Lyapunov exponents at bandedges

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Contents

Chapter 1

Bandedge of Anderson model - main result

In this chapter the basic objects of interest are introduced and the main result, a detailed analysis of the Lyapunov exponent and the rotation number at the bandedge of the one dimensional Anderson model is presented.

1.1 Model

Let u be an element of the Hilbert space $l^2(\mathbb{Z})$, i.e. $\|u\|^2 := \sum_{n\in\mathbb{Z}} |u(n)|^2 < \infty$. A discrete version Δ on $l^2(\mathbb{Z})$ of the Laplacian on $L^2(\mathbb{R})$ can be defined by

$$(\Delta\, u)(n) \;=\; [u(n+1) \;-\; u(n)] - [u(n) - u(n-1)] \;=\; u(n+1) + u(n-1) - 2u(n)\;.$$

If a function V on $\mathbb{Z}$ is interpreted as a potential, a discrete version of the Hamilton operator would be

$$[(-\Delta \,+\, V)\, u](n) \;=\; -u(n+1) \,-\, u(n-1) \,+\, (V(n) \,+\, 2)\, u(n)\;.$$

As the constant 2 only moves the spectrum we will neglect it. Thus we consider the following operators

$$(H_0\, u)(n) \;=\; -u(n+1) - u(n-1) \qquad \text{and} \tag{1.1}$$

$$(H\, u)(n) \;=\; (H_0\, u)(n) \,+\, V(n)u(n)\;, \quad \text{short notation: } H \;=\; H_0 \,+\, V\;. \tag{1.2}$$

For convenience let us introduce the following notation:

$$\delta_m \,\in\, l^2(\mathbb{Z}), \quad \delta_m(n) \,=\, 0 \quad \text{if} \;\; m \neq n \;\; \text{and} \;\; \delta_m(m) \,=\, 1$$

Proposition 1.1. *The spectrum of H_0 is $\mathrm{spec}(H_0) = [-2, 2]$, i.e. the bandedges for H_0 are the energies $E = \pm 2$.*

Proof. As $\|H_0\,\delta_0\| = 2 = 2\,\|\delta_0\|$ and as one has[1]

$$\|H_0\,u\|^2 \;=\; \sum_{n\in\mathbb{Z}} |u(n+1)+u(n-1)|^2 \;\leq\; \sum_{n\in\mathbb{Z}} 2|u(n+1)|^2 \;+\; 2|u(n-1)|^2 \;=\; 4\|u\|^2$$

it follows that $\|H_0\| = 2$ which implies $\mathrm{spec}(H_0) \subset [-2,2]$.
For $E \in [-2,2]$ let $e^{i\eta} = \frac{1}{2}(-E + i\sqrt{4-E^2})$ which fulfills the identity

$$e^{2i\eta} + Ee^{i\eta} + 1 \;=\; 0\;,$$

thus $u(n) = e^{in\eta}$ is a formal eigenvector of H_0. Let us define

$$x_m(n) \;=\; \frac{1}{2m+1}\,e^{in\eta},\;\;\text{if}\;\;|n|\leq m,\;\;x_m(n)\;=\;0\;\;\text{else},\quad m\;\in\;\mathbb{N}\;,$$

to get $\|x_m\| = 1$. Then one has

$$(E\mathbf{1} - H_0)x_m \;=\; \frac{1}{2m+1}(-e^{i(m+1)\eta}\delta_m - e^{-i(m+1)\eta}\delta_{-m} + e^{im\eta}\delta_{m+1} + e^{-im\eta}\delta_{-m-1})$$

which leads to $\lim_{m\to\infty}\|(E\mathbf{1}-H_0)x_m\| = 0$. Thus E is an approximate eigenvalue of H_0 and therefore lies in the spectrum. (This is the Weyl criterion which can be found in any book about functional analysis.) □

For our purpose we will have a coupling constant $\lambda \geq 0$ and a random potential where all values are independent, identically distributed. Therefore let $(\Sigma, \mathbf{p})$ be a probability space[2] and V_σ a real valued random variable. To avoid technical difficulties we also assume that V_σ is (essentially) compact supported. For most of the theorems a detailed analysis shows that this assumption can be relaxed by finite moment conditions. Furthermore let $\mathbb{E}(V_\sigma) = 0$ which is the most interesting case. As we want to have a real random potential we also assume $\mathbb{E}(V_\sigma^2) > 0$. Consider the product space $\hat{\Omega} := \Sigma^{\mathbb{Z}}$ with product measure $\mathbf{p}^{\otimes\,\mathbb{Z}}$ and for $\omega \in \hat{\Omega}$ and $\lambda \geq 0$ define the Hamiltonian

$$H_{\lambda,\omega} \;:=\; H_0 \;+\; \lambda\,V_\omega, \quad \text{where} \quad V_\omega(n) \;=\; V_{\omega_n}\;. \tag{1.3}$$

Expectation values w.r.t. $\mathbf{p}$ (sometimes $\mathbf{p}^{\otimes\mathbb{Z}}$ or $\mathbf{p}^{\otimes\mathbb{N}}$) will be denoted by $\mathbb{E}$.

1.2 Transfer matrices and Lyapunov exponents

Using the transfer matrices

$$T^E_{\lambda,\sigma} \;:=\; \begin{pmatrix} \lambda V_\sigma - E & -1 \\ 1 & 0 \end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{R}) \tag{1.4}$$

[1] This inequality follows from the arithmetic-quadratic mean inequality $\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$ for real numbers.

[2] We neglect the σ-Algebra in the notation as one may assume that $\Sigma \subset \mathbb{R}$ with the Borel σ-algebra and $V_\sigma = \sigma$.

one can write the stationary Schrödinger equation at Energy E

$$H_{\lambda,\omega}\, u \;=\; E\, u \;\Leftrightarrow\; -u(n+1) \;-\; u(n-1) \;+\; \lambda\, V_{\omega_n} u(n) \;=\; E\, u(n)$$

as

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} \;=\; T^E_{\lambda,\omega_n} \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} .$$

The asymptotic behaviour of (most of) the formal solutions u can be characterized by the Lyapunov exponent.

Definition 1.2. *The Lyapunov exponent for the sequence T^E_{λ,ω_n} of $\mathrm{SL}(2,\mathbb{R})$ matrices, if existent, is defined as*

$$\gamma(E,\lambda,\omega) \;:=\; \lim_{N\to\infty} \frac{1}{N} \log \left\| \prod_{n=0}^{N-1} T^E_{\lambda,\omega_{N-n}} \right\| \;=\; \lim_{N\to\infty} \frac{1}{N} \log \| T^E_{\lambda,\omega_N}\, T^E_{\lambda,\omega_{N-1}} \,\dots\, T^E_{\lambda,\omega_1} \| \;.$$

One can show that this limit exists for $\mathbf{p}^{\otimes\mathbb{Z}}$ almost all $\omega \in \hat{\Omega}$ and it is almost surely constant (see Theorem 2.2). Thus we can define

$$\gamma(E,\lambda) \;:=\; \mathbb{E}_\omega \left[\gamma(E,\lambda,\omega)\right] .$$

Remark: One can also define a Lyapunov exponent for the asymptotic to $-\infty$ by

$$\gamma^-(E,\lambda,\omega) \;\;:=\;\; \lim_{N\to-\infty} \frac{1}{|N|} \log \left\| (T^E_{\lambda,\omega_{-N+1}})^{-1}\, (T^E_{\lambda,\omega_{-N+2}})^{-1} \,\dots\, (T^E_{\lambda,\omega_0})^{-1} \right\| ,$$

but one has $\mathbf{p}^{\otimes\mathbb{Z}}$ almost surely $\gamma = \gamma^-$ as it is shown in Theorem 2.2. For this reason most of the time we will actually consider the space $\Omega := \Sigma^{\mathbb{N}}$ with product measure $\mathbf{p}^{\otimes\mathbb{N}}$.

Because of the equivalence of norms on $\mathbb{R}^{2\times 2}$ the Lyapunov exponent does not depend on the choice of the norm and therefore it is also invariant under a basis change, i.e. for $M \in \mathrm{GL}(2,\mathbb{R})$ one can use the matrices $M\, T^E_{\lambda,\sigma} M^{-1}$ instead of $T^E_{\lambda,\sigma}$ to get the same result. Furthermore defining $e_\theta := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ one has (Theorem 2.4 or [JSS, Lemma 3])

$$\gamma(E,\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N} \int_0^\pi \mathbb{E}(\log \| M\, T^E_{\lambda,\omega_N} \dots T^E_{\lambda,\omega_1}\, M^{-1}\, e_\theta \|)\, \mathrm{d}\mu(\theta)$$

for any continuous measure μ on $\mathbb{RP}(1)$, $M \in \mathrm{GL}(2,\mathbb{R})$. If V_σ has some randomness (for details see Theorem 2.7) one even gets

$$\gamma(E,\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}(\log \| M\, T^E_{\lambda,\omega_N} \dots T^E_{\lambda,\omega_1}\, M^{-1}\, e_\theta \|) \tag{1.5}$$

uniformly for all $\theta \in \mathbb{RP}(1)$. Again we can use any norm on $\mathbb{R}^2$ we want, but for our calculation $\|\cdot\|$ will denote the euclidean norm on $\mathbb{R}^2$ (and the corresponding operator norm for matrices) from now on.

1.3 Phase shift dynamics and rotation number

The natural action s_M of an invertible matrix $M \in \mathrm{GL}(2,\mathbb{R})$ on $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ is given by

$$e_{s_M(\theta)} \;=\; \frac{M\, e_\theta}{\|M\, e_\theta\|}\;.$$

Actually the matrices even act on the linear subspaces of $\mathbb{R}^2$, thus one can project s_M to an action on $\mathbb{RP}(1) \cong \mathbb{R}/\pi\mathbb{Z}$, for convenience this will be considered in Section 3.3. To get simpler notations let $s^E_{\lambda,\sigma} := s_{T^E_{\lambda,\sigma}}$.

These functions can be lifted to smooth functions $S^E_{\lambda,\sigma}$ from $\mathbb{R}$ to $\mathbb{R}$. As $\det(T^E_{\lambda,\sigma}) > 0$ one finds by differentiating that $S^E_{\lambda,\sigma}$ is monotonic increasing w.r.t. θ. By the condition

$$-\frac{1}{2}\pi \;<\; S^E_{\lambda,\sigma}(\theta) - \theta \;<\; \frac{3}{2}\pi$$

which can be satisfied[3] the lifted functions $S^E_{\lambda,\sigma}$ are uniquely determined. Together with an initial value θ_0 they define a Markov process $\theta^E_n(\lambda,\theta_0)$ on $\Omega = \Sigma^{\mathbb{N}}$ by conjunction, i.e. for $\omega \in \Omega$ define iteratively

$$\theta^E_0(\lambda,\theta_0)(\omega) \;:=\; \theta_0, \quad \theta^E_n(\lambda,\theta_0)(\omega) \;:=\; S^E_{\lambda,\omega_n}(\theta^E_{n-1}(\lambda,\theta_0)(\omega))\;. \tag{1.6}$$

One may interpret θ^E_n as a discrete time random dynamical system on $\mathbb{R}$.

Definition 1.3. *The S^1 rotation number[4] $\mathcal{N}$ for the sequence T^E_{λ,ω_n}, if existent, is defined as*

$$\mathcal{N}(E,\lambda,\omega) \;:=\; \frac{1}{\pi}\lim_{N\to\infty}\frac{1}{N}\,\theta^E_N(\lambda,\theta_0)(\omega)\;.$$

Similar as for the Lyapunov exponent one can show that this limit exists almost surely and it is almost surely constant w.r.t. ω (see Theorem 2.8). Thus we may define

$$\mathcal{N}(E,\lambda) \;:=\; \mathbb{E}_\omega\left[\mathcal{N}(E,\lambda,\omega)\right]\;.$$

Remarks:

1. The functions $S^E_{\lambda,\sigma}$ are monotonic increasing and they can be projected to bijective functions on $\mathbb{RP}(1)$, thus the phase shifts $\Delta^E_{\lambda,\sigma}(\theta) := S^E_{\lambda,\sigma}(\theta) - \theta$ are π-periodic functions. Therefore $\theta_0 < \tilde{\theta}_0 < \theta_0 + m\pi$ leads to $\theta^E_n(\lambda,\theta_0)(\omega) < \theta_n(\lambda,\tilde{\theta}_0)(\omega) < \theta_n(\lambda,\theta_0)(\omega) + m\pi$, where m is a natural number. Thus the limit of $\frac{1}{N}\theta^E_N(\lambda,\theta_0)(\omega)$ and $\frac{1}{N}\theta^E_N(\lambda,\tilde{\theta}_0)(\omega)$, if existent, is the same which means that the rotation number does not depend on the initial phase θ_0.

[3] As $T^E_{\lambda,\sigma} e_{\frac{\pi}{2}} = e_\pi \,\forall\, E,\lambda,\sigma$ one sets $S^E_{\lambda,\sigma}(\frac{\pi}{2}) = \pi$ and as $S^E_{\lambda,\sigma}$ projected on $\mathbb{RP}(1)$ is bijective the inequality is satisfied.

[4] For the proofs later we will consider another rotation number $\mathcal{R}$ using a slightly different definition for the lifted functions S.

2. If we define a rotation number $\mathcal{N}^-$ for the asymptotic behaviour to $-\infty$ we would almost surely get $\mathcal{N}^- = -\mathcal{N}$, because we have the same dynamical system on S^1 backward.

3. Theorem 2.9 shows that this S^1 rotation number equals the Integrated Density of States. That is actually the reason why we use $\mathcal{N}$ for the notation.

Let us consider what happens after a basis change. Therefore let $M \in \mathrm{GL}(2,\mathbb{R})$ and consider $\tilde{T}^E_{\lambda,\sigma} := M T^E_{\lambda,\sigma} M^{-1}$. For these matrices we get the actions $\tilde{s}^E_{\lambda,\sigma} := s_{\tilde{T}^E_{\lambda,\sigma}} = s_M \circ s^E_{\lambda,\sigma} \circ s_M^{-1}$. We lift these functions to $\tilde{S}^E_{\lambda,\sigma} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{S}^E_{\lambda,\sigma} = S_M \circ S^E_{\lambda,\sigma} \circ S_M^{-1}$, where S_M is any continuous lift of s_M. This leads to the Markov process $\tilde{\theta}^E_n(\lambda, \tilde{\theta}_0)$ by

$$\tilde{\theta}^E_0(\lambda, \tilde{\theta}_0) \;:=\; \tilde{\theta}_0, \quad \tilde{\theta}^E_n(\lambda, \tilde{\theta}_0) \;:=\; \tilde{S}^E_{\lambda,\omega_n}(\tilde{\theta}^E(\lambda, \tilde{\theta}_0)) \,.$$

The corresponding rotation number is defined by $\tilde{\mathcal{N}} = \frac{1}{\pi}\,\mathbb{E}_\omega \lim_{N\to\infty} \frac{1}{N}\,\tilde{\theta}^E_N$. Using the definition of $\tilde{S}^E_{\lambda,\sigma}$ above one can show by induction that

$$\tilde{\theta}^E_n(\lambda, S_M(\theta_0)) \;=\; S_M[\theta^E_n(\lambda, \theta_0)]$$

which leads to $\tilde{\mathcal{N}} = \mathrm{sgn}(\det M)\,\mathcal{N}$, because S_M is monotonic increasing iff $\det M > 0$ and it is monotonic decreasing iff $\det M < 0$.

Using the phase shifts $\tilde{\Delta}^E_{\lambda,\sigma}(\theta) := \tilde{S}^E_{\lambda,\sigma}(\theta) - \theta$ one gets

$$\tilde{\theta}^E_N(\lambda, \tilde{\theta}_0) \;=\; \tilde{\theta}_0 + \sum_{n=1}^{N} \tilde{\Delta}^E_{\lambda,\omega_n}(\tilde{\theta}^E_{n-1})$$

which leads to

$$\tilde{\mathcal{N}}(E,\lambda) \;=\; \frac{1}{\pi}\,\mathbb{E}_\omega \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \tilde{\Delta}^E_{\lambda,\omega_n}(\tilde{\theta}^E_{n-1}) \,. \tag{1.7}$$

As $\tilde{\theta}_{n-1}$ does not depend on ω_n it follows that

$$\mathbb{E}_\omega \left[\tilde{\Delta}^E_{\lambda,\omega_n}(\tilde{\theta}^E_{n-1})\right] \;=\; \mathbb{E}_\omega \left[\mathbb{E}_\sigma \left(\tilde{\Delta}^E_{\lambda,\sigma}(\tilde{\theta}^E_{n-1})\right)\right] \,.$$

As this expression lies for all $n \in \mathbb{N}$ in the compact interval $\mathbb{E}(\tilde{\Delta}^E_{\lambda,\sigma})([0,\pi])$ one can exchange the limit and the expectation value in (1.7) to get

$$\mathcal{N}(E,\lambda) \;=\; \mathrm{sgn}(\det M)\,\frac{1}{\pi} \lim_{N\to\infty} \frac{1}{N}\mathbb{E}_\omega \sum_{n=0}^{N-1} \mathbb{E}_\sigma(\tilde{\Delta}^E_{\lambda,\sigma}(\tilde{\theta}^E_n)) \,. \tag{1.8}$$

The right hand side of equation (1.5) can be expanded to such a Birkhoff sum as well: (for details see the calculation of Corollary 2.5)

$$\gamma(E,\lambda) \;=\; \mathbb{E}_\omega \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}_\sigma(\log \|\tilde{T}^E_{\lambda,\sigma}\, e_{\tilde{\theta}^E_n}\|) \tag{1.9}$$

Note that the matrix M may depend on λ and E.

