Index pairings with symmetries and applications to topological insulators

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What is a topological insulator?

- d-dimensional disordered system of independent Fermions with a combination of basic symmetries

TRS, PHS, CHS = time reversal, particle hole, chiral symmetry

- Fermi level in a Gap or Anderson localization regime
- Topology of bulk (in Bloch bundles over Brillouin torus):

winding numbers, Chern numbers, $\mathbb{Z}_2$-invariants, higher invariants

- Delocalized edge modes with non-trivial topology
- Bulk-edge correspondence
- Topological bound states at defects (zero modes)
- Toy models: tight-binding

Aim: index theory for invariants also for disordered systems

Periodic table of topological insulators

Schnyder-Ryu-Furusaki-Ludwig, Kitaev 2008

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
1	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Aims of the talk

Construct operator algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ and K-group elements

Realize entries of periodic table as indices of Fredholm ops

First complex cases:

even d: index pairing $K_0(\mathcal{A})$ with even K-cycle

odd d: index pairing $K_1(\mathcal{A})$ with odd K-cycle

Then real cases:

Implement symmetries on K-groups and K-cycles

8 KR-groups and 8 KR-cycles $=$ 64 pairings

K-cycles = spectral triples $\Rightarrow$ index pairings

Suppose $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ unital algebra (often C*-algebra)

odd K-cycle is Dirac operator $D = D^*$ on $\mathcal{H}$ with

(i) compact resolvent (ii) $[D, A]$ bounded $\forall\, A \in \mathcal{A}$

even K-cycle if exists grading $\Gamma = \Gamma^*$ with $\Gamma^2 = \mathbf{1}$ and

(iii) $\Gamma D \Gamma = -D$ (iv) $A\Gamma = \Gamma A \;\; \forall\, A \in \mathcal{A}$

odd K-cycle: set Hardy $E = \chi(D > 0)$ then (Atiyah, Connes...)

$$T \;=\; EAE + \mathbf{1} - E \qquad \text{Fredholm for } [A]_1 \in K_1(\mathcal{A})$$

even K-cycle: use Dirac phase $\frac{D}{|D|} = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}$ for $\Gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$. Then

$$T \;=\; PFP + \mathbf{1} - P \qquad \text{Fredholm for } [P]_0 \in K_0(\mathcal{A})$$

Reminder on index pairings

If $[E, A]$ compact, then $R = EA^{-1}E + \mathbf{1} - E$ pseudo-inverse for T:

$$\begin{aligned} TR &= (EAE + \mathbf{1} - E)(EA^{-1}E + \mathbf{1} - E) \\ &= EAE\, EA^{-1}E + \mathbf{1} - E \;=\; \mathbf{1} + K \end{aligned}$$

Hence T Fredholm. Now

$$2E - \mathbf{1} \;=\; D|D|^{-1} \;=\; D\,\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^{\frac{1}{2}}}\,(\lambda + D^2)^{-1}$$

Thus

$$[E, A] \;=\; \frac{1}{2}\,[D|D|^{-1}, A] \;=\; \frac{1}{2\pi}\int_0^\infty \frac{d\lambda}{\lambda^{\frac{1}{2}}}\,[D(\lambda + D^2)^{-1}, A]$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{d\lambda}{\lambda^{\frac{1}{2}}}\,\left([D, A](\lambda + D^2)^{-1} + D(\lambda + D^2)^{-1}[D^2, A](\lambda + D^2)^{-1}\right)$$

Tight-binding toy models in dimension d

Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L$

Fiber $\mathbb{C}^L = \mathbb{C}^{2s+1} \otimes \mathbb{C}^r$ with spin s and r internal degrees

e.g. $\mathbb{C}^r = \mathbb{C}^2_{\rm ph} \otimes \mathbb{C}^2_{\rm sl}$ particle-hole space and sublattice space

Typical Hamiltonian

$$H_\omega \;=\; \Delta^B \;+\; W_\omega \;=\; \sum_{i=1}^{d} \left(t_i^* S_i^B + t_i (S_i^B)^*\right) \;+\; W_\omega$$

Magnetic translations $S_j^B S_i^B = e^{iB_{i,j}} S_i^B S_j^B$ in Laudau gauge:

$$S_1^B = S_1 \qquad S_2^B = e^{iB_{1,2}X_1} S_2 \qquad S_3^B = e^{iB_{1,3}X_1 + iB_{2,3}X_2} S_3$$

t_i matrices $L \times L$, e.g. spin orbit coupling, (anti)particle creation

matrix potential $W_\omega = W_\omega^* = \sum_{n \in \mathbb{Z}^d} |n\rangle \omega_n \langle n|$ with matrices ω_n

