Flat bands of edge states via weak invariants of semimetals

Hermann Schulz-Baldes, Tom Stoiber
Erlangen

December 2020

Bulk-boundary correspondence for $d = 1$

SSH model (Su-Schrieffer-Heeger 1980, polyacetelyn polymer)

$$H \;=\; \tfrac{1}{2}(\sigma_1 + i\sigma_2) \otimes S + \tfrac{1}{2}(\sigma_1 - i\sigma_2) \otimes S^* + \mu\,\sigma_2 \otimes \mathbf{1}$$

where S bilateral shift on $\ell^2(\mathbb{Z})$, $\mu \in \mathbb{R}$ mass and Pauli matrices

In their grading

$$H \;=\; \begin{pmatrix} 0 & S - i\mu \\ S^* + i\mu & 0 \end{pmatrix} \qquad \text{on } \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

Off-diagonal $\cong$ chiral symmetry $\sigma_3^* H \sigma_3 = -H$. In Fourier space:

$$H = \int_{[-\pi,\pi)}^{\oplus} dk\, H_k \qquad , \qquad H_k = \begin{pmatrix} 0 & e^{-ik} - i\mu \\ e^{ik} + i\mu & 0 \end{pmatrix}$$

Topological invariant for $\mu \neq -1, 1$

$$\text{Wind}(k \in [-\pi,\pi) \mapsto e^{ik} + i\mu) \;=\; \delta(\mu \in (-1,1))$$

Chiral bound states

Half-space Hamiltonian

$$\widehat{H} = \begin{pmatrix} 0 & \widehat{S} - i\mu \\ \widehat{S}^* + i\mu & 0 \end{pmatrix} \qquad \text{on } \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$$

where $\widehat{S}$ unilateral right shift on $\ell^2(\mathbb{N})$

Still chiral symmetry $\sigma_3^* \widehat{H} \sigma_3 = -\widehat{H}$

If $\mu = 0$, simple bound state at $E = 0$ with eigenvector $\psi_0 = \binom{|0\rangle}{0}$

Perturbations, *e.g.* in μ, cannot move or lift this bound state ψ_μ!

Positive chirality conserved: $\sigma_3 \psi_\mu = \psi_\mu$

Theorem (Basic bulk-boundary correspondence (BBC))

If $\widehat{P}$ projection on bound states of $\widehat{H}$, then

$$\mathrm{Wind}(k \mapsto e^{ik} + i\mu) = \mathrm{Tr}(\widehat{P}\sigma_3)$$

Generalizations

Theorem (Disordered Noether-Gohberg-Krein Theorem)

If $H = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$ *and* Π *Hardy projection on half-line, then* $\mathbb{P}$*-almost surely*

$$i\,\mathbf{E}_\omega\,\mathrm{Tr}\,\langle 0|A_\omega^{-1}\,i[X, A_\omega]|0\rangle \;=\; -\,\mathrm{Ind}(\Pi A_\omega \Pi) \;=\; \mathrm{Tr}(\widehat{P}\sigma_3)$$

- Even mobility gap regime can be attained (Graf-Shapiro)
- K-theoretic interpretation via boundary maps of Toeplitz extension
- Leads to higher dimensional generalizations (book with Prodan)

In this talk:

$2d$ graphene Hamiltonian is also chiral

but has only pseudogap (vanshing DOS at Fermi level)

This semimetal can have flat band of edge states!

Similar BBC? **How about even higher dimension?**

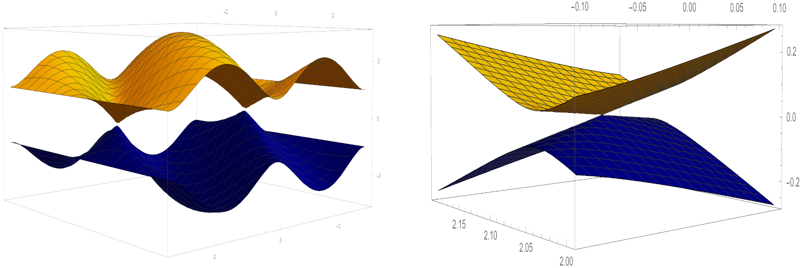
Model for graphene

On honeycomb lattice = decorated triangular lattice, so on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$

$$H = \begin{pmatrix} 0 & S_1 + S_1^* S_2 + 1 \\ S_1^* + S_2^* S_1 + 1 & 0 \end{pmatrix}$$

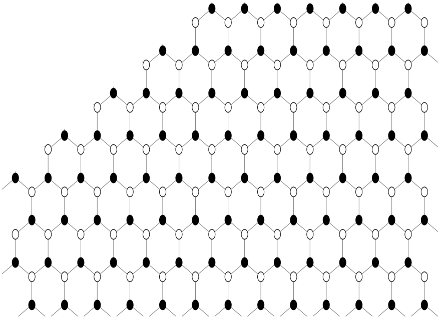
where S_1, S_2 shifts on $\ell^2(\mathbb{Z}^2)$. Clearly chiral $\sigma_3 H \sigma_3 = -H$. Fourier:

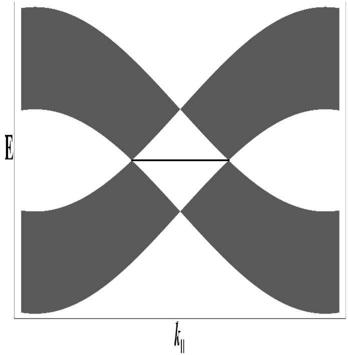
$$H \cong \int_{\mathbb{T}^2}^{\oplus} dk \begin{pmatrix} 0 & e^{ik_1} + e^{i(k_2-k_1)} + 1 \\ e^{-ik_1} + e^{-i(k_2-k_1)} + 1 & 0 \end{pmatrix}$$