1.4 Phase diagram at a bandedge

From Proposition 1.1 we know that the bandedges of $H_{0,\omega} = H_0$ are $E = \pm 2$. The aim is a detailed pertubative (w.r.t. λ) analysis at these energies where we scale E with λ^α. As the calculation is analog at $E = 2$ we only consider $E = -2$. To be more precisely we want to calculate $\gamma_{\alpha,x}(\lambda) := \gamma(-2+\lambda^\alpha x, \lambda)$ and $\mathcal{N}_{\alpha,x}(\lambda) := \mathcal{N}(-2+\lambda^\alpha x, \lambda)$, $x \in \mathbb{R}$, pertubatively w.r.t. λ. (Note that the case $\alpha = 1$ corresponds to the calculation of $\gamma(-2,\lambda)$ and $\mathcal{N}(-2,\lambda)$ where $\mathbb{E}(V_\sigma) = x$.)
Now the functions $\mathbb{E}_\sigma(\tilde{\Delta}^E_{\lambda,\sigma}(\theta))$ and $\mathbb{E}_\sigma(\log \|\tilde{T}^E_{\lambda,\sigma}\, e_\theta\|)$ occurring in (1.8) and (1.9) are π-periodic, thus it is sufficient to consider the projections of $\tilde{\theta}^E_n$ on $\mathbb{R}\mathrm{P}(1) \cong \mathbb{R}/\pi\mathbb{Z}$ which we will call the projected Markov process from now on. The trick is to find a (λ-dependent) basis change M such that one can pertubatively calculate the lowest order ν of the invariant measure ν_λ of the projected Markov process $\tilde{\theta}^E_n$. Such an invariant measure is given by

$$\mathbb{E}_\sigma \int_0^\pi f(\tilde{S}^{-2+\lambda^\alpha x}_{\lambda,\sigma}(\tilde{\theta}))\,\mathrm{d}\nu_\lambda(\tilde{\theta}) \;=\; \int_0^\pi f(\tilde{\theta})\,\mathrm{d}\nu_\lambda(\tilde{\theta})\,.$$

Furstenberg states a criterion when such an invariant measure is unique (see Theorem 2.6). Clearly ν_λ depends on λ, x and α, the latter two ones are suppressed in the notation. Our result will be

$$\mathbb{E}_\omega \frac{1}{N}\sum_{n=0}^{N-1} f(\tilde{\theta}^E_n) \quad \overset{N\to\infty}{\longrightarrow} \quad \int_0^\pi f(\tilde{\theta})\,\mathrm{d}\nu(\tilde{\theta}) \,+\, \mathcal{O}(\lambda^\delta)$$

for a suitable $\delta > 0$ and we will have rigorous control of the error terms. Let us define

$$T_{\lambda,\sigma} \;:=\; M_0\, T^{-2+\lambda^\alpha x}_{\lambda,\sigma}\, M_0^{-1} \;=\; \begin{pmatrix} 1+\lambda V_\sigma - \lambda^\alpha x & 1 \\ \lambda V_\sigma - \lambda^\alpha x & 1 \end{pmatrix}, \quad \text{where} \quad M_0 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

the dependence on x and α is suppressed. We see that for $\lambda = 0$ the transfer matrices are equivalent to a Jordan matrix that cannot be diagonalised. This situation is characteristic for bandedges.

Using these matrices the corresponding projected dynamical system θ_n (on $\mathbb{R}\mathrm{P}(1)$) would have one fix point, $\theta = 0$, for $\lambda = 0$. Thus the invariant measure to lowest order would be a Dirac measure. But this fix point is not stable. Let $s_0 := s_{T_{0,\sigma}}$ (note that $T_{0,\sigma}$ is independent of σ), we lift this function to S_0 such that $S_0(0) = 0$. Then one has $S_0(\theta) - \theta \leq 0$ as S has only multiples of π as fix points. Figure 1.1 shows the function s_0, where the fundamental domain $(-\frac{\pi}{2}, \frac{\pi}{2}]$ is used.
From one side the fix point of the deterministic part is attractive, but once you pass $m\pi$ from above by random movement, the deterministic dynamics attracts $(m-1)\pi$. This produces a negative rotation number which is fine because $\det(M_0) < 0$ and a Dirac measure might be the wrong answer. Thus we need to consider this dynamics in much more detail. Therefore we blow up the neighbourhood of this fix point with a rate λ^δ for some suitable exponent δ, i.e. we consider

$$M_{\lambda,\delta}\, T_{\lambda,\sigma}\, M_{\lambda,\delta}^{-1} \;=\; \begin{pmatrix} 1+\lambda V_\sigma - \lambda^\alpha x & \lambda^\delta \\ \lambda^{1-\delta} V_\sigma - \lambda^{\alpha-\delta} x & 1 \end{pmatrix}, \qquad M_{\lambda,\delta} \;:=\; \begin{pmatrix} \lambda^\delta & 0 \\ 0 & 1 \end{pmatrix}.$$

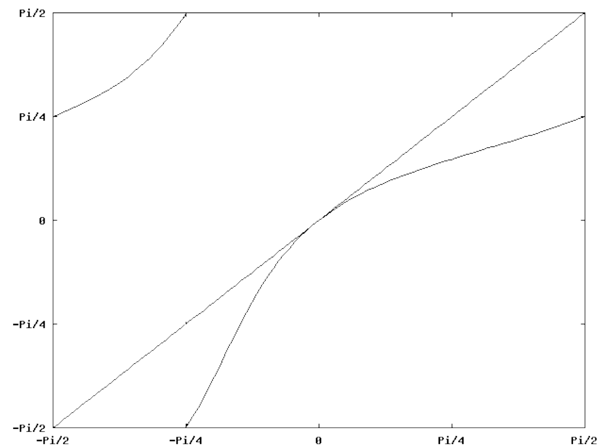


Figure 1.1: free dynamics

We want to choose δ in such a way that we get a non parabolic first degree or a second degree anomaly (classified in chapter 3), therefore we get three regimes.

For $x > 0$ and $\alpha < \frac{4}{3}$ we have an elliptic regime. We choose $\delta = \frac{\alpha}{2}$ and conjugate the result by the matrix $M_1 := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{x}} \end{pmatrix}$ to get the matrices

$$\mathbf{1} + \lambda^{1-\frac{\alpha}{2}} \begin{pmatrix} 0 & 0 \\ \frac{V_\sigma}{\sqrt{x}} & 0 \end{pmatrix} + \lambda^{\frac{\alpha}{2}} \begin{pmatrix} 0 & \sqrt{x} \\ -\sqrt{x} & 0 \end{pmatrix} + \begin{pmatrix} -\lambda^\alpha \sqrt{x} + \lambda\, V_\sigma & 0 \\ 0 & 0 \end{pmatrix} .$$

It is important to note that the condition on α is equivalent to $\frac{\alpha}{2} < 2(1-\frac{\alpha}{2})$ (that is actually the way how this condition is derived). Thus we have a first degree, elliptic anomaly and the invariant measure to lowest order is the Lebesgue measure (up to the normalization constant $\frac{1}{\pi}$, see Theorem 3.8). For $1-\frac{\alpha}{2} \leq \frac{\alpha}{2} \;\Leftrightarrow\; \alpha \geq 1$ Corollary 3.9 leads[5] to (note that $\det(M_{\lambda,\delta}\, M_0) < 0$)

$$\begin{aligned} \mathcal{N}_{\alpha,x}(\lambda) &= \lambda^{\frac{\alpha}{2}} \frac{\sqrt{x}}{\pi} + \mathcal{O}(\lambda^{1-\frac{\alpha}{2}}) \\ \gamma_{\alpha,x}(\lambda) &= \lambda^{2-\alpha} \frac{1}{8} \frac{\mathbb{E}(V_\sigma)}{x} + \mathcal{O}(\lambda^{4-\frac{5}{2}\alpha}, \lambda) \end{aligned}$$

Note that[6] $4 - \frac{5}{2}\alpha = 2 - \alpha + 2 - \frac{3}{2}\alpha > 2 - \alpha$.

[5] The lifted functions $S_{\lambda,\sigma}$ in Section 3.2 and therefore the rotation number $\mathcal{R}$ used in Corollary 3.9 differ from the definitions in Section 1.3. But as for the transfer matrix $T_{0,\sigma}^{-2}$ the lifted function $S_{0,\sigma}^{-2}$ has fix points and as V_σ is compact supported both definitions coincide for small λ, even after the basis change (which does not work for $\lambda = 0$). According to the considerations after Definition 1.3 and Lemma 3.5 one has $\mathcal{N} = \operatorname{sgn}(\det M)\,\mathcal{R}$. That would not be the case if we would analyse the bandedge at energy $E = 2$, because $T_{0,\sigma}^{2}$ has a negative eigenvalue and one would have $S_{0,\sigma}^{E}(0) = \pi$ leading to $\mathcal{N} = \operatorname{sgn}(\det M)\,\mathcal{R}+1$.

[6] This term corresponds to $2\alpha_{L+1} - \alpha_L$ in Corollary 3.9

For $\alpha < 1$ the used Corollary 3.9 would give us $\gamma = 0\lambda^\alpha + \mathcal{O}(\lambda^\alpha, \lambda^{2-\frac{3}{2}\alpha})$. To get the right first non-vanishing coefficient and scaling, we need again a little basis change which works just for this very special case.

Therefore define $\eta_\lambda \in [0,\pi]$ by $\cos(\eta_\lambda) = 1 - \lambda^\alpha \frac{x}{2}$ and let $M_2 = \begin{pmatrix} \sin(\eta_\lambda)\,\lambda^{-\frac{\alpha}{2}} & 0 \\ -\lambda^{\frac{\alpha}{2}}\frac{x}{2} & \sqrt{x} \end{pmatrix}$. Then one has $\sin(\eta_\lambda) = \sqrt{\lambda^\alpha x - \lambda^{2\alpha}\frac{x^2}{4}} = \lambda^{\frac{\alpha}{2}}\sqrt{x} + \mathcal{O}(\lambda^{\frac{3}{2}\alpha})$ and a conjugation by M_2 gives

$$\begin{pmatrix} \cos(\eta_\lambda) & \sin(\eta_\lambda) \\ -\sin(\eta_\lambda) & \cos(\eta_\lambda) \end{pmatrix} \left[\mathbf{1} + \frac{\lambda V_\sigma}{\sin(\eta_\lambda)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right]$$

which coincides with the matrix above up to order less than λ^α, but this form is better to calculate γ. For $M = M_2\, M_1\, M_{\lambda,\frac{\alpha}{2}}\, M_0$ one finds after a little calculation

$$\mathbb{E}\,\log \|M\, T_{\lambda,\sigma}\, M^{-1}\, e_\theta\| \;=\; \frac{\lambda^2\,\mathbb{E}(V_\sigma^2)}{\sin^2(\eta_\lambda)} \left(\frac{1}{8} + \frac{1}{4}\cos(2\theta) + \frac{1}{8}\cos(4\theta) \right) + \mathcal{O}\left(\frac{\lambda^3}{\sin(\eta_\lambda)^3} \right)$$

and by equation 1.5 and Theorem 3.8 one has

$$\begin{aligned} \mathcal{N}_{\alpha,x}(\lambda) &= \lambda^{\frac{\alpha}{2}} \frac{\sqrt{x}}{\pi} + \mathcal{O}(\lambda^{1-\frac{\alpha}{2}}, \lambda^\alpha) \\ \gamma_{\alpha,x}(\lambda) &= \lambda^{2-\alpha}\,\frac{1}{8}\,\frac{\mathbb{E}(V_\sigma)}{x} + \mathcal{O}(\lambda^{3-\frac{3}{2}\alpha}, \lambda^2, \lambda^{3-2\alpha}) \end{aligned}$$

Now consider again $\alpha < \frac{4}{3}$, but this time let $x < 0$. This is the hyperbolic regime. We choose again $\delta = \frac{\alpha}{2}$ and conjugate by the matrix $M_3 = \begin{pmatrix} \sqrt{-x} & -1 \\ \sqrt{-x} & 1 \end{pmatrix}$ to get

$$\mathbf{1} + \lambda^{1-\frac{\alpha}{2}} \frac{V_\sigma}{2\sqrt{-x}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \lambda^{\frac{\alpha}{2}} \begin{pmatrix} -\sqrt{-x} & 0 \\ 0 & \sqrt{-x} \end{pmatrix} + \mathcal{O}(\lambda^\alpha, \lambda)$$

and Theorem 3.15 gives

$$\begin{aligned} \mathcal{N}_{\alpha,x}(\lambda) &= \mathcal{O}(\lambda^\delta) \\ \gamma_{\alpha,x}(\lambda) &= \lambda^{\frac{\alpha}{2}}\sqrt{-x} + \mathcal{O}(\lambda^\delta) \end{aligned}$$

for a suitable $\delta > \frac{\alpha}{2}$.

For $\alpha \geq \frac{4}{3}$ we have a 'second degree' regime, because choosing $\delta = \frac{2}{3}$ leads to a second degree anomaly. As for $\alpha > \frac{4}{3}$ there will be no dependence on x to lowest order[7] we consider $\alpha = \frac{2}{3}$. Then one has

$$M_{\lambda,\frac{2}{3}}\, T_{\lambda,\sigma}\, M_{\lambda,\frac{2}{3}}^{-1} \;=\; \exp\left[\lambda^{\frac{1}{3}} \begin{pmatrix} 0 & 0 \\ V_\sigma & 0 \end{pmatrix} + \lambda^{\frac{2}{3}} \begin{pmatrix} 0 & 1 \\ -x & 0 \end{pmatrix} + \mathcal{O}(\lambda) \right].$$

[7]For $\alpha > \frac{4}{3}$ the lowest order coefficient and scaling coincides with the case $x = 0$ which is independent of α. The only thing that would be different is the error term, instead of $\mathcal{O}(\lambda)$ one would have $\mathcal{O}(\lambda, \lambda^{\alpha-\frac{2}{3}})$.

For this case one can calculate the invariant measure to lowest order. We will just state the result here. For $|\theta| < \frac{\pi}{2}$ let

$$\kappa(\theta) \;:=\; \int_0^\theta \frac{-2x\cos^2(\theta) - 2\sin^2(\tilde\theta)}{\mathbb{E}(V_\sigma^2)\,\cos^4(\tilde\theta)}\,\mathrm{d}\tilde\theta, \qquad H(\theta) \;:=\; \int_{-\frac{\pi}{2}}^\theta e^{-\kappa(\tilde\theta)}\,\mathrm{d}\tilde\theta$$

then the invariant measure to lowest order has the density

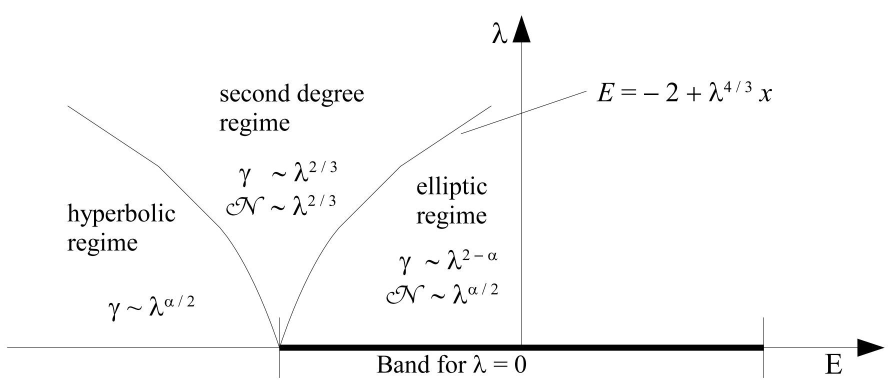
$$\rho(\theta) \;:=\; \frac{c\,e^{\kappa(\theta)}}{\mathbb{E}(V_\sigma^2)\,\cos^4(\theta)}\,H(\theta)$$

where c is a normalisation constant such that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho(\theta)\,\mathrm{d}\theta = 1$. Theorem 3.10 then gives

$$\begin{aligned}
\mathcal{N}_{\alpha,x}(\lambda) &= \frac{1}{\pi}\,\lambda^{\frac{2}{3}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x\cos^2(\theta) + \sin^2(\theta)\right)\,\rho(\theta)\,\mathrm{d}\theta \;+\; \mathcal{O}(\lambda)\\
\gamma_{\alpha,x}(\lambda) &= \lambda^{\frac{2}{3}} \left[\frac{\mathbb{E}(V_\sigma^2)}{8} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1-x}{2}\sin(2\theta) \;-\; \frac{\mathbb{E}(V_\sigma^2)}{4}\cos(4\theta)\right)\right] \rho(\theta)\,\mathrm{d}\theta \;+\; \mathcal{O}(\lambda)\;.
\end{aligned}$$

This result coincides with the result of Derrida and Gardener [DG], but they could not control the error terms. They wrote that the best reason for the transformation with $\lambda^{\frac{2}{3}}$ and the choice to vary the energy with $\lambda^{\frac{4}{3}}$ is that one can solve the lowest order equation for the density of the invariant measure. This is a very bad argument as there is no reason for the invariant measure to be absolutely continuous. Thus a priori there is no reason why one should be able to calculate a density for it pertubatively and even if one could there is no reason why the result should have a meaning. This work shows, that this is really the case. Furthermore it is more conceptual and therefore gives a better understanding for the appearance of $\frac{2}{3}$ and $\frac{4}{3}$ as exponents. In [DG] the transformation was not made at the level of matrices.

The considerations above are summarised in the following phase diagram.



