathcal{T}\big(\rho(t)\,[\partial_t \rho(t), [X, \rho(t)]]\big)$$

However, $\rho(t)$ unknown. So adiabatic limit of slow time changes:

Theorem 7.5 (Kingsmith-Vanderbuilt and [ST])

$t \in \mathbb{S}^1 \mapsto H(t)$ smooth with gap open for all t

With $\rho(0) = P_0(0)$ and $\varepsilon\, \partial_t \rho(t) = \imath[\rho(t), H(t)]$, for any $N \in \mathbb{N}$

$$\Delta P \;=\; i \int_0^{2\pi} dt\, \mathcal{T}\big(P_0(t)\,[\partial_t P_0(t), [X, P_0(t)]]\big) + \mathcal{O}(\varepsilon^N)$$

Now add time to algebra: $C(\mathbb{S}^1, \mathcal{A}_d)$ is like $\mathcal{A}_{d+1}$

0th component is time and $\nabla_0 = \partial_t$

Also trace on $C(\mathbb{S}^1, \mathcal{A}_d)$ is $\frac{1}{2\pi} \int_0^{2\pi} dt\, \mathcal{T}$

Corollary 7.6

Polarization of periodically driven system is topological:

$$\Delta P_j \;=\; 2\pi \,\mathrm{Ch}_{\{0,j\}} + \mathcal{O}(\varepsilon^N)$$

For $d = 1, 2$ and $j = 1$, one hence has $\Delta P_1 \in 2\pi\,\mathbb{Z}$ up to $\mathcal{O}(\varepsilon^N)$

However, in $d = 3$ one does **not** have $\Delta P_j \in 2\pi\,\mathbb{Z}$, but due to generalized Streda formula, magneto-electric response satisfies

$$\alpha_{1,2,3} \;=\; \partial_{B_{2,3}} \Delta P_1 \;=\; 2\pi\,\mathrm{Ch}_{\{0,1,2,3\}} \;\in\; 2\pi\,\mathbb{Z}$$

Similarly: IDOS on gaps satisfies gap labelling

Chiral polarization

Chiral Hamiltonian $H = -\sigma_3 H \sigma_3$, typically due to sub-lattice symmetry

chiral polarization = difference between two electric dipole moments

$$P_{\mathrm{c}} \;=\; \mathbf{E}\,\mathrm{Tr}\,\langle 0|P\sigma_3\, X\, P|0\rangle \;=\; i\,\mathcal{T}(P\sigma_3 \nabla P)$$

due to $X|0\rangle = 0$. Let U be Fermi unitary of P

Proposition 7.7 ([PS])

$$P_{\mathrm{c},j} \;=\; -\tfrac{1}{2}\,\mathrm{Ch}_{\{j\}}(U) \qquad , \qquad j = 1, \ldots, d$$

Proof. Expressing P in terms of U

$$P_{\mathrm{c}} \;=\; \frac{i}{4}\,\mathcal{T}\left(\begin{pmatrix} \mathbf{1} & U^* \\ -U & -\mathbf{1} \end{pmatrix}\begin{pmatrix} 0 & -\nabla U^* \\ -\nabla U & 0 \end{pmatrix}\right) \;=\; \frac{i}{4}\,\mathcal{T}(-U^*\nabla U + U\nabla U^*)$$

Now use $U\nabla U^* = -(\nabla U)U^*$ and cyclicity $\square$

8 Bulk-boundary correspondence and applications

Toeplitz extension $\mathcal{T}(\mathcal{A}_d) = C^*(S_1^B, \ldots, S_{d-1}^B, \widehat{S}_d^B, W_\omega)$

$$\begin{array}{ccccccccc} & & \text{edge} & & \text{half-space} & & \text{bulk} & & \\ 0 & \to & \mathcal{E}_d & \to & \mathcal{T}(\mathcal{A}_d) & \to & \mathcal{A}_d & \to & 0 \end{array}$$

Moreover: $$\mathcal{E}_d \cong \mathcal{A}_{d-1} \otimes \mathcal{K}(\ell^2(\mathbb{N}))$$

$$\begin{array}{ccccc} K_0(\mathcal{A}_{d-1}) & \xrightarrow{i_*} & K_0(\mathcal{T}(\mathcal{A}_d)) & \xrightarrow{\pi_*} & K_0(\mathcal{A}_d) \\ \uparrow \text{Ind} & & & & \downarrow \text{Exp} \\ K_1(\mathcal{A}_d) & \xleftarrow{\pi_*} & K_1(\mathcal{T}(\mathcal{A}_d)) & \xleftarrow{i_*} & K_1(\mathcal{A}_{d-1}) \end{array}$$

Theorem 8.1 ([KRS, PS])

$$\mathrm{Ch}_{I\cup\{d\}}(A) \;=\; -\,\widehat{\mathrm{Ch}}_I(\mathrm{Ind}(A)) \qquad |I| \text{ even}\,,\; [A] \in K_1(\mathcal{A}_d)$$

$$\mathrm{Ch}_{I\cup\{d\}}(P) \;=\; \widehat{\mathrm{Ch}}_I(\mathrm{Exp}(P)) \qquad |I| \text{ odd}\,,\; [P] \in K_0(\mathcal{A}_d)$$

Here $\widehat{\mathrm{Ch}}_I = \mathrm{Tr} \otimes \mathrm{Ch}_I$ **Proof:** loooong **Example:** $d = 1$ as for SSH

Proof of BBC using *KK*-theory [BCR, BKR]

Bulk-boundary exact sequence $0 \to \mathcal{E}_d \to \mathcal{T}(\mathcal{A}_d) \to \mathcal{A}_d \to 0$ gives

$$[\mathrm{ext}] \;\in\; \mathrm{Ext}^{-1}(\mathcal{A}_d, \mathcal{E}_d) \;\cong\; KK^1(\mathcal{A}_d, \mathcal{E}_d)$$

(see Kasparov 1981). Further view, with $j = |I| \bmod 2$,

$$[\widehat{\mathrm{Ch}}_I] \;\in\; KK^j(\mathcal{E}_d, \mathbb{C}) \qquad , \qquad [\mathrm{Ch}_{I\cup\{d\}}] \;\in\; KK^{j+1}(\mathcal{A}_d, \mathbb{C})$$

Theorem 8.2 ([BKR])

For Kasparov product $KK^1(\mathcal{A}_d, \mathcal{E}_d) \times KK^j(\mathcal{E}_d, \mathbb{C}) \to KK^{j+1}(\mathcal{A}_d, \mathbb{C})$

$$[\mathrm{ext}]\, \hat{\otimes}_{\mathcal{E}_d}\, [\widehat{\mathrm{Ch}}_I] \;=\; (-1)^d [\mathrm{Ch}_{I\cup\{d\}}]$$

For d even and $|I| = d-1$, let $[P]_0 \in K_0(\mathcal{A}_d) = KK^0(\mathbb{C}, \mathcal{A}_d)$. *Thus*

$$\begin{aligned}
\widehat{\mathrm{Ch}}_I(\mathrm{Exp}(P)) \;&=\; [\mathrm{Exp}(P)]_1\, \hat{\otimes}_{\mathcal{E}_d}\, [\widehat{\mathrm{Ch}}_I] \\
&=\; [P]_0\, \hat{\otimes}_{\mathcal{A}_d}\, [\mathrm{ext}]\, \hat{\otimes}_{\mathcal{E}_d}\, [\widehat{\mathrm{Ch}}_I] \\
&=\; [P]_0\, \hat{\otimes}_{\mathcal{A}_d}\, [\mathrm{Ch}_{I\cup\{d\}}] \;=\; \mathrm{Ch}_{I\cup\{d\}}(P)
\end{aligned}$$

Boundary maps in terms of Hamiltonians

Theorem 8.3 ([KRS, PS])

Let $H \in M_L(\mathcal{A}_d)$ with gap $\Delta \ni \mu$ and $P = \chi(H \leqslant \mu) \in M_L(\mathcal{A}_d)$
With continuous $g(E) = 1$ for $E < \Delta$ and $g(E) = 0$ for $E > \Delta$:

$$\mathrm{Exp}([P]_0) \;=\; [\exp(-2\pi\imath\, g(\widehat{H}))]_1 \;\in\; K_1(\mathcal{E}_d)$$

Proof: $g(\widehat{H}) \in \mathcal{T}(\mathcal{A}_d)$ is a selfadjoint lift of P □

Theorem 8.4 ([PS])