Observable algebra

Configurations $\omega = (\omega_n)_{n\in\mathbb{Z}^d} \in \Omega$ compact probability space $(\Omega, \mathbb{P})$

$\mathbb{P}$ invariant and ergodic w.r.t. $T : \mathbb{Z}^d \times \Omega \to \Omega$

Covariance w.r.t. to dual magnetic translations $V_a = S_j^B V_a (S_j^B)^*$

$$V_a H_\omega V_a^* \;=\; H_{T_a\omega} \qquad a \in \mathbb{Z}^d$$

$\|A\| = \sup_\omega \|A_\omega\|$ is C*-norm on

$$\mathcal{A}_d \;=\; \mathrm{C}^*\left\{A = (A_\omega)_{\omega\in\Omega} \text{ finite range covariant operators}\right\}$$
$$\cong\; \text{twisted crossed product } C(\Omega) \rtimes_B \mathbb{Z}^d$$

Fact: Suppose Ω contractible

$\Longrightarrow$ rotation algebra $\mathrm{C}^*(S_j^B)$ is deformation retract of $\mathcal{A}_d$

Pimsner-Voiculescu: $K_0(\mathcal{A}_d) = \mathbb{Z}^{2^{d-1}}$ and $K_1(\mathcal{A}_d) = \mathbb{Z}^{2^{d-1}}$

K-group elements of interest

Fermi level $\mu \in \mathbb{R}$ in spectral gap of H_ω (or Anderson localization)

$$P_\omega \;=\; \chi(H_\omega \leq \mu) \qquad \text{covariant Fermi projection}$$

Hence: $P = (P_\omega)_{\omega\in\Omega} \in \mathcal{A}_d$ fixes element in $K_0(\mathcal{A}_d)$

Suppose furthermore in odd d: chiral symmetry (grading)

$$H_\omega \;=\; -\,J_{\mathrm{ch}}^* H_\omega J_{\mathrm{ch}} \qquad J_{\mathrm{ch}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

Then

$$H_\omega \;=\; \begin{pmatrix} 0 & A_\omega \\ A_\omega^* & 0 \end{pmatrix}$$

If $\mu = 0$ in gap, $A = (A_\omega)_{\omega\in\Omega} \in \mathcal{A}_d$ invertible and $[A]_1 \in K_1(\mathcal{A}_d)$

Local index theorem for odd dimension d

$\Gamma_1, \ldots, \Gamma_d$ irrep of Clifford C_d on $\mathbb{C}^{2^{(d-1)/2}}$, $x \in (\mathbb{R} \setminus \mathbb{Z})^d$

$$D \;=\; \sum_{j=1}^{d} (X_j + x_j) \otimes \mathbf{1} \otimes \Gamma_j \qquad \text{Dirac operator on } \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L \otimes \mathbb{C}^{2^{(d-1)/2}}$$

odd K-cycle and Hardy projection $E = \chi(D > 0)$ satisfies

$$[E, A_\omega] \;\text{ compact and in } \mathcal{L}^{d+\epsilon} \;\text{ für } A = (A_\omega)_{\omega \in \Omega} \in \mathcal{A}_d$$

Theorem (Prodan, S-B 2014)

Almost sure index $\mathrm{Ind}(E\, A_\omega E + \mathbf{1} - E)$ *equal to odd Chern number*

$$\mathrm{Ch}_d(A) = \frac{(-i\pi)^{\frac{d-1}{2}}}{i\, d!!} \sum_{\rho \in S_d} (-1)^\rho\, \mathcal{T}\left(\prod_{j=1}^{d} A^{-1} \nabla_{\rho_j} A \right)$$

where

$$\mathcal{T}(A) \;=\; \mathbf{E}_{\mathbb{P}}\, \mathrm{Tr}_L\, \langle 0 | A_\omega | 0 \rangle \qquad \nabla_j A_\omega = i[X_j, A_\omega]$$

Local index theorem for even dimension d

As above $\Gamma_1, \ldots, \Gamma_d$ Clifford, now grading $\Gamma = -i^{-d/2}\Gamma_1 \cdots \Gamma_d$

Even K-cycle with $D = -\Gamma D \Gamma = |D| \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}$ for $\Gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Theorem (Connes $d = 2$, Prodan, Leung, Bellissard 2013)

