Propagation of singularities for solutions to Hamilton-Jacobi equations

Piermarco Cannarsa

University of Rome “Tor Vergata”

Webinar

Department of Mathematics, FAU, Germany  June 3, 2020
Propagation of singularities for solutions to Hamilton-Jacobi equations

Piermarco Cannarsa
University of Rome “Tor Vergata”

Webinar

Department of Mathematics, FAU, Germany June 3, 2020
The evolutionary Hamilton-Jacobi equation,

\[
(HJ) \quad \frac{\partial \phi}{\partial t} + H(t, x, \nabla \phi) = 0
\]

appears in diverse mathematical models ranging from analytical mechanics to combinatorics, condensed matter, turbulence, and cosmology . . . In many of these applications the objects of interest are described by singularity of solutions, which inevitably appear for generic initial data after a finite time due to the nonlinearity of (HJ). Therefore one of the central issues both for theory and applications is to understand the behaviour of the system after singularities form.
Overview

\[ \Omega \subset \mathbb{R}^n \text{ bounded} \]

\[ H(x, u, Du) = 0 \quad \text{a.e. in } \Omega \]

- \( u : \Omega \rightarrow \mathbb{R} \) Lipschitz viscosity solution
- \( p \mapsto H(x, u, p) \) is convex

The object of our study

\[ \text{Sing}(u) = \{ x \in \Omega \mid \overline{\partial}Du(x) \} \]

Examples

1. Hamilton-Jacobi equation
   \[
   \begin{cases}
   u_t + H(t, x, D_x u) = 0 & \forall t \in [0, T], x \in \mathbb{R}^n \\
   u(0, x) = u_0(x) & x \in \mathbb{R}^n
   \end{cases}
   \]

2. Eikonal-type equation
   \[
   \begin{cases}
   \frac{1}{2} \langle A(x) Du, Du \rangle + V(x) = 0 & x \in \Omega \subset \mathbb{R}^n \\
   u(x) = 0 & x \in \partial \Omega
   \end{cases}
   \]

3. Weak KAM theory
   \[ H(x, Du) = c \quad (x \in \mathbb{T}^n) \]
The distance function from a set $S \subset \mathbb{R}^n$

$$d_S(x) = \inf_{y \in S} |x - y|$$

is locally semiconcave on $\mathbb{R}^n \setminus \overline{S}$
by using characteristics:

on \( [0, T] \times \mathbb{R}^n \setminus \text{Sing}(u) \)

\( u \) is as smooth as the data (maximal regularity)
Denote by $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Legendre transform

$$L(t, x, q) = \max_{p \in \mathbb{R}^n} [\langle q, p \rangle - H(t, x, p)]$$

The value function

$$u(t, x) = \inf_{\xi(t) = x} \left\{ \int_0^t L(s, \xi(s), \xi'(s)) \, dt + u_0(\xi(0)) \right\}$$

gives the viscosity solution of

$$\begin{cases} u_t(t, x) + H(t, x, D_x u(t, x)) = 0 \\ u(0, x) = u_0(x) \end{cases}$$

$\exists Du(t, x) \iff$ unique minimizer at $(t, x)$
Outline

1. The beginning
2. Use of (some) geometric measure theory
3. The discovery of singular dynamics
4. The distance function and applications to topology
5. Weak KAM solutions and applications to topology
6. Uniqueness of singular characteristics in 2D
Outline

1. The beginning
2. Use of (some) geometric measure theory
3. The discovery of singular dynamics
4. The distance function and applications to topology
5. Weak KAM solutions and applications to topology
6. Uniqueness of singular characteristics in 2D
How this story began

On the Singularities of the Viscosity Solutions to Hamilton–Jacobi–Bellman Equations

PIERMARCO CANNARSA & HALIL METE SONER

3. Introduction. This paper is concerned with the local structure of the first-order singularities of the solutions of the Hamilton–Jacobi–Bellman equation

\[ \frac{\partial}{\partial t} u(x,t) + H(x,t,\nabla u(x,t)) = 0, \quad (x,t) \in \Omega \times (0,T) \]