Dirac points $k_\pm = (\frac{(3\pm1)\pi}{3}, 0)$ DOS vanishes at $E = 0$ (pseudogap)

Edges





Zigzag boundary $\cong$ replace S_1 by unilateral shift $\widehat{S}_1$

Armchair boundary $\cong$ replace S_2 by unilateral shift $\widehat{S}_2$

Fact (Saito, Dresselhaus ***et al.*** **1988):** edge states only for Zigzag

Edge states and BBC for surface DOS

$\xi = \binom{\xi_1}{\xi_2} \in \mathbb{S}^1$ direction perpendicular to boundary (possibly irrational)

$\widehat{H} = \Pi_\xi \, H \, \Pi_\xi$ half-space restriction of graphene Hamiltonian

Kernel projection $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$ on flat band of surface states

$\widehat{\mathcal{T}}$ trace per unit volume along the boundary

bulk Fermi unitary $U = (S_1 + S_1^* S_2 + 1)|S_1 + S_1^* S_2 + 1|^{-1}$

Theorem (with Tom Stoiber)

$$i\,\mathcal{T}(U^{-1}\nabla_\xi U) \;=\; \widehat{\mathcal{T}}(\widehat{P}_+) - \widehat{\mathcal{T}}(\widehat{P}_-)$$

where $\mathcal{T}(B) = \mathbf{E}\,\mathrm{Tr}(\langle 0|B|0\rangle)$ *and* $\nabla_\xi = \xi \cdot \nabla$ *with* $\nabla_j B = i[X_j, B]$

Moreover, result stable under chiral surface disorder

Proves existence of edge states (generalizes Feffermann, Weinstein)

Singularities of Fermi unitary and Besov spaces

Fourier $U \cong \int dk\ U(k)$ with

$$U(k) \;=\; \frac{e^{ik_1} + e^{i(k_2-k_1)} + 1}{|e^{ik_1} + e^{i(k_2-k_1)} + 1|}$$

Vorticities at Dirac points, not even continuous, so $U \notin C(\mathbb{T}^2)$

But U lies in Besov $B^1_{1,1}(\mathbb{T}^2)$, namely for all ξ:

$$\int_0^1 \frac{dt}{t^2} \int dk \, \big| U(k+\xi t) + U(k-\xi t) - 2\,U(k) \big| \;<\; \infty$$

Similarly $U \in B^{1/2}_{2,2}(\mathbb{T}^2) = H^{1/2}(\mathbb{T}^2)$. Enough for index theory because:

Peller (1980's):

Hankel operators on $\mathbb{T}^1$ with Besov symbols have traceclass properties

$f \in B^{1/p}_{p,p}(\mathbb{T}^1) \iff \Pi f(\mathbf{1} - \Pi) \in \mathcal{L}^p$ Schatten ideal (Π Hardy proj.)

Implication: weak invariant $i\,\mathcal{T}(U^{-1}\nabla U)$ well-defined and index thm

Further remarks

$$i\,\mathcal{T}(U^{-1}\nabla_\xi U) \;=\; i\,\mathcal{T}(U^{-1}\nabla_1 U)\,\xi_1 \;+\; i\,\mathcal{T}(U^{-1}\nabla_2 U)\,\xi_2 \;=\; \frac{1}{3}\,\xi_2$$

Latter in graphene ($\Longrightarrow$ difference zigzag $\xi_2 = 1$ and armchair $\xi_2 = 0$)

Value $\frac{1}{3}$ is **not** topological (it is relative distance between Dirac points)!

Pairing $\langle[\xi \cdot X], [U]_1\rangle = i\,\mathcal{T}(U^{-1}\nabla_\xi U)$ over huge algebra $C^*(B^{1/2}_{2,2} \cap L^\infty)$

Thus values **not** in discrete range of $[U]_1 \in K_1(\mathcal{A}) \mapsto \langle[\xi \cdot X], [U]_1\rangle$

Changing H continuously, changes value of $i\,\mathcal{T}(U^{-1}\nabla_\xi U)$ continuously

Only BBC equality always holds and is hence topological

Similar situation: Levinson's theorem for scattering on hypersurfaces

Next: extension to disordered chiral systems and higher dimension

Hypothesis: pseudo-gap or Anderson localization at $E = 0$

Higher dimension and disorder

Disordered d-dimensional rotation C^*-algebra $\mathcal{A} = C(\Omega) \rtimes_{\mathbf{B}} \mathbb{Z}^d$

Trace per unit volume $\mathcal{T} \rightsquigarrow \mathcal{M} = L^\infty(\mathcal{A}, \mathcal{T})$. Derivations $(\nabla_1, \ldots, \nabla_d)$

$\xi \in \mathbb{S}^{d-1}$ direction perpendicular to hypersurface, $\widehat{\mathcal{T}}$ trace along it

Theorem (with Tom Stoiber)