Almost sure index $\mathrm{Ind}(P_\omega F P_\omega + \mathbf{1} - P_\omega)$ *equal to*

$$\mathrm{Ch}_d(P) \;=\; \frac{(-2i\pi)^{\frac{d}{2}}}{\frac{d}{2}!} \sum_{\rho \in S_d} (-1)^\rho \, \mathcal{T}\left(P \prod_{j=1}^{d} \nabla_{\rho_j} P \right)$$

Special case $d = 2$: $F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$ and

$$\mathrm{Ind}(PFP + \mathbf{1} - P) \;=\; \mathrm{Ind}(PFP) \;=\; 2\pi\imath \, \mathcal{T}(P[[X_1, P], [X_2, P]])$$

Periodic table again

Index theorem for complex cases (strong inv. = pairing d-cocycle)

$j\backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
1	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Discrete symmetries

$$\begin{aligned} \text{chiral symmetry (CHS)} : \quad & J_{\rm ch}^* \, H \, J_{\rm ch} \;=\; -H \\ \text{time reversal symmetry (TRS)} : \quad & S_{\rm tr}^* \, \overline{H} \, S_{\rm tr} \;=\; H \\ \text{particle-hole symmetry (PHS)} : \quad & S_{\rm ph}^* \, \overline{H} \, S_{\rm ph} \;=\; -H \end{aligned}$$

$S_{\rm tr} = e^{i\pi s^y}$ orthogonal on $\mathbb{C}^{2s+1}$ with $S_{\rm tr}^2 = \pm\mathbf{1}$ even or odd

$S_{\rm ph}$ orthogonal on $\mathbb{C}^2_{\rm ph}$ with $S_{\rm ph}^2 = \pm\mathbf{1}$ even or odd

Note: TRS + PHS $\Longrightarrow$ CHS with $J_{\rm ch} = S_{\rm tr} S_{\rm ph}$

10 combinations of symmetries: none (1), one (5), three (4)

10 Cartan-Altland-Zirnbauer classes (CAZ): 2 complex, 8 real

Further distinction in each of the 10 classes: topological insulators

Periodic table: real classes only

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Focus on system in $d = 2$ with odd TRS $S = S_{\mathrm{tr}}$:

$$S^2 = -\mathbf{1} \qquad S^* \overline{H} S = H$$

$\mathbb{Z}_2$ index for odd TRS and $d = 2$

Rewrite $S^*\overline{H}S = H = S^*H^tS$ with $H^t = (\overline{H})^*$

$\Longrightarrow \quad S^*(H^n)^tS = H^n$ for $n \in \mathbb{N} \quad \Longrightarrow \quad S^*P^tS = P$

For $d = 2$ now $F = \frac{X_1+iX_2}{|X_1+iX_2|} = F^t$ and $[S, F] = 0$

Hence Fredholm operator $T = PFP + \mathbf{1} - P$ of type

Definition T odd symmetric $\Longleftrightarrow S^*T^tS = T \Longleftrightarrow (TS)^t = -TS$

Theorem (Atiyah-Singer 1969, S-B 2013)

$\mathbb{F}_2(\mathcal{H}) = \{$odd symmetric Fredholm operators$\}$ has 2 connected components labelled by compactly stable homotopy invariant

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2 \;\in\; \mathbb{Z}_2$$

Application: $\mathbb{Z}_2$ phase label for Kane-Mele model if dyn. localized

Proof via Kramers degeneracy:

First of all: $\mathrm{Ind}(T) = 0$ because $\mathrm{Ker}(T^*) = S\,\overline{\mathrm{Ker}(T)}$

Idea: $\mathrm{Ker}(T) = \mathrm{Ker}(T^*T)$

and positive eigenvalues of T^*T have even multiplicity

Let $T^*Tv = \lambda v$ and $w = S\,\overline{Tv}$ (N.B. $\lambda \neq 0$). Then

$$\begin{aligned} T^*T\,w &= S\,(S^*T^*S)\,(S^*TS)\,\overline{Tv} \\ &= S\,\overline{T}\,\overline{T^*\,T\,v} = \lambda\,S\,\overline{T}\,\overline{v} = \lambda\,w\,. \end{aligned}$$

Suppose now $\mu \in \mathbb{C}$ with $v = \mu\,w$. Then

$$v = \mu\,S\,\overline{T}\,\overline{v} = \mu\,S\,\overline{T}\,\overline{\mu}\,S\,T\,v = -|\mu|^2\,T^*T\,v = -|\mu|^2\,\lambda\,v$$

Contradiction to $v \neq 0$.