Unfortunately we are not able to calculate the leading coefficient for the IDS (rotation number) in the hyperbolic regime so far. For $\alpha < 1$ and $x < 0$ we know, as the potential is compact supported, that the energy $E = -2 + \lambda^\alpha x$ lies outside the spectrum of $H_{\lambda,\omega}$ for small λ, (spectral radius is of order $2 + \mathcal{O}(\lambda)$), thus one has $\mathcal{N} = 0$. But we do not know exactly what happens in the hyperbolic regime for $1 < \alpha < \frac{4}{3}$.
However something is known about the IDS at the edge of the spectrum of H_ω - *which lies in the hyperbolic regime* - if V_σ is supported by a finite set. For this case we may assume w.l.o.g. that Σ is finite, $V_\sigma \neq V_{\tilde{\sigma}}$ if $\sigma \neq \tilde{\sigma}$ and $\mathbf{p}(\sigma) \neq 0\ \forall \sigma \in \Sigma$. According to [PF, II.4.D] we have $\mathrm{spec}(H_{\lambda,\omega}) = [-2, 2] + \mathrm{supp}\, \lambda\, V_\sigma$ almost surely. Now let $V_{\hat{\sigma}}$ be the minimum of $\{V_\sigma : \sigma \in \Sigma\}$, then $-2 + \lambda V_{\hat{\sigma}}$ is the lower bound of $\mathrm{spec}(H_{\lambda,\omega})$. (Note that $V_{\hat{\sigma}} < 0$ as we have $\mathbb{E}(V_\sigma) = 0, \mathbb{E}(V_\sigma^2) \neq 0$.) Furthermore let $H_{\lambda,\hat{\sigma}} = H_0 + \lambda\, V_{\hat{\sigma}}$ (i.e. the potential is deterministic and equals $\lambda V_{\hat{\sigma}}$ at each point) and let $\mathcal{N}_{\hat{\sigma}}(E, \lambda)$ denote the IDS (rotation number) for the operator $H_{\lambda,\hat{\sigma}}$. For convenience define for fixed λ

$$\delta\mathcal{N}(\epsilon) \;:=\; \mathcal{N}(-2 + \lambda V_{\hat{\sigma}} + \epsilon, \lambda); \quad \delta\mathcal{N}_{\hat{\sigma}}(\epsilon) \;:=\; \mathcal{N}_{\hat{\sigma}}(-2 + \lambda V_{\hat{\sigma}} + \epsilon, \lambda)\ .$$

Then a result from Schulz-Baldes [Sch2] gives us that (for fixed λ) there exist constants $C > 0$, $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$ one has

$$\frac{1}{2}\, \delta\mathcal{N}_{\hat{\sigma}}(\epsilon)\, \mathbf{p}(\hat{\sigma})^{\frac{1}{\delta\mathcal{N}_{\hat{\sigma}}(\epsilon)}+1} \;\leq\; \delta\mathcal{N}(\epsilon) \;\leq\; C\, \delta\mathcal{N}_{\hat{\sigma}}(\epsilon)\, \mathbf{p}(\hat{\sigma})^{\frac{1}{\delta\mathcal{N}_{\hat{\sigma}}(\epsilon)}}\ .$$

Using the deterministic transfer matrix for $H_{\lambda,\hat{\sigma}}$ and calculating the eigenvalues (the transfer matrix is equivalent to a rotation matrix in this case) one gets for $\epsilon < 4$

$$\delta\mathcal{N}_{\hat{\sigma}}(\epsilon) \;=\; \frac{1}{\pi}\, \arccos\left(1 \,-\, \frac{\epsilon}{2}\right)\ .$$

Chapter 2

Preliminaries

In this chapter we state some important theorems for the Lyapunov exponent and the rotation number. Especially the existence of both is proven. The oscillation theorem tells us that the Integrated Density of States and the rotation number coincide. For more background information I recommend to read [CFKS, 9].

2.1 The Lyapunov exponent

We consider random Hamilton Operators as in equation (1.3). The corresponding transfer matrices as defined in (1.4) are denoted by T_σ where the dependence on λ and the Energy E is suppressed in the notation as λ and E will be fixed in the following theorems.

For convenience for $\omega \in \hat{\Omega} = \Sigma^{\mathbb{Z}}$ and $N \in \mathbb{N}$ define

$$\begin{aligned} \mathcal{T}_N(\omega) &:= \prod_{n=0}^{N-1} T_{\omega_{N-n}} = T_{\omega_N} T_{\omega_{N-1}} \dots T_{\omega_1} \\ \mathcal{T}_{-N}(\omega) &:= \left(\prod_{n=0}^{N-1} T_{\omega_n} \right)^{-1} = \left(T_{\omega_{-N+1}} \right)^{-1} \dots \left(T_{\omega_0} \right)^{-1} . \end{aligned}$$

The Lyapunov exponents for the asymptotic to $\pm\infty$ will be denoted by $\gamma^\pm$, i.e.

$$\gamma^\pm := \lim_{N\to\infty} \frac{1}{|N|} \log \|\mathcal{T}_N(\omega)\| .$$

We need the following result from probability theory. A nice proof for a more general situation can be found in [Kal] or any other book about probability theory.

Theorem 2.1. *(Subadditive Ergodic Theorem). Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{S}$ be a measure preserving transformation, i.e. $P(\mathcal{S}^{-1}(A)) = P(A), \forall\, A \in \mathcal{A}$. If ξ_N is a real valued, subadditive process, i.e.*

$$\xi_{N+M}(\omega) \le \xi_N(\omega) + \xi_M(\mathcal{S}^N \omega) \quad \textit{almost surely,}$$

satisfying $\mathbb{E}(|\xi_N|) < \infty$ *for each* N *and* $\Gamma(\xi) := \inf \mathbb{E}(\xi_N)/N > -\infty$, *then* $\xi_N(\omega)/N$ *converges almost surely. If* $\mathcal{S}$ *is ergodic, then*

$$\lim_{N\to\infty} \frac{1}{N}\,\xi_N(\omega) \;=\; \Gamma(\xi) \;=\; \inf \frac{\mathbb{E}(\xi_N)}{N} \quad \textit{almost surely.}$$

Now we can prove the convergence for the Lyapunov exponents as it is done in [CFKS, 9.3].

Theorem 2.2. *For fixed* λ *and fixed energy* E *and* $\mathbf{p}^{\otimes\mathbb{Z}}$*-almost all* ω *the Lyapunov exponents* $\gamma^{\pm}$ *exist, are independent of* ω *and one has* $\gamma^- = \gamma^+ =: \gamma$.

Remark: The subset $\Omega_1 \subset \Omega$ with measure 1 where the expression for the Lyapunov exponent converges depends on λ and E.

Proof. We do the first part only for γ^+ as everything works analog for γ^-. For the space $\hat{\Omega} = \Sigma^{\mathbb{Z}}$ (or $\Omega = \Sigma^{\mathbb{N}}$) with product measure the shift operator $(\mathcal{S}\omega)_n := \omega_{n+1}$ is ergodic. Let $\xi_N(\omega) := \log \|\mathcal{T}_N(\omega)\|$. Then one has

$$\begin{aligned} \xi_{N+M}(\omega) \;&=\; \log \|\mathcal{T}_{N+M}(\omega)\| \;=\; \log \|\mathcal{T}_M(\mathcal{S}^N\omega)\,\mathcal{T}_N(\omega)\| \\ &\leq\; \log\big(\|\mathcal{T}_M(\mathcal{S}^N\omega)\|\,\|\mathcal{T}_N(\omega)\|\big) \;=\; \xi_N(\omega) \,+\, \xi_M(\mathcal{S}^N\omega)\;, \end{aligned}$$

thus the process is subadditive. Furthermore we have

$$\mathbb{E}_\omega(|\xi_N(\omega)|) \;\leq\; \mathbb{E}_\omega \sum_{n=1}^{N} \log \|T_{\omega_n}\| \;=\; N\,\mathbb{E}_\sigma \log \|T_\sigma\| \;<\; \infty$$

because T_σ is compact supported as the potential V_σ was assumed to have compact support. Furthermore as $\det \mathcal{T}_N = 1$ one has $\|\mathcal{T}_N\| \geq 1 \;\Rightarrow\; \Gamma(\xi) = \inf \mathbb{E}(\xi_N)/N \geq 0 > -\infty$. Thus one can apply Theorem 2.1 to get the desired result.

Now let $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then by equation (1.4) one gets $AT_\sigma A^{-1} = T_\sigma^{-1}$. Thus the matrix valued random variables $A\mathcal{T}_N A^{-1}$ and $\mathcal{T}_{-N}$ have the same distribution. As different norms are equivalent the matrices $\mathcal{T}_N$ and $A\mathcal{T}_N A^{-1}$ lead to the same Lyapunov exponent and one finds

$$\gamma^- \;=\; \mathbb{E}(\gamma^-) \;=\; \mathbb{E}(\gamma^+) \;=\; \gamma^+$$

almost surely. □

Osceledec's 'multiplicative ergodic theorem' connects the asymptotic behaviour of $\mathcal{T}_N$ with that of $\mathcal{T}_N\, e_\theta$. For a proof see [Rue].

Theorem 2.3. *(Multiplicative Ergodic Theorem, Osceledec). Let* $T_n \in \mathrm{SL}(2,\mathbb{R})$ *satisfy* $\lim_{N\to\infty} \frac{1}{N} \log \|T_N\| = 0$. *If* $\gamma := \lim_{N\to\infty} \frac{1}{N} \log \|T_N \ldots T_1\| > 0$, *then there exists a one-dimensional subspace* $V^- \subset \mathbb{R}^2$ *such that*

$$\lim_{N\to\infty} \frac{1}{N} \log \|T_N \ldots T_1\, v\| \;=\; -\gamma \qquad \textit{for } v \in V^- \textit{ and} \tag{2.1}$$

$$\lim_{N\to\infty} \frac{1}{N} \log \|T_N \ldots T_1\, v\| \;=\; \gamma \qquad \textit{for } v \notin V^-. \tag{2.2}$$

Using this theorem one can prove the following as it was done in [JSS, Lemma 3].

Theorem 2.4. *For any continuous*[1] *probability measure* μ *on* $\mathbb{R}\mathrm{P}(1) = [0, \pi)$ *one has*

$$\gamma \;=\; \lim_{N\to\infty} \frac{1}{N} \int_0^\pi \mathbb{E}\left(\log \|\mathcal{T}_N\, e_\theta\| \,\mathrm{d}\mu(\theta)\right)$$

The right hand side of the equation above can be expanded into a sum. Therefore set $\theta = \theta_0$ and define θ_n as in Section 1.3 by $\theta_n := S_{\omega_n}(\theta_{n-1})$ to get $e_{\theta_n} = \frac{\mathcal{T}_N e_\theta}{\|\mathcal{T}_N e_\theta\|}$. Then one has

$$\begin{aligned}
\mathbb{E}\,\log \|\mathcal{T}_N\, e_{\theta_0}\| &= \mathbb{E}\,\log\left[\left(\prod_{n=0}^{N-2} \frac{\|\mathcal{T}_{N-n}\, e_{\theta_0}\|}{\|\mathcal{T}_{N-n-1}\, e_{\theta_0}\|}\right)\|\mathcal{T}_1\, e_{\theta_0}\|\right] \\
&= \mathbb{E}\,\log\left[\prod_{n=1}^{N} \|T_{\omega_n}\, e_{\theta_{n-1}}\|\right] \;=\; \mathbb{E}\sum_{n=0}^{N-1} \mathbb{E}_\sigma\,\log\left(\|T_\sigma\, e_{\theta_n}\|\right)\;,
\end{aligned}$$

for the last equation note that θ_{n-1} is independent of ω_n. This sum would collapse if we integrate over θ_0 w.r.t. an invariant measure ν for the projected Markov process θ_n on $\mathbb{R}\mathrm{P}(1)$. As such an invariant measure is given by

$$\int_0^\pi f(\theta)\,\mathrm{d}\nu(\theta) \;=\; \mathbb{E}_\sigma \int_0^\pi f(S_\sigma(\theta))\,\mathrm{d}\nu(\theta)\,, \qquad f \in C^\infty(\mathbb{R}\mathrm{P}(1))$$

one can deduce by induction

$$\int_0^\pi \mathbb{E}_\omega[f(\theta_n(\theta_0,\omega))]\,\mathrm{d}\nu(\theta_0) \;=\; \int_0^\pi f(\theta)\,\mathrm{d}\nu(\theta)\;.$$

Thus Theorem 2.4 leads to the following corollary.

Corollary 2.5. *If* ν *is a continuous, invariant probability measure on* $\mathbb{R}\mathrm{P}(1)$ *of the Markov process* θ_n *then we get*

$$\gamma \;=\; \int_0^\pi \mathbb{E}_\sigma(\log\|T_\sigma\, e_\theta\|)\,\mathrm{d}\nu(\theta)$$

Furtstenberg's theorem [BL, A.II.4] gives certain conditions where such a continuous invariant measure ν exists.

Theorem 2.6. *(Furstenberg) Let* G *be the smallest closed subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *which contains the (essential) support of* $(T_\sigma)_{\sigma\in\Sigma}$ *(w.r.t.* $\mathbf{p}$*). Let us assume that*

(i) G *is not compact,*

(ii) for any $\overline{x} \in \mathbb{R}\mathrm{P}(1) : \{\overline{Mx}, M \in G\}$ *has more than two elements, where* $\overline{x}$ *denotes the projection of* $x \in \mathbb{R}^2$ *on* $\mathbb{R}\mathrm{P}(1)$.

[1] A measure μ is continuous if $\mu(\{\theta\}) = 0$ for all θ. This does not mean that there exists a Lebesgue density for this measure, that would be an absolutely continuous measure.

Then there exists a unique invariant distribution ν on $\mathbb{R}\mathrm{P}(1)$ for the process θ_n and ν is continuous. Furthermore for this case the Lyapunov exponent γ is positive.

Remark: It can be shown that if (i) holds, the second condition is equivalent to G being strongly irreducible, i.e. there is no finite family of proper linear subspaces $V_1, \ldots, V_k$ of $\mathbb{R}^2$ such that

$$T(V_1 \cup V_2 \ldots \cup V_k) \;=\; V_1 \cup V_2 \ldots \cup V_k$$

for any $T \in G$.

For many cases it is not necessary to average over θ_0 w.r.t. a continuous measure like it is done in Theorem 2.4. To be more precisely we get the following [BL, A.III.3.4].

Theorem 2.7. *Let G be the smallest closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ containing the support of T_σ. Suppose that*

(i) $\mathbb{E}(\log \|T_\sigma\|) \;<\; \infty, \quad \mathbb{E}(\log \|T_\sigma^{-1}\|) \;<\; \infty$ *and*

(ii) G is strongly irreducible.

Then $\mathbb{E}\,\frac{1}{N}\,\log\|\mathcal{T}_N\, e_\theta\|$ converges to γ uniformly on $\mathbb{R}\mathrm{P}(1)$.

2.2 Rotation number and IDS

Recall that we defined the functions $S^E_{\lambda,\sigma}$ by the actions of $T^E_{\lambda,\sigma}$ on S^1 and the condition $-\frac{\pi}{2} < \Delta^E_{\lambda,\sigma}(\theta) < \frac{3}{2}\pi$ where $\Delta^E_{\lambda,\sigma}(\theta) = S^E_{\lambda,\sigma}(\theta) - \theta$ is a π-periodic function. Furthermore we defined the Markov process $\theta^E_n = S^E_{\lambda,\omega_n}(\theta^E_{n-1})$ on Ω which depends on λ and θ_0. As λ is fixed from now on we set $\lambda = 1$ and neglect it in the notation (a fixed λ can be absorbed into the definition of the potential). The rotation number was given by

$$\mathcal{N}(E) \;=\; \frac{1}{\pi}\lim_{N\to\infty}\frac{1}{N}\,\theta^E_N \;=\; \frac{1}{\pi}\lim_{N\to\infty}\frac{1}{N}\left(\theta_0 + \sum_{n=0}^{N-1}\Delta^E_{\omega_n}(\theta^E_n)\right),$$

where θ_0 is any initial condition.

Theorem 2.8. *For fixed E and fixed λ the rotation number $\mathcal{N}$ is defined for almost all ω and it is almost surely constant.*

Proof. Let $\xi_N(\omega) = \theta^E_N(\pi,\omega)$. $\mathcal{S}$ denotes the ergodic shift operator on Ω. Furthermore let $\tilde{\theta}$ denote the projection of θ modulo π, i.e. $\tilde{\theta} \in [0,\pi)$ such that $\theta - \tilde{\theta} \in \pi\mathbb{Z}$. Then one has

$$\begin{aligned}
\xi_{N+M}(\omega) \;&=\; \pi + \sum_{n=0}^{N+M-1}\Delta^E_{\omega_n}(\theta^E_n) \;=\; \xi_N(\omega) + \sum_{n=0}^{M-1}\Delta^E_{(\mathcal{S}^N\omega)_n}(\tilde{\theta}^E_{N+n}) \\
&\le\; \xi_N(\omega) + \tilde{\theta}^E_N + \sum_{n=0}^{M-1}\Delta^E_{(\mathcal{S}^N\omega)_n}(\tilde{\theta}^E_{N+n}) \;=\; \xi_N(\omega) + \theta_M(\tilde{\theta}^E_N, \mathcal{S}^N\omega) \\
&\le\; \xi_N(\omega) + \xi_M(\mathcal{S}^N\omega)\,.
\end{aligned}$$

The last step follows from the monotony of the functions S_σ^E leading to the monotony of $\theta_N^E(\theta_0, \omega)$ w.r.t. θ_0. Thus the process ξ_N is subadditive. Furthermore we have $\mathbb{E}(|\xi_N|) \leq \pi + N\frac{3}{2}\pi < \infty$ and $\Gamma(\xi) = \inf\left(\mathbb{E}(\xi_N)/N\right) \geq -\frac{\pi}{2} > -\infty$, therefore Theorem 2.1 completes the proof. □

The Integrated Density of States (short IDS) (for fixed λ) at energy E of the family $(H_{\lambda,\omega})_{\omega\in\hat{\Omega}}$ can almost surely be defined by [PF]

$$\text{IDS} \;=\; \lim_{N\to\infty} \frac{1}{N}\,\text{Tr}(\Pi_N\, 1_{(-\infty,E]}(\Pi_N\, H_\omega\, \Pi_N))\;.$$

where Π_N denotes the projection of $l^2(\mathbb{Z})$ on $l^2(\{1,\ldots,N\}) \subset l^2(\mathbb{Z})$, i.e. $\Pi_N u(n) = u(n)$ for $n = 1,\ldots,N$ and $\Pi_N u(n) = 0$ else. $1_{(-\infty,E]}$ denotes the characteristic function of $(-\infty, E]$. Thus the trace counts the number of independent eigenvectors (sum of dimensions of eigenspaces) in $l^2(\{1,\ldots,N\})$ of the operator $\Pi_N H_\omega \Pi_N$ that correspond to eigenvalues less or equal than E. (The projection operator before the characteristic function ensures that for $E \geq 0$ the infinitely many eignevectors not lying in $l^2(\{1,\ldots,N\})$ are not counted.) For convenience we will denote this number by C_N^E and the projected Hamiltonian by H_N, i.e.

$$H_N \;:=\; \Pi_N H_\omega \Pi_N, \quad C_N^E \;:=\; \text{Tr}(\Pi_N\, 1_{(-\infty,E]}(H_N))\;.$$

For a more formal definition of the IDS see [CFKS, 9.2]. One can deduce the remarkable fact, that the IDS and the S^1 rotation number coincide. The following proof can be found for a bigger class of Hamilton operators in [JSS].