(1.1)

\[ u(x,t) = \phi(x), \quad (x,t) \in \partial \Omega \times (0,T), \]

where \( \Omega \) is an open domain in \( \mathbb{R}^n \). It is known that this equation does not have classical solutions regardless of how smooth the data is. Also, there may be many solutions satisfying (1.1) almost everywhere. However, M. G. Crandall and P.-L. Lions introduced the notion of viscosity solutions to resolve this problem. This solution was proved to be unique under some very general assumptions [1]. Then, the properties of viscosity solutions have been studied by many authors: P.-L. Lions [12], L. C. Evans, M. G. Crandall and P.-L. Lions [6], P. E. Souganidis [18], M. G. Crandall and P. E. Souganidis [6], B. Ishii [13], …

When the Hamiltonian \( H(x,t,p) \) is known in \( p \), equation (1.1) is related to a variational problem—see W. H. Fleming and R. Rishel [14]. In fact, under some assumptions, the viscosity solutions \( u(x,t) \) of (1.1) is the value function of the following variational problem:

(1.2) \[ u(x,t) = \inf \left\{ \int_{t}^{T} L(x,s,u_s,s_\nu(s)) ds + \phi(\xi(T)) : (\xi) \in \xi \right\}, \]

where \( L(x,t,p) \) is the Legendre transform of \( H(x,t,p) \) in the \( p \)-variable.

Indiana University Mathematics Journal 45, Vol. 38, No. 3 (1987)

Figure: how, where, and whom with...
The “discovery” of semiconcave functions

Ω ⊆ \( \mathbb{R}^n \) open
\( u : \Omega \to \mathbb{R} \) semiconcave with modulus \( \omega : [0, \infty] \to [0, \infty] \) if

\[
\lambda u(x) + (1 - \lambda) u(y) - u(\lambda x + (1 - \lambda) y) \leq \lambda (1 - \lambda) |x - y| \omega(|x - y|)
\]

for all \( x, y \) such that \([x, y] \subset \Omega \) and \( \lambda \in [0, 1] \)

Special cases:
- \( \omega(s) \equiv 0 \) \( \rightarrow \) concave
- \( \omega(s) = Cs \ (C > 0) \) \( \rightarrow \) linearly semiconcave
  - In this case, there is a concave function \( v \) such that
    \[
    u(x) = v(x) + \frac{C}{2} |x|^2
    \]
- \( \omega(s) = Cs^\alpha \ (C > 0, 0 < \alpha < 1) \) \( \rightarrow \) fractionally semiconcave
  - In this case, \((\star)\) is no longer valid
Further references on semiconcave functions

- **control theory and sensitivity analysis**
  - Fleming – McEneaney 2000
  - Rifford 2000, 2002

- **nonsmooth and variational analysis**
  - Rockafellar 1982
  - Colombo – Marigonda 2006, Nguyen 2010

- **differential geometry**
  - Perelman 1995, Petrunin 2007

- **monographs**
  - C – Sinestrari (Birkhäuser 2004)
  - Villani (Springer 2009)
For any semiconcave $u : \Omega \to \mathbb{R}$

- the superdifferential at $x \in \Omega$ coincides with Clarke’s gradient

$$D^+ u(x) = \text{co } D^* u(x) = \partial u(x)$$

where $D^* u(x) = \{ \lim_{i \to \infty} Du(x_i) \mid x_i \to x \}$ reachable gradients

- $D^+ u(x) = \{ p \} \iff u$ differentiable
For $u : \Omega \to \mathbb{R}$ semiconcave and $H \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$

- if $u$ is a viscosity solution of $H(x, u, Du) = 0$ in $\Omega$, then
  
  $$H(x, u(x), p) = 0 \quad \forall x \in \Omega, p \in D^* u(x)$$

- if $H(x, u, \cdot)$ convex, then
  
  $$H(x, u, Du) = 0 \text{ a.e. } \iff H(x, u, Du) = 0 \text{ viscosity}$$

- if $H(x, u, \cdot)$ strictly quasi-convex, then
  
  $$x \in \text{Sing}(u) \iff \min_{p \in D^+ u(x)} H(x, u(x), p) < 0$$
Our first propagation result

\[ u_t + H(t, x, D_x u) = 0 \quad \text{in} \ (0, T) \times \mathbb{R}^n \]  