$H \in M_{2L}(\mathcal{A})$ with chiral symmetry $\sigma_3 H \sigma_3 = -H$

Suppose bulk H either has pseudo-gap at 0, namely $\gamma > 1$ with

$$\mathcal{T}\big(\chi(|H| \leqslant \epsilon)\big) \;\leqslant\; C_\gamma \, \epsilon^\gamma$$

and or has mobility gap in $(-\epsilon_0, \epsilon_0)$, that is, for some $s \in (0, 1)$

$$\sup_{|\epsilon| \leqslant \epsilon_0} \mathbf{E} \, \| \langle 0 | (H - \epsilon + i\,0)^{-1} | m \rangle \|^s \;\leqslant\; C_s \, e^{-\beta_s |m|}$$

Then, for Fermi unitary U and kernel projection $\widehat{P} = \widehat{P}_+ + \widehat{P}_-$ as above,

$$i\,\mathcal{T}(U^{-1} \nabla_\xi U) \;=\; \widehat{\mathcal{T}}(\widehat{P}_+) - \widehat{\mathcal{T}}(\widehat{P}_-)$$

Structural elements of proof

- harmonic analysis on semifinite W*-algebras with $\mathbb{R}^d$-action
- based on (dynamical) Arveson spectrum
- construction of non-commutative Besov spaces
- Peller criteria for traceclass properties
- index theorems for invariants with Besov symbols (beyond C^k)
- reformulation of Breuer-Fredholm indices as boundary invariants
- pseudogap and mobility gap imply Besov properties

rather heavy artillery - but intrinsically interesting pure math

Covers gapped models, but applies to others, *e.g.* Weyl semimetals

10 slides on proof, first numerics and spectral localizer in semimetals

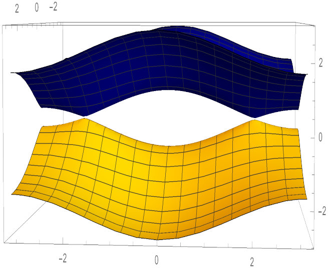
Stacked SSH as chiral $2d$ toy model

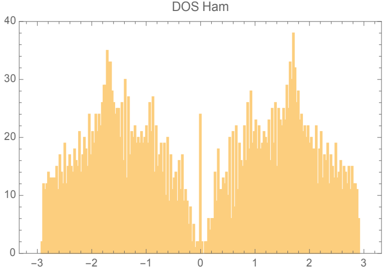
SSH in direction 1 with coupling in direction 2 and chiral randomness

$$H = \begin{pmatrix} 0 & S_1 - \mu \\ S_1^* - \mu & 0 \end{pmatrix} - \delta \begin{pmatrix} 0 & S_2 + S_2^* \\ S_2 + S_2^* & 0 \end{pmatrix} + \lambda \sum_{n \in \mathbb{Z}^2} v_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where v_n i.i.d. random variables with uniform distribution in $[-\frac{1}{2}, \frac{1}{2}]$

(2 or 4) Dirac points for periodic model if $k_1 = 0, \pi$, $2\delta \cos(k_2) + \mu = \pm 1$

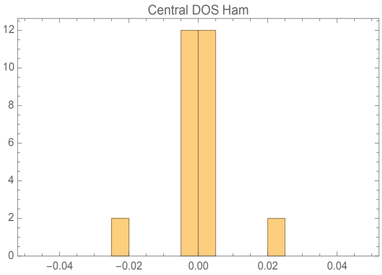




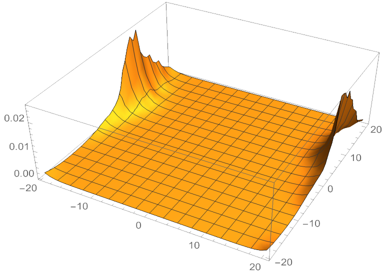
$\lambda = 0.2$, $\mu = 1.3$, $\delta = 0.3$ and volume $[-\rho, \rho]^2$ with $\rho = 20$

Central DOS and one of the edge states

Zoom into the central DOS



Same parameters as above



There are $28 = 2 \cdot 14$ (approximate) zero modes of H

Corresponding eigenstates only on two opposite edges

(edges weakly coupled, edge states vanish on other edges!)

Edge state dens. $= \frac{14}{41} \approx i\mathcal{T}(U^{-1}\nabla_1 U) = \int \frac{dk_2}{2\pi}\, \chi(\mu + 2\delta\cos(k_2) < 1) \approx \frac{1}{3}$

Here first $\approx$ is precisely the equality in the theorem (1 chiral sector)

Spectral localizer (with Terry Loring)

$$L_\kappa \;=\; \begin{pmatrix} -H & \kappa\, D^* \\ \kappa\, D & H \end{pmatrix}$$

where $D = \sum_{j=1}^{d} \Gamma_j X_j$ Dirac and $\kappa > 0$ tuning parameter

$L_{\kappa,\rho}$ finite volume restriction to $[-\rho,\rho]^d$ with Dirichlet conditions

Theorem (with Terry Loring)

d even and $\Gamma_d = i\,\mathbf{1}$ (so $D = X_1 + iX_2$ for $d = 2$)

Suppose H has gap g around 0 and let $P = \chi(H < 0)$ Fermi projection

$$\mathrm{Ch}_d(P) \;=\; \frac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho})$$

provided that

$$\kappa \;<\; \frac{12\,g^3}{\|H\|\,\|[D,H]\|} \qquad\qquad \rho \;>\; \frac{2\,g}{\kappa}$$