Let $H \in M_{2L}(\mathcal{A}_d)$ chiral with gap $\Delta \ni 0$ and Fermi unitary $U \in M_L(\mathcal{A}_d)$
With odd continuous $f(E) = -1$ for $E < \Delta$ and $f(E) = 1$ for $E > \Delta$:

$$\mathrm{Ind}([U]_1) \;=\; [e^{-\imath\frac{\pi}{2}f(\widehat{H})}\mathrm{diag}(\mathbf{1},0)e^{\imath\frac{\pi}{2}f(\widehat{H})}]_0 - [\mathrm{diag}(\mathbf{1},0)]_0 \;\in\; K_0(\mathcal{E}_d)$$

If central band of edge states gapped with projection $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$,

$$\mathrm{Ind}([U]_1) \;=\; [\widehat{P}_+]_0 - [\widehat{P}_-]_0 \;\in\; K_0(\mathcal{E}_d)$$

Physical implication in $d = 2$: QHE

P Fermi projection below a bulk gap $\Delta \subset \mathbb{R}$. Kubo formula:

$$\text{Hall conductance} \;=\; \text{Ch}_{\{1,2\}}(P)$$

Bulk-boundary:

$$\text{Ch}_{\{1,2\}}(P) \;=\; \text{Ch}_{\{1\}}(\text{Exp}(P)) \;=\; \text{Wind}(\text{Exp}(P))$$

With continuous $g(E) = 1$ for $E < \Delta$ and $g(E) = 0$ for $E > \Delta$:

$$\text{Exp}(P) \;=\; \exp(-2\pi i\, g(\widehat{H})) \in \mathcal{T}(\mathcal{A}_2)$$

as indeed $\pi(g(\widehat{H})) = g(H) = P$ so that $\pi(\text{Exp}(P)) = \mathbf{1}$ trivial

Theorem 8.5 (Quantization of boundary currents [KRS, PS])

$$\text{Ch}_{\{1,2\}}(P) \;=\; \mathbb{E} \sum_{n_2 \geqslant 0} \langle 0, n_2 | g'(\widehat{H}) i[X_1, \widehat{H}] | 0, n_2 \rangle$$

The r.h.s. is current density flowing along the boundary

Proof: With $\widehat{\mathcal{T}}(A) = \mathcal{T}_1\,\mathrm{Tr}_2(A) = \mathbf{E}_{\mathbb{P}} \sum_{n_2 \geqslant 0} \langle 0, n_2 | \widehat{A}_\omega | 0, n_2 \rangle$, r.h.s. is

$$j^{\mathrm{e}}(g) \;=\; \mathbb{E} \sum_{n_2 \geqslant 0} \langle 0, n_2 | g'(\widehat{H}) i[X_1, \widehat{H}] | 0, n_2 \rangle \;=\; \widehat{\mathcal{T}}\big(\widehat{J}_1\, g'(\widehat{H})\big)$$

Summability in n_2 has to be checked

Let $\Pi : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z} \times \mathbb{N})$ surjective partial isometry,

namely $\Pi\Pi^*$ identity on $\ell^2(\mathbb{Z} \times \mathbb{N})$

Then $\widehat{H} = \Pi H \Pi^*$

Proposition 8.6

For $G \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(G) \cap \sigma(H) = \emptyset$

Then the operator $G(\widehat{H})$ is $\widehat{\mathcal{T}}$-traceclass

Proof based on functional calculus often attributed to Helffer-Sjorstrand

Proposition 8.7 (Functional calculus à la Dynkin 1972)

$\chi \in C_0^\infty((-1,1),[0,1])$ even and equal to 1 on $[-\delta,\delta]$

For $N \geqslant 1$ let quasi-analytic extension $\widetilde{G} : \mathbb{C} \to \mathbb{C}$ of G by

$$\widetilde{G}(x,y) \;=\; \sum_{n=0,\ldots,N} G^{(n)}(x)\,\frac{(iy)^n}{n!}\,\chi(y) \qquad , \qquad z = x + iy$$

Then with norm-convergent Riemann sum

$$G(H) \;=\; \frac{-1}{2\pi}\int_{\mathbb{R}^2} dx\,dy\;\partial_{\overline{z}}\widetilde{G}(x,y)\;(z-H)^{-1}$$

Proof. Crucial identity is

$$\partial_{\overline{z}}\widetilde{G}(x,y) \;=\; G^{(N+1)}(x)\,\frac{(iy)^N}{N!}\,\chi(y) + i\sum_{n=0,\ldots,N} G^{(n)}(x)\,\frac{(iy)^n}{n!}\,\chi'(y)$$

In particular, uniformly in x, y, one has $|\partial_{\overline{z}}\widetilde{G}(x,y)| \leqslant C\,|y|^N$

Hence also $\partial_{\overline{z}}\widetilde{G}(x,0) = 0$. Now resolvent bound. Details.... □

Proof of Proposition 8.6. Geometric resolvent identity

$$\frac{1}{z-\widehat{H}} \;=\; \Pi\,\frac{1}{z-H}\,\Pi^* \;+\; \frac{1}{z-\widehat{H}}\,(\widehat{H}\,\Pi^* - \Pi\,H)\,\frac{1}{z-H}\,\Pi^*$$

in Dynkin for $G(\widehat{H})$ together with $G(H)=0$ leads to

$$\begin{aligned} G(\widehat{H}) \;&=\; \Pi\,G(H)\,\Pi^* \;+\; \widehat{K} \\ &=\; \frac{-1}{2\pi}\int_{\mathbb{R}^2} dx\,dy\;\partial_{\overline{z}}\widetilde{G}(x,y)\;\frac{1}{z-\widehat{H}}\,(\widehat{H}\,\Pi^* - \Pi\,H)\,\frac{1}{z-H}\,\Pi^* \end{aligned}$$

Resolvents have fall-off of their matrix elements off the diagonal:

$$(n_j-m_j)^k\langle n|(z-H)^{-1}|m\rangle \;=\; i^k\,\langle n|\nabla_j^k(z-H)^{-1}|m\rangle \qquad , \qquad k\in\mathbb{N}$$

Expand $\nabla^k(z-H)^{-1}$ by Leibniz rule. As $\|\nabla^k H\|\leqslant C$

$$|\langle n|(z-H)^{-1}|m\rangle| \;\leqslant\; \frac{1}{|y|^{k+1}}\,\frac{C_k}{1+|n_j-m_j|^k}$$

Same bound holds for resolvent of $\widehat{H}$ (improvement: Combes-Thomas)

If finite range, $\widehat{H}\Pi^* - \Pi H$ has matrix elements only on boundary. Then

$$
\begin{aligned}
&|\langle 0, n_2|\widehat{K}|0, n_2\rangle| \\
&\quad \leqslant \sum_{m\in\mathbb{Z}\times\mathbb{N}} \sum_{k\in\mathbb{Z}^2} \frac{1}{2\pi} \int_{\mathbb{R}^2} dx\, dy\, |\partial_{\overline{z}}\widetilde{G}(x,y)|\, |\langle 0, n_2|(z-H)^{-1}|m\rangle| \\
&\qquad\qquad\qquad |\langle m|\widehat{H}\,\Pi^* - \Pi\, H|k\rangle|\, |\langle k|(z-H)^{-1}|0, n_2\rangle| \\
&\quad \leqslant C \sum_{m_1\geqslant 0} \int_{\mathbb{R}^2} dx\, dy\, |\partial_{\overline{z}}\widetilde{G}(x,y)|\, \frac{1}{|y|^{2k+2}}\, \frac{1}{1+|n_2|^{2k}}\, \frac{1}{1+|m_1|^{2k}}
\end{aligned}
$$

Now above bound on resolvent for $N \geqslant 2k+2$

As integral over bounded region, sum can be carried out

$$
|\langle 0, n_2|\widehat{K}|0, n_2\rangle| \;\leqslant\; \frac{C}{1+|n_2|^{2k}}
$$

But this implies desired $\widehat{\mathcal{T}}$-traceclass estimate $\square$

Proof of Theorem 8.5. Set $\widehat{U} = \mathrm{Exp}(P) = \exp(-2\pi i\, g(\widehat{H}))$ and

$$\mathrm{Ind} \;=\; i\,\widehat{\mathcal{T}}\big((\widehat{U}^* - \mathbf{1})\nabla_1 \widehat{U}\big)$$

Express $\widehat{U}$ as exponential series and use Leibniz rule:

$$\mathrm{Ind} \;=\; \sum_{m=0}^{\infty} \frac{(2\pi i)^m}{m!} \sum_{l=0}^{m-1} \widehat{\mathcal{T}}\Big((\widehat{U}^* - \mathbf{1})\, g(\widehat{H})^l\, \nabla_1 g(\widehat{H})\, g(\widehat{H})^{m-l-1}\Big)$$

where trace and sum exchange by $\widehat{\mathcal{T}}$-traceclass property of $\widehat{U} - \mathbf{1}$