(HJ)

**Theorem (C – Soner 1987)**

Let

- \( u \) be a semiconcave a viscosity solution of (HJ)
- \((t_0, x_0) \in (0, T) \times \mathbb{R}^n \) and \( \tau > 0 \) be such that
  
  \[ u \in C^1([t_0, t_0 + \tau] \times B_\tau(x_0)) \]

Then

\[ u \in C^1([t_0, t_0 + \tau] \times B_\tau(x_0)) \]

This shows that \((t_0, x_0) \in \text{Sing}(u)\) propagates along a discrete set

**Problem:** how to connect these singular points with a singular line?
Towards the use of measure theory


Outline

1. The beginning
2. Use of (some) geometric measure theory
3. The discovery of singular dynamics
4. The distance function and applications to topology
5. Weak KAM solutions and applications to topology
6. Uniqueness of singular characteristics in 2D
Semiconcave functions and rectifiability

Ω ⊆ \( \mathbb{R}^n \) open \( u : \Omega \rightarrow \mathbb{R} \) semiconcave

**Singular set**

\[
\text{Sing}(u) = \left\{ x \in \Omega \mid \mathcal{A}Du(x) \right\} = \left\{ x \in \Omega \mid \dim D^+ u(x) \geq 1 \right\}
\]

can be stratified by looking at singular magnitude

\[
\text{Sing}(u) = \bigcup_{j=1}^{n} \text{Sing}_j(u) \text{ with } \text{Sing}_j(u) := \left\{ x \in \Omega \mid \dim D^+ u(x) = j \right\}
\]

**Theorem**

\( \text{Sing}_j(u) \) countably \((n - j)\)-rectifiable

\( \text{Sing}(u) \) countably \((n - 1)\)-rectifiable

- Zajíček (1978), Veselý (1979) concave functions
- Alberti – Ambrosio – C (1992) general semiconcave functions
Closure of the singular set

\[ u : (0, T) \times \mathbb{R}^n \to \mathbb{R} \text{ semiconcave} \]

\[
\begin{aligned}
  u_t(t, x) + H(t, x, D_x u(t, x)) &= 0 \quad (t, x) \in (0, T) \times \mathbb{R}^n \\
u(0, x) &= u_0(x) \quad x \in \mathbb{R}^n
\end{aligned}
\]

(HJ)

where, for some \( k \geq 1 \),

- \( H = H(t, x, p) \in C^{k+1} \) strictly convex and superlinear in \( p \)
- \( u_0 \in C^{k+1}(\mathbb{R}^n) \)

Then \( \text{Sing}(u) \) countably \( n \)-rectifiable: what about \( \overline{\text{Sing}(u)} \)?

By characteristics: \( u \in C^{k+1}([0, T] \times \mathbb{R}^n \setminus \overline{\text{Sing}(u)}) \)

Fleming 1969 (by a Sard-type argument)

\[
\overline{\text{Sing}(u)} \subseteq \text{Sing}(u) \cup \text{Conj}(u) \text{ and } \mathcal{H}^{n+1/k}(\overline{\text{Conj}(u)}) = 0
\]

This is not enough to derive \( n \)-rectifiability
Rectifiability of the cut set

Theorem (C – Mennucci – Sinestrari 1997)

- $\overline{\text{Sing}(u)} = \text{Sing}(u) \cup \text{Conj}(u)$
- $\text{Conj}(u)$ is countably $\mathcal{H}^n$-rectifiable (and so is $\overline{\text{Sing}(u)}$)
- $\mathcal{H}^{n-1+2/k}(\text{Conj}(u) \setminus \text{Sing}(u)) = 0$  \((k \geq 2)\)
- $\mathcal{H} - \text{dim} (\text{Conj}(u) \setminus \text{Sing}(u)) \leq n - 1$  \((k = \infty)\)
Outline

1. The beginning

2. Use of (some) geometric measure theory

3. The discovery of singular dynamics

4. The distance function and applications to topology

5. Weak KAM solutions and applications to topology

6. Uniqueness of singular characteristics in 2D
An example

Do singularities of lower magnitude propagate?