Localizes topological information in spectral manner by Dirac trap

Variations of spectral localizer

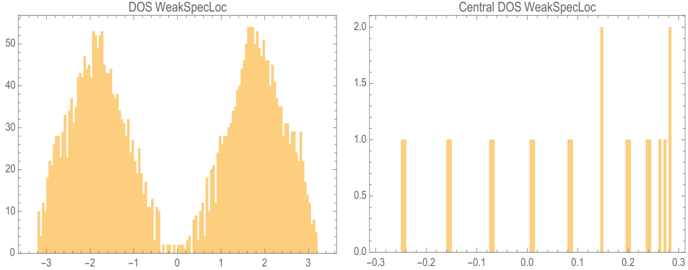
- In odd dimension for chiral systems (first paper with Terry Loring)
- Also works in mobility gap regime
  (numerics with Lozano-Viesca and Schober, no proof yet)
- Works for spin Chern numbers and alike
  (”approximate conservation laws and symmetries”, with Nora Doll)
- Works for $\mathbb{Z}_2$-indices (guessed by Terry Loring)
  (”signs of Pfaffians of skew spectral localizer”, with Nora Doll)
- Extends to semifinite setting for weak invariants (with Tom Stoiber)
- Allows to study invariants in semimetals (explained next):
  Compute weak winding numbers for above theorem
  Counts Dirac and Weyl points in disordered semimetals

Weak spectral localizer for weak winding numbers

$$L_\kappa^{\text{weak}} = \begin{pmatrix} \kappa X_1 & A_{\text{per}}^* \\ A_{\text{per}} & -\kappa X_1 \end{pmatrix} \qquad H_{\text{per}} = \begin{pmatrix} 0 & A_{\text{per}}^* \\ A_{\text{per}} & 0 \end{pmatrix}$$

H_{per} stacked SSH H periodized in 2-direction $\qquad \kappa = 0.1$

As above $\lambda = 0.2$, $\mu = 1.3$, $\delta = 0.3$ and volume $[-\rho, \rho]^2$ with $\rho = 20$



Half-signature of $L_{\kappa,\rho}^{\text{weak}} \approx 14$

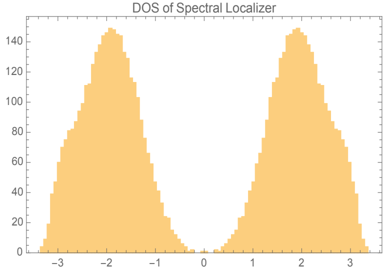
weak winding number $i\mathcal{T}(U^{-1}\nabla_1 U) =$ half-signature density $\approx \frac{14}{41} \approx \frac{1}{3}$

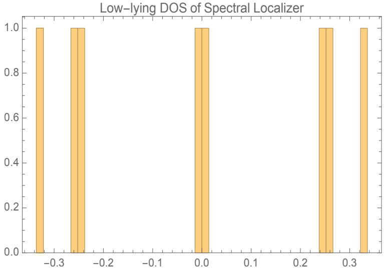
Approximate zero modes of spectral localizer

$$L_\kappa \;=\; \begin{pmatrix} -H & \kappa\,\mathbf{1}_2\otimes(X_1+\imath X_2) \\ \kappa\,\mathbf{1}_2\otimes(X_1-\imath X_2) & H \end{pmatrix} \;=\; -\,\sigma_1\otimes\mathbf{1}\,L_\kappa\,\sigma_1\otimes\mathbf{1}$$

Vanishing signature (Chern number vanishes due to chiral symmetry)

$L_{\kappa,\rho}$ restriction to $[-\rho,\rho]^2$ Stacked SSH as above and $\kappa = 0.07$





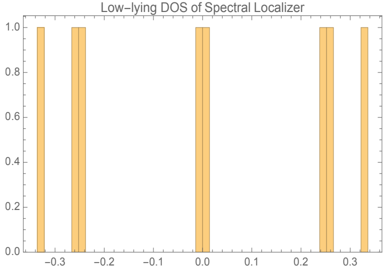
Approximate kernel of multiplicity 2 = number of Dirac points

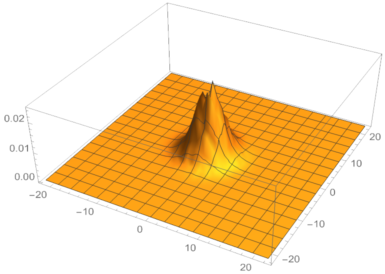
Very large gap to first excited $\approx \sqrt{\kappa} \approx 0.26$ (as for Dirac Ham.)

Gap above groundstate as for Dirac Hamiltonian (explicit computation)

Ground states of spectral localizer

Plot of modulus (over 4-dim fiber) of one of the two ground states:





lowest eigenvalue $\nu \approx C\,\lambda$ with C very small (perturbation theory)

For $\lambda = 0$, one has $\nu \approx e^{-1/\kappa}$ (phase space tunnelling)

Approximate kernel dimension counts number of Dirac points

Conclusion: Concept of number of Dirac points stable under disorder

Moreover: existence of Dirac points $\Longrightarrow$ non-vanishing weak windings

Why it works so well (for general dimension d):

In Fourier space:

$$\mathcal{F} L_\kappa^2 \mathcal{F}^* \;=\; -\kappa^2 \sum_{j=1}^{d} \partial_{k_j}^2 \;+\; \begin{pmatrix} (H_k)^2 & \kappa \sum_{j=1}^{d} \Gamma_j (\partial_{k_j} H_k) \\ \kappa \sum_{j=1}^{d} \Gamma_j (\partial_{k_j} H_k) & (H_k)^2 \end{pmatrix}$$

