Minicourse: Multiscale behaviour in selection-mutation systems

D. Dawson, A. Greven

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Part 3: Duality

1. Generalities on duality
2. Dualities for population models
3. Incorporating selection and mutation

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Goals

We want to study the *longtime behaviour* of

population models (Fleming-Viot type)

via *duality* which works, *both* on the level of

- configurations,
- genealogies,

based on the same **moment dual** with dual based on **coalescents.**

Good duals for configurations and genealogies are sometimes conflicting.

Definition (Duality)

A process $(X_t)_{t\geq 0}$ with state space E and $(X'_t)_{t\geq 0}$ with state space E' with E and E' both Polish spaces satisfies a *duality relation* if there exists a function

$$H : E \times E' \to \mathbb{R}, \quad H \in C_b(E \times E', \mathbb{R}), \tag{0.1}$$

such that

$$E[H(X_t, X'_0)] = E[H(X_0, X'_t)] \text{ for all } (X_0, X'_0) \in E \times E'. \tag{0.2}$$

Usually with dualities we assume that the family of functions

$$\{H(\cdot, X'_0), X'_0 \in E'\}, \tag{0.3}$$

is distribution and convergence-determining on the state space E. □

Generator criterion for duality

The duality relation between two Markov processes and associated semigroups with generators G_1 respectively G_2 w.r.t. to the function H is implied by the properties :

$$H(\cdot, X_0') \in \mathcal{D}(G_1), \quad H(X_0, \cdot) \in \mathcal{D}(G_2) \quad \forall\, (X_0, X_0') \in E \times E', \tag{0.4}$$

$$(G_1(H(\cdot, X_0')))(X_0) = (G_2(H(X_0, \cdot)))(X_0'), \quad \forall\, X_0 \in E, \quad X_0' \in E'. \tag{0.5}$$

Representation of f.d.d. (Fleischmann & Greven, 1996) [FG96]

We can represent for Markov processes with a *dual for the time-space process* based on *frozen states* the f.d.d. Consider initial states for dual at times

$$0 = t_0 < t_1 < \cdots < t_n = t$$

denoted

$$X'^{,1}, X'^{,2}, \cdots, X'^{,n}$$

which are frozen till time

$$t - t_1, t - t_2, \cdots, t - t_{n-1}, 0.$$

Then *space-time processes*

$$(s, X_s)_{s\in[0,t]}, (t-s, X'_{t-s})_{s\in[0,t]}$$

are dual w.r.t. θH, in case of moment duals

$$\theta H(\theta X, \theta X') = \Pi_{i=1}^{n} H((t_i, X_i), (t - t_i, X'^{,i})).$$

Definition (Feynman-Kac duality)

The processes are in Feynman-Kac duality with respect to H, satisfying (0.1) and (0.3), if there exists a function V on E' which is bounded and continuous such that:

$$E\left[H(X_t, X_0')\right] = E\left[H(X_0, X_t')exp\left(\int_0^t V(X_s')ds\right)\right]. \quad \square \tag{0.6}$$

Checking for Feynman-Kac duality entails to show:

$$(G_1 H(\cdot, X_0'))(X_0) = (G_2 H(X_0, \cdot))(X_0') + V(X_0')H(X_0, X_0'), \quad \forall\, (X_0, X_0') \in E \times E'. \tag{0.7}$$

This is a criterion one can typically check by explicit calculation.

1. Generalities on duality

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Dual process

We introduce now the dual process for the neutral population model. The Kingman coalescent is a process with values in the partitions of $\mathbb{N}$ with the rule,

$$\text{two partition elements coalesce (i.e. } \pi_1, \pi_2 \to \pi_1 \cup \pi_2\text{) at rate } d. \quad (0.8)$$

Choose f a bounded continuous function on $\mathbb{K}^{\mathbb{N}}$ which depends only on the L first entries for some $L \in \mathbb{N}$ and this object does not change under the dual dynamic (for the neutral model).

$$E = \mathcal{P}(\mathbb{K}), \quad E' = \{ \text{ partitions of } \mathbb{N}\} \cup C_b(\mathbb{K}^{\mathbb{N}}, \mathbb{R}), \tag{0.9}$$

$$H(X,Z) = \int_{\mathbb{K}^{\mathbb{N}}} f(u_{\pi(1)}, \cdots) X^{\otimes \mathbb{N}}(du_1, du_2, \cdots), \qquad Z = (\pi, f), \quad X \in \mathcal{P}(\mathbb{K}), \tag{0.10}$$

where the partition Π of $\mathbb{N}$ in Z is represented by a map. Partition elements are mapped on their rank, where they are ordered by their least element:

$$\pi : \mathbb{N} \to \mathbb{N}. \tag{0.11}$$

The case $\mathbb{K} = \{0,1\}, f(u_1, \cdots) = \prod_{i=1}^{n} 1_{\{1\}}(u_i)$ gives classical statement for FW-diffusion.