Due to cyclicity and $[\widehat{U}, g(\widehat{H})] = 0$, each summand equal to

$$\widehat{\mathcal{T}}((\widehat{U}^* - 1)\, g(\widehat{H})^{m-1}\, \nabla_1 g(\widehat{H}))$$

Exchanging sum and trace, summing up again:

$$\mathrm{Ind} \;=\; -2\pi\, \widehat{\mathcal{T}}\Big((\mathbf{1} - \widehat{U})\, \nabla_1 g(\widehat{H})\Big)$$

Now same argument for $\widehat{U}^k = \exp(-2\pi i\, k\, g(\widehat{H}))$ for $k \neq 0$,

$$\mathrm{Ind} \;=\; \frac{i}{k}\, \widehat{\mathcal{T}}\big((\widehat{U}^k - \mathbf{1})^* \nabla_1 \widehat{U}^k\big) \;=\; -2\pi\, \widehat{\mathcal{T}}\Big((\mathbf{1} - \widehat{U}^k)\, \nabla_1 g(\widehat{H})\Big)$$

Writing $g(E) = \int dt\, \tilde{g}(t)\, e^{-E(1+it)}$ with adequate $\tilde{g}$, by DuHamel

$$\mathrm{Ind} = 2\pi \int dt\, \tilde{g}(t)\, (1+it) \int_0^1 dq\, \widehat{\mathcal{T}} \left((\widehat{U}^k - \mathbf{1})\, e^{-(1-q)(1+it)\widehat{H}} (\nabla_1 \widehat{H}) e^{-q(1+it)\widehat{H}} \right)$$

With $g'(E) = -\int dt\, (1+it)\, \tilde{g}(t)\, e^{-E(1+it)}$ for $k \neq 0$,

$$\mathrm{Ind} \;=\; 2\pi\, \widehat{\mathcal{T}} \left((\widehat{U}^k - \mathbf{1})\, g'(\widehat{H})\, \nabla_1 \widehat{H} \right)$$

For $k = 0$, the r.h.s. vanishes. To conclude, let $\phi \in C_0^\infty((0,1),\mathbb{R})$

Fourier coefficients $a_k = \int_0^1 dx\, e^{-2\pi ikx} \phi(x)$ satsify $\sum_k a_k e^{2\pi ikx} = \phi(x)$

In particular, $\sum_k a_k = 0$ and

$$\begin{aligned} a_0\, \mathrm{Ind} \;=\; -\sum_{k\neq 0} a_k\, \mathrm{Ind} \quad &= \quad 2\pi \sum_k a_k\, \widehat{\mathcal{T}} \left((\mathbf{1} - \widehat{U}^k)\, g'(\widehat{H})\, \nabla_1 \widehat{H} \right) \\ &= \quad 2\pi\, \widehat{\mathcal{T}} \big((0 - \phi(g(\widehat{H})))\, g'(\widehat{H})\, \nabla_1 \widehat{H} \big) \end{aligned}$$

As $\phi \to \chi_{[0,1]}$ also $a_0 \to 1$ and $\phi(g(\widehat{H}))g'(\widehat{H}) \to g'(\widehat{H})$ (no Gibbs)

As $J_1 = \nabla_1 \widehat{H}$ proof is concluded □

Chiral system in $d = 3$: anomalous surface QHE

Chiral Fermi projection P (off-diagonal) $\Longrightarrow$ Fermi unitary A

$$\mathrm{Ch}_{\{1,2,3\}}(A) \;=\; \mathrm{Ch}_{\{1,2\}}(\mathrm{Ind}(A))$$

Magnetic field perpendicular to surface opens gap in surface spec.

With $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$ projection on central surface band, as in SSH:

$$\mathrm{Ind}(A) \;=\; [\widehat{P}_+] - [\widehat{P}_-]$$

Theorem 8.8 ([PS])

Suppose either $\widehat{P}_+ = 0$ *or* $\widehat{P}_- = 0$ (conjectured to hold). *Then:*

$\mathrm{Ch}_{\{1,2,3\}}(A) \neq 0 \Longrightarrow$ *surface QHE, Hall cond. imposed by bulk*

Actually only approximate chiral symmetry needed

Experiment? No (approximate) chiral topological material known

Delocalization of boundary states

Hypothesis: bulk gap at Fermi level μ

Disorder: in arbitrary finite strip along boundary hypersurface

Theorem 8.9 ([PS])

For even d, if strong invariant $\mathrm{Ch}_{\{1,\dots,d\}}(P) \neq 0$,
then no Anderson localization of boundary states in bulk gap
Technically: Aizenman-Molcanov bound for no energy in bulk gap

Theorem 8.10 ([PS])

For odd $d \geqslant 3$, *if strong invariant* $\mathrm{Ch}_{\{1,\dots,d\}}(A) \neq 0$,
then no Anderson localization at $\mu = 0$

BBC for continuously periodically driven systems

BBC in time direction: stroboscopics   Here: BBC in spacial direction

Lift $t \in \mathbb{S}^1 \cong [0, 2\pi) \mapsto \widehat{H}(t)$ of continuous gapped $t \in \mathbb{S}^1 \mapsto H(t)$ in

$$0 \longrightarrow C(\mathbb{S}^1, \mathcal{E}_d) \overset{i}{\longrightarrow} C(\mathbb{S}^1, \widehat{\mathcal{A}}_d) \overset{\mathrm{ev}}{\longrightarrow} C(\mathbb{S}^1, \mathcal{A}_d) \longrightarrow 0$$

Then for polarization in direction d with adiabatic projection P_A:

$$\Delta P_d \;=\; 2\pi\,\mathrm{Ch}_{\{0,d\}}(P_A) \;=\; 2\pi\,\mathrm{Ch}_{\{0\}}(U_\Delta)$$

where 0-th component still time and $[U_\Delta]_1 = \mathrm{Exp}[P_A]_0$. Now

$$\mathrm{Ch}_{\{0\}}(U_\Delta) \;=\; -\,2\pi \int_0^{2\pi} dt\; \widehat{\mathcal{T}}\Big(g'\big(\widehat{H}(t)\big)\,\partial_t \widehat{H}(t)\Big)$$

For $d = 1$, this is 2π times spectral flow of boundary eigenvalues. Thus

$$\Delta P_1 \;=\; -\,2\pi\;\mathrm{SF}\big(t \in \mathbb{S}^1 \mapsto \widehat{H}(t) \text{ by } \mu\big)$$

namely charge pumped from valence to conduction states

For $d > 1$, spectral flow is in sense of Breuer-Fredholm operators

Application to topological Floquet systems

Given $t \mapsto H(t) = H(t)^* \in \mathcal{A}_d$ piecewise continuous 2π-periodic family

Differentiable path of unitaries $t \mapsto U(t) \in \mathcal{A}_d$ from

$$i\,\partial_t U(t) \;=\; H(t)\,U(t) \qquad , \qquad U(0) \;=\; \mathbf{1}$$

Evolution $U = U(2\pi)$ over period 2π called Floquet operator

Suppose $e^{\imath\theta} \notin \sigma(U)$ quasi-energy spectrum for $\theta \in [0, 2\pi)$ and set

$$h_\theta \;=\; -\,(2\pi i)^{-1}\;\log_\theta(U)$$

Here $\log_\theta$ natural logarithm with branch cut along $r \in [0, \infty) \mapsto re^{\imath\theta}$

By construction, $U = e^{-2\pi i h_\theta}$. Set

$$H_\theta(t) \;=\; \begin{cases} 2\,H(2t)\,, & t \in [0, \pi] \\ -\,2\,h_\theta\,, & t \in (\pi, 2\pi] \end{cases}$$

Now periodized time evolution V_θ with $V_\theta(0) = V_\theta(2\pi) = \mathbf{1}$

$$i\,\partial_t V_\theta(t) \;=\; H_\theta(t)\,V_\theta(t)\,, \qquad V_\theta(0) \;=\; \mathbf{1}$$

Invariants and BBC

There are new bulk invariants involving the time $t = x_0$, *e.g.* strong inv.

$$\mathrm{Ch}_{\{0,1,\dots,d\}}(V_\theta)$$

Consider now boundary evolution:

$$i\,\partial_t \widehat{U}(t) \;=\; \widehat{H}(t)\,\widehat{U}(t) \qquad , \qquad \widehat{U}(0) \;=\; \widehat{\mathbf{1}}$$