Theorem 2.9. *(oscillation theorem) The IDS and the S^1 rotation number coincide, i.e. one has almost surely*

$$\mathcal{N}(E) \;=\; \lim_{N\to\infty} \frac{1}{\pi}\frac{1}{N}\,\mathbb{E}\left(\theta_N^E(\omega)\right) \;=\; \lim_{N\to\infty}\frac{1}{N}\,C_N^E\;.$$

To be more precisely one has $\left|\frac{1}{\pi}\theta_N^E - C_N^E\right| \leq \frac{1}{2}$ if we begin with the initial phase $\theta_0 = 0$.

To prove this we first introduce some more notations. Let u^E be a formal solution of the Schrödinger equation at energy E, i.e. let

$$\begin{pmatrix} u^E(n+1) \\ u^E(n) \end{pmatrix} \;=\; T_{\omega_n}^E \begin{pmatrix} u^E(n) \\ u^E(n-1) \end{pmatrix}, \quad T_\sigma^E \;=\; \begin{pmatrix} V_\sigma - E & -1 \\ 1 & 0 \end{pmatrix}\;.$$

with initial values $u^E(1) = \cos(\theta_0)$ and $u^E(0) = \sin(\theta_0)$. Furthermore define $R_n^E > 0$ by

$$R_n^E \begin{pmatrix} \cos\theta_n^E \\ \sin\theta_n^E \end{pmatrix} \;=\; \begin{pmatrix} u^E(n+1) \\ u^E(n) \end{pmatrix}\;.$$

Lemma 2.10. *Starting with the same initial angle θ_0 for all energies E one gets the following:*

$$\left(R_N^E\right)^2\,\partial_E\,\theta_N^E \;=\; \begin{cases} \sum_{n=1}^N (u^E(n))^2 & \text{if } N > 0 \\ -\sum_{n=N+1}^0 (u^E(n))^2 & \text{if } N < 0 \end{cases} \tag{2.3}$$

Proof. If $u^E(n) \neq 0$ one gets from the recurrence relation above

$$\cot(\theta_n^E) \;=\; -\tan(\theta_{n-1}^E) \,+\, V_{\omega_n} \,-\, E\;.$$

As the condition $-\frac{\pi}{2} < \Delta_\sigma^E < \frac{3}{2}\pi$ implies $\Delta_\sigma^E(\frac{\pi}{2}) = \frac{\pi}{2}$ for all E it follows that the functions Δ_σ^E and therefore θ_n^E are smooth w.r.t. E. Thus one can differentiate the equation above w.r.t. E to get

$$\partial_E\,\theta_n^E \;=\; \frac{\sin^2(\theta_n^E)}{\cos^2(\theta_{n-1}^E)}\,\partial_E\,\theta_{n-1}^E \;+\; \sin^2(\theta_n^E)\;.$$

Multiplying this with $(R_n^E)^2$ and using $u(n) = \sin(\theta_n^E)R_n^E = \cos(\theta_{n-1}^E)R_{n-1}^E$ leads to

$$(R_n^E)^2\,\partial_E\,\theta_n^E \;=\; (R_{n-1}^E)^2\,\partial_E\,\theta_{n-1}^E \;+\; (u^E(n))^2\;. \tag{2.4}$$

If $u^E(n) = 0$ one can consider $\tan(\theta_n^E)$ to get the same result using similar steps. Iterating (2.4) gives (2.3) as $\partial_E\,\theta_0^E = 0$ by definition. □

Note that $\partial_E\,\theta_n^E$ is strictly positive for $n \geq 2$.

Proof of Theorem 2.9. An eigenvector u of H_N in $l^2(\{1,\ldots,N\})$ has to fulfill the boundary conditions $u(0) = 0$ leading to an initial phase $\theta_0 = 0$ and $u(N+1) = 0$ leading to $\theta_N \equiv \frac{\pi}{2} \mod \pi$. For any $c > 0$ one can choose E small enough to get $V_{\omega_n} - E > c$ for all $n = 1,\ldots,N$. Using this one finds iteratively that $u^E(N) > 0$ for E sufficiently close to $-\infty$ as well as $\lim_{E\to-\infty} \frac{u^E(N)}{u^E(N+1)} = 0$ which means $\lim_{E\to-\infty} \tan(\theta_N^E) = 0$. From $-\frac{\pi}{2} < \Delta_\sigma^E(\theta) < \frac{3}{2}\pi$ one gets $\lim_{E\to-\infty} \Delta_\sigma^E(0) = 0$ for all σ. Thus we find $\lim_{E\to-\infty} \theta_N^E = 0$ for all $N \geq 0$. As θ_N^E is continuous and monotonic increasing w.r.t. E it follows that the j-th eigenvalue of H_N (counted from below, $E_1 < E_2 < \ldots < E_N$) with an eigenvector in $l^2(\{1,\ldots,N\})$ satisfies

$$\theta_N^{E_j} \;=\; \frac{\pi}{2} \;+\; \pi(j-1), \quad \theta_0 \;=\; 0\;.$$

This oscillation theorem implies immediately

$$\left|\frac{1}{\pi}\,\theta_N^E \;-\; C_N^E\right| \;\leq\; \frac{1}{2}\;.$$

□

Chapter 3

Anomalies

This chapter generalises some of the ideas of [Sch1] and handles some more cases, especially the ones needed to analyse a bandedge of the Anderson model in detail.

3.1 Definition of Anomalies

We will consider families $(T_{\lambda,\sigma})_{\lambda\in\mathbb{R}_+,\sigma\in\Sigma}$ of matrices in $\mathrm{SL}(2,\mathbb{R})$ depending on a random variable σ in some probability space $(\Sigma,\mathbf{p})$ and a coupling parameter $\lambda\in\mathbb{R}_+$. Let us assume that $T_{\lambda,\sigma}$ has compact support for small λ, i.e. the closure of $\{T_{\lambda,\sigma}:\lambda\in U,\sigma\in\Sigma\}$ shall be compact for some neighbourhood U of 0. We further suppose that we can expand the dependence on λ in some power series, not necessarily with integer or rational exponents. The expectation values with respect to $\mathbf{p}$ (sometimes $\mathbf{p}^{\otimes\mathbb{N}}$) will be denoted by $\mathbb{E}$ (sometimes $\mathbb{E}_\sigma$ to emphasize the integration variable). Notations like $\mathcal{O}(\lambda^m)$ for rest terms R do always mean, that $\lambda^{-m}R$ is uniformly bounded, i.e. if R depends on λ and some variable a then $R=\mathcal{O}(\lambda^m)$ iff $\exists\, C,\epsilon>0:|R|<\lambda^m C$ for all a and $\lambda<\epsilon$.

Definition 3.1. *The value $\lambda=0$ is an anomaly of first order of the family $(T_{\lambda,\sigma})_{\lambda\in\mathbb{R},\sigma\in\Sigma}$ if for all $\sigma\in\Sigma$:*

$$T_{0,\sigma} \;=\; \pm\mathbf{1}\;, \tag{3.1}$$

with a sign that may depend on $\sigma\in\Sigma$. In order to further classify the anomalies and for later use, let us introduce $P_{k,\sigma}\in\mathrm{sl}(2,\mathbb{R})$ and α_k for $k=1,\dots,K+1$ such that $0<\alpha_1<\dots<\alpha_{K+1}=2\alpha_1<\beta\le\alpha_1+\alpha_2$ and $\mathbf{p}(P_{k,\sigma}=0)<1$ for $k=1,\dots,K$ (which means none of these matrices shall be identically 0 for almost all σ), by

$$MT_{\lambda,\sigma}M^{-1} \;=\; \pm\exp\left(\sum_{k=1}^{K+1}\lambda^{\alpha_k}\,P_{k,\sigma}\;+\;\mathcal{O}(\lambda^\beta)\right)\quad\text{almost surely,} \tag{3.2}$$

where $M\in\mathrm{SL}(2,\mathbb{R})$ is a λ- and σ-independent basis change to be chosen later. An anomaly is said to be of first degree and L-th kind if $L\le K$, $\mathbb{E}(P_{k,\sigma})=0$ for $k=1,\dots,L-1$ and $\mathbb{E}(P_{L,\sigma})$ is non-vanishing. An anomaly of first degree and L-th kind *is called elliptic if* $\det(\mathbb{E}(P_{L,\sigma})) > 0$*, hyperbolic if* $\det(\mathbb{E}(P_{L,\sigma})) < 0$ *and parabolic if* $\det(\mathbb{E}(P_{L,\sigma})) = 0$*. Note that all these notions are independent of the choice of* M*.*

If $\mathbb{E}(P_{k,\sigma}) = 0$*, for* $k = 1, \ldots, K$ *(then the variance of* $P_{1,\sigma}$ *is non-vanishing) then an anomaly is said to be of second degree and* K*-th kind.*

Furthermore, for $m \in \mathbb{N}$*, set* $\hat{\sigma} = (\sigma(m), \ldots, \sigma(1)) \in \hat{\Sigma} = \Sigma^m$*, as well as* $\hat{\mathbf{p}} = \mathbf{p}^{\otimes m}$ *and* $T_{\lambda,\hat{\sigma}} = T_{\lambda,\sigma(m)} \ldots T_{\lambda,\sigma(1)}$*. Then* $\lambda = 0$ *is an anomaly of* m*-th order of the family* $(T_{\lambda,\sigma})_{\lambda\in\mathbb{R},\sigma\in\Sigma}$ *if the family* $(T_{\lambda,\hat{\sigma}})_{\lambda\in\mathbb{R},\hat{\sigma}\in\hat{\Sigma}}$ *has an anomaly of first order at* $\lambda = 0$ *in the above sense. The definitions of degree, kind and nature transpose to* m*-th order anomalies.*

If there is a mapping $\lambda \mapsto M_\lambda \in \mathrm{GL}(2,\mathbb{R})$ *for* $\lambda > 0$ *where* M_λ *is independent of* σ*, such that for*

$$\tilde{T}_{\lambda,\sigma} := M_\lambda \, T_{\lambda,\sigma} \, M_\lambda^{-1}$$

the limit $\tilde{T}_{0,\sigma} := \lim_{\lambda\downarrow 0} \tilde{T}_{\lambda,\sigma}$ *exists (for almost all* σ*) and such that* $\tilde{T}_{\lambda,\sigma}$ *has an anomaly in the above sense, then we say that* $T_{\lambda,\sigma}$ *is transformed to an anomaly by* M_λ*.*
Note that the limits $\lim_{\lambda\downarrow 0} M_\lambda$ *and* $\lim_{\lambda\downarrow 0} M_\lambda^{-1}$ *may not exist.*

Examples

1. At a bandedge of the Anderson model we have already seen that we can transform the occurring transfer matrices to first order anomalies of first or second degree using suitable λ-dependent basis changes. Furthermore to treat the hyperbolic case we will see that a transformation to a second degree anomaly is done. For the cases analysed in Section 1.4 this anomaly could be of first, second or even third kind. Thus we need such general considerations.

2. At the bandcenter of the Anderson model ($E = 1$) one has $T_{0,\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and it follows that $T_{0,\sigma_1} T_{0,\sigma_2} = -\mathbf{1}$ and we get an anomaly of second order.

3.2 Phase shift dynamics

Definition 3.2. *The group action* s_T *of a matrix* $T \in \mathrm{GL}(2,\mathbb{R})$ *on* $\mathbb{RP}(1) \cong \mathbb{R}/\pi\mathbb{Z}$ *is given by*[1]

$$e_{s_T(\theta)} = \pm \frac{T e_\theta}{\|T e_\theta\|}, \qquad e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad s_T(\theta),\, \theta \in [0, \pi) .$$

Note that this is a group action, i.e. one has $s_T \circ s_M = s_{TM}$ and $s_{\mathbf{1}} = \mathrm{id}$.

Let $T \in \mathrm{SL}(2,\mathbb{R})$ then for any integer n you get the same counterclockwise going path $s_T([n\pi, (n+1)\pi])$ that generates the fundamental group. Thus you can lift s_T to a monotonic increasing, analytic function S_T mapping from $\mathbb{R}$ to $\mathbb{R}$, such that $S_T(\theta) - \theta$ is π-periodic. (If $M \in \mathrm{GL}(2,\mathbb{R}), \det M < 0$ then the lifted function S_M is monotonic

[1] Note that the definition for the functions s_T here differs from that in Section 1.3, actually the definition here gives the projected functions of those from Section 1.3.

decreasing.) This function is determined up to an integer multiple of π. If T has real eigenvectors, then the map s_T has fix points and the lift is done in such a way, that S_T also has fix points. For the other case one realises the following.

Proposition 3.3. *If $T \in \mathrm{SL}(2,\mathbb{R})$ has no real eigenvectors then there exists a matrix M with* ***positive*** *determinant, such that $MTM^{-1} = \pm R_\eta := \pm\left(\begin{smallmatrix}\cos\eta & -\sin\eta\\ \sin\eta & \cos\eta\end{smallmatrix}\right)$ for $\eta \in (-\pi/2,\pi/2], \eta \neq 0$, η is uniquely determined by T.*

Proof. Working with the complex field one realises, that there exists $\hat\eta \in (0,\pi)$ such that $e^{i\hat\eta}$ and $e^{-i\hat\eta}$ are eigenvalues of T, thus T is equivalent to $R_{\hat\eta} = \hat M T \hat M^{-1}$. If $\det \hat M > 0$ define $M := \hat M$, if $\det \hat M < 0$ let $M = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\hat M$ then $\det M > 0$ and $MTM^{-1} = R_{-\hat\eta}$. If $M = \hat M$ and $\pi > \hat\eta > \pi/2$ one realizes that $R_{\hat\eta} = -R_{\hat\eta-\pi}$ and similar for $M = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\hat M$ and $-\pi < -\hat\eta \leq -\pi/2$ one has $R_{-\hat\eta} = -R_{\pi-\hat\eta}$. Thus we found $\eta \in (-\pi/2,\pi/2], \eta \neq 0$ and $e \in \{1,-1\}$ such that $MTM^{-1} = eR_\eta$.

To show uniqueness of η note (by comparing eigenvalues) that two **different** matrices R_η and $eR_{\tilde\eta}$ are equivalent if and only if $\eta = \tilde\eta = \pi/2$ and $e = -1$ or $\eta = -\tilde\eta$ and $e = 1$. For $\eta \in (-\pi/2,\pi/2)$ and $\eta \neq 0$, there is no real matrix M with positive determinant, such that $MR_\eta M^{-1} = R_{-\eta}$. Thus you can not get both, R_η and $R_{-\eta}$, by conjugation of T with a matrix M with positive determinant. □

If T has no real eigenvalues, let $MTM^{-1} = \pm R_\eta$, $\det M > 0$ and $\eta \in (-\pi/2,\pi/2]\setminus\{0\}$ like in Proposition 3.3. For $\eta < 0$ the lift is done in such a way that $-\pi < S_T(\theta) - \theta < 0$ and, for $\eta > 0$, such that $0 < S(\theta) - \theta < \pi$. (Conjugation by a matrix M with negative determinant changes a 'forward rotating' matrix to a 'backward rotating' one, e.g. $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)R_\eta\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = R_{-\eta}$.)

Our aim is to calculate the (expected) rotation number and Lyapunov exponent for a random sequence of independent, identically distributed matrices pertubatively. Therefore we consider the probability space $(\Sigma^{\mathbb{N}}, \mathbf{p}^{\otimes\mathbb{N}})$. For $\omega \in \Omega := \Sigma^{\mathbb{N}}$ we get the sequence T_{λ,ω_n}. Now let $\lambda = 0$ be an anomaly of first order of the family $T_{\lambda,\sigma}$ and let $M, P_{k,\sigma}, \alpha_k$ and β be defined as in Definition 3.1, i.e.

$$M\,T_{\lambda,\sigma}\,M^{-1} \;=\; \exp\left[\sum_{k=1}^{K+1} P_{k,\sigma} + \mathcal{O}(\lambda^\beta)\right] .$$

Given an initial angle θ_0 the sequence

$$\theta_n(\omega) \;:=\; S_{\lambda,\omega_n}(\theta_{n-1}(\omega)) \;:=\; S_{MT_{\lambda,\omega_n}M^{-1}}(\theta_{n-1}(\omega))$$

is a Markov process which depends on λ and θ_0 and is described by the phase shift dynamics $\Delta_{\lambda,\sigma}(\theta) := S_{\lambda,\sigma}(\theta) - \theta$. Note that these functions are π-periodic in θ.