Now $\mathrm{span}\{v, w\}$ is invariant subspace of T^*T.

Go on to orthogonal complement

Periodic table again

$j\backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	$+1$	0	0				$\boxed{2\,\mathbb{Z}}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Example of a $2\,\mathbb{Z}$ index: $d = 4$ and even TRS $H = \overline{H}$

$2\,\mathbb{Z}$ indices

Proposition

Let $S = \overline{S} = (S^*)^{-1}$ satisfy $S^2 = -\mathbf{1}$.
On $\mathbb{F}_4(\mathcal{H}) = \{T = S^*\overline{T}S$ quaternionic Fredholm operator$\}$

$$\mathrm{Ind}(T) \;\in\; 2\,\mathbb{Z}$$

Proof. Suppose $T^*Tv = \lambda v$. Then $w = S\overline{v}$ also $T^*Tw = \lambda w$.

Applies to $d = 4$ and even TRS

Then Fermi projection satisfies $P = \overline{P}$

Dirac phase $D = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}$ satisfies $F = S^*\overline{F}S$ with $S^2 = -\mathbf{1}$
(more on that shortly)

Atiyah-Singer classifying spaces for Real K-theory

$\mathbb{F}_k^{\mathbb{R}} =$ anti-s.a. Freds on $\mathcal{H}_{\mathbb{R}}$ commuting with C_{k-1} and $\pm i \in \sigma_{\mathrm{ess}}$

Fact: $\mathbb{F}_1^{\mathbb{R}}$ and O of same homotopy type and $\pi_k(O) = \pi_0(\mathbb{F}_k^{\mathbb{R}})$

Theorem

Bijections to Freds on complex Hilbert space with $S^2 = -\mathbf{1}$

$\mathbb{F}_0^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, \overline{T} = T\}$

$\mathbb{F}_1^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F} \,|\, \overline{T} = -T\}$

$\mathbb{F}_2^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, S^* T^t S = T\}$

$\mathbb{F}_3^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F}_* \,|\, S^* \overline{T} S = T\}$

$\mathbb{F}_4^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, S^* \overline{T} S = T\}$

$\mathbb{F}_5^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F} \,|\, S^* \overline{T} S = -T\}$

$\mathbb{F}_6^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, T^t = T\}$

$\mathbb{F}_7^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F}_* \,|\, \overline{T} = T\}$

Above: for $d = 2$ and odd TRS, $T = PFP + \mathbf{1} - P \in \mathbb{F}_2^{\mathbb{R}}$

Above: for $d = 4$ and even TRS, $T = PFP + \mathbf{1} - P \in \mathbb{F}_4^{\mathbb{R}}$

Symmetries of the Dirac operator

$$D = \sum_{j=1}^{d} (X_j + x) \otimes \mathbf{1} \otimes \Gamma_j$$

$\Gamma_1, \ldots, \Gamma_d$ irrep of C_d with $\Gamma_{2j} = -\overline{\Gamma_{2j}}$ and $\Gamma_{2j+1} = \overline{\Gamma_{2j+1}}$

In even d exists grading $\Gamma = \Gamma^*$ with $D = -\Gamma D \Gamma$ and $\Gamma^2 = \mathbf{1}$

Moreover, exists real unitary Σ (essentially unique) with

$d = 8 - i$	8	7	6	5	4	3	2	1
Σ^2	$\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\Sigma^* \overline{D} \Sigma$	D	$-D$	D	D	D	$-D$	D	D
$\Gamma \Sigma \Gamma$	Σ		$-\Sigma$		Σ		$-\Sigma$	

(D, Γ, Σ) defines a KR^i-cycle (spectral triple with real structure)

(Kasparov 1981, Connes 1995, Gracia-Varilly-Figueroa 2000)

Periodic table again

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Focus on the 8 cases in dimension $d = 3$

even j Fermi projection P, odd j with chiral sym. and invertible A

Dirac operator in $d = 3$

no grading and $\Sigma^* \overline{D} \Sigma = -D$ with $\Sigma^2 = -\mathbf{1}$

Hence Hardy is complex symplectic (Lagrangian): $\Sigma^* \overline{E} \Sigma = \mathbf{1} - E$

(D, Σ) specifies element of $KR^5(\mathcal{A})$ pairs with any $KR_j(\mathcal{A})$