Floquet operator $\widehat{U} = \widehat{U}(2\pi) \in \mathcal{T}(\mathcal{A}_d)$ is unitary lift of U

Theorem 8.11 ([SS])

Let $e^{\imath\theta} \notin \sigma(U)$ and $e^{\imath\theta'}$ not in the same gap as $e^{\imath\theta}$
$g_\theta : \mathbb{S}^1 \to [0,1]$ smooth increasing with jump down by 1 at $e^{\imath\theta'}$

$$\Theta^{-1}(\mathrm{Ind}([V_\theta]_1)) \;=\; [e^{-2\pi i\, g_\theta(\widehat{U})}]_1$$

If $d = 2$ reformulation as counting of edge channels

9 Implementation of symmetries

This invokes real structure simply denoted by bar on $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$

$$\begin{aligned} &\text{chiral symmetry (CHS)}: && J_{\rm ch}^* \, H \, J_{\rm ch} \;=\; -H \\ &\text{time reversal symmetry (TRS)}: && S_{\rm tr}^* \, \overline{H} \, S_{\rm tr} \;=\; H \\ &\text{particle-hole symmetry (PHS)}: && S_{\rm ph}^* \, \overline{H} \, S_{\rm ph} \;=\; -H \end{aligned}$$

$S_{\rm tr} = e^{i\pi s^y}$ orthogonal on $\mathbb{C}^{2s+1}$ with $S_{\rm tr}^2 = \pm\mathbf{1}$ even or odd

$S_{\rm ph}$ orthogonal on $\mathbb{C}^2_{\rm ph}$ with $S_{\rm ph}^2 = \pm\mathbf{1}$ even or odd

Note: TRS + PHS $\implies$ CHS with $J_{\rm ch} = S_{\rm tr} S_{\rm ph}$

10 combinations of symmetries: none (1), one (5), three (4)

10 Cartan-Altland-Zirnbauer classes (CAZ): 2 complex, 8 real

Further distinction in each of the 10 classes: topological insulators

Periodic table of topological insulators

Schnyder-Ryu-Furusaki-Ludwig,Kitaev 2008: just strong invariants

$j\backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
1	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Periodic table: real classes only

64 pairings = 8 KR-cycles paired with 8 KR-groups

$j\backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Focus on system in $d = 2$ with odd TRS $S = S_{\mathrm{tr}}$:

$$S^2 = -\mathbf{1} \qquad S^*\overline{H}S = H$$

$\mathbb{Z}_2$ index for odd TRS and $d = 2$

Rewrite $S^*\overline{H}S = H = S^* H^t S$ with $H^t = (\overline{H})^*$

$\implies S^*(H^n)^t S = H^n$ for $n \in \mathbb{N} \implies S^* P^t S = P$

For $d = 2$, Dirac phase $F = \frac{X_1 + iX_2}{|X_1 + iX_2|} = F^t$ and $[S, F] = 0$

Hence Fredholm operator $T = PFP$ of following type

Definition T odd symmetric $\iff S^* T^t S = T \iff (TS)^t = -TS$

Theorem 9.1 (Atiyah-Singer 1969)

$\mathbb{F}_2(\mathcal{H}) = \{$*odd symmetric Fredholm operators*$\}$ *has 2 connected components labelled by compactly stable homotopy invariant*

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2 \in \mathbb{Z}_2$$

Application: $\mathbb{Z}_2$ phase label for Kane-Mele model if dyn. localized

Existence proof of $\mathbb{Z}_2$-indices via Kramers arg.

First of all: $\mathrm{Ind}(T) = 0$ because $\mathrm{Ker}(T^*) = S\,\overline{\mathrm{Ker}(T)}$

Idea: $\mathrm{Ker}(T) = \mathrm{Ker}(T^*T)$

and positive eigenvalues of T^*T have even multiplicity

Let $T^*Tv = \lambda v$ and $w = S\,\overline{Tv}$ (N.B. $\lambda \neq 0$). Then

$$
\begin{aligned}
T^*T\,w &= S\,(S^*T^*S)\,(S^*TS)\,\overline{Tv} \\
&= S\,\overline{T}\,\overline{T^*\,T\,v} = \lambda\,S\,\overline{T}\,\overline{v} = \lambda\,w\,.
\end{aligned}
$$

Suppose now $\mu \in \mathbb{C}$ with $v = \mu\,w$. Then

$$
v = \mu\,S\,\overline{T}\,\overline{v} = \mu\,S\,\overline{T}\,\overline{\mu}\,S\,T\,v = -|\mu|^2\,T^*T\,v = -|\mu|^2\,\lambda\,v
$$

Contradiction to $v \neq 0$.

Now $\mathrm{span}\{v, w\}$ is invariant subspace of T^*T.

Go on to orthogonal complement

Symmetries of the Dirac operator

$$D \;=\; \sum_{j=1}^{d} X_j \otimes \mathbf{1} \otimes \gamma_j$$

$\gamma_1, \ldots, \gamma_d$ irrep of C_d with $\gamma_{2j} = -\overline{\gamma_{2j}}$ and $\gamma_{2j+1} = \overline{\gamma_{2j+1}}$

In even d exists grading $\Gamma = \Gamma^*$ with $D = -\Gamma D \Gamma$ and $\Gamma^2 = \mathbf{1}$

Moreover, exists real unitary Σ (essentially unique) with

$d = 8 - i$	8	7	6	5	4	3	2	1
Σ^2	$\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\Sigma^* \overline{D} \Sigma$	D	$-D$	D	D	D	$-D$	D	D
$\Gamma \Sigma \Gamma$	Σ		$-\Sigma$		Σ		$-\Sigma$	

(D, Γ, Σ) defines a KR^i-cycle (spectral triple with real structure)

(Kasparov 1981, Connes 1995, Gracia-Varilly-Figueroa 2000)

Index theorems for periodic table

Symmetries of *KR*-cycles **and** Fermi projection/unitary lead to:

Theorem 9.2

Index theorems for all strong invariants in periodic table

Remarks:

Result holds also in the regime of strong Anderson localization

$2\,\mathbb{Z}$ entries result from quaternionic Fredholm (even Ker, CoKer)

Links to Atiyah-Singer classifying spaces

Formulation as Clifford valued index theorem possible

Physical implications: case by case study necessary!

Example: focus on TRS $d = 2$ quantum spin Hall system (QSH)

Spin Chern numbers [Pro]

Approximate spin conservation $\Longrightarrow$ spin Chern numbers $\mathrm{SCh}(P)$

Kane-Mele Hamiltonian has small commutator $[H, s_z]$

Also $[P, s_z]$ small and thus $Ps_zP|_{\mathrm{Ran}(P)}$ spectrum close to $\{-1, 1\}$

$\Longrightarrow$ spectral gap! Let $P_\pm$ be two associated spectral projections

Proposition 9.3 ([Pro])

$P_\pm$ have off-diagonal decay so that Chern numbers can be defined

Hence $P = P_+ + P_-$ decomposes in two *smooth* projections

Definition 9.4

Spin Chern number of P is $\mathrm{SCh}(P) = \mathrm{Ch}(P_+)$

By TRS, $\mathrm{Ch}(P) = 0$ and thus $\mathrm{SCh}(P) = -\mathrm{Ch}(P_-)$

Theorem 9.5 ([SB1])

$\mathrm{Ind}_2(PFP) = \mathrm{SCh}(P) \bmod 2$

Spin filtered helical edge channels for QSH

Remarkable: Non-trivial topology $\mathrm{SCh}(P)$ persists TRS breaking!