Figure: singularities of magnitude 1 do propagate along straight lines.
A counterexample

Figure: an isolated singularity of magnitude 1 at the origin

\[ u(x, y) = 3 - \sqrt{\left(\frac{3x}{2}\right)^2 + \left(\frac{2y}{3}\right)^4} \]
Back to first example

Figure: here

\[ D^* u(0, 0) \subsetneq \partial D^+ u(0, 0) \]
The propagation principle

Ω ⊆ \( \mathbb{R}^n \) open \( u : \Omega \to \mathbb{R} \) semiconcave

**Theorem (Albano – C 1999)**

Let \( x_0 \in \text{Sing}(u) \) be such that \( \partial D^+ u(x_0) \setminus D^* u(x_0) \neq \emptyset \)

Fix any \( p_0 \in \partial D^+ u(x_0) \setminus D^* u(x_0) \) and \( q_0 \in \mathbb{R}^n \setminus \{0\} \) such that

\[
q_0 \cdot (p - p_0) \geq 0 \quad \forall p \in D^+ u(x_0)
\]

Then \( \exists x : [0, \tau] \to \Omega \) Lipschitz such that

\[
\begin{align*}
\dot{x}(t) &\in q_0 - p_0 + D^+ u(x(t)) & t \in [0, \tau] \text{ a.e.} \\
x(0) &= x_0 \\
x(t) &\in \text{Sing}(u) & \forall t \in [0, \tau] \\
\dot{x}^+(0) &= q_0
\end{align*}
\]
The role of generalized characteristics

Let \( u : \Omega \rightarrow \mathbb{R} \) be a semiconcave solution \( H(x, u, Du) = 0 \)

Definition

\( x : [0, \tau] \rightarrow \Omega \) (\( 0 < \tau \leq \infty \)) is a \textit{generalized characteristic} for \((u, H)\)

\[ \dot{x}(t) \in \text{co} \ D_pH\left(x(t), u(x(t)), D^+u(x(t))\right) \quad \text{for a.e. } t \in [0, \tau[ \]

In general, there may be more than one generalized characteristic starting from a given point. However, for \( H(x, p) = \frac{1}{2}|p|^2 + V(x) \),

\[ \forall x \in \Omega \exists! \text{ generalized characteristic for } (u, H) \text{ starting from } x \]

Indeed, if \( x_1 \) and \( x_2 \) are two generalized characteristics

\[ \dot{x}_i(t) \in D^+u(x_i(t)) \quad (t \in [0, \sigma] \text{ a.e.}). \]

starting from the same point then

\[ \frac{1}{2} \frac{d}{dt} |x_2(t) - x_1(t)|^2 = \langle \dot{x}_2(t) - \dot{x}_1(t), x_2(t) - x_1(t) \rangle \leq K |x_2(t) - x_1(t)|^2 \]

So, \( x_1 \equiv x_2 \) by Gronwall
Strict characteristics – Uniqueness?

Let \( u : \Omega \to \mathbb{R} \) be a semiconcave solution \( H(x, u, Du) = 0 \)

Definition

A Lipschitz arc \( y : [0, \tau[ \to \Omega \) (\( 0 < \tau \leq \infty \)) is a strict (or broken) characteristic for \((u, H)\) if there is a measurable selection \( p(t) \in D^+ u(y(t)) \) such that

\[
\dot{y}(t) = D_pH(y(t), u(y(t)), p(t)) \quad \text{for a.e. } t \in [0, \tau[.
\]