As the lift from the function s_T to S_T is continuous in some neighbourhood of $\mathbf{1}$ and $-\mathbf{1}$ (non-continuity appears for matrices T which are equivalent to $\pm R_{\pm\frac{\pi}{2}}$) one can expand the dependence from $S_{\lambda,\sigma}$ on λ. Therefore we introduce the unit vector $v = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\-i\end{smallmatrix}\right)$ and the first order trigonometric polynomials

$$p_{k,\sigma}(\theta) \;:=\; \Im m\left(\frac{\langle v|P_{k,\sigma}|e_\theta\rangle}{\langle v|e_\theta\rangle}\right) \qquad k = 1,\ldots,K+1 \;. \tag{3.3}$$

Using the identity[2] $e^{2i\theta} = \frac{\langle v|e_\theta\rangle}{\langle \bar{v}|e_\theta\rangle}$ one deduces

$$
\begin{aligned}
e^{2iS_{\lambda,\sigma}(\theta)} &= \frac{\left\langle v \middle| \mathbf{1} + \sum_{k=1}^{K+1} \lambda^{\alpha_k} P_{k,\sigma} + \frac{1}{2}\lambda^{2\alpha_1} P_{1,\sigma}^2 + \mathcal{O}(\lambda^\beta) \middle| e_\theta \right\rangle}{\left\langle \bar{v} \middle| \mathbf{1} + \sum_{k=1}^{K+1} \lambda^{\alpha_k} P_{k,\sigma} + \frac{1}{2}\lambda^{2\alpha_1} P_{1,\sigma}^2 + \mathcal{O}(\lambda^\beta) \middle| e_\theta \right\rangle} \\
&= \frac{\langle v|e_\theta\rangle}{\langle \bar{v}|e_\theta\rangle} \left[1 + \sum_{k=1}^{K+1} \lambda^{\alpha_k} \frac{\langle v|P_{k,\sigma}|e_\theta\rangle}{\langle v|e_\theta\rangle} + \frac{1}{2}\lambda^{2\alpha_1} \frac{\langle v|P_{1,\sigma}^2|e_\theta\rangle}{\langle v|e_\theta\rangle} \right] \cdot \\
&\quad \left[1 - \sum_{k=1}^{K+1} \lambda^{\alpha_k} \frac{\langle \bar{v}|P_{k,\sigma}|e_\theta\rangle}{\langle \bar{v}|e_\theta\rangle} - \frac{1}{2}\lambda^{2\alpha_1} \frac{\langle \bar{v}|P_{1,\sigma}^2|e_\theta\rangle}{\langle \bar{v}|e_\theta\rangle} + \lambda^{2\alpha_1} \frac{\langle \bar{v}|P_{1,\sigma}|e_\theta\rangle^2}{\langle \bar{v}|e_\theta\rangle^2} \right] + \mathcal{O}(\lambda^\beta) \\
&= e^{2i\theta} \cdot \exp\left[\sum_{k=1}^{K+1} 2i\lambda^{\alpha_k} p_{k,\sigma}(\theta) + 2i\lambda^{2\alpha_1} \frac{1}{2} p_{1,\sigma}\, \partial_\theta\, p_{1,\sigma} \right] + \mathcal{O}(\lambda^\beta)
\end{aligned}
$$

You get the last line by noting that the exponent has to be imaginary together with the identities

$$
P_{1,\sigma}^2 = -\det(P_{1,\sigma})\, \mathbf{1}, \qquad \Im m\, \frac{\langle \bar{v}|P_{1,\sigma}|e_\theta\rangle^2}{\langle \bar{v}|e_\theta\rangle^2} = p_{1,\sigma}\, \partial_\theta\, p_{1,\sigma} \ .
$$

Defining $q_\sigma := p_{K+1,\sigma} + \frac{1}{2} p_{1,\sigma} \partial_\theta p_{1,\sigma}$ we get

$$
S_{\lambda,\sigma}(\theta) = \theta + \sum_{k=1}^{K} \lambda^{\alpha_k} p_{k,\sigma} + \lambda^{2\alpha_1} q_\sigma + \mathcal{O}(\lambda^\beta) \ . \tag{3.4}
$$

Finally note that $S_{\lambda,\sigma}^{-1} = S_{(MT_{\lambda,\sigma}M^{-1})^{-1}}$ for small λ, which yields

$$
S_{\lambda,\sigma}^{-1}(\theta) = \theta - \sum_{k=1}^{K} \lambda^{\alpha_k} p_{k,\sigma} - \lambda^{2\alpha_1} q_\sigma + \lambda^{2\alpha_1} p_{1,\sigma}\, \partial_\theta\, p_{1,\sigma} + \mathcal{O}(\lambda^\beta) \ . \tag{3.5}
$$

3.3 Rotation number and Lyapunov exponent

Definition 3.4. *The* $\mathbb{R}\mathrm{P}(1)$ *rotation number* $\mathcal{R}$ *for the random sequence* $MT_{\lambda,\omega_n}M^{-1}$ *leading to the Markov process* $\theta_n(\lambda,\theta_0)(\omega)$ *is given by*

$$
\mathcal{R}(\lambda) := \frac{1}{\pi} \lim_{N\to\infty} \frac{1}{N}\theta_N = \frac{1}{\pi} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta_{\lambda,\omega_n}(\theta_{n-1}) \tag{3.6}
$$

[2] $\langle\cdot|\cdot\rangle$ denotes the scalar product, where we use the convention that it is anti-linear in the first and linear in the second component, i.e. $\langle v|w\rangle = \bar{v}^\top w$.

Remark: We call this rotation number $\mathcal{R}$ to emphasize the (possible) difference to the S^1 rotation number $\mathcal{N}$ of Definition 1.3 which is equal to the Integrated Density of States.

Analog to Theorem 2.8 this limit exists $\mathbf{p}^{\otimes\mathbb{N}}$-almost surely w.r.t. ω and it is almost surely constant. Furthermore it is independent of θ_0 (Definition 1.3, Remark 1) and one can deduce as in (1.8) that

$$\mathcal{R}(\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}_\omega\, \frac{1}{\pi} \sum_{n=0}^{N-1} \mathbb{E}_\sigma(\Delta_{\lambda,\sigma}(\theta_n))\,. \tag{3.7}$$

One can show that for the lifted maps for $M \in \mathrm{GL}(2,\mathbb{R})$ one has $S_{MTM^{-1}} = S_M \circ S_T \circ S_M^{-1}$ if $\det M > 0$ or T is not equivalent to $\pm R_{\frac{\pi}{2}}$, where you can actually use any lift of the function s_M. (The addition of an integer multiple of π for the lifted function S_M will be equalized by using S_M^{-1} and not $S_{M^{-1}}$). If S_T has a fix point we see this immediately. Otherwise w.l.o.g. let T be forward rotating and S_M be monotonic increasing. Then one has $0 < S_T \circ S_M^{-1}(\theta) - S_M^{-1}(\theta) < \pi$. By monotony of S_M and the fact that $S_M(\theta+\pi) = S_M(\theta)+\pi$ we get $0 < S_M \circ S_T \circ S_M^{-1}(\theta) - \theta < \pi$. The only problem occurs for $\det M < 0$ and T being equivalent to $R_{\frac{\pi}{2}}$, then $S_{MTM^{-1}}$ and $S_M \circ S_T \circ S_M^{-1}$ differ by π, but it is a question of convention if one considers such a T as forward or backward rotating as the mean effect is a half rotation around $\mathbb{R}\mathrm{P}(1)$. But as $T_{0,\sigma} = \pm\mathbf{1}$ and as the $T_{\lambda,\sigma}$ are compact supported, we won't get such a T for small λ, thus we get the following important Lemma.

Lemma 3.5. *Let $M \in \mathrm{GL}(2,\mathbb{R})$. Then the absolute value of the rotation number is invariant under a basis change by M for small λ, i.e. the sequences T_{λ,ω_n} and $MT_{\lambda,\omega_n}M^{-1}$ give rise to the same absolute rotation number (even if M is λ-dependent). The sign changes iff S_M is decreasing, i.e. iff* $\det M < 0$.

The asymptotic behavior of the products of the random sequence of matrices $(T_{\lambda,\sigma_n})_{n\geq 1}$ can also be characterized by the Lyapunov exponent.

$$\begin{aligned}
\gamma(\lambda) \;&=\; \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}_\omega\, \log\left(\left\|\prod_{n=0}^{N-1} M\,T_{\lambda,\omega_{N-n}}\,M^{-1}\,e_{\theta_0}\right\|\right) \\
&=\; \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}_\omega \sum_{n=0}^{N-1} \mathbb{E}_\sigma(\log\|MT_{\lambda,\sigma}M^{-1}e_{\theta_n}\|)\,,
\end{aligned} \tag{3.8}$$

where θ_0 is an arbitrary initial condition. According to Theorem 2.4 one might expect that we have to average over θ_0 w.r.t. an arbitrary continuous measure before taking the limit, but for the cases where we will derive a pertubative formula for γ we will see that at least to the lowest order the result is independent of θ_0 (compare with Theorem 2.7). A result of Furstenberg states a criterion for having a positive Lyapunov exponent (Theorem 2.6, [BL, A.II.4]). A quantitative control of the Lyapunov exponent in the vicinity of a critical point is given in [SSS, Proposition 1], however, only in the case where the critical point is not an anomaly of first or second order. The latter two cases are dealt with in the present work.

Let us define $\gamma_{k,\sigma} := \langle \bar{v} | P_{k,\sigma} | v \rangle$ and $\gamma_{1,1,\sigma} := \langle \bar{v} | P_{1,\sigma}^\top P_{1,\sigma} | v \rangle$ then we get

$$\begin{aligned} \|MT_{\lambda,\sigma}M^{-1}\, e_\theta\|^2 &= \big\langle e_\theta \,\big|\, (MT_{\lambda,\sigma}M^{-1})^\top\, MT_{\lambda,\sigma}M^{-1} \,\big|\, e_\theta \big\rangle + \mathcal{O}(\lambda^\beta) \\ &= \left\langle e_\theta \left| \mathbf{1} + 2\sum_{k=1}^{K+1} \lambda^{\alpha_k} P_{k,\sigma} + \lambda^{2\alpha_1}(P_{1,\sigma}^2 + P_{1,\sigma}^\top P_{1,\sigma}) \right| e_\theta \right\rangle + \mathcal{O}(\lambda^\beta) \\ &= 1 + \Re e \left[2\sum_{k=1}^{K+1} \lambda^{\alpha_k} \gamma_{k,\sigma} e^{2i\theta} + \lambda^{2\alpha_1} \left(2|\gamma_{1,\sigma}|^2 + \gamma_{1,1,\sigma} e^{2i\theta} \right) \right] + \mathcal{O}(\lambda^\beta) \end{aligned}$$

where we used the identities

$$\langle e_\theta | T | e_\theta \rangle \;=\; \frac{1}{2}\mathrm{Tr}(T) \;+\; \Re e\left(\langle \bar{v} | T | v \rangle e^{2i\theta} \right), \quad \mathrm{Tr}(P_{k,\sigma}) \;=\; 0,$$

$$\langle \bar{v} | P_{1,\sigma}^2 | v \rangle \;=\; 0 \quad \text{and} \quad \mathrm{Tr}(P_{1,\sigma}^\top P_{1,\sigma} + P_{1,\sigma}^2) \;=\; 4|\gamma_{1,\sigma}|^2 \,.$$

Thus we finally obtain for an anomaly of first order

$$\begin{aligned} \log \|MT_{\lambda,\sigma}M^{-1} e_\theta\| \;&=\; \sum_{k=1}^{K+1} \lambda^{\alpha_k}\, \Re e\left(\gamma_{k,\sigma}\, e^{2i\theta} \right) + \\ &+\; \frac{1}{2}\lambda^{2\alpha_1}\, \Re e\left(|\gamma_{1,\sigma}|^2 + \gamma_{1,1,\sigma}\, e^{2i\theta} - \gamma_{1,\sigma}^2\, e^{4i\theta} \right) + \mathcal{O}(\lambda^\beta) \end{aligned} \tag{3.9}$$

where we used

$$2\left[\Re e\left(\gamma_{1,\sigma}\, e^{2i\theta} \right) \right]^2 \;=\; |\gamma_{1,\sigma}|^2 \;+\; \Re e\left(\gamma_{1,\sigma}^2 e^{4i\theta} \right) \,.$$

3.4 Formal pertubative formula for the invariant measure

In this section we present how to derive the formal pertubative formula for the invariant measure. Such an invariant measure ν_λ on $\mathbb{R}\mathrm{P}(1)$ for the projected, stationary Markov process θ_n is given by the equation

$$\int_0^\pi f(\theta)\mathrm{d}\nu_\lambda(\theta) \;=\; \mathbb{E}\int_0^\pi f(S_{\lambda,\sigma}(\theta))\mathrm{d}\nu_\lambda(\theta)\,, \qquad f \,\in\, C^\infty(\mathbb{R}\mathrm{P}(1))\,. \tag{3.10}$$

Furstenberg gave a condition where this invariant measure is unique and the Lyapunov exponent of the associated product of random matrices is positive (Theorem 2.6). Then ν_λ is also Hölder continuous, so it does not contain a point component. Nevertheless there is absolutely no reason why this measure should be absolutely continuous and there will be no proof or discussion about this in this text. But to get a formal formula we may assume, that $d\nu_\lambda(\theta) = \rho_\lambda(\theta)\, d\theta$ where ρ_λ is twice differentiable (w.r.t. θ) and the dependence on λ can be expanded in some power series with some exponents (of course depending on $\alpha_1, \ldots, \alpha_m$). With this assumption a substitution $S_{\lambda,\sigma}(\theta) \to \theta$ on the right hand side of (3.10) leads to

$$\mathbb{E}\big(\partial_\theta S^{-1}_{\lambda,\sigma}(\theta)\rho_\lambda(S^{-1}_{\lambda,\sigma}(\theta))\big) \;=\; \rho_\lambda(\theta)$$

which with equation (3.5) gives

$$\mathbb{E}\left[\rho_\lambda - \sum_{k=1}^{K} \lambda^{\alpha_k}\, \partial_\theta(p_{k,\sigma}\rho_\lambda) + \lambda^{2\alpha_1}\frac{1}{2}\partial_\theta\left[p^2_{1,\sigma}\partial_\theta\rho_\lambda + 2p_{1,\sigma}\partial_\theta p_{1,\sigma}\rho_\lambda - 2q_\sigma\rho_\lambda\right] + \mathcal{O}(\lambda^\beta)\right] = \rho_\lambda$$

$$\Leftrightarrow\; 0 = \sum_{k=1}^{K} \lambda^{\alpha_k}\, \partial_\theta(\mathbb{E}(p_{k,\sigma})\rho_\lambda) - \lambda^{2\alpha_1}\frac{1}{2}\partial_\theta\left[\partial_\theta(\mathbb{E}(p^2_{1,\sigma}\rho_\lambda)) - 2\mathbb{E}(q_\sigma)\rho_\lambda\right] + \mathcal{O}(\lambda^\beta)$$

Thus if ρ is the lowest order invariant measure one has for a first degree anomaly of L-th kind

$$\rho \;=\; \frac{c}{\mathbb{E}(p_{L,\sigma})}$$

for a suitable constant c. This makes sense if $\mathbb{E}(p_{L,\sigma})$ is never zero which happens iff $\det(P_\sigma) > 0$, i.e. iff the anomaly is elliptic. The parabolic and hyperbolic case will be transformed to a second degree anomaly by a λ-dependent basis change.

If the anomaly is of second degree and K-th kind, one finds

$$\frac{1}{2}\partial_\theta\big(\mathbb{E}(p^2_{1,\sigma})\rho\big) - \mathbb{E}(q_\sigma)\rho \;=\; c \tag{3.11}$$

where the constant c has to be chosen such that this equation admits a π-periodic, normalized solution ρ. The case where $\mathbb{E}(p^2_{1,\sigma}) > 0$ for all θ was already treated by Schulz-Baldes [Sch1]. For this case you can easily solve the equation. Setting

$$\kappa(\theta) \;:=\; \int_0^\theta \frac{2\mathbb{E}(q_\sigma(\tilde\theta))}{\mathbb{E}(p^2_{1,\sigma}(\tilde\theta))}\, d\tilde\theta, \quad H(\theta) \;:=\; \int_0^\theta e^{-\kappa(\tilde\theta)}\, d\tilde\theta, \quad C = \frac{e^{-\kappa(\pi)} - 1}{H(\pi)}$$

the solution is given by

$$\rho(\theta) \;=\; \frac{\tilde c e^{\kappa(\theta)}}{\mathbb{E}(p^2_{1,\sigma}(\theta))}\,(CH(\theta) + 1) \tag{3.12}$$

where $\tilde c$ is a normalization constant. Comparing this equation with (3.11) gives $c = \frac{1}{2}\tilde c C$. Note the important fact, that $\rho(\theta)$ is an analytic function of θ.

One can have $\mathbb{E}(p^2_{1,\sigma}(\hat\theta)) = 0 \;\Leftrightarrow\; p_{1,\sigma}(\hat\theta) = 0$ (for almost all σ) for at most two different angles in a fundamental domain of $\mathbb{R}\mathrm{P}(1)$ (if this happens for three different angles then $\mathbb{E}(p^2_{1,\sigma}) = 0$ for all θ). The case where this happens only for one angle $\hat\theta$ can be treated with the same methods, if $\mathbb{E}(q_\sigma)(\hat\theta) \neq 0 \;\Leftrightarrow\; \mathbb{E}(p_{K+1,\sigma}(\hat\theta)) \neq 0$, because of the following lemma which coincides with Lemma 4.5 and is proven there.

Lemma 3.6. *Let $p, q, f \in C^\infty(\mathbb{R}), d \in \mathbb{R}$ such that*

$$p(d) \;=\; 0, \qquad \forall\, n \in \mathbb{N}_0 : \quad q(d) + np'(d) \neq 0$$

and such that there is a neighbourhood U of d, where $p(x) \neq 0$ except for $x = d$. Let Q be an antiderivative of $\frac{q}{p}$ for $x \in U, x \neq d$ such that

$$\lim_{x \downarrow d} \frac{\exp(Q)}{p} \;=\; 0, \quad \forall\, n \in \mathbb{N} : \lim_{x \uparrow d} |p|^n \exp(Q) \;=\; \infty\,.$$

Then there exist (in U) smooth (C^∞) solutions F for the differential equation

$$pF' + qF \;=\; f\,.$$

To define such a smooth solution ρ in our case we use the interval $[\hat\theta, \hat\theta + \pi)$. Let

$$\kappa(\theta) \;:=\; \int_{\hat\theta+\pi/2}^{\theta} \frac{2\mathbb{E}(q_\sigma(\tilde\theta))}{\mathbb{E}(p_{1,\sigma}^2(\tilde\theta))}\, \mathrm{d}\tilde\theta, \quad H(\theta) \;:=\; \int_\xi^\theta e^{-\kappa(\tilde\theta)}\, \mathrm{d}\tilde\theta, \quad \theta \,\in\, (\hat\theta, \hat\theta+\pi)$$

where the lower boundary ξ of the integral for H is set to be $\xi = \hat\theta$ if $\lim_{\theta\downarrow\hat\theta} \kappa(\theta) = \infty$, that happens whenever $\mathbb{E}(q_\sigma(\hat\theta)) < 0$, and $\xi = \hat\theta + \pi$ if $\lim_{\theta\uparrow\hat\theta+\pi} \kappa(\theta) = \infty$ which is the case for $\mathbb{E}(q_\sigma(\hat\theta)) > 0$. Then the solution is the periodized smooth function

$$\rho(\theta) \;=\; \frac{2ce^{\kappa(\theta)}}{\mathbb{E}(p_{1,\sigma}^2(\theta))} H(\theta) \tag{3.13}$$

where c is a normalization constant. It is important to notice that $c \neq 0$ in this case. The (proof of the) lemma stated above will give us that the limit to $\hat\theta$ of this expression exists and that this function is infinitely often differentiable, even at $\hat\theta$.