General strategy: approximately conserved quantities lead to integer-valued invariants which persist breaking of real symmetry

Further example:

Kitaev chain (Class D with $\mathbb{Z}_2$-invariant) has a winding number

Theorem 9.6

If $\mathrm{SCh}(P) \neq 0$, spin filtered edge currents in $\Delta \subset$ gap are stable w.r.t. perturbations by magnetic field and disorder:

$$\mathbf{E}\,\mathrm{Tr}\,\langle 0|\chi_\Delta(\widehat{H})\,\tfrac{1}{2}\{i[\widehat{H},X_1],s_z\}|0\rangle \;=\; |\Delta|\;\mathrm{SCh}(P)\;+\;\textit{correct.}$$

Resumé: $\mathrm{Ind}_2(PFP) = 1 \Longrightarrow$ no Anderson loc. for edge states

Rice group of Du (since 2011): QSH stable w.r.t. magnetic field

10 Spectral flow in topological insulators

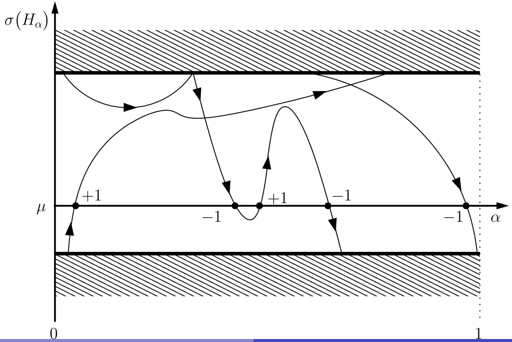
Theorem 10.1 (Laughlin 1983, Avron, Punelli 1992, Macris, [DS])

H disordered Harper-like operator on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^L$ with $\mu \in$ gap

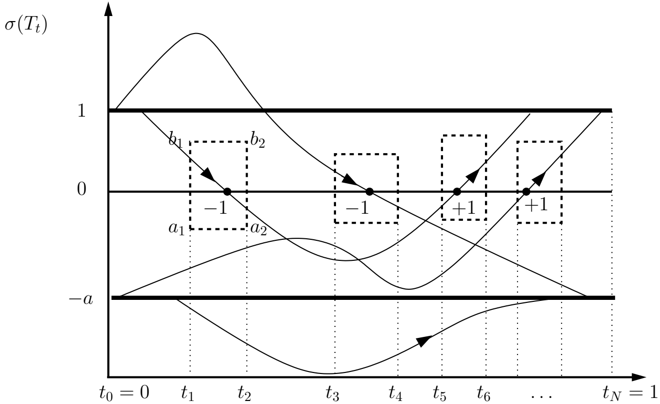
H_α Hamiltonian with extra flux $\alpha \in [0,1]$ through 1 cell of $\mathbb{Z}^2$

Then for $P = \chi(H \leqslant \mu)$

$$\mathrm{SF}\Big(\alpha \in [0,1] \mapsto H_\alpha \ \text{ through } \ \mu\Big) \;=\; -\,\mathrm{Ch}_{\{1,2\}}(P)$$



Phillips' analytic definition (1996)



$\exists$ finite partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ of $[0, 1]$ and $a_n < 0 < b_n$ with $t \in [t_{n-1}, t_n] \mapsto \chi(T_t \in [a_n, b_n])$ continuous. Set:

$$\mathrm{SF}(t \in [0,1] \mapsto T_t) \;=\; \sum_{n=1}^{N} \mathrm{Tr}_{\mathcal{H}} \left(\chi(T_{t_{n-1}} \in [a_n, 0]) \;-\; \chi(T_{t_n} \in [a_n, 0])\right)$$

Theorem 10.2 (Phillips 1996)

$\mathrm{SF}(t \in [0,1] \mapsto T_t)$ *independent of partition and* $a_n < 0 < b_n$*.*

It is a homotopy invariant when end points are kept fixed.

It satisfies concatenation and normalization:

$\mathrm{SF}(t \in [0,1] \mapsto T + (1-2t)P) = -\dim(P)$ *for* $TP = P$

Theorem 10.3 (Lesch 2004)

Homotopy invariance, concatenation, normalization characterize SF

Theorem 10.4 (Perera 1993, Phillips 1996)

SF *on loops establishes isomorphism* $\pi_1(\mathbb{F}^*_{\mathrm{sa}}) = \mathbb{Z}$

Theorem 10.5 (Phillips 1996)

0 gap of $H = H^*$ and $P = \chi(H \leqslant 0)$. If $t \in [0,1] \mapsto H_t = H_t^*$ with

(i) *$H_1 = UH_0U^*$ for unitary U*

(ii) *0 in essential gap of H_t for all $t \in [0,1]$*

then

$$\mathrm{SF}\Big(t \in [0,1] \mapsto H_t \;\text{ through }\; 0\Big) \;=\; -\,\mathrm{Ind}(PUP)$$

Exact sequence interpretation: Mapping cone associated to U:

$$\mathcal{M} \;=\; \{t \in [0,1] \mapsto A_t \in \mathcal{A} + \mathcal{K} \;:\; A_0 = U^* A_1 U,\; A_t - A_0 \in \mathcal{K}\,\}$$

with $0 \to S\mathcal{K} \hookrightarrow \mathcal{M} \overset{\text{ev}}{\to} \mathcal{A} \to 0$. Now $K_1(S\mathcal{K}) = K_0(\mathcal{K}) = \mathbb{Z}$ and

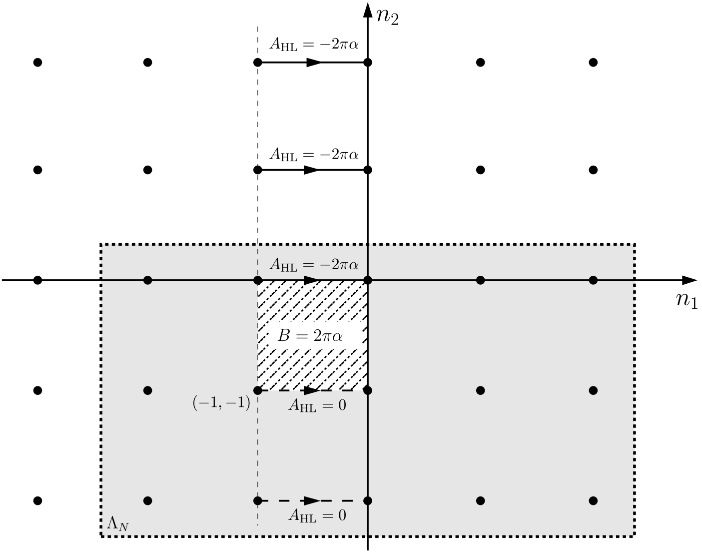
$$\mathrm{Exp}[P]_0 \;=\; [\exp(2\pi i\,\mathrm{Lift}(P)_t)]_1 \;=\; [\exp(2\pi i(P + t\,U^*[P,U]))]_1$$

Then for pairing with odd Fredholm module $(\mathcal{H}, U)$

$$\langle(\mathcal{H},U)\,,\,[P]_0\rangle \;=\; \langle(\int dt \otimes \mathrm{Tr}, \partial_t)\,,\,\mathrm{Exp}[P]_0\rangle \;=\; \mathrm{SF}(2P - \mathbf{1} + t\,U^*[2P - \mathbf{1}, U])$$

Proof of bulk-boundary in $d = 2$ (idea Macris 2002)

Based on gauge invariance and compact stability



Exact sequence behind the Laughlin argument

Theorem 10.6

With $\mathcal{E}(\mathcal{A}_2) = C^*(S_1^B, S_2^B, P_0 = |0\rangle\langle 0|)$*, split exact sequence*

$$0 \longrightarrow \mathcal{K} \overset{i}{\hookrightarrow} \mathcal{E}(\mathcal{A}_2) \underset{\underset{j}{\hookleftarrow}}{\overset{\pi}{\to}} \mathcal{A}_2 \longrightarrow 0$$

Moreover, $\mathcal{E}(\mathcal{A}_2) = C^*(S_1^{B,\alpha}, S_2^{B,\alpha})$ *for* $\alpha \in \mathbb{R}\backslash\mathbb{Z}$ *where* $S_j^{B,\alpha}$ *extra flux*

Thus $\mathrm{Ind} = 0$ and $\mathrm{Exp} = 0$ so that

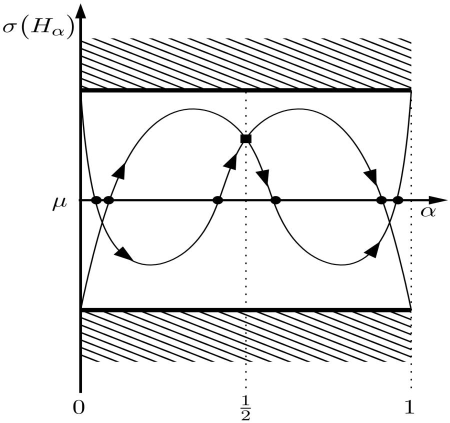
$$\begin{array}{ccccc} K_0(\mathcal{K}) = \mathbb{Z} & \xrightarrow{i_*} & K_0(\mathcal{E}(\mathcal{A}_2)) = \mathbb{Z}^3 & \xrightarrow{\pi_*} & K_0(\mathcal{A}_2) = \mathbb{Z}^2 \\ \uparrow \mathrm{Ind} & & & & \downarrow \mathrm{Exp} \\ K_1(\mathcal{A}_2) = \mathbb{Z}^2 & \xleftarrow{\pi_*} & K_1(\mathcal{E}(\mathcal{A}_2)) = \mathbb{Z}^2 & \xleftarrow{i_*} & K_1(\mathcal{K}) = 0 \end{array}$$