Second oder differential operator on $L^2(\mathbb{T}^2, \mathbb{C}^{2L})$

As in semi-classical analysis with $\hbar = \kappa$

IMS localization isolates Dirac points

At each Dirac point explicitly solvable Dirac Hamiltonians

Each Dirac Hamiltonian has simple zero mode and a gap of order κ

Theorem

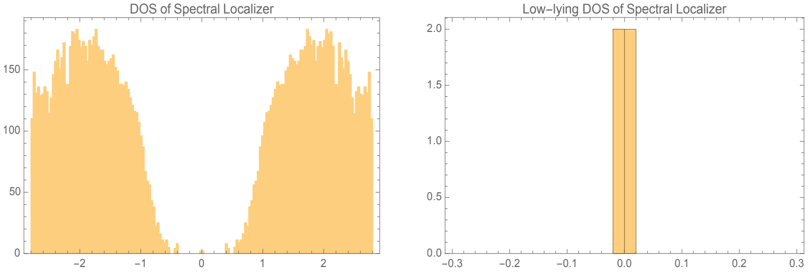
L_κ has as many eigenvalues $\leqslant \kappa$ as H has Dirac points

Next excited level is $\mathcal{O}(\sqrt{\kappa})$

Weyl points of $3d$ systems (same strategy)

$$H = H_{p+ip} + \delta \begin{pmatrix} 0 & S_3 + S_3^* \\ S_3 + S_3^* & 0 \end{pmatrix} + H_{\text{Weyl shift}} + \lambda H_{\text{dis}}$$

$H_{\text{Weyl shift}}$ shifts Weyl points to different energies (no pseudogap)



$\rho = 7$, so cube of size 15, $\delta = 0.6$, $\mu = 1.2$, $\lambda = 0.5$, $\kappa = 0.1$

Approximate kernel dimension counts number of Weyl points

Existence of Weyl points $\Longrightarrow$ non-vanishing weak Chern numbers

$\Longrightarrow$ surface currents (as in QHE, extension of Besov techniques)

Constructions for definition of Besov spaces:

Semifinite trace $\mathcal{T}$ gives von Neumann algebra $\mathcal{M}$

Non-commutative spaces $X = L^p(\mathcal{M})$, $p \geqslant 1$, Banach spaces

$L^2(\mathcal{M})$ is GNS-Hilbert space of $\mathcal{T}$

$\mathbb{R}^n$-action α on $\mathcal{M}$ which leaves $\mathcal{T}$ invariant

$\mathcal{T}$-invariance $\implies$ α extends isometrically to action β on $X = L^p(\mathcal{M})$

For $f \in L^1(\mathbb{R}^n)$ and $x \in X$ define $\beta_f(x)$ as Riemann integral

$$\beta_f(x) \;=\; \int_{\mathbb{R}^n} f(-t)\,\beta_t(x)\,dt$$

Then for $f \in FA(\mathbb{R}^n) = \mathcal{F}L^1(\mathbb{R}^n)$ define Fourier multiplier $\widehat{f}\,* \in \mathcal{B}(X)$ by

$$\widehat{f} * x \;=\; \beta_{\mathcal{F}^{-1}f}(x)$$

$\sigma(x) =$ Arveson spectrum $= \{\lambda \in \hat{\mathbb{R}} : f(\lambda) = 0 \;\text{ if }\; \widehat{f} * x = 0,\; f \in \mathcal{F}L^1\}$

Non-commutative Besov spaces:

X Banach space with isometric $\mathbb{R}^n$-action β on X (above $X = L^p(\mathcal{M})$)

Given smooth $\varphi : \mathbb{R} \to [0,1]$ supported by $[-2,-2^{-1}] \cup [2^{-1},2]$ and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) \;=\; 1$$

Littlewood-Payley dyadic decomposition $(W_j)_{j\in\mathbb{N}}$ by

$$W_j \;=\; \varphi(|2^{-j}\cdot|) \quad \text{for } j>0\,, \qquad W_0 \;=\; 1-\sum_{j>0} W_j$$

Now:

$$B^s_q(X) \;=\; \left\{ x\in X \;:\; \|x\|_{B^s_q(X)} = \Big(\sum_{j\geqslant 0} 2^{qsj}\,\|\widehat{W_j}*x\|^q_X\Big)^{\frac{1}{q}} < \infty \right\}$$

Set

$$B^s_{p,q}(\mathcal{M}) \;=\; B^s_q(L^p(\mathcal{M}))$$

Properties of Besov spaces:

Proposition

Definition of $B^s_q(X)$ independent of choice of φ

$(B^s_q(X), \|\,.\,\|_{B^s_q(X)})$ *Banach space for $s \in \mathbb{R}$ and $q \in [1, \infty)$*

An equivalent norm is given by

$$\|x\|_{\widetilde{B}^s_q(X)} \;=\; \|x\|_X \;+\; \left(\int_{[0,1]} t^{-sq}\, \omega^N_X(x,t)^q \,\frac{dt}{t} \right)^{\frac{1}{q}}$$

where

$$\omega^N_X(x,t) \;=\; \sup_{|r|\leqslant t} \|\Delta^N_r(x)\|_X \qquad , \qquad N \geqslant s$$

with finite difference operator $\Delta_t : X \to X$ given by

$$\Delta_t(x) \;=\; \beta_t(x) - x$$

Corollary

For $B^s_{p,q}(\mathcal{M}) = B^s_q(L^p(\mathcal{M}))$ and $s \in [0,1]$, $B^s_{p,q}(\mathcal{M}) \cap \mathcal{M}$ is a $*$-algebra