Pairing with invertible A (odd K-group) if chiral symmetry

$$\langle [A]_{\mathrm{odd}} \, , \, [(D, \Sigma)]_5 \rangle \;=\; \mathrm{Ind}_*(EAE - \mathbf{1} - E)$$

Pairing with invertible $\mathbf{1} - 2P$ (P even K-group) if no chiral sym.

$$\begin{aligned} \langle [P]_{\mathrm{even}} \, , \, [(D, \Sigma)]_5 \rangle \;&=\; \mathrm{Ind}_2(E(\mathbf{1} - 2P)E - \mathbf{1} - E) \;\in\; \mathbb{Z}_2 \\ &=\; \mathrm{Ind}_2(P(\mathbf{1} - 2E)P - \mathbf{1} - P) \;\in\; \mathbb{Z}_2 \end{aligned}$$

For even-odd pairing only secondary $\mathbb{Z}_2$ index!

Periodic table: even-odd and odd-even

The only non-vanishing entries for odd $j+d$ are boxed:

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$
2	0	$+1$	0	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$	

Focus now on odd TRS for $d=3$

Odd TRS in $d = 3$

$\Sigma^* \overline{D} \Sigma = -D$ with $\Sigma^2 = -\mathbf{1} \quad \Longrightarrow \quad \Sigma^* \overline{E}\, \Sigma = \mathbf{1} - E$

Unitary $A = \mathbf{1} - 2P$ satisfies $S^* \overline{A}\, S = A$ with $S^2 = -\mathbf{1}$

$[S, \Sigma] = 0$ and $T = EAE + \mathbf{1} - E$ Fredholm operator

Claim: $\mathrm{Ind}_2(T) \in \mathbb{Z}_2$ well-defined

Non-standard Kramers degeneracy argument (even for matrices!)

Proposition

Let $T^* T v = \lambda v$ with $\lambda > 0$ and $v \in \mathcal{H}$. Introduce $w \in \mathcal{H}$ by

$$\overline{w} \;=\; RSA^* EAEv$$

where $R = \Phi(\Sigma\overline{\Phi})^*$ built from frame Φ for $E = \Phi\Phi^*$

Then v and w are linearly independent and $T^* T w = \lambda w$.

Résumé

- For $2\,\mathbb{Z}$ indices invoking symplectic projections 4th Kramers arg.
- Also index parings for two unitaries A and F by doubling

$$T \;=\; P\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} P + \mathbf{1} - P \qquad\qquad P \;=\; \frac{1}{2}\begin{pmatrix} \mathbf{1} & F \\ F^* & \mathbf{1} \end{pmatrix}$$

  Only $\mathbb{Z}_2$ as for two projections and exchange $A \leftrightarrow F$ allowed

- Index theorem for **every** entry of periodic table
- For vanishing (empty) entry in table, homotopy to trivial model
- Non-trivial examples given by tight-binding Hamiltonians, built upon complex examples (*e.g.* Kane-Mele $=$ 2 Haldane)
- Implications of non-trivial invariants to be examined

Periodic table of topological insulators (2008)

Schnyder-Ryu-Furusaki-Ludwig, Kitaev $KR_j(\mathbb{R}^d_\tau) \cong \pi_{j-1-d}(O)$

$j \backslash d$	TRS	PHS	CHS	1	2	3	4	5	6	7	8
0	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
1	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
0	$+1$	0	0				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	$+1$	$+1$	1	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
3	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
4	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
5	-1	-1	1	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
6	0	-1	0		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
7	$+1$	-1	1			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Symmetries of the Dirac operator

$$D \;=\; \sum_{j=1}^{d} (X_j + x_j) \otimes \mathbf{1} \otimes \Gamma_j$$

$\Gamma_1, \ldots, \Gamma_d$ irrep of C_d with $\Gamma_{2j} = -\overline{\Gamma_{2j}}$ and $\Gamma_{2j+1} = \overline{\Gamma_{2j+1}}$

In even d exists grading $\Gamma = \Gamma^*$ with $D = -\Gamma D \Gamma$ and $\Gamma^2 = \mathbf{1}$

Moreover, exists real unitary Σ (essentially unique) with

$d = 8 - i$	8	7	6	5	4	3	2	1
Σ^2	$\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\Sigma^* \, \overline{D} \, \Sigma$	D	$-D$	D	D	D	$-D$	D	D
$\Gamma \, \Sigma \, \Gamma$	Σ		$-\Sigma$		Σ		$-\Sigma$	