$\mathbb{Z}_2$ invariant and half-spectral flow for QSH

Theorem 10.7

$\alpha \in [0,1] \mapsto H(\alpha)$ *inserted flux in Kane-Mele model (breaks TRS)*

$\mathrm{Ind}_2(PFP) = 1 \implies$ *half-spectral flow* $\mathrm{SF}(\alpha \in [0, \frac{1}{2}] \mapsto H(\alpha)) \bmod 2 = 1$



Spectral flow in higher dimensions

For d even, index theorem used Dirac (even Fredholm module)

$$D \;=\; \langle\gamma|X\rangle \;=\; -\Gamma D\Gamma \;=\; |D| \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} \;=\; |D|G$$

Then strong invariants:

$$\mathrm{Ch}_{\{1,\ldots,d\}}(P) \;=\; \mathrm{Ind}(P_\omega F P_\omega)$$

Aim: Calculate this as a spectral flow upon inserting monopole

Introduce non-abelian skew-adjoint gauge potential for $k = 1, \ldots, d$:

$$A_k^\alpha \;=\; \alpha\, G\partial_k G \;=\; \frac{\alpha}{2R^2}\,[D,\gamma_k] \;\sim\; R^{-1}$$

where $R^2 = D^2 = X^2$. One has $A_k^\alpha = \Gamma A_k^\alpha \Gamma$ diagonal. Set

$$\nabla_k^\alpha \;=\; \partial_k - A_k^\alpha \qquad \text{on } L^2(\mathbb{R}^d, \mathbb{C}^N)$$

Monopole translations

Proposition 10.8

For $v \in \mathbb{R}^d$, $i\nabla_v^\alpha = i\sum_k v_k \nabla_k^\alpha$ *is essentially selfadjoint and*

$$(e^{\nabla_v^\alpha}\psi)(x) \;=\; M_v^\alpha(x)\,\psi(x+v)\;, \qquad \psi \in L^2(\mathbb{R}^d,\mathbb{C}^{2N})$$

where $x \in \mathbb{R}^d \backslash \{tv : t \in [-1,0]\} \mapsto M_v^\alpha(x) \in \mathrm{U}(2N)$ *is continuous with*

$$\lim_{|x|\to\infty} M_v^\alpha(x) \;=\; \mathbf{1}_{2N}$$

Phase factor has rotation covariance w.r.t. Pin Group representation:

$$g_O\, M_v^\alpha(O^* x)\, g_O^* \;=\; M_{Ov}^\alpha(x)$$

and

$$G\, e^{\nabla_v^\alpha}\, G \;=\; e^{\nabla_v^{1-\alpha}}$$

Restriction $e^{\nabla_k^\alpha}$ to $\ell^2(\mathbb{Z}^d,\mathbb{C}^N)$ gives monopole translations S_k^α

Proposition 10.9

$S_k^\alpha - S_k^0$ compact operator

Suppose Hamiltonian given by polynominal in shifts and potential

$$H = P(S_1, \dots, S_d) + W$$

Insertion of monopole into Hamiltonian gives

$$H_\alpha = P(S_1^\alpha, \dots, S_d^\alpha) + W$$

Facts: $\alpha \mapsto H_\alpha - \mu$ path of selfadjoint Fredholms and $H_1 = G^* H_0 G$

Theorem 10.10 ([CS])

Let d bei even

$$\mathrm{SF}\Big(\alpha \in [0,1] \mapsto H_\alpha \ \text{ through } \ \mu\Big) = -\,\mathrm{Ch}_{\{1,\dots,d\}}(P)$$

Odd dimensional version involves ”chirality flow”

11 Dirty superconductors

Disordered one-electron Hamiltonian h on $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2s+1}$

$\mathfrak{c} = (\mathfrak{c}_{n,l})$ anhilation operators on fermionic Fock space $\mathcal{F}_-(\mathcal{H})$

Hamilt. on $\mathcal{F}_-(\mathcal{H})$ with mean field pair creation $\Delta^* = -\overline{\Delta} \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} \mathbf{H} - \mu\,\mathbf{N} &= \mathfrak{c}^*\,(h - \mu\,\mathbf{1})\,\mathfrak{c} \;+\; \frac{1}{2}\,\mathfrak{c}^*\,\Delta\,\mathfrak{c}^* \;-\; \frac{1}{2}\,\mathfrak{c}\,\overline{\Delta}\,\mathfrak{c} \\ &= \frac{1}{2}\begin{pmatrix}\mathfrak{c}\\ \mathfrak{c}^*\end{pmatrix}^* \begin{pmatrix} h-\mu & \Delta \\ -\overline{\Delta} & -\overline{h}+\mu\end{pmatrix}\begin{pmatrix}\mathfrak{c}\\ \mathfrak{c}^*\end{pmatrix} \end{aligned}$$

Hence BdG Hamiltonian on $\mathcal{H}_{\rm ph} = \mathcal{H} \otimes \mathbb{C}^2_{\rm ph}$

$$H_\mu \;=\; \begin{pmatrix} h-\mu & \Delta \\ -\overline{\Delta} & -\overline{h}+\mu\end{pmatrix}$$

Even PHS (Class D)

$$S_{\rm ph}^*\,\overline{H_\mu}\,S_{\rm ph} \;=\; -H_\mu \qquad , \qquad S_{\rm ph} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

Class D systems

$\mathrm{spec}(H_\mu) = -\mathrm{spec}(H_\mu)$ and generically gap or pseudo-gap at 0

Theorem 11.1

Gibbs (KMS) state for observable $\mathbf{Q} = d\Gamma(Q)$

$$\frac{1}{Z_{\beta,\mu}} \mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})} \left(\mathbf{Q}\, e^{-\beta(\mathbf{H}-\mu\,\mathbf{N})} \right) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{ph}}}(f_\beta(H_\mu)\, Q)$$

Example $p + ip$ wave superconductor with $\mathcal{H} = \ell^2(\mathbb{Z}^2)$

$$h = S_1 + S_1^* + S_2 + S_2^* \qquad \Delta_{p+ip} = \delta\,(S_1 - S_1^* + i(S_2 - S_2^*))$$

Then $P = \chi(H_\mu \leqslant 0)$ satisfies $\mathrm{Ch}(P) = 1$ for $\mu > 0$ and $\delta > 0$

Conjecture (Kubo missing) Quantized Wiedemann-Franz

$$\kappa_H = \frac{\pi}{8}\,\mathrm{Ch}(P)\; T + \mathcal{O}(T^2)$$

Spectral flow in a BdG-Hamiltonian

Flux tube in two-dimensional BdG Hamiltonian

$$S_{\rm ph}^* \,\overline{H_\alpha}\, S_{\rm ph} \;=\; -H_{-\alpha} \qquad , \qquad S_{\rm ph}^2 = \pm\mathbf{1}$$

Then $S_{\rm ph}^* \,\overline{H_\alpha}\, S_{\rm ph} = -U^* H_{1-\alpha} U$ so that

$$\sigma(H_\alpha) \;=\; -\sigma(H_{-\alpha}) \;=\; -\sigma(H_{1-\alpha})$$

PHS only for $\alpha = 0, \frac{1}{2}, 1$ and thus $\mathrm{Ind}_2(H_{\frac{1}{2}})$ wel-defined

Theorem 11.2 ([DS])

$\mathrm{Ind}(PUP) \bmod 2 = \mathrm{Ind}_2(H_{\frac{1}{2}})$

or: odd Chern number implies existence of zero mode at defect

These zero modes are Majorana fermions (Read-Green 2000)

Worth noting: $S_{\rm ph}^2 = -\mathbf{1} \implies \mathrm{Ind}(PUP)$ even $\implies$ no zero mode

Spin quantum Hall effect in Class C

Theorem 11.3 (Altland-Zirnbauer 1997)

$SU(2)$ spin rotation invariance $[\mathbf{H}, \mathbf{s}] = 0$

$\Longrightarrow$ $H = H_{\rm red} \otimes \mathbf{1}$ *with odd PHS (Class C)*

$$S_{\rm ph}^* \, \overline{H_{\rm red}} \, S_{\rm ph} = -H_{\rm red} \qquad , \qquad S_{\rm ph} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