More constructions:

Crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}^n$ with semifinite trace $\widehat{\mathcal{T}}$ (via Hilbert algebras)

In application $n = 1$: $\widehat{\mathcal{T}}$ is trace per unit volume along the boundary

W^*-crossed product defined in regular representation on $L^2(\mathbb{R}^n, \mathcal{H})$

$$\mathcal{N} \;=\; L^\infty(\mathcal{M} \rtimes_\alpha \mathbb{R}^n, \widehat{\mathcal{T}}) \;=\; \mathcal{M} \rtimes_\alpha \mathbb{R}^n \;\subset\; \mathcal{B}(L^2(\mathbb{R}^n, \mathcal{H}))$$

Contains bd. Borel functions of $D = (D_1, \ldots, D_n) = i\partial_t$ on $L^2(\mathbb{R}^n, \mathcal{H})$

Furthermore: L^p-spaces $L^p(\mathcal{N}, \widehat{\mathcal{T}})$ for $p \geqslant 1$

Irrep of complex Clifford algebra generated by $\Gamma_1, \ldots, \Gamma_n \in M_{2N}$ with

$$\{\Gamma_i, \Gamma_j\} \;=\; 0 \qquad , \qquad \Gamma_j^2 = \mathbf{1}$$

Introduce Dirac operator affiliated with $M_{2N}(\mathcal{N})$

$$\mathbf{D} \;=\; \sum_{j=1}^{n} \Gamma_j \otimes D_j$$

Peller criterion for Hankel operators:

Hardy projection $\Pi = \chi(\mathbf{D} > 0)$ in $M_{2N}(\mathcal{N})$, but not $L^p(M_{2N}(\mathcal{N}), \mathrm{Tr} \otimes \hat{\mathcal{T}})$

Now for "symbol" $A \in \mathcal{M}$, Toeplitz and Hankel operators in $M_{2N}(\mathcal{N})$ are

$$T_A = \Pi A \Pi \quad , \quad H_A = \Pi A (\mathbf{1} - \Pi)$$

Theorem

For all $p > n$ and $A \in \mathcal{M} \cap B^{n/p}_{p,p}(\mathcal{M})$, one has $H_A \in L^p(M_{2N}(\mathcal{N}), \mathrm{Tr} \otimes \hat{\mathcal{T}})$

For $n = 1$, also $p = 1$ is sufficient

Proof: explicit calculations for $p = 1$

L^2-estimates for weighted Hankels with symbol $B^{p/2}_{2,2}$ for $p > 2$

Involved estimates on weighted Hankels for $p = \infty$

Intricate application *à la Peller* of analytic interpolation (*e.g.* Lunardi) □

Classical case is $n = 1$ and $\mathcal{M} = L^\infty(\mathbb{R})$ with $\alpha_t(f)(y) = f(y + t)$

Index theorem

Theorem

$(\mathcal{M},\mathcal{T})$ semifinite von Neumann with $\mathbb{R}^n$-action α leaving $\mathcal{T}$ invariant

Generators of α on $\mathcal{M}$ denoted by $\nabla_1,\ldots,\nabla_n$

Let n be odd and unitary $U\in\mathcal{M}$ with $U-\mathbf{1}\in B^{n/(n+1)}_{n+1,n+1}$, then

$$c_n\sum_{\sigma\in S_n}(-1)^\sigma\,\mathcal{T}\Big(\prod_{j=1}^{n}U^{-1}\nabla_{\sigma(j)}U\Big)\;=\;\widehat{\mathcal{T}}\text{-Ind}\big(\Pi\,U\,\Pi+(\mathbf{1}-\Pi)\big)$$

where semifinite index of $\widehat{\mathcal{T}}$-Breuer-Fredholm $T\in M_{2N}(\mathcal{N})$ is

$$\widehat{\mathcal{T}}\text{-Ind}(T)\;=\;\widehat{\mathcal{T}}(\mathrm{Ker}(T))\;-\;\widehat{\mathcal{T}}(\mathrm{Ker}(T^*))$$

Similar results for n even

Important: no differentiability assumption (as Lesch, Wahl for $n=1$)

Comments:

Proof: uses Peller criterion

Geometric identities like Connes' triangle identity

Semi-finite Calderon-Fedosov formula for index □

Application: solid state systems with disorder probability space

$$\mathcal{M} = L^{\infty}(C(\Omega) \rtimes_{\mathbf{B}} \mathbb{Z}^d, \mathbb{P})$$

with action α given by n-dimensional subgroup of dual $\mathbb{T}^d$-action

Case $n = d$: previous index theorem for strong invariants
(uses Takai-Takesaki duality)

For $n = 1$: geometric interpretation of Π as half-space projection
$\Longrightarrow$ surface states (next slide)

For $1 < n < d$: still under investigation, but likely general BBC
(as in Prodan/Schulz-Baldes)

Surface states via index theorem

H chiral Hamiltonian and $\widehat{H} = \Pi H \Pi$ with polar decompositions

$$\mathrm{sgn}(H) = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix} \quad , \quad \mathrm{sgn}(\widehat{H}) = \begin{pmatrix} 0 & \widehat{U} \\ \widehat{U}^* & 0 \end{pmatrix}$$