Although this was a formal calculation to get the invariant measure ν_λ to lowest order, for those cases where we could solve the differential equation for ρ the theorems stated in the next chapter will give us the following.

Theorem 3.7. *If f is a π-periodic, twice continuously differentiable function one has*

$$\mathbb{E}_\omega \frac{1}{N} \sum_{n=0}^{N} f(\theta_n) \;=\; \int_0^\pi f(\theta)\, \rho(\theta)\, \mathrm{d}\theta \,+\, \mathcal{O}((\lambda^{\tilde\delta} N)^{-1}, \lambda^\delta)$$

for suitable $\tilde\delta, \delta > 0$. Integrating this equation over θ_0 w.r.t. the invariant measure $\mathrm{d}\nu_\lambda$ and letting $N \to \infty$ leads to

$$\int_0^\pi f(\theta)\, \mathrm{d}\nu_\lambda(\theta) \;=\; \int_0^\pi f(\theta)\, \rho(\theta)\, \mathrm{d}\theta \,+\, \mathcal{O}(\lambda^\delta)\,.$$

3.5 Results

In this chapter we summerise the main theorems we are going to prove in detail. To get the formulas for the rotation number we use equations (3.4) and (3.7) and for the Lyapunov exponent we use equations (3.8) and (3.9). The family $T_{\lambda,\sigma}$ is always supposed to have an anomaly of first order. For anomalies of m-th order use $\hat{\Sigma} := \Sigma^m$ with product measure to calculate $\hat{\mathcal{R}}$ and $\hat{\gamma}$. Then one has $\gamma = \frac{1}{m}\hat{\gamma}$. The calculation of the rotation number is a little bit more subtle. Each matrix $T_{0,\sigma}$ has a mean rotation of $\frac{k}{m}\pi$ for a certain integer k. From this you have to calculate the rotation number for $\lambda = 0$ first and then $\frac{1}{m}\hat{\mathcal{R}}$ is the pertubative correction.

Theorem 3.8. *If the family $T_{\lambda,\sigma}$ has an elliptic anomaly of first order, first degree and L-th kind ($L \leq K$) and if f is any π-periodic, continuously differentiable function one has*

$$\mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \frac{\int_0^\pi \frac{f(\theta)}{\mathbb{E}(p_{L,\sigma}(\theta))}\,\mathrm{d}\theta}{\int_0^\pi \frac{1}{\mathbb{E}(p_{L,\sigma}(\theta))}\,\mathrm{d}\theta} \;+\; \mathcal{O}((\lambda^{\alpha_L}N)^{-1}, \lambda^{\alpha_{L+1}-\alpha_L})\;.$$

To calculate the Lyapunov exponent for an elliptic anomaly of first degree and L-th kind it turns out that one should use a special basis change. Therefore let

$$T_{\lambda,\sigma} \;=\; \exp\left[\sum_{k=1}^{K+1} \lambda^{\alpha_k}\,\hat{P}_{k,\sigma} \;+\; \mathcal{O}(\lambda^\beta)\right], \quad \mathbb{E}(\hat{P}_{L,\sigma}) \;=\; \begin{pmatrix} \hat{a}_L & \hat{b}_L \\ \hat{c}_L & -\hat{a}_L \end{pmatrix}\;.$$

As this anomaly should be elliptic one has $\det \mathbb{E}(\hat{P}_{L,\sigma}) \;=\; -\hat{a}_L^2 - \hat{b}_L\hat{c}_L > 0$ which implies $\hat{c}_L \neq 0$. Define η and M by

$$\eta^2 \;=\; -\hat{a}_L - \hat{b}_L\hat{c}_L\;, \qquad M \;:=\; \begin{pmatrix} \hat{c}_L & -\hat{a}_L \\ 0 & \eta \end{pmatrix} \;\in\; \mathrm{GL}(2,\mathbb{R})$$

to get

$$\mathbb{E}(P_{L,\sigma}) \;=\; \mathbb{E}(M\,\hat{P}_{L,\sigma}\,M^{-1}) \;=\; \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}\;.$$

From this one deduces $p_{L,\sigma}(\theta) = \eta$ for all θ. Thus by Theorem 3.8 the invariant measure to lowest order is given by the Haar measure on $\mathbb{R}\mathrm{P}(1)$. From (3.4) one gets $\mathbb{E}(\Delta_{\lambda,\sigma}(\theta)) = \lambda^{\alpha_L}\eta + \mathcal{O}(\lambda^{\alpha_{L+1}})$ and

$$\mathbb{E}(\gamma_{L,\sigma}) \;=\; \frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} \;=\; 0\;.$$

Thus Theorem 3.8 and the equations (3.7), (3.8) and (3.9) lead to the following.

Corollary 3.9. *If $T_{\lambda,\sigma}$ has an elliptic anomaly of first order, first degree and L-th kind, we find $\eta \in \mathbb{R}$ and a certain basis change $MT_{\lambda,\sigma}M^{-1}$ such that $\mathbb{E}(P_{L,\sigma}) = \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}$. This basis change gives $\mathbb{E}(p_{L,\sigma}) = \eta$ which leads to*

$$\mathcal{R}(\lambda) \;=\; \lambda^{\alpha_L}\,\frac{\eta}{\pi} \;+\; \mathcal{O}(\lambda^{\alpha_{L+1}})\;.$$

Furthermore we have $\mathbb{E}(\gamma_{k,\sigma}) = 0$ *for* $k = 1, \ldots, L$ *(for* $k = L$ *this holds only after the used basis change). Thus we get*

$$\gamma(\lambda) \;=\; \frac{1}{2}\lambda^{2\alpha_1}\,\mathbb{E}(|\gamma_{1,\sigma}|^2) \;+\; \mathcal{O}(\lambda^{2\alpha_{L+1}-\alpha_L}, \lambda^{\beta}) \;.$$

Remark: This only gives us the leading term coefficient for γ if $2\alpha_{L+1} > 2\alpha_1 + \alpha_L$ which holds for example if $L = K$ (then one has $\alpha_{L+1} = 2\alpha_1$). Otherwise we only get an upper bound for the correct scaling of γ.

For anomalies of second degree we get the following.

Theorem 3.10. *Let the family* $T_{\lambda,\sigma}$ *have an anomaly of first order and second degree (and* K*-th kind) such that* $\mathbb{E}(p_{1,\sigma}^2(\hat{\theta})) = 0$ *happens for at most one angle* $\hat{\theta}$*, where* $q_\sigma(\hat{\theta}) \neq 0$ *for that case. Then one can find a constant* c *and a smooth,* π*-periodic solution* ρ *for*

$$\frac{1}{2}\partial_\theta(\mathbb{E}(p_{1,\sigma}^2)\rho) \;-\; \mathbb{E}(q_\sigma)\rho \;=\; c$$

such that $\int_0^\pi \rho(\theta)\,\mathrm{d}\theta = 1$ *(given with equation (3.12) or (3.13)). For a* π*-periodic, twice continuously differentiable function* f *one has*

$$\mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \int_0^\pi f(\theta)\,\rho(\theta)\,\mathrm{d}\theta \;+\; \mathcal{O}((\lambda^{2\alpha_1}N)^{-1}, \lambda^{\beta-2\alpha_1}) \;.$$

Corollary 3.11. *If* $T_{\lambda,\sigma}$ *has an anomaly of first order and second degree fulfilling the conditions of Theorem 3.10 and if* ρ *is defined as in that theorem one has*

$$\mathcal{R}(\lambda) \;=\; \lambda^{2\alpha_1}\int_0^\pi q_\sigma(\theta)\,\rho(\theta)\,\mathrm{d}\theta \;+\; \mathcal{O}(\lambda^{\beta}) \;.$$

and

$$\gamma(\lambda) \;=\; \lambda^{2\alpha_1}\frac{1}{2}\,\Re e\left[\mathbb{E}(|\gamma_{1,\sigma}|^2) + \mathbb{E}\left(\gamma_{K+1,\sigma} + \gamma_{1,1,\sigma}\right) I_1 - \mathbb{E}(\gamma_{1,\sigma}^2) I_2\right] \;+\; \mathcal{O}(\lambda^{\beta})$$

where $I_j := \int_0^\pi e^{2ij\theta}\,\rho(\theta)\,\mathrm{d}\theta$*.*

There is one special case for second degree anomalies where we will show, that the invariant measure is to lowest order a Dirac measure.

Theorem 3.12. *Let* $T_{\lambda,\sigma}$ *have an anomaly of first order and second degree. If there exists one angle* $\hat{\theta}$ *such that*

$$\mathbb{E}(p_{1,\sigma}^2(\hat{\theta})) \;=\; \mathbb{E}\left(\left[\partial_\theta p_{1,\sigma}(\hat{\theta})\right]^2\right) \;=\; 0, \quad \mathbb{E}(q_\sigma(\hat{\theta})) \;=\; 0, \quad \partial_\theta\mathbb{E}(q_\sigma(\hat{\theta})) \;<\; 0 \;,$$

then one finds for a π*-periodic and 3 times continuously differentiable function* f *that*

$$\mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; f(\hat{\theta}) \;+\; \mathcal{O}((\lambda^{2\alpha_1}N)^{-1}, \lambda^{\beta-2\alpha_1}) \;.$$

Corollary 3.13. *If $T_{\lambda,\sigma}$ has an anomaly of first order and second degree fulfilling the conditions of Theorem 3.12 then one has*

$$\mathcal{R} \;=\; \mathcal{O}(\lambda^{\beta})$$

and

$$\gamma(\lambda) \;=\; \lambda^{2\alpha_1}\,\Re e\left(\mathbb{E}(\gamma_{K+1,\sigma})e^{2i\hat{\theta}}\right) \;+\; \mathcal{O}((\lambda^{2\alpha_1}N)^{-1},\lambda^{\beta})\;.$$

For the last equation note that the conditions on $p_{1,\sigma}$ lead to

$$\Re e\left[|\gamma_{1,\sigma}|^2 \;+\; \gamma_{1,1,\sigma}\,e^{2i\hat{\theta}} \;-\; \gamma_{1,\sigma}^2 e^{4i\hat{\theta}}\right] \;=\; 0$$

which is easily seen if $\hat{\theta} = \frac{\pi}{2}$ (this is the case after a suitable rotation basis change). For that case one deduces (see equation (4.5))

$$P_{1,\sigma} \;=\; \begin{pmatrix} 0 & 0 \\ c_{1,\sigma} & 0 \end{pmatrix} \;\Rightarrow\; \gamma_{1,\sigma} \;=\; \frac{-ic_{\sigma}}{2}, \quad \gamma_{1,1,\sigma} \;=\; \frac{c_{\sigma}^2}{2}\;.$$

Some parabolic anomalies of first degree can be transformed to a second degree anomaly like in Theorem 3.10. The next theorem states the right order of the rotation number and Lyapunov exponent for that case. It will become clear how to calculate the leading coefficient in section 4.3. For convenience we use the following notations. If the exponent $\xi > 0$ occurs inside the exponential in the expansion of $MT_{\lambda,\sigma}M^{-1}$ then let $P_{\xi,\sigma}$ be the corresponding matrix (i.e. the term $\lambda^{\xi}P_{\xi,\sigma}$ occurs inside the exponential).

Theorem 3.14. *Let $T_{\lambda,\sigma}$ have an parabolic anomaly of first order, first degree and L-th kind where $\mathbb{E}(P_{L,\sigma})e_{\hat{\theta}} = 0$. Let χ be the lowest exponent such that $e_{\hat{\theta}}$ is* ***not*** *for almost all σ an eigenvector of $P_{\chi,\sigma}$ (such a χ is supposed to exist).*
If $\frac{1}{3}(\chi+\alpha_L) \leq \alpha_1$, $e_{\hat{\theta}}$ is an eigenvector of $\mathbb{E}(P_{\xi,\sigma})$ for all $\xi < \frac{1}{3}(4\chi+\alpha_L)$ and furthermore $\mathbb{E}(P_{\xi,\sigma})e_{\hat{\theta}} = 0$ for all $\xi < \frac{2}{3}(\chi+\alpha_L)$ (all this conditions hold if e.g. $\chi = \alpha_1$ and $L = K = 1$)[3] then one has

$$\mathcal{R}(\lambda) \;=\; \mathcal{O}(\lambda^{\frac{2}{3}(\chi+\alpha_L)}), \quad \gamma(\lambda) \;=\; \mathcal{O}(\lambda^{\frac{2}{3}(\chi+\alpha_L)})\;.$$

Remark: We will see that if $\frac{1}{3}(\chi+\alpha_L) = \alpha_1$ and if $a_{1,\sigma}$ is not zero for almost all σ and linear dependent to $c_{\chi,\sigma}$ - *which denotes the left down entry of $P_{\chi,\sigma}$* - we are not able to calculate the coefficient. This is due to the fact that $\mathbb{E}(p_{1,\sigma}^2(\theta))$ should be zero for at most one angle θ in Theorem 3.10.

Some hyperbolic anomalies of first degree can be transformed to a second degree anomaly like in Theorem 3.12. For $L = K = 1$ the result after this transformation coincides with the considerations in [Sch1].

[3] These conditions lead to $\frac{1}{3}(\chi+\alpha_L) = \frac{2}{3}\alpha_1 < \alpha_1$ and $\frac{4}{3}\chi + \frac{1}{3}\alpha_L = \frac{5}{3}\alpha_1 < 2\alpha_1 = \alpha_2$. Thus one has $\mathbb{E}(P_{\xi,\sigma})e_{\hat{\theta}} = 0$ for all $\xi < \frac{4}{3}\chi + \frac{1}{3}\alpha_L$.

Theorem 3.15. *Let $T_{\lambda,\sigma}$ have an hyperbolic anomaly of first order, first degree and L-th kind. Then one finds a basis change such that $\mathbb{E}(P_{L,\sigma}) = \begin{pmatrix} -\eta & 0 \\ 0 & \eta \end{pmatrix}$ where $\eta > 0$. After this basis change let χ be the lowest exponent such that e_0 is **not** for almost all σ an eigenvector of $P_{\chi,\sigma}$. If e_0 is an eigenvector of $\mathbb{E}(P_{\xi,\sigma})$ for all $\xi < \frac{1}{2}\alpha_L + \chi$ (this holds e.g. if $\chi = \alpha_1$ and $L = K$)[4] then one has*

$$\mathcal{R}(\lambda) \;=\; \mathcal{O}(\lambda^{\delta})$$

and

$$\gamma(\lambda) \;=\; \lambda^{\alpha_L}\,\eta \;+\; \mathcal{O}(\lambda^{\delta})$$

for a suitable $\delta > \alpha_L$.

[4]These conditions lead to $\chi + \frac{1}{2}\alpha_L < 2\alpha_1 = \alpha_{L+1}$ and for $k = 1, \ldots, L$ we know that e_0 is an eigenvector of $\mathbb{E}(P_{k,\sigma})$.

Chapter 4

Proofs

In this chapter we prove the main theorems stated in Section 3.5. For the whole chapter let $T_{\lambda,\sigma}$ have an anomaly of first order. Recall the whole situation, $(\Sigma, \mathbf{p})$ is a probability space, $S_{\lambda,\sigma}(\theta)$ are real functions, $\sigma \in \Sigma, \lambda \geq 0$ and $\theta \in \mathbb{R}$, such that

$$S_{\lambda,\sigma}(\theta) = \theta + \sum_{k=1}^{K} \lambda^{\alpha_k} p_{k,\sigma} + \lambda^{2\alpha_1} q_\sigma + \lambda^\beta r_\sigma, \quad 0 < \alpha_1 < \ldots < \alpha_K < 2\alpha_1 < \beta \leq \alpha_1 + \alpha_2$$

where $p_{k,\sigma}$ and q_σ are π-periodic smooth functions of θ and independent of λ, r_σ depends on λ and θ, but for all λ it is also π-periodic w.r.t. θ. For convenience we also defined $\alpha_{K+1} = 2\alpha_1$. The Markov process θ_n on $(\Sigma^{\mathbb{N}}, \mathbf{p}^{\otimes\mathbb{N}})$ is generated by $S_{\lambda,\sigma}$, i.e. $\theta_n := S_{\lambda,\omega_n}(\theta_{n-1})$. It depends on λ and the initial phase θ_0.

4.1 Main idea

The main idea for the proofs is to expand a Birkhoff sum $\mathbb{E}\,\frac{1}{N}\sum_{n=1}^{N} F(\theta_n)$ w.r.t. λ to get $\mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1} F(\theta_n) + \lambda^\delta f(\theta_n) +$ higher order terms in λ, where f fulfills some differential equation in F. Subtracting the first term gives that a Birkhoff sum $\mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n)$ is of order $\mathcal{O}((\lambda^\delta N)^{-1}, \lambda^{\tilde{\delta}})$ for suitable $\delta, \tilde{\delta} > 0$. To be more precisely we get the following.

But first a little definition. We say that all mixed moments up to n-th order of $p_{k,\sigma}, q_\sigma$ and r_σ are uniformly bounded iff

$$\exists C, \epsilon > 0\ \forall\, \lambda \leq \epsilon,\ \forall\, \theta \in \mathbb{R}\ :\ \left|\mathbb{E}\left(\prod_{k=1}^{K} p_{k,\sigma}^{g_k} q_\sigma^h r_\sigma^j\right)\right| \leq C, \quad \sum_{k=1}^{K} g_k + h + j \leq n$$

where g_k, h, j and n are non negative integers.
In our situation these mixed moments are uniformly bounded up to order n for any $n \in \mathbb{N}$ as $T_{\lambda,\sigma}$ was supposed to have compact support for small λ. Nevertheless let us state the following lemmas for a more general situation.