Example $d + id$ wave superconductor with h as above and

$$\Delta_{d+id} = \delta \left(i(S_1 + S_1^* - S_2 - S_2^*) + (S_1 - S_1^*)(S_2 - S_2^*) \right) s^2$$

Again $\mathrm{Ch}(P) = 2$ for $\delta > 0$ and $\mu > 0$

Theorem 11.4

Spin Hall conductance (Kubo) and spin edge currents quantized

12 Semimetals

Recall Bulk-Boundary-Correspondence (BBC) for $1D$ chiral systems:

Theorem 12.1

Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^{2L})$ *with chiral symmetry* $J = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$

Gapped chiral Hamiltonian $H = -JHJ$ *off-diagonal:* $H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$

Half-space restriction $\widehat{H}$ *on* $\ell^2(\mathbb{N}, \mathbb{C}^{2L})$ *has kernel projection* $\widehat{P}$ *with*

$$\widehat{P} \;=\; \widehat{P}_+ \;+\; \widehat{P}_- \qquad , \qquad J\widehat{P}_\pm \;=\; \pm\widehat{P}_\pm$$

Then

$$i\,\mathcal{T}(A^{-1}\nabla A) \;=\; \mathrm{Tr}(\widehat{P}_+) \;-\; \mathrm{Tr}(\widehat{P}_-)$$

where $\mathcal{T}(B) = \mathbf{E}\,\mathrm{Tr}(\langle 0|B|\rangle)$ *and* $\nabla B = i[X, B]$

Now: $2d$ graphene Hamiltonian also chiral, but only pseudogap

This semimetal can have flat band of edge states! **Similar BBC?**

Model for graphene

On honeycomb lattice = decorated triangular lattice, so on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$

$$H \;=\; \begin{pmatrix} 0 & S_1 + S_1^* S_2 + 1 \\ S_1^* + S_2^* S_1 + 1 & 0 \end{pmatrix}$$

where S_1, S_2 shifts on $\ell^2(\mathbb{Z}^2)$. Clearly chiral $JHJ = -H$. After Fourier:

$$H \;\cong\; \int_{\mathbb{T}^2}^{\oplus} dk \; \begin{pmatrix} 0 & e^{ik_1} + e^{i(k_2-k_1)} + 1 \\ e^{-ik_1} + e^{-i(k_2-k_1)} + 1 & 0 \end{pmatrix}$$

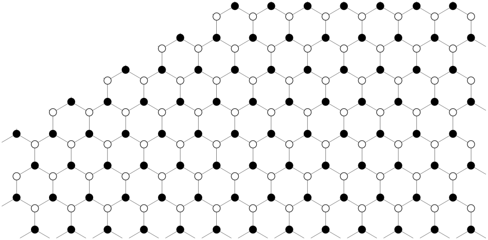
DOS vanishes at $E = 0$ (pseudogap). Dirac points $k_{\pm} = (\frac{(3\pm 1)\pi}{3}, 0)$

Zigzag boundary $\cong$ replace S_1 by unilateral shift $\widehat{S}_1$

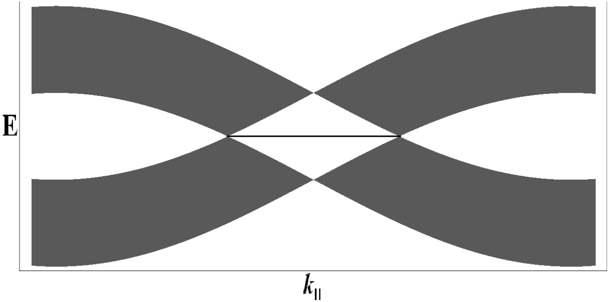
Armchair boundary $\cong$ replace S_2 by unilateral shift $\widehat{S}_2$

Fact (Saito, Dresselhaus ***et al.*** **1988):** edge states only for Zigzag

Illustration



Energy bands for half-space $\widehat{H}$ with zigzag edge:



Surface DOS

$\xi = \binom{\xi_1}{\xi_2} \in \mathbb{S}^1$ direction perpendicular to boundary

$\widehat{H} = \Pi_\xi \, H \, \Pi_\xi$ half-space restriction of graphene Hamiltonian

Kernel projection $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$ on flat band of surface states

$\widehat{\mathcal{T}}$ trace per unit volume along the boundary

Fermi unitary $U =$ phase of off-diagonal $(S_1 + S_1^* S_2 + 1)$

Theorem 12.2 ([SSt])

$$i\,\mathcal{T}(U^{-1}\nabla_\xi U) \;=\; \widehat{\mathcal{T}}(\widehat{P}_+) - \widehat{\mathcal{T}}(\widehat{P}_-)$$

where $\mathcal{T}(B) = \mathbf{E}\,\mathrm{Tr}(\langle 0|B|0\rangle)$ *and* $\nabla_\xi = \xi \cdot \nabla$ *with* $\nabla_j B = i[X_j, B]$

Moreover: $i\,\mathcal{T}(U^{-1}\nabla_1 U) = 0$ and $i\,\mathcal{T}(U^{-1}\nabla_2 U) = \frac{1}{3}$

Explains difference zigzag / armchair

Proves existence of edge states (generalizes Feffermann, Weinstein)

Singularities of Fermi unitary and Besov spaces

Fourier $U \cong \int dk\; U(k)$ with

$$U(k) \;=\; \frac{e^{ik_1} + e^{i(k_2-k_1)} + 1}{|e^{ik_1} + e^{i(k_2-k_1)} + 1|}$$

Vorticities at Dirac points, not even continuous, so $U \notin \mathcal{A}_2 = C(\mathbb{T}^2)$

But U lies in Besov $B^1_{1,1}$, namely for all ξ:

$$\int_0^1 \frac{dt}{t^2} \int dk \,\big|U(k+\xi t) + U(k-\xi t) - 2\,U(k)\big| \;<\; \infty$$

Similarly $U \in B^{1/2}_{2,2}$. Both enough to push index theorem through as:

Peller (1980's):

Toeplitz operators with Besov symbols have traceclass properties

$f \in B^{1/p}_{p,p}(\mathbb{T}^1) \;\;\Longrightarrow\;\; \Pi f(\mathbf{1}-\Pi) \in \mathcal{L}^p$ Schatten ideal

Remarks

Pairing $\langle [\xi \cdot X], [U]_1 \rangle = i\mathcal{T}(U^{-1}\nabla_\xi U)$ over huge algebra $C^*(B^1_{1,1} \cap L^\infty)$

Thus values **not** in discrete range of $[U]_1 \in K_1(\mathcal{A}_2) \mapsto \langle [\xi \cdot X], [U]_1 \rangle$

Index theory for sufficiently smooth elements

Changing H continuously, changes value of $i\mathcal{T}(U^{-1}\nabla_\xi U)$ continuously

$\Longrightarrow$ surface state density changes continuous (even for fixed ξ)

Only equality and thus BBC always holds and is hence topological

Similar situation: Levinson's theorem for scattering on hypersurfaces

In the following:

extension to disordered chiral systems and higher dimension

Hypothesis: pseudo-gap and Anderson localization at $E = 0$

Higher dimension and disorder

Disordered d-dimensional rotation C^*-algebra $\mathcal{A}_d = C(\Omega) \rtimes_B \mathbb{Z}^d$

Trace $\mathcal{T}$ and derivations $\nabla = (\nabla_1, \ldots, \nabla_d)$

$\xi \in \mathbb{S}^{d-1}$ direction perpendicular to hypersurface, $\widehat{\mathcal{T}}$ trace along it

Theorem 12.3 ([SSt])

$H \in M_{2L}(\mathcal{A}_d)$ with chiral symmetry $JHJ = -H$

Suppose pseudo-gap at 0, namely there is $\gamma > 1$ with

$$\mathcal{T}\big(\chi(|H| \leqslant \epsilon)\big) \;\leqslant\; C_\gamma\, \epsilon^\gamma$$

and a mobility gap in $(-\epsilon_0 \epsilon_0)$, that is, for some $s \in (0,1)$

$$\sup_{|\epsilon| \leqslant \epsilon_0} \mathbf{E}\, \|\langle 0|(H - \epsilon + \imath 0)^{-1}|n\rangle\|^s \;\leqslant\; C_s\, e^{-\beta_s |n|}$$

Then, for Fermi unitary U and kernel projection $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$ as above,

$$i\, \mathcal{T}(U^{-1} \nabla_\xi U) \;=\; \widehat{\mathcal{T}}(\widehat{P}_+) - \widehat{\mathcal{T}}(\widehat{P}_-)$$

Constructions:

Finite trace $\mathcal{T}$ gives von Neumann algebra $\mathcal{M} = L^\infty(\mathcal{A}_d, \mathcal{T})$