If (i) $U \in B^{1/2}_{2,2}(\mathcal{M})$, (ii) $\widehat{U} - \Pi U \Pi$ is $\widehat{\mathcal{T}}$-compact , (iii) physical rep.,

$$\widehat{\mathcal{T}}(\widehat{P}_+ - \widehat{P}_-) = \widehat{\mathcal{T}}(\sigma_3 \,\mathrm{Ker}(\widehat{H})) = \widehat{\mathcal{T}}\text{-Ind}(\widehat{U}) = \widehat{\mathcal{T}}\text{-Ind}(\Pi U \Pi)$$

and then index theorem implies Theorem in first part

Tough analytical issue: pseudogap or mobility gap imply (i) and (ii)

One main idea is that γ-pseudogap condition implies for $p > 0$

$$H^{-1} \in L^p(\mathcal{M}) \quad \text{and} \quad \|H^{-1} - (H+z)^{-1}\|_p \leqslant C|\Im m(z)|^{(\gamma - p)/p}$$

Used to estimate $\Pi\,\mathrm{sgn}(H)\,\Pi - \mathrm{sgn}(\widehat{H})$ after functional calculus

Constructions for definition of Besov spaces:

Semifinite trace $\mathcal{T}$ gives von Neumann algebra $\mathcal{M}$

Non-commutative spaces $X = L^p(\mathcal{M})$, $p \geqslant 1$, Banach spaces

$L^2(\mathcal{M})$ is GNS-Hilbert space of $\mathcal{T}$

$\mathbb{R}^n$-action α on $\mathcal{M}$ which leaves $\mathcal{T}$ invariant

$\mathcal{T}$-invariance $\implies$ α extends isometrically to action β on $X = L^p(\mathcal{M})$

For $f \in L^1(\mathbb{R}^n)$ and $x \in X$ define $\beta_f(x)$ as Riemann integral

$$\beta_f(x) \; = \; \int_{\mathbb{R}^n} f(-t)\, \beta_t(x)\, dt$$

Then for $f \in FA(\mathbb{R}^n) = \mathcal{F}L^1(\mathbb{R}^n)$ define Fourier multiplier $\widehat{f} * \in \mathcal{B}(X)$ by

$$\widehat{f} * x \; = \; \beta_{\mathcal{F}^{-1}f}(x)$$

$\sigma(x) =$ Arveson spectrum $= \{\lambda \in \hat{\mathbb{R}} : f(\lambda) = 0 \;\text{ if }\; \widehat{f} * x = 0,\; f \in \mathcal{F}L^1\}$

Non-commutative Besov spaces:

X Banach space with isometric $\mathbb{R}^n$-action β on X (above $X = L^p(\mathcal{M})$)

Given smooth $\varphi : \mathbb{R} \to [0,1]$ supported by $[-2,-2^{-1}] \cup [2^{-1},2]$ and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) \;=\; 1$$

Littlewood-Payley dyadic decomposition $(W_j)_{j\in\mathbb{N}}$ by

$$W_j \;=\; \varphi(|2^{-j}\cdot|) \quad \text{for } j>0\,, \qquad W_0 \;=\; 1-\sum_{j>0} W_j$$

Now:

$$B^s_q(X) \;=\; \left\{ x\in X \;:\; \|x\|_{B^s_q(X)} = \Big(\sum_{j\geqslant 0} 2^{qsj}\,\|\widehat{W_j} * x\|^q_X\Big)^{\frac{1}{q}} < \infty \right\}$$

Set

$$B^s_{p,q}(\mathcal{M}) \;=\; B^s_q(L^p(\mathcal{M}))$$

Properties of Besov spaces:

Proposition

Definition of $B^s_q(X)$ independent of choice of φ

$(B^s_q(X), \| \,.\, \|_{B^s_q(X)})$ *Banach space for $s \in \mathbb{R}$ and $q \in [1,\infty)$*

An equivalent norm is given by

$$\|x\|_{\tilde{B}^s_q(X)} \;=\; \|x\|_X \;+\; \left(\int_{[0,1]} t^{-sq}\, \omega^N_X(x,t)^q \,\frac{dt}{t} \right)^{\frac{1}{q}}$$

where

$$\omega^N_X(x,t) \;=\; \sup_{|r| \leqslant t} \|\Delta^N_r(x)\|_X \qquad , \qquad N \geqslant s$$

with finite difference operator $\Delta_t : X \to X$ given by

$$\Delta_t(x) \;=\; \beta_t(x) - x$$

Corollary

For $B^s_{p,q}(\mathcal{M}) = B^s_q(L^p(\mathcal{M}))$ and $s \in [0,1]$, $B^s_{p,q}(\mathcal{M}) \cap \mathcal{M}$ is a $*$-algebra

More constructions:

Crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}^n$ with semifinite trace $\widehat{\mathcal{T}}$ (via Hilbert algebras)

In application $n = 1$: $\widehat{\mathcal{T}}$ is trace per unit volume along the boundary

W^*-crossed product defined in regular representation on $L^2(\mathbb{R}^n, \mathcal{H})$

$$\mathcal{N} \;=\; L^\infty(\mathcal{M} \rtimes_\alpha \mathbb{R}^n, \widehat{\mathcal{T}}) \;=\; \mathcal{M} \rtimes_\alpha \mathbb{R}^n \;\subset\; \mathcal{B}(L^2(\mathbb{R}^n, \mathcal{H}))$$

Contains bd. Borel functions of $D = (D_1, \ldots, D_n) = i\partial_t$ on $L^2(\mathbb{R}^n, \mathcal{H})$