Lemma 4.1. *Let all mixed moments up to 2nd order of* $p_{k,\sigma}, q_\sigma$ *and* r_σ *be uniformly bounded. Furthermore let* $T_{\lambda,\sigma}$ *have an anomaly of first degree and (at least) L-th kind,* $L \le K$*, i.e.* $\mathbb{E}(p_{k,\sigma}) = 0$ *for* $k = 1, \ldots, L-1$*. If F is any* π*-periodic, twice continuously differentiable function, then one has*

$$f \;=\; \mathbb{E}(p_{L,\sigma})F' \;\;\Rightarrow\;\; \mathbb{E}_\omega \frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \mathcal{O}\left(\lambda^{\alpha_{L+1}-\alpha_L}, \frac{\lambda^{-\alpha_L}}{N}\right) .$$

Proof. Let $\mathbb{E}(p_{k,\sigma}) = 0$ for $k = 1, \ldots, L-1$ and define

$$u_\sigma \;:=\; \sum_{k=L+1}^{K} \lambda^{\alpha_k - \alpha_{L+1}} p_{k,\sigma} + \lambda^{2\alpha_1 - \alpha_{L+1}} q_\sigma + \lambda^{\beta - \alpha_{L+1}} r_\sigma \;=\; \mathcal{O}(1) .$$

We claim

$$\mathbb{E}\left[F(S_{\lambda,\sigma}(\theta))\right] \;=\; F(\theta) \,+\, F'(\theta)\,\mathbb{E}(p_{L,\sigma}(\theta)) \,+\, R(\lambda,\theta) \tag{4.1}$$

such that $\exists\, \delta, C > 0 \quad \forall\, \lambda \le \delta \quad \forall\, \theta : \quad |R(\lambda,\theta)| \;\le\; C\lambda^{\alpha_{L+1}}$.
To show this we do a Taylor expansion

$$F(S_{\lambda,\sigma}(\theta)) \;=\; F\left(\theta \,+\, \sum_{k=1}^{L} \lambda^{\alpha_k}\, p_{k,\sigma}(\theta) \,+\, \lambda^{\alpha_{L+1}}\, u_\sigma(\theta)\right) \;=\; F(\theta) +$$

$$F'(\theta)\left(\sum_{k=1}^{L} \lambda^{\alpha_k}\, p_{k,\sigma}(\theta) + \lambda^{\alpha_{L+1}} u_\sigma(\theta)\right) \,+\, \frac{1}{2}F''(\xi)\left(\sum_{k=1}^{L} \lambda^{\alpha_k}\, p_{k,\sigma}(\theta) + \lambda^{\alpha_{L+1}} u_\sigma(\theta)\right)^2$$

where ξ is some σ, λ and θ dependent real number for the error term. Comparing this with (4.1) and using $\mathbb{E}(p_{k,\sigma}) = 0$ for $k = 1, \ldots, L$ leads to

$$R(\lambda,\theta) \;=\; F'(\theta)\,\mathbb{E}\left(\lambda^{\alpha_{L+1}} u_\sigma(\theta)\right) \,+\, \frac{1}{2}F''(\xi)\,\mathbb{E}\left(\sum_{k=1}^{L} \lambda^{\alpha_k} p_{k,\sigma}(\theta) \,+\, \lambda^{\alpha_{L+1}} u_\sigma(\theta)\right)^2$$

As F' and F'' are bounded, the occurring mixed moments are uniformly bounded and as $\alpha_{L+1} \le \alpha_{K+1} = 2\alpha_1$ one deduces that there exist a constant C such that for **all** θ and λ small enough one has

$$|R(\lambda,\theta)| \;\le\; C\lambda^{\alpha_{L+1}} .$$

This is exactly what we claimed in (4.1) which leads to

$$\mathbb{E}_\omega \frac{1}{N}\sum_{n=1}^{N} F(\theta_n) \;=\; \mathbb{E}_\omega \frac{1}{N}\sum_{n=0}^{N-1}\left[F(\theta_n) \,+\, \lambda^{\alpha_L}\, F'(\theta_n)\mathbb{E}_\sigma(p_{L,\sigma}(\theta_n)) \,+\, R(\lambda,\theta_n)\right] \;\; .$$

Thus if λ is small enough one has for all θ_0

$$\left|\mathbb{E}_\omega \frac{1}{N}\sum_{n=0}^{N-1} F'(\theta_n)\,\mathbb{E}_\sigma(p_{L,\sigma}(\theta_n))\right| \;\le\; \frac{\tilde{C}\lambda^{-\alpha_L}}{N} + C\lambda^{\alpha_{L+1}-\alpha_L}$$

where $\tilde{C} = 2\max F(\theta)$. This inequality is exactly what we wanted to show. □

Lemma 4.2. *Let all mixed moments up to 3rd order of $p_{k,\sigma}, q_\sigma$ and r_σ be uniformly bounded. Furthermore let $T_{\lambda,\sigma}$ have an anomaly of second degree (and K-th kind), i.e. $\mathbb{E}(p_{k,\sigma}) = 0$ for $k = 1, \ldots, K$. If F is any π-periodic, 3 times continuously differentiable function, then one has*

$$f \;=\; \frac{1}{2}\mathbb{E}(p_{1,\sigma}^2)F'' \,+\, \mathbb{E}(q_\sigma)F' \;\Rightarrow\; \mathbb{E}_\omega \frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \mathcal{O}\left(\lambda^{\beta-2\alpha_1}, \frac{\lambda^{-2\alpha_1}}{N}\right).$$

Proof. Let $\mathbb{E}(p_{k,\sigma}) = 0$ for $k = 1, \ldots, K$, we claim

$$\mathbb{E}\left[F(S_{\lambda,\sigma}(\theta))\right] \;=\; F(\theta) \,+\, \lambda^{2\alpha_1}\left[F'(\theta)\,\mathbb{E}(q_\sigma(\theta)) \,+\, \frac{1}{2}F''(\theta)\mathbb{E}(p_{1,\sigma}^2(\theta))\right] \,+\, R(\lambda,\theta) \tag{4.2}$$

such that $\quad \exists\,\delta, C \in \mathbb{R}_+ \quad \forall\,\lambda \le \delta \quad \forall\,\theta \,:\quad |R(\lambda,\theta)| \;\le\; \lambda^\beta C.$
This is again shown by a Taylor expansion.

$$F(S_{\lambda,\sigma}(\theta)) \;=\; F\left(\theta + \sum_{k=1}^{K}\lambda^{\alpha_k}p_{k,\sigma}(\theta) \,+\, \lambda^{2\alpha_1}q_\sigma(\theta) \,+\, \lambda^\beta r_\sigma(\theta)\right) \;=$$

$$F(\theta) \,+\, F'(\theta)\left(\sum_{k=1}^{K}\lambda^{\alpha_k}p_{k,\sigma}(\theta) \,+\, \lambda^{2\alpha_1}q_\sigma(\theta) \,+\, \lambda^\beta r_\sigma(\theta)\right) +$$

$$\frac{F''(\theta)}{2}\left(\sum_{k=1}^{K}\lambda^{\alpha_k}p_{k,\sigma}(\theta) \,+\, \lambda^{2\alpha_1}q_\sigma(\theta) \,+\, \lambda^\beta r_\sigma(\theta)\right)^2 \,+\, R_1(\lambda,\theta,\sigma)$$

The rest term of the Taylor expansion can be written as

$$R_1(\lambda,\theta,\sigma) \;=\; \frac{1}{6}F'''(\xi)\left(\sum_{k=1}^{K}\lambda^{\alpha_k}p_{k,\sigma}(\theta) \,+\, \lambda^{2\alpha_1}q_\sigma(\theta) \,+\, \lambda^\beta r_\sigma(\theta)\right)^3$$

where ξ depends on λ, σ and θ. As the occurring mixed moments are uniformly bounded and as F''' is bounded one finds a constant C_1 such that $|\mathbb{E}(R_1(\lambda,\theta,\sigma))| \le \lambda^{3\alpha_1}C_1$ for all θ and λ small enough. Comparing the Taylor expansion with (4.2) and using $\mathbb{E}(p_{k,\sigma}) = 0$ one deduces

$$R(\lambda,\theta) \;=\; \lambda^\beta\, F'(\theta)\mathbb{E}(r_\sigma(\theta)) \,+\, \frac{1}{2}F''(\theta)\,\mathbb{E}\left(\sum_{k=2}^{K}\lambda^{\alpha_k}p_{k,\sigma}(\theta) + \lambda^{2\alpha_1}q_\sigma(\theta) + \lambda^\beta r_\sigma(\theta)\right)^2$$

$$+\,F''(\theta)\,\mathbb{E}\left(\lambda^{\alpha_1}p_{1,\sigma}(\theta)\left(\sum_{k=2}^{K}\lambda^{\alpha_k}p_{k,\sigma}(\theta) + \lambda^{2\alpha_1}q_\sigma(\theta) + \lambda^\beta r_\sigma(\theta)\right)\right) \,+\, \mathbb{E}(R_1(\lambda,\theta,\sigma))\,.$$

Similar arguments show now, that there exist constants such that for all θ and λ small enough one has

$$|R(\lambda,\theta)| \;\le\; \lambda^\beta C_2 \,+\, \lambda^{\alpha_1+\alpha_2}C_3 \,+\, \lambda^{3\alpha_1}C_1 \;\le\; \lambda^\beta C$$

where we used that $\beta \le \alpha_1 + \alpha_2 \le 3\alpha_1$. That was our claim. Using (4.2) leads to

$$\begin{aligned}\mathbb{E}\,\frac{1}{N}\sum_{n=1}^{N} F(\theta_n) \;&=\; \mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1}\left(F(\theta_n) \,+\, R(\lambda,\theta_n)\right) \,+\\ &\lambda^{2\alpha_1}\,\mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1}\left[F'(\theta_n)\mathbb{E}_\sigma(q_\sigma(\theta_n)) + \frac{F''(\theta_n)}{2}\mathbb{E}_\sigma(p_{1,\sigma}^2(\theta_n))\right]\;.\end{aligned}$$

From there it follows that for λ small enough and for all θ_0 one has

$$\left|\mathbb{E}\,\frac{1}{N}\sum_{n=0}^{N-1}\left(F'(\theta_n)\mathbb{E}_\sigma(q_\sigma(\theta_n)) \,+\, \frac{1}{2}F''(\theta_n)\mathbb{E}_\sigma(p_{1,\sigma}^2(\theta_n))\right)\right| \;\le\; \frac{\tilde{C}\lambda^{-2\alpha_1}}{N} + C_0\lambda^{\beta-2\alpha_1}$$

where $\tilde{C} = 2\max F(\theta)$. This proves the lemma. □

Connecting this lemma with the formula for the invariant measure will give us the mentioned results (Theorem 3.8 and Theorem 3.10). As for some cases we need to consider the functions $p_{k,\sigma}$ and q_σ in more detail and for convenience let us introduce the following notations:

$$P_{k,\sigma} \;=:\; \begin{pmatrix} a_{k,\sigma} & b_{k,\sigma} \\ c_{k,\sigma} & -a_{k,\sigma}\end{pmatrix}\;,\qquad k = 1,\ldots,K+1 \tag{4.3}$$

If for $\xi > 0$ the term $\lambda^\xi P_{\xi,\sigma}$ occurs inside the exponential in the expansion of $MT_{\lambda,\sigma}M^{-1}$ then we define

$$P_{\xi,\sigma} \;=:\; \begin{pmatrix} a_{\xi,\sigma} & b_{\xi,\sigma} \\ c_{\xi,\sigma} & -a_{\xi,\sigma}\end{pmatrix}\;, \tag{4.4}$$

otherwise let $a_{\xi,\sigma} = b_{\xi,\sigma} = c_{\xi,\sigma} = 0$ for all σ. Note that with these notations we have $a_{\alpha_k,\sigma} = a_{k,\sigma}$ and so on. Some variables might have two meanings. To get not confused let us set the following convention. For latin letters or numbers as index, like $a_{k,\sigma}$ or $a_{1,\sigma}$, we mean definition (4.3), for greek letters and formulas of greek letters, like $a_{\xi,\sigma}, a_{\chi-\delta,\sigma}$ we mean definition (4.4).

From equation (3.3) one finds

$$p_{k,\sigma}(\theta) \;=\; c_{k,\sigma}\cos^2(\theta) - b_{k,\sigma}\sin^2(\theta) - 2a_{k,\sigma}\sin(\theta)\cos(\theta)\;\;. \tag{4.5}$$

4.2 Elliptic first degree anomaly

If $T_{\lambda,\sigma}$ has an elliptic anomaly of first degree and L-th kind, then $\mathbb{E}(p_{L,\sigma})$ is never zero. If it would be zero for one θ then by a rotation basis change we can assume that this happens for $\theta = 0$ and using (4.5) leads to $\mathbb{E}(P_{L,\sigma}) = \begin{pmatrix} a_L & b_L \\ 0 & -a_L\end{pmatrix} \;\Rightarrow\; \det\mathbb{E}(P_{L,\sigma}) \le 0$, but this means the anomaly would not be elliptic. The following is equivalent to Theorem 3.8.

Theorem 4.3. *Let $T_{\lambda,\sigma}$ have an elliptic anomaly of first degree and L-th kind, i.e. $\mathbb{E}(p_{k,\sigma}) = 0$ for $k = 1,\ldots,L-1$ and $\forall\,\theta$: $\mathbb{E}(p_{L,\sigma}(\theta)) \neq 0$. We define $\rho = \frac{c}{\mathbb{E}(p_{L,\sigma})}$ such that $\int_0^\pi \rho(\theta)\,\mathrm{d}\theta = 1$. Let f be any π periodic, continuously differentiable function. Then one has*

$$\mathbb{E}\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \int_0^\pi f(\theta)\rho(\theta)\mathrm{d}\theta + \mathcal{O}\left(\frac{\lambda^{-\alpha_L}}{N}, \lambda^{\alpha_{L+1}-\alpha_L}\right) .$$

Proof. W.l.o.g. let $\int_0^\pi f(\theta)\rho(\theta)\mathrm{d}\theta = 0$, otherwise consider $f - \int_0^\pi f(\theta)\,\rho(\theta)\,\mathrm{d}\theta$ instead of f. Define $F(\theta) := \int_0^\theta \frac{f(\tilde\theta)}{\mathbb{E}(p_{L,\sigma}(\tilde\theta))}\,\mathrm{d}\tilde\theta$, then F is twice continuously differentiable and π-periodic as f and $\mathbb{E}(p_{L,\sigma})$ are π periodic and $F(0) = F(\pi) = 0$.
Thus one has $f = \mathbb{E}(p_{L,\sigma})F'$ and Lemma 4.1 proves the theorem. □

4.3 Parabolic first degree anomaly

Under certain conditions a parabolic anomaly of first degree can be transformed to an anomaly of second degree as in Theorem 3.10 by a λ dependent basis change. Therefore we define

$$M_{\lambda,\delta} := \begin{pmatrix} \lambda^\delta & 0 \\ 0 & 1 \end{pmatrix} , \quad \text{then one has} \quad M_{\lambda,\delta}P_{k,\sigma}M_{\lambda,\delta}^{-1} \;=\; \begin{pmatrix} a_{k,\sigma} & \lambda^\delta b_{k,\sigma} \\ \lambda^{-\delta}c_{k,\sigma} & -a_{k,\sigma} \end{pmatrix} .$$

Now if we have a parabolic anomaly of first degree and L-th kind we get after a rotation basis change $\mathbb{E}(P_{1,\sigma})e_0 = 0$ which means $\mathbb{E}(c_{L,\sigma}) = \mathbb{E}(a_{L,\sigma}) = 0, \mathbb{E}(b_{L,\sigma}) = b_L \neq 0$.
Let χ be the smallest exponent such that $\mathbb{E}(c^2_{\chi,\sigma}) > 0$. If $\mathbb{E}(c_{\xi,\sigma}) = 0$ for all $\xi < \frac{4}{3}\chi + \frac{1}{3}\alpha_L$, $\frac{1}{3}(\chi+\alpha_L) \leq \alpha_1$ and $\mathbb{E}(a_{\xi,\sigma}) = 0$ for all $\xi < \frac{2}{3}(\chi+\alpha_L)$ (this condition together with that for $\mathbb{E}(c_{\xi,\sigma})$ above leads to $\mathbb{E}(P_{\xi,\sigma})e_0 = 0$ as it is stated in Theorem 3.14; all these conditions hold e.g. if $\chi = \alpha_1$ and $L = K = 1$), then choose $\delta := \frac{1}{3}(2\chi - \alpha_L) > 0$ to get

$$2(\chi - \delta) \;=\; \frac{2}{3}(\chi + \alpha_L) \;=\; \alpha_L + \delta \;.$$

For this case one finds that $M_{\lambda,\delta}\,MT_{\lambda,\sigma}M^{-1}\,M_{\lambda,\delta}^{-1}$ has an anomaly of second degree of the form

$$\pm\exp\left[\lambda^{\frac{1}{3}(\chi+\alpha_L)}\begin{pmatrix} a_{\chi-\delta,\sigma} & 0 \\ c_{\chi,\sigma} & -a_{\chi-\delta,\sigma} \end{pmatrix} + \ldots + \lambda^{\frac{2}{3}(\chi+\alpha_L)}\begin{pmatrix} a_{\alpha_L+\delta,\sigma} & b_{L,\sigma} \\ c_{\alpha_L+2\delta,\sigma} & -a_{\alpha_L+\delta,\sigma} \end{pmatrix} + \ldots\right] .$$

Note that $\alpha_L + 2\delta = \frac{4}{3}\chi + \frac{1}{3}\alpha_L$, thus the condition on $\mathbb{E}(c_{\xi,\sigma})$ assures that this is a second degree anomaly. Furthermore note that $a_{\chi-\delta,\sigma} = 0$ if $\chi - \delta = \frac{1}{3}(\chi + \alpha_L) < \alpha_1$ and $a_{\chi-\delta,\sigma} = a_{\alpha_1,\sigma} = a_{1,\sigma}$ for $\frac{1}{3}(\chi + \alpha_L) = \alpha_1$. As the upper right entry of the matrix inside the exponential is 0 to lowest order - *this corresponds to* $\mathbb{E}(p^2_{1,\sigma}(\frac{\pi}{2})) = 0$ - it is important to note that $\mathbb{E}(b_{L,\sigma}) \neq 0$ - *this corresponds to* $\mathbb{E}(q_\sigma(\frac{\pi}{2})) \neq 0$ *after the transformation.* If $a_{\chi-\delta,\sigma} = 0$ for (almost) all σ or if $a_{\chi-\delta,\sigma}$ and $c_{\chi,\sigma}$ are not linear dependent then $\hat\theta = \frac{\pi}{2}$

is the only angle where $\mathbb{E}(p_{1,\sigma}^2(\hat{\theta})) = 0$ after the transformation which means we can apply Theorem 3.10 to prove Theorem 3.14. The only case where we get a problem is if $\chi - \delta = \alpha_1$ and $a_{\alpha_1,\sigma}$ is not zero for almost all σ and linear dependent to $c_{\chi,\sigma}$.