(D, Γ, Σ) defines a KR^i-cycle (spectral triple with real structure)

(Kasparov 1981, Connes 1995, Gracia-Varilly-Figueroa 2000)

Odd *KR*-groups - paired with *KR*-cycles

$$W_1(\mathcal{A}) \;=\; \left\{ U \in \cup_{n\geq 1} M_n(\mathcal{A}^+) \;\middle|\; U^{-1} \;=\; U^* \right\} \qquad \mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$$

Equivalence relation $\sim$ by homotopy and $U \sim \left(\begin{smallmatrix} U & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$

$$K_1(\mathcal{A}) \;=\; W_1(\mathcal{A})/\sim$$

Now $\tau(A) = \overline{A}$ anti-linear involutive $*$-autmorphism on $\mathcal{A}$

τ extended to $\mathcal{A}^+$ as $\tau(A,t) = (\tau(A), \overline{t})$. With $\Sigma = \left(\begin{smallmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$

$$\begin{aligned}
W_1(\mathcal{A},\tau) \;&=\; \left\{ U \in W_1(\mathcal{A}) \;\middle|\; \overline{U} \;=\; U \right\} \\
W_3(\mathcal{A},\tau) \;&=\; \left\{ U \in W_{1,\mathrm{ev}}(\mathcal{A}) \;\middle|\; \Sigma^* \,\overline{U}\, \Sigma \;=\; U^* \right\} \\
W_5(\mathcal{A},\tau) \;&=\; \left\{ U \in W_{1,\mathrm{ev}}(\mathcal{A}) \;\middle|\; \Sigma^* \,\overline{U}\, \Sigma \;=\; U \right\} \\
W_7(\mathcal{A},\tau) \;&=\; \left\{ U \in W_1(\mathcal{A}) \;\middle|\; \overline{U} \;=\; U^* \right\}
\end{aligned}$$

$$KR_{2i+1}(\mathcal{A},\tau) \;=\; W_{2i+1}(\mathcal{A},\tau)/\sim$$

Even KR-groups - paired with KR-cycles

Instead of projections P work with selfadjoints $Q = \mathbf{1} - 2P$

$$V_0(\mathcal{A}) \;=\; \left\{ Q \in \cup_{n\geq 1} M_{2n}(\mathcal{A}^+) \;\middle|\; Q^* \;=\; Q\;,\;\; Q^2 \;=\; \mathbf{1} \right\}\;,$$

Equiv. relation $\sim$ by homotopy and $Q \sim \begin{pmatrix} Q & 0 \\ 0 & E_2 \end{pmatrix}$ with $E_2 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Then $K_0(\mathcal{A}) = V_0(\mathcal{A})/\sim$ abelian group via $[Q] + [Q'] = [\begin{pmatrix} Q & 0 \\ 0 & Q' \end{pmatrix}]$

$$\begin{aligned}
V_0(\mathcal{A},\tau) \;&=\; \left\{ Q \in V_0(\mathcal{A}) \;\middle|\; \overline{Q} \;=\; Q \right\} \\
V_2(\mathcal{A},\tau) \;&=\; \left\{ Q \in V_{0,\mathrm{ev}}(\mathcal{A}) \;\middle|\; S_2^*\,\overline{Q}\,S_2 \;=\; -\,Q \right\} \qquad S_2 = \imath\sigma_2 \otimes \Sigma \\
V_4(\mathcal{A},\tau) \;&=\; \left\{ Q \in V_{0,\mathrm{ev}}(\mathcal{A}) \;\middle|\; S_4^*\,\overline{Q}\,S_4 \;=\; Q \right\} \qquad S_4 = \mathbf{1} \otimes \Sigma \\
V_6(\mathcal{A},\tau) \;&=\; \left\{ Q \in V_0(\mathcal{A}) \;\middle|\; S_6^*\,\overline{Q}\,S_6 \;=\; -\,Q \right\} \qquad S_6 = \imath\sigma_2 \otimes \mathbf{1}
\end{aligned}$$

$$K_{2i}(\mathcal{A},\tau) \;=\; V_{2i}(\mathcal{A},\tau)/\sim$$

Current projects and questions

Above index parings show that KR-groups pair with KR-cycles

Aim: direct proofs for exactness of connecting maps in KR-theory

Related recent preprint: Boersema and Loring 2015

Aim: spell out bulk-edge correspondence in real cases

Aim: further investigate physical implications of invariants
(zero modes, surface states, surface currents, Hall effects, polarization, magnetization,...)