The existence of a strict characteristic starting from a given \( x \in \Omega \) is no longer guaranteed by the general theory, but for a Tonelli Hamiltonian \( H \) we have:

Theorem (Khanin – Sobolevski 2014)

For any \( x \in \Omega \) there exists a strict characteristic \( y : [0, \tau[ \to \Omega \) such that \( y(0) = x \) and

\[
\dot{y}^+(t) = D_pH(y(t), u(y(t)), p(t)) \quad \text{for every } t \in [0, \tau[ \text{ with}
\]

\[
H(y(t), u(y(t)), p(t)) = \min_{p \in D^+ u(y(t))} H(y(t), u(y(t)), p)
\]

Problem

For a Tonelli Hamiltonian, is the strict characteristic starting from \( x \in \Omega \) unique?
Singular dynamics

Singular characteristics at noncritical points


- \( u : \Omega \rightarrow \mathbb{R} \) semiconcave solution \( H(x, u, Du) = 0 \)
- \( x_0 \in \text{Sing}(u) \) such that \( 0 \notin D_p H(x_0, u(x_0), D^+ u(x_0)) \)
  \( x_0 \) not a critical point

Then \( \exists \dot{x} : [0, \tau[ \rightarrow \Omega \) generalized characteristic such that

\[
\begin{align*}
\dot{x}(0) &= x_0 \\
x(t) &\in \text{Sing}(u) \quad \forall t \in [0, \tau[ \\
\dot{x}^+(0) &= D_p H(x_0, u(x_0), p_0), \quad p_0 = \arg \min_{p \in D^+ u(x_0)} H(x_0, u(x_0), p) \\
\lim_{t \rightarrow 0^+} \text{ess sup}_{s \in [0, t]} |\dot{x}(s) - \dot{x}^+(0)| &= 0
\end{align*}
\]

Any generalized characteristics as above will be called a singular characteristic (for \( u \))
Outline

1. The beginning
2. Use of (some) geometric measure theory
3. The discovery of singular dynamics
4. The distance function and applications to topology
5. Weak KAM solutions and applications to topology
6. Uniqueness of singular characteristics in 2D
The Euclidean distance function

\[ \Omega \subset \mathbb{R}^n \text{ bounded open set} \]

\[ d_\Omega(x) = \min_{y \in \partial\Omega} |x - y| \quad (x \in \overline{\Omega}) \]

\[ \text{Sing}(d_\Omega) = \{ x \in \Omega \mid \text{proj}_{\partial\Omega}(x) \text{ multivalued} \} \neq \emptyset \quad \text{medial axis} \]

\[ |Dd_\Omega(x)|^2 = 1 \quad x \in \Omega \text{ a.e.} \]

\[ \dot{x} = Dd_\Omega(x) \]

\[ d_\Omega(x) \]
Classification by homotopy equivalence

Problem

Are $\Omega$ and $\text{Sing}(d\Omega)$ homotopy equivalent?
F. Wolter (1993): deformation retract technique works if

- $\Omega \subset \mathbb{R}^n$ and $\partial \Omega \in C^2$
- $\Omega \subset \mathbb{R}^2$ and $\partial \Omega$ is piecewise $C^2$

**Figure:** $r(\cdot)$ discontinuous along edges above $A$, $B$, $C$ and $D$
A result in computer graphics


$\Omega$ has the same homotopy type as $\text{Sing}(d_\Omega)$

Proof.

Suppose one can construct a global generalized gradient flow $X(t, x)$

$$
\begin{cases}
\dot{X}(t) \in D^+d_\Omega(X(t)) & t \in [0, \infty) \text{ a.e.} \\
X(0) = x
\end{cases}
$$

such that

$$
X(t_0, x) \in \text{Sing}(d_\Omega) \implies X(t, x) \in \text{Sing}(d_\Omega) \quad \forall t \in [t_0, \infty)
$$

Define homotopy $\mathbb{H} : \Omega \times [0, 1] \rightarrow \Omega$ by $\mathbb{H}(x, t) = X(tT, x)$ where $T > 0$ is such that $X(T, x) \in \text{Sing}(d_\Omega)$ $\forall x \in \Omega$
Global propagation for the distance function