Furthermore: L^p-spaces $L^p(\mathcal{N}, \widehat{\mathcal{T}})$ for $p \geqslant 1$

Irrep of complex Clifford algebra generated by $\Gamma_1, \ldots, \Gamma_n \in M_{2N}$ with

$$\{\Gamma_i, \Gamma_j\} \;=\; 0 \qquad , \qquad \Gamma_j^2 = \mathbf{1}$$

Introduce Dirac operator affiliated with $M_{2N}(\mathcal{N})$

$$\mathbf{D} \;=\; \sum_{j=1}^{n} \Gamma_j \otimes D_j$$

Peller criterion for Hankel operators:

Hardy projection $\Pi = \chi(\mathbf{D} > 0)$ in $M_{2N}(\mathcal{N})$, but not $L^p(M_{2N}(\mathcal{N}), \mathrm{Tr} \otimes \hat{\mathcal{T}})$

Now for "symbol" $A \in \mathcal{M}$, Toeplitz and Hankel operators in $M_{2N}(\mathcal{N})$ are

$$T_A = \Pi A \Pi \qquad , \qquad H_A = \Pi A (\mathbf{1} - \Pi)$$

Theorem

For all $p > n$ and $A \in \mathcal{M} \cap B^{n/p}_{p,p}(\mathcal{M})$, one has $H_A \in L^p(M_{2N}(\mathcal{N}), \mathrm{Tr} \otimes \hat{\mathcal{T}})$

For $n = 1$, also $p = 1$ is sufficient

Proof: explicit calculations for $p = 1$

L^2-estimates for weighted Hankels with symbol $B^{p/2}_{2,2}$ for $p > 2$

Involved estimates on weighted Hankels for $p = \infty$

Intricate application *à la Peller* of analytic interpolation (*e.g.* Lunardi) □

Classical case is $n = 1$ and $\mathcal{M} = L^\infty(\mathbb{R})$ with $\alpha_t(f)(y) = f(y + t)$

Index theorem

Theorem

$(\mathcal{M}, \mathcal{T})$ semifinite von Neumann with $\mathbb{R}^n$-action α leaving $\mathcal{T}$ invariant

Generators of α on $\mathcal{M}$ denoted by $\nabla_1, \ldots, \nabla_n$

Let n be odd and unitary $U \in \mathcal{M}$ with $U - \mathbf{1} \in B^{n/(n+1)}_{n+1,n+1}$, then

$$c_n \sum_{\sigma \in S_n} (-1)^{\sigma} \, \mathcal{T}\Big(\prod_{j=1}^{n} U^{-1} \nabla_{\sigma(j)} U \Big) \;=\; \widehat{\mathcal{T}}\text{-Ind}\big(\Pi \, U \, \Pi + (\mathbf{1} - \Pi) \big)$$

where semifinite index of $\widehat{\mathcal{T}}$-Breuer-Fredholm $T \in M_{2N}(\mathcal{N})$ is

$$\widehat{\mathcal{T}}\text{-Ind}(T) \;=\; \widehat{\mathcal{T}}(\text{Ker}(T)) \,-\, \widehat{\mathcal{T}}(\text{Ker}(T^*))$$

Similar results for n even

Important: no differentiability assumption (as Lesch, Wahl for $n = 1$)

Comments:

Proof: uses Peller criterion

Geometric identities like Connes' triangle identity

Semi-finite Calderon-Fedosov formula for index $\square$

Application: solid state systems with disorder probability space

$$\mathcal{M} \;=\; L^\infty(C(\Omega) \rtimes_{\mathbf{B}} \mathbb{Z}^d, \mathbb{P})$$

with action α given by n-dimensional subgroup of dual $\mathbb{T}^d$-action

Case $n = d$: previous index theorem for strong invariants

(uses Takai-Takesaki duality)

For $n = 1$: geometric interpretation of Π as half-space projection

$\Longrightarrow$ surface states (next slide)

For $1 < n < d$: still under investigation, but likely general BBC

(as in Prodan/Schulz-Baldes)

Surface states via index theorem

H chiral Hamiltonian and $\widehat{H} = \Pi H \Pi$ with polar decompositions

$$\mathrm{sgn}(H) = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix} \quad , \quad \mathrm{sgn}(\widehat{H}) = \begin{pmatrix} 0 & \widehat{U} \\ \widehat{U}^* & 0 \end{pmatrix}$$

If (i) $U \in B^{1/2}_{2,2}(\mathcal{M})$, (ii) $\widehat{U} - \Pi U \Pi$ is $\widehat{\mathcal{T}}$-compact , (iii) physical rep.,

$$\widehat{\mathcal{T}}(\widehat{P}_+ - \widehat{P}_-) = \widehat{\mathcal{T}}(\sigma_3 \,\mathrm{Ker}(\widehat{H})) = \widehat{\mathcal{T}}\text{-Ind}(\widehat{U}) = \widehat{\mathcal{T}}\text{-Ind}(\Pi U \Pi)$$

and then index theorem implies Theorem in first part

Tough analytical issue: pseudogap or mobility gap imply (i) and (ii)

One main idea is that γ-pseudogap condition implies for $p > 0$

$$H^{-1} \in L^p(\mathcal{M}) \quad \text{and} \quad \|H^{-1} - (H+z)^{-1}\|_p \leqslant C|\Im m(z)|^{(\gamma - p)/p}$$

Used to estimate $\Pi\,\mathrm{sgn}(H)\,\Pi - \mathrm{sgn}(\widehat{H})$ after functional calculus