Non-commutative spaces $L^p(\mathcal{M})$, $p > 0$, Banach or quasi-Banach

$L^2(\mathcal{M}) = L^2(\Omega, \mathbb{P}) \otimes \ell^2(\mathbb{Z}^d)$ is GNS-Hilbert space of $\mathcal{T}$

Suppose components of ξ not rationally related. $\mathbb{R}$-action α on $\mathcal{A}_d$:

$$\alpha_t(A) \;=\; e^{t\,\xi\cdot\nabla}(A)$$

$\mathcal{T}$-invariance $\implies$ α extends isometrically to $L^p(\mathcal{M})$

On GNS unitary with generator $D = \xi \cdot X$ and spectral decomposition:

$$L^2(\mathcal{M}) \;=\; \int_{\sigma(D)}^{\oplus} \mathcal{H}_\lambda \,\mu(d\lambda)$$

So "Fourier"-decomposition of $A \in \mathcal{M} \subset L^2(\mathcal{M})$:

$$A \;=\; \int_{\sigma(D)}^{\oplus} A_\lambda \,\mu(d\lambda)$$

Here: Fourier spectrum $=$ Averson spectum

Besov spaces:

X Banach space with isometric $\mathbb{R}$-action α (here $X = L^p(\mathcal{M})$)

For $f \in L^1(\mathbb{R})$ and $x \in X$ define $\alpha_f(x)$ as Riemann integral

$$\alpha_f(x) \;=\; \int_{\mathbb{R}} f(-t)\,\alpha_t(x)\,dt$$

Then for $f \in FA(\mathbb{R}) = \mathcal{F}L^1(\mathbb{R})$ define Fourier multiplier $\widehat{f}* \in \mathcal{B}(X)$ by

$$\widehat{f} * x \;=\; \alpha_{\mathcal{F}^{-1}f}(x)$$

Given smooth $\varphi : \mathbb{R} \to [0,1]$ supported by $[-2,-2^{-1}] \cup [2^{-1},2]$ and

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}x) \;=\; 1$$

Littlewood-Payley dyadic decomposition $(W_k)_{k\in\mathbb{N}}$ by

$$W_k \;=\; \varphi(|2^{-k}\cdot|) \quad \text{for } k > 0\;, \qquad W_0 \;=\; 1 - \sum_{k>0} W_k$$

Now: $B^s_q(X) \;=\; \Big\{ x \in X \;:\; \|x\|_{B^s_q(X)} = \Big(\sum_{k\geqslant 0} 2^{qsk}\,\|\widehat{W}_k * x\|^q_X\Big)^{\frac{1}{q}} < \infty \Big\}$

Properties of Besov spaces:

Proposition 12.4

Definition of $B^s_q(X)$ independent of choice of φ

$(B^s_q(X), \|\,.\,\|_{B^s_q(X)})$ *Banach space for $s \in \mathbb{R}$ and $q \in [1,\infty)$*

An equivalent norm is given by

$$\|x\|_{\tilde{B}^s_q(X)} \;=\; \|x\|_X \,+\, \left(\int_{[0,1]} t^{-sq}\, \omega^N_X(x,t)^q \,\frac{dt}{t} \right)^{\frac{1}{q}}$$

where

$$\omega^N_X(x,t) \;=\; \sup_{|r| \leqslant t} \|\Delta^N_r(x)\|_X$$

with finite difference operator $\Delta_t : X \to X$ given by

$$\Delta_t(x) \;=\; \alpha_t(x) - x$$

More constructions:

Set

$$B^s_{p,q}(\mathcal{M}) \;=\; B^s_q(L^p(\mathcal{M}))$$

Elements have "differentiability properties perp. to hypersurface"

Crossed product $\mathcal{A}_d \rtimes_\alpha \mathbb{R}$ with semifinite trace $\widehat{\mathcal{T}}$ (via Hilbert algebras)

$\widehat{\mathcal{T}}$ is trace per unit volume along the boundary

It gives von Neumann $\mathcal{N} = L^\infty(\mathcal{A}_d \rtimes_\alpha \mathbb{R}, \widehat{\mathcal{T}}) = \mathcal{M} \rtimes_\alpha \mathbb{R}$

Furthermore: L^p-spaces $L^p(\mathcal{N}, \widehat{\mathcal{T}})$ for $p > 0$

Half-space projection $\Pi = \chi(D > 0)$ in $\mathcal{N}$, but not $L^p(\mathcal{N}, \widehat{\mathcal{T}})$ for $p < \infty$

Now for "symbol" $A \in \mathcal{M}$, Toeplitz and Hankel operators are

$$T_A \;=\; \Pi A \Pi \qquad , \qquad H_A \;=\; \Pi A(\mathbf{1} - \Pi)$$

These are operators in $\mathcal{N}$

Peller criterion and index theorem

Theorem 12.5 ([SSt])

For all $p \geqslant 1$ and $A \in \mathcal{M} \cap B^{1/p}_{p,p}(\mathcal{M})$, one has $H_A \in L^p(\mathcal{M})$

Proof: explicit calculations for $p = 1, 2, \infty$, then analytic interpolation

Classical commutative case is $\mathcal{M} = C_0(\mathbb{R})$ with $\alpha_t(f)(y) = f(y + t)$

In this case Peller even proved inverse implication

Theorem 12.6 ([SSt])

For $U \in \mathcal{M}$ with $U - \mathbf{1} \in B^{1/2}_{2,2}$,

$$i\,\mathcal{T}(U^{-1}\nabla_\xi U) \;=\; \widehat{\mathcal{T}}\text{-}\mathrm{Ind}\big(\Pi\, U\, \Pi + (\mathbf{1} - \Pi)\big)$$

where semifinite index of $\widehat{\mathcal{T}}$-Breuer-Fredholm $T \in \mathcal{N}$ is defined by

$$\widehat{\mathcal{T}}\text{-}\mathrm{Ind}(T) \;=\; \widehat{\mathcal{T}}(\mathrm{Ker}(T)) \;-\; \widehat{\mathcal{T}}(\mathrm{Ker}(T^*))$$

Application of index theorem

H chiral Hamiltonian and $\widehat{H} = \Pi H \Pi$ with polar decompositions

$$\mathrm{sgn}(H) = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}, \qquad , \qquad \mathrm{sgn}(\widehat{H}) = \begin{pmatrix} 0 & \widehat{U} \\ \widehat{U}^* & 0 \end{pmatrix}.$$

If (i) $U \in B^{1/2}_{2,2}(\mathcal{M})$ and (ii) $\widehat{U} - \Pi U \Pi$ is $\widehat{\mathcal{T}}$-compact, then

$$\widehat{\mathcal{T}}(\widehat{P}_+ - \widehat{P}_-) = \widehat{\mathcal{T}}(J\,\mathrm{Ker}(\widehat{H})) = \widehat{\mathcal{T}}\text{-Ind}(\widehat{U}) = \widehat{\mathcal{T}}\text{-Ind}(\Pi U \Pi)$$

and the index theorem implies the Theorem

Tough analytical issue: pseudogap and mobility gap imply (i) and (ii)

Main idea is that γ-pseudogap condition implies for $p > 0$

$$H^{-1} \in L^p(\mathcal{M}) \quad \text{and} \quad \|H^{-1} - (H+z)^{-1}\|_p \leqslant C|\Im m(z)|^{(\gamma - p)/p}$$

Used to estimate $\Pi\,\mathrm{sgn}(H)\,\Pi - \mathrm{sgn}(\widehat{H})$ after functional calculus

13 Further results and bibliography

- topology associated to spacial reflections, etc. (Gomi, Thiang)
- weak invariants via *KK*-theory (Prodan, Schulz-Baldes)
- BBC in real cases (Bourne, Carey, Rennie, Kellendonk)
- effects of corners (Hayashi, Thiang)
- analysis of bosonic systems (Peano, Schulz-Baldes)
- analysis of photonic crystals (De Nittis, Lein)
- stability of invariants w.r.t. interactions
  (Bachmann, de Roeck, Fraas, *et al*)

Other groups (each with personal point of view)

- Bourne, Carey, Rennie, Kellendonk
- Mathai, Thiang, Hanabus
- Zirnbauer, Kennedy, Alldridge, Max
- Panati, Monaco, Teufel, Cornean, Moscolari
- Katsura, Koma, Gomi
- Hayashi, Furuta, Kotani
- Graf, Porta
- Gawedzki, Delplace, Tauber, Fruchart
- Kaufmann's, Li

- many theoretical physics groups

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