**Theorem** (Albano – C – Khai T. Nguyen – Sinestrari 2013)

For any given \( x_0 \in \Omega \) let \( X : [0, \infty) \rightarrow \Omega \) be the unique solution of

\[
\begin{align*}
\dot{X}(t) &\in D^+d_\Omega(X(t)) & t &\in [0, \infty) \text{ a.e.} \\
X(0) &= x_0
\end{align*}
\]

Then

\[
x_0 \in \text{Sing}(d_\Omega) \quad \implies \quad X(t) \in \text{Sing}(d_\Omega) \quad \forall t \in [0, \infty)
\]
Outline

1. The beginning
2. Use of (some) geometric measure theory
3. The discovery of singular dynamics
4. The distance function and applications to topology
5. Weak KAM solutions and applications to topology
6. Uniqueness of singular characteristics in 2D
Weak KAM solutions on manifolds

$(M, g)$ compact connected Riemannian manifold

$u : M \to \mathbb{R}$ solution of

$$H(x, Du(x)) = 0 \quad (x \in M)$$

**Figure:** W. Cheng and A. Fathi

$\gamma : [a, b] \to M$ is $u$-calibrating or absolute minimizer if

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s))ds$$

where

$$L(x, v) = \sup_{p \in T^*_x M} [p(v) - H(x, p)]$$
The Cut and Aubry sets

\[
\text{Cut}(u) = \text{cut set of } u \text{ consists of all } x \in M \text{ such that } x \in \gamma([a, b]) \text{ for some } u\text{-calibrating } \gamma \implies x = \gamma(b)
\]

\[
\mathcal{I}(u) = \text{Aubry set of } u \text{ consists of all } x \in M \text{ such that } x = \gamma(0) \text{ for some } u\text{-calibrating } \gamma : \mathbb{R} \to M
\]

Observe \( \text{Sing}(u) \subseteq \text{Cut}(u) \subseteq M \setminus \mathcal{I}(u) \)
Topology of singular sets

\( u : M \to \mathbb{R} \) solution of \( H(x, Du(x)) = 0 \)

**Theorem (C – Cheng – Fathi 2017)**

All the inclusions

\[
\text{Sing}(u) \subseteq \text{Cut}(u) \subseteq M \setminus \mathcal{I}(u)
\]

are homotopy equivalences

http://dx.doi.org/10.1016/j.crma.2016.12.004
Lax-Oleinik operators

Proof based on homotopy map \( \mathbb{H} : (M \setminus \mathcal{I}(u)) \times [0, 1] \rightarrow M \setminus \mathcal{I}(u) \)

\[
\begin{align*}
\mathbb{H}(x, 0) &= x \quad \forall x \in M \setminus \mathcal{I}(u) \\
\mathbb{H}(M \setminus \mathcal{I}(u), 1) &\subset \text{Sing}(u) \\
\mathbb{H}(\text{Cut}(u), t) &\subset \text{Sing}(u) \quad \forall t \in ]0, 1]
\end{align*}
\]

Lax-Oleinik operators

\[
T_t^- u(x) = \inf_{y \in M} \left\{ u(y) + A_t(x, y) \right\}, \quad T_t^+ u(x) = \sup_{y \in M} \left\{ u(y) - A_t(x, y) \right\}
\]

where \( A_t(x, y) \) is the minimal action

\[
A_t(x, y) = \inf_{\xi} \left\{ \int_0^t L(\xi(s), \dot{\xi}(s)) \, ds \mid \xi(0) = x, \, \xi(t) = y \right\}
\]
The intrinsic characteristic

Construction of $H$ based on intrinsic characteristic $X : [0, t_0] \times M \to M$

\[ X(t, x) = \arg \max_{y \in M} \{ u(y) - A_t(x, y) \} \]

Properties of intrinsic characteristic:

(a) $Y$ is single-valued and continuous on $[0, t_0] \times M$

(b) $x \in \text{Sing}(u) \implies X(t, x) \in \text{Sing}(u) \forall t \in [0, t_0]$
   (propagation of singularities)

(c) $X(t, x) \notin \text{Sing}(u) \implies s \mapsto X(s, x)$ is $u$-calibrating on $[0, t]$

Extend $X : [0, \infty[ \times M \to M$ by induction on $n \geq 1$

\[ X(t, x) = X(t - nt_0, X(nt_0, x)) \quad \forall t \in [nt_0, (n + 1)t_0] \]

Construction is also used to obtain global propagation of singularities
(C – Cheng 2017)
Outline

1. The beginning
2. Use of (some) geometric measure theory
3. The discovery of singular dynamics
4. The distance function and applications to topology
5. Weak KAM solutions and applications to topology
6. Uniqueness of singular characteristics in 2D

P. Cannarsa (Rome Tor Vergata)
Three types of singular characteristics

Let $u : \Omega \rightarrow \mathbb{R}$ be a semiconcave solution of $H(x, u, Du) = 0$ and let $x_0 \in \text{Sing}(u)$ be such that $0 \not\in D_pH(x_0, u(x_0), D^+u(x_0))$

Three notions of singular characteristics with initial point $x_0$:

1. **generalized characteristics**

$$\dot{x}(t) \in \text{co} D_pH(x(t), u(x(t)), D^+u(x(t)))$$

2. **strict characteristics**

$$\dot{y}^+(t) \in D_pH(y(t), u(y(t)), D^+u(y(t)))$$

3. **intrinsic characteristics**

$$X(t, x_0) = \arg\max_{y \in M} \{u(y) - A_t(x_0, y)\}$$

**Problem**

How are these notions related?

**Mechanical systems:**

$$H(x, u, p) = \frac{1}{2} \langle A(x)p, p \rangle + V(x) \quad \implies \quad x = y = X(\cdot, x_0)$$
Uniqueness of singular characteristics \((n = 2)\)

**Theorem (C – Cheng)**

Let \(u : \Omega \rightarrow \mathbb{R}\) be a semiconcave solution of \(H(x, Du) = 0\) in \(\Omega \subset \mathbb{R}^2\) and

- \(x_0 \in \text{Sing}(u)\) be such that \(0 \notin D_pH(x_0, D^+u(x_0))\)

Let \(x_i : [0, T_i] \rightarrow \Omega\) \((i = 1, 2)\) be Lipschitz arcs such that

1. \(x_1(0) = x_0 = x_2(0)\)
2. \(x_i(t) \in \text{Sing}(u)\) \(\forall t \in [0, T_i]\) \((i = 1, 2)\)
3. \(\dot{x}_1^+(0) = D_pH(x_0, p_0) = \dot{x}_2^+(0),\) \(p_0 = \arg \min_{p \in D^+u(x_0)} H(x_0, p)\)
4. \(\lim_{t \to 0^+} \text{ess sup}_{s \in [0, t]} |\dot{x}_i(s) - \dot{x}_i^+(0)| = 0\) \((i = 1, 2)\)

Then, there exists \(\tau \in (0, T_2]\) and a bi-Lipschitz homeomorphism

\[
\phi : [0, \tau] \rightarrow [0, \phi(\tau)] \subset [0, T_1]
\]

such that, \(x_1(\phi(t)) = x_2(t)\) for all \(t \in [0, \tau]\).

Therefore, all singular characteristics coincide up to a bi-Lipschitz reparameterization.
Concluding remarks

The study of singularities of solutions to HJ equation has made remarkable progress:

- results that seemed impossible have been obtained
- connections with other domains have been established

Nevertheless, many interesting problems remain open

- more general convex Hamiltonians (not Tonelli)? (technical but doable)
- uniqueness of singular characteristics in higher dimension? (no clue except for mechanical systems... look at higher dimensional singular sets?)
- nonconvex Hamiltonians?
  new ideas and tools needed (Pinezich, SICON 2019)
Thank You

Figure: Pavia, San Michele Maggiore