Lemma (Duality Fleming-Viot)

Assume that $s = m = c = 0$ *for* $(Y_t)_{t\geq 0}$ *and* $(\eta_t)_{t\geq 0}$ *w.r.t. H in (0.10), relations (0.2) and (0.3) hold.* □

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Function-valued dual

The dual process for the *Fleming-Viot diffusion with selection and mutation* is a *function-valued process* driven by a *particle system* denoted $(\eta_t, \mathcal{H}_t)_{t\geq 0}$,

$$\begin{aligned} &\eta \text{ is a (partition-valued)-Kingman coalescent (at rate } d) \\ &\text{with birth of new individual at rate } s \text{ per individual,} \end{aligned} \tag{0.12}$$

$$(\mathcal{H}_t)_{t\geq 0} \text{ is a } \textit{function-valued process} \text{ driven by } \eta. \tag{0.13}$$

$$E' = \bigcup_{N\in\mathbb{N}} \mathbb{P}(\lfloor N \rfloor)\times\left(\bigcup_{n=1}^{\infty} C_b((\mathbb{K})^n,\mathbb{R})\right) = (\bigcup_{N\in\mathbb{N}} \mathbb{P}\lfloor N \rfloor)\times\mathbb{S} \subseteq \mathbb{P}(\mathbb{N})\times C_b(\mathbb{K}^{\mathbb{N}},\mathbb{R}). \tag{0.14}$$

Start with n individuals creating the partition $\{\{1\},\cdots,\{n\}\}$ and with every individual (partition element) we associate a distinct variable and then $\mathcal{H}_0 = f$ is a function of n-variables defined in $(\mathbb{K})^n$ which is positive and bounded.

The dynamics of the system is as follows.

- Two transitions occur in the *particle system* (i.e. η),
  (1) at rate d two partition elements *coalesce* into one new partition element and
  (2) at rate s independently for each partition element a *birth* occurs, i.e. every partition element creates independent of everything else a new individual in the basic set which forms its own partition element.
- The *function-valued part* (i.e. f) makes the transitions corresponding to resampling, selection and mutation:

If a *coalescence* event occurs, the variables corresponding to the partition elements are set equal,

$$(0.15)$$

if a *birth* occurs for individual i and $\mathcal{H}_t = f$, with $f : (\mathbb{K})^n \to \mathbb{R}$, then $f \to \chi_i f \otimes 1 + f \otimes (1 - \chi_{n+1})$, an element of $C_b(\mathbb{K}^{n+1}, \mathbb{R})$,

$$(0.16)$$

at rate m for every variable u_i in $\mathcal{H}_t = f$, $f : (\mathbb{K})^n \to \mathbb{R}$, the following transition occurs:
$f \longrightarrow \int_{\mathbb{K}} f(u_1, \cdots, u_n)(M(v, du_i))$.

$$(0.17)$$

Note for alternative version:

$$\chi_i f \otimes 1 + f \otimes (1 - \chi_{n+1})) - f = (\chi_i f - f \otimes \chi_{n+1}) - f + f. \qquad (0.18)$$

Theorem

The FV-process with mutation and selection $(Y_t)_{t \geq 0}$ *and* $(\eta_t, \mathcal{H}_t)_{t \geq 0}$ *are in duality w.r.t.* $H(\cdot, \cdot)$. □

Remark

A classical result by Donnelly, Kurtz, Ethier results in a Feynman-Kac duality.
Here the transition is

$$f \to \chi(u_i)f - \chi(u_{m+1})f \tag{0.19}$$

and a Feynman-Kac term arises and the duality relation holds only for $t \leq t_0$ *for* some positive but finite t_0.
The Feynman-Kac potential V *is given by*

$$V(\eta_s) = s \mid \Pi_s \mid \tag{0.20}$$

Alternative Mutation transition

Introduce the following jumps in function space for $g \in L_\infty(\mathbb{K}^k)$:

- where M_i denotes the application of the operator M to the i-th variable of the function,
- jumps occur for a function $f \in L_\infty(\mathbb{K}^k)$ independently at rate m for each $i = 1, 2, \cdots, k$.

$$g(u_1, u_2, \cdots, u_k) \longrightarrow (M_i g)(u_1, u_2, \cdots, u_k), \; k \in \mathbb{N}, \qquad (0.21)$$

Selection mechanisms

Definition (Conditioned evolution of $\mathcal{F}$ and $\mathcal{F}^+, \mathcal{G}^+$: the selection mechanisms)

(i) If a birth occurs in the process (η_t) due to the partition element $\pi_i, i \in \{1, \dots, \mid \pi_t \mid\}$, then for $\mathcal{F}_{t-} = g$ the following transition occurs from an element in $L_\infty((\mathbb{K})^m)$ to elements in $L_\infty((\mathbb{K})^{m+1})$:

$$g(u_1, \dots, u_m) \longrightarrow \chi(u_i) g(u_1, \dots, u_m) - \chi(u_{m+1}) g(u_1, \dots, u_m), \tag{0.22}$$

where the new variable is associated with the partition element of the newly born individual

Definition

(i′) For $\mathcal{F}_t^+$ the transition (0.22) is replaced by

$$g(u_1,\cdots,u_m) \longrightarrow \widehat{g}(u_1,\ldots,u_{m+1}) = (\chi(u_i)+(1-\chi(u_{m+1})))g(u_1,\cdots,u_m), \tag{0.23}$$

where the new variable is associated with the newly born individual.

$$\parallel \widehat{g} \parallel_\infty \leq 2 \parallel g \parallel_\infty, \text{ and } \widehat{g} \geq 0, \text{ if } g \geq 0. \tag{0.24}$$

(i″) For $(\mathcal{G}_t^+)_{t\geq 0}$ we use the following transition:

$$\begin{aligned} g(u_1,\cdots,u_m) \quad &\longrightarrow \quad (\chi(u_i)1(u_{m+1})g(u_1,\cdots,u_m) \\ &+ \quad (1-\chi(u_i))g(u_1,\cdots,u_{i-1},u_{m+1},u_{i+1},\cdots,u_m), \end{aligned} \tag{0.25}$$

in which case

$$\parallel \widehat{g} \parallel_\infty \leq \parallel g \parallel_\infty .\square \tag{0.26}$$

Remark

The mechanism (i) will lead to a signed-function-valued process and requires a Feynman-Kac factor, while the mechanism (i') leads to a non-negative function-valued process and in this case NO Feynman-Kac factor is needed.
The mechanism (i'') results in a norm-preserving transition.

We refer to the respective function valued process as

$$\mathcal{F}, \mathcal{F}^+, \mathcal{G}^+. \tag{0.27}$$

Spatial case

To incorporate space we have to modify the duality function and the coalescent as follows:

- The spatial coalescent now has partition elements which have locations $\xi(1),\cdots,\xi(\mid \pi_t \mid)$ in Ω, which carry out $c \cdot a(\cdot,\cdot)$ *independent random walks* up to coalescence $(\bar{a}(i,j) = a(j,i))$.
- The duality function is now:
  $H : E \times E' \to \mathbb{R}$ by

$$H(X,(\eta,\mathcal{F})) = \int\limits_{\mathbb{K}} \cdots \int\limits_{\mathbb{K}} \mathcal{F}(u_1,\cdots,u_{|\pi|}) x_{\xi(1)}(du_1)\cdots x_{\xi(|\pi|)}(du_m). \tag{0.28}$$

Then the collections of bounded measureable functions

$$\{H(\cdot,(\eta,\mathcal{F})),(\eta,\mathcal{F}) \in E'\}, \quad (\{H(X,\cdot), \quad X \in (\mathcal{P}(\mathbb{K}))^{\Omega}\}) \tag{0.29}$$

are measure-determining on $(E,\mathcal{B}(E))$ respectively $(E',\mathcal{B}(E'))$.

Definition (Modified mutation dual)

We define the *modified dual process* $(\widehat{\eta}_t, \mathcal{F}_t)_{t \geq 0}$resp.$(\widehat{\eta}, \mathcal{F}_t^+)_{t \geq 0}(\widehat{\eta}_t, \mathcal{G}_t^+)_{t \geq 0}$ for a mutation matrix M satisfying $mM \geq \bar{m} 1 \otimes \rho$ as follows.

Enlarge the space Ω to

$$\Omega \cup \{*\}. \tag{0.30}$$

The dynamics of the process $(\widehat{\eta}_t, \mathcal{F}_t)$ resp. $(\widehat{\eta}_t, \mathcal{F}_t^+)_{t \geq 0}, (\widehat{\eta}_t, \mathcal{G}_t^+)_{t \geq 0}$ is now obtained by

- The transition rates of the dual particle process η in $*$ are 0.
- In addition the function-valued part does not change for variables associated with partition elements located on $\{*\}$.

Definition

- adding to the mechanism of η for partition elements still located on Ω (rather than $*$):

  jumps of the partition elements to $\{*\}$ after exponential waiting times at rate $\bar{m}$

  independently of each other (and independent of those of all other transitions)

- changing the dynamics of $\mathcal{F}_t, \mathcal{F}_t^+, \mathcal{G}_t^+$ by

  replacing mM_t generated by the mutation kernel M by the semigroup M_t^* corresponding to the transition rates $(mM - \bar{m}1 \otimes \rho)$. □

We can use both constructions presented in the previous slides at once:

Definition (Pure jump process dual)

In particular in combining the new representations of the state-dependent part $mM - \bar{m}1 \otimes \rho$ and the state-independent part in $\bar{m}1 \otimes \rho$ we get a pure Markov jump process

$$(\widehat{\eta}_t, \widehat{\mathcal{F}}_t)_{t \geq 0}, \ (\widehat{\eta}_t, \widehat{\mathcal{F}}^+_t)_{t \geq 0}, (\widehat{\eta}_t, \widehat{\mathcal{G}}^+_t)_{t \geq 0}. \qquad \square \qquad (0.31)$$

Concrete dualities

Now we have to tune the general construction to the case of finitely many types and concrete constructions:

- set-valued duals
- modified duals for selection transition
- taste of the application is in the last action.

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