If $\mathbb{E}(c_{\xi,\sigma}) \neq 0$ for some $\xi < \frac{4}{3}\chi + \frac{1}{3}\alpha_L$ then let $\tilde{\chi}$ be the smallest exponent such that $\mathbb{E}(c_{\tilde{\chi},\sigma}) \neq 0$. Furthermore one finds $\tilde{\chi} > \alpha_L$. Now if $\frac{1}{2}(\alpha_L + \tilde{\chi}) < 2\alpha_1$ and $\mathbb{E}(a_{\xi,\sigma}) = 0$ for all $\xi < \frac{1}{2}(\alpha_L + \tilde{\chi})$ then choose $\delta = \frac{1}{2}(\tilde{\chi} - \alpha_L)$ and the basis change by $M_{\lambda,\delta}$ leads to

$$\pm \exp\left[\lambda^{\zeta}\begin{pmatrix} a_{\zeta,\sigma} & 0 \\ c_{\zeta+\delta,\sigma} & -a_{\zeta,\sigma}\end{pmatrix} + \ldots + \lambda^{\frac{1}{2}(\alpha_L+\tilde{\chi})}\begin{pmatrix} a_{\alpha_L+\delta,\sigma} & b_{L,\sigma} \\ c_{\tilde{\chi},\sigma} & a_{\alpha_L+\delta,\sigma}\end{pmatrix} + \ldots\right]$$

where the lowest occurring exponent ζ inside the exponential is either α_1 or $\chi - \delta = \chi - \frac{1}{2}\tilde{\chi} + \frac{1}{2}\alpha_L$. According to the definitions above we have $\tilde{\chi} < \frac{4}{3}\chi + \frac{1}{3}\alpha_L$ which is equivalent to $2(\chi - \delta) > \frac{1}{2}(\alpha_L + \tilde{\chi})$. Together with the condition $2\alpha_1 > \frac{1}{2}(\alpha_L + \tilde{\chi})$ this implies that we have an anomaly of first degree that can be elliptic, hyperbolic or parabolic. If the anomaly is parabolic we can proceed with similar basis changes as above. For a lot of cases we will be able to transform it to some anomaly where we can calculate the leading coefficients for the oscillating sums.

The first condition on χ ($\mathbb{E}(c_{\xi,\sigma}^2) = 0, \forall\, \xi < \chi$) means that all the functions in the expansion for $S_{\lambda,\sigma}$ with exponent less than χ have the value zero for $\theta = 0$ for almost all σ. Comparing the parabolic case with the elliptic one shows, that for small λ the invariant measure somehow is very close to a Dirac measure at the parabolic fix point (fix point of expansion of $S_{\lambda,\sigma}$ up to order α_L, that is $\theta = 0$ after our chosen basis change). But this fix point is not stable, once the sequence θ_n passed it, it goes around $\mathbb{RP}(1)$ almost deterministically (and not back to the fix point), therefore the Dirac measure might be the wrong answer for the lowest order. Nevertheless most of the time the dynamical system θ_n will be around this fix point. By the λ dependent basis change we 'blow up' the neighbourhood for small λ with a rate, such that we can specify the nature of this anomaly in more detail. Unfortunately there remain some cases that we are not able to handle with these methods so far.

If we do not find such a χ, i.e. if the left down entry of all the matrices in the expansion for $MT_{\lambda,\sigma}M^{-1}$ is identically zero for almost all σ, then $\theta = 0$ is a fix point of $S_{\lambda,\sigma}$ to all orders. If we further assume that the expansion converges against $T_{\lambda,\sigma}$ for (almost) all σ and λ small enough then e_0 is almost surely an eigenvector of $T_{\lambda,\sigma}$ for small λ. Thus the Dirac measure at this angle really is an invariant measure.

Finally let us note a remarkable fact. Such a transformation even works sometimes for parabolic anomalies of 'zeroth degree', i.e. if

$$MT_{0,\sigma}M^{-1} = \pm\begin{pmatrix} 1 & b_{0,\sigma} \\ 0 & 1\end{pmatrix} = \pm\exp\left[\begin{pmatrix} 0 & b_{\sigma} \\ 0 & 0\end{pmatrix}\right], \quad \mathbb{E}(b_{0,\sigma}) \neq 0\,.$$

For this case define χ as above. If $\mathbb{E}(c_{\xi,\sigma}) = 0$ for all $\xi < \frac{4}{3}\chi$, $\mathbb{E}(a_{\xi,\sigma}) = 0$ for all $\xi < \frac{2}{3}\chi$

and $\frac{1}{3}\chi \le \alpha_1$ then let $\delta = \frac{2}{3}\chi$ and one finds that

$$M_{\lambda,\delta}\, MT_{\lambda,\sigma}M^{-1}\, M_{\lambda,\delta}^{-1} \;=\; \pm\exp\left[\lambda^{\frac{1}{3}\chi}\begin{pmatrix} a_{\frac{1}{3}\chi,\sigma} & 0 \\ c_{\chi,\sigma} & -a_{\frac{1}{3}\chi,\sigma}\end{pmatrix} + \ldots + \lambda^{\frac{2}{3}\chi}\begin{pmatrix} a_{\delta,\sigma} & b_\sigma \\ c_{2\delta,\sigma} & -a_{\delta,\sigma}\end{pmatrix} + \ldots\right]$$

has an anomaly of second degree. Similar as above if $\mathbb{E}(c_{\xi,\sigma}) \neq 0$ for some $\xi < \frac{4}{3}\chi$ let $\tilde{\chi}$ be the smallest exponent such that $\mathbb{E}(c_{\tilde{\chi},\sigma}) \neq 0$. If $\mathbb{E}(a_{\xi,\sigma}) = 0$ for all $\xi < \frac{1}{2}\tilde{\chi}$ then choose $\delta = \frac{1}{2}\tilde{\chi}$ to get the matrices

$$\pm\exp\left[\lambda^{\zeta}\begin{pmatrix} a_{\zeta,\sigma} & 0 \\ c_{\zeta+\delta,\sigma} & -a_{\zeta,\sigma}\end{pmatrix} + \,\ldots\, + \lambda^{\frac{1}{2}\tilde{\chi}}\begin{pmatrix} a_{\delta,\sigma} & b_{0,\sigma} \\ c_{\tilde{\chi},\sigma} & -a_{\delta,\sigma}\end{pmatrix} + \,\ldots\right]$$

where the lowest exponent ζ is either α_1 or $\chi - \delta$. For both cases we find $2\zeta > \tilde{\chi}$ and we get an anomaly of first degree.
All this was actually done to handle the bandedges of the Anderson model. There we obtained after a λ-independent basis change the matrices (see page 10)

$$\begin{pmatrix} 1+\lambda V_\sigma - \lambda^\alpha x & 1 \\ \lambda V_\sigma - \lambda^\alpha x & 1\end{pmatrix} \;=\; \exp\left[\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + \lambda\begin{pmatrix} \frac{V_\sigma}{2} & 0 \\ V_\sigma & -\frac{V_\sigma}{2}\end{pmatrix} + \lambda^\alpha\begin{pmatrix} -\frac{x}{2} & 0 \\ -x & \frac{x}{2}\end{pmatrix} + \ldots\right].$$

Therefore we have $\chi = \min(1,\alpha)$ and $\tilde{\chi} = \alpha$ (for $\alpha < 2$ and $x \neq 0$). The condition $\tilde{\chi} > \frac{4}{3}\chi$ (which corresponds to $\mathbb{E}(c_{\xi,\sigma}) = 0$ for $\xi < \frac{4}{3}\chi$) is equivalent to $\alpha \geq \frac{4}{3}$. Thus we chose $\delta = \frac{1}{2}\tilde{\chi} = \frac{1}{2}\alpha$ if $\alpha < \frac{4}{3}$ and $\delta = \frac{2}{3}\chi = \frac{2}{3}$ if $\alpha \geq \frac{4}{3}$ for the transformation. In the first case we got a first degree anomaly and in the second case we got a second degree anomaly.

4.4 Hyperbolic first degree anomaly

Let us assume that $T_{\lambda,\sigma}$ has an hyperbolic anomaly of first degree and L-th kind. Let

$$T_{\lambda,\sigma} = \pm\exp\left[\sum_{k=1}^{K+1}\lambda^{\alpha_k}\hat{P}_{k,\sigma} + \mathcal{O}(\lambda^\beta)\right], \quad \mathbb{E}(\hat{P}_{L,\sigma}) \;=\; \begin{pmatrix} \hat{a}_L & \hat{b}_L \\ \hat{c}_L & -\hat{a}_L\end{pmatrix}.$$

As the anomaly is hyperbolic we find $\eta > 0$ such that $\eta^2 = \hat{a}_L^2 + \hat{b}_L\hat{c}_L > 0$. Furthermore define

$$M \;:=\; \begin{pmatrix} -\hat{c}_L & \hat{a}_L+\eta \\ -\hat{c}_L & \hat{a}_L-\eta\end{pmatrix} \quad \text{if} \quad \hat{c}_L \neq 0\,, \qquad M \;:=\; \begin{pmatrix} 2\hat{a}_L & \hat{b}_L \\ 0 & 1\end{pmatrix} \quad \text{if} \quad \hat{c}_L = 0\,.$$

Then one deduces

$$\mathbb{E}(P_{L,\sigma}) \;=\; \mathbb{E}\left[M\,\hat{P}_{L,\sigma}\,M^{-1}\right] \;=\; \begin{pmatrix} -\eta & 0 \\ 0 & \eta\end{pmatrix}.$$

Thus we have $\mathbb{E}(c_{L,\sigma}) = \mathbb{E}(b_{L,\sigma}) = 0$ and $\mathbb{E}(a_{L,\sigma}) = -\eta < 0$. $S_{\lambda,\sigma}$ expanded up to order $\lambda_1^{\alpha_L}$ has two fix points on $\mathbb{R}/\pi\mathbb{Z}$, one is unstable ($\theta = 0$) and one is stable ($\theta = \frac{\pi}{2}$). For

the case $L = K = 1$ Schulz-Baldes [Sch1] showed that we will have to lowest order a Dirac distribution at the stable fix point. We do now a similar trick as in the parabolic case. We blow up a neighbourhood around the unstable fix point, which means we compress the neighbourhood around the stable fix point such that we get a second degree anomaly as in Theorem 3.12 and we will have a Dirac distribution to lowest order. One could say that we analyse the unstable fix point in more detail after the basis change, to see that it really has no influence. Unfortunately we need a sufficient rate of random movement for the dynamical system θ_n at the unstable fix point (the system should not stay there very long). Thus there will remain some cases which cannot be treated with these methods.
As in the parabolic case let χ be the smallest exponent such that $\mathbb{E}(c^2_{\chi,\sigma}) > 0$ (such a χ is supposed to exist). If we have $\mathbb{E}(c_{\xi,\sigma}) = 0$ for all $\xi < \frac{1}{2}\alpha_L + \chi$ then let $\delta := \chi - \frac{1}{2}\alpha_L > 0$ to get $2(\chi - \delta) = \alpha_L$. Thus we find that

$$M_{\lambda,\delta}\, MT_{\lambda,\sigma}M^{-1}\, M_{\lambda,\delta}^{-1} \;=\; \pm\exp\left[\lambda^{\frac{1}{2}\alpha_L}\begin{pmatrix} 0 & 0 \\ c_{\chi,\sigma} & 0 \end{pmatrix} + \ldots + \lambda^{\alpha_L}\begin{pmatrix} a_{L,\sigma} & b_{\alpha_L-\delta,\sigma} \\ c_{\alpha_L+\delta,\sigma} & -a_{L,\sigma} \end{pmatrix} + \ldots\right]$$

has an anomaly of second degree. Note that $\mathbb{E}(b_{\alpha_L-\delta,\sigma}) = 0$ and $\mathbb{E}(a_{L,\sigma}) < 0$, thus this situation corresponds to Theorem 3.12 which now shows Theorem 3.15.

If there is a $\xi < \frac{1}{2}\alpha_L + \chi$ such that $\mathbb{E}(c_{\xi,\sigma}) \neq 0$ then let again $\tilde{\chi}$ be the smallest exponent such that $\mathbb{E}(c_{\tilde{\chi},\sigma}) \neq 0$ and choose $\delta := \tilde{\chi} - \alpha_L > 0$ to get

$$M_{\lambda,\delta}\, MTM^{-1}\, M_{\lambda,\delta}^{-1} \;=\; \pm\exp\left[\lambda^{\zeta}\begin{pmatrix} a_{\zeta,\sigma} & 0 \\ c_{\zeta+\delta,\sigma} & -a_{\zeta,\sigma} \end{pmatrix} + \ldots + \lambda^{\alpha_L}\begin{pmatrix} a_{L,\sigma} & b_{\alpha_L-\delta,\sigma} \\ c_{\tilde{\chi},\sigma} & -a_{L,\sigma} \end{pmatrix}\right]$$

where ζ is either α_1 or $\chi - \delta$. For both cases we have $2\zeta > \alpha_L$. Thus we have again a hyperbolic first degree anomaly (as $\mathbb{E}(b_{\alpha_L-\delta,\sigma}) = 0$) and one can proceed with similar basis changes as above.

4.5 Some results from analysis

Before we start with the main theorems for anomalies of second degree we will prove two lemmas we will need in some situations to prove the demanded smoothness of certain functions. But first let us note some version of L'Hospital's rule which unfortunately is not always stated like this in analysis books.

Lemma 4.4. *Let $d, l \in \mathbb{R}, l < d$ and let f, g be continuous differentiable functions mapping from the interval (l, d) to $\mathbb{R}$. If*

$$\lim_{x\uparrow d} g(x) \;=\; \pm\infty, \quad \lim_{x\uparrow d}\frac{f'(x)}{g'(x)} \;=\; C \in \mathbb{R}$$

then one has

$$\lim_{x\uparrow d}\frac{f(x)}{g(x)} \;=\; C\;.$$

Proof. In most analysis books one finds this rule with the additional condition, that also $f(x)$ converges to $\pm\infty$. We will see, that this is not necessary. W.l.o.g. let $\lim_{x\uparrow d} g(x) = \infty$. The case for $-\infty$ goes analog. Consider $h(x) = f(x) + (1-C)g(x)$. Clearly h is also continuous differentiable and we have

$$\lim_{x\uparrow d} \frac{h'(x)}{g'(x)} \;=\; \lim_{x\uparrow d} \frac{f'(x)}{g'(x)} + (1-C) \;=\; 1$$

Thus there exists some constant m with $l < m < d$ such that $h'(x) > \frac{1}{2}g'(x)$ for all $x \in [m, d)$. By the monotony of the integral we get $h(x) > \frac{1}{2}[g(x) - g(m)] + h(m)$ for all $x \in [m, d)$. Therefore one deduces that $\lim_{x\uparrow d} h(x) = \infty$. Using now the version of L'Hospital stated in most books leads to

$$\lim_{x\uparrow d} \frac{f(x)}{g(x)} \;=\; \lim_{x\uparrow d} \frac{h(x) + (C-1)g(x)}{g(x)} \;=\; 1 \,+\, (C \,-\, 1) \;=\; C\,.$$

□

Lemma 4.5. *Let $p, q, f \in C^\infty(\mathbb{R}), d \in \mathbb{R}$, such that*

$$p(d) \;=\; 0, \qquad \forall\, n \in \mathbb{N}_0 : \quad q(d) + np'(d) \neq 0 \tag{4.6}$$

and such that there exists a neighbourhood U of d, where $p(x) \neq 0\; \forall\, x \in U, x \neq d$. Let Q be an antiderivative of $\frac{q}{p}$ for $x \in U, x \neq d$, such that

$$\lim_{x\downarrow d} \frac{\exp(Q(x))}{p(x)} \;=\; 0, \qquad \forall\, n \in \mathbb{N} : \quad \lim_{x\uparrow d} |p|^n(x) \exp(Q(x)) \;=\; \infty\,. \tag{4.7}$$

Then there exist (in U) smooth (C^∞) solutions for the differential equation

$$pF' + qF = f\,. \tag{4.8}$$

For $x > d$ the smooth solution is unique, i.e. two different smooth solutions coincide for $x > d$, but for $x < d$ any solution can be taken.
If you only need F to be m times continuously differentiable it is enough to postulate this for p, q and f (instead of C^∞) and equations (4.6) and (4.7) are only needed for $n = 0, 1, \ldots, m$.

Proof. The homogeneous solution is $\exp(-Q)$ as $p(x)\,\partial_x \exp(-Q(x)) + q(x)\exp(-Q(x)) = 0$ for $x \in U$, $x \neq d$. Assume that certain solutions, F_1 for $x < d$ and F_2 for $x > d$ of (4.8) can be smooth continued at $x = d$. By the definition of Q one gets $\partial_x^n \exp(-Q) = \frac{h}{p^n}\exp(-Q)$ where h is some polynomial in p and q and ∂_x^n denotes the n-th derivative w.r.t. x. Thus equation (4.7) leads to $\lim_{x\uparrow d} \partial_x^n \exp(-Q(x)) = 0$ for all n. Therefore the solutions $F_1 + C\exp(-Q)$ for $x < d$ and F_2 for $x > d$ can also be smooth continued.
According to (4.7) we have $\lim_{x\downarrow d} \exp(-Q) = \infty$, thus there is only one solution for $x > d$ which might be able to be smooth continued at $x = d$, and that is

$$F(x) \;:=\; \exp(-Q(x)) \int_d^x \exp Q(s) \frac{f(s)}{p(s)} \mathrm{d}s \;=:\; \exp(-Q(x))w(x), \quad x > d, x \in U. \tag{4.9}$$

The integral exists by the first part of (4.7). For $x \in U$ and $x < d$ let w be any antiderivative of $\exp(Q)f/p$ and let $F(x) := \exp(-Q(x))w(x)$. Now F is a solution of (4.8) for $x \neq d, x \in U$. We want to show that it can be smooth continued for $x = d$ by $F(d) := \frac{f(d)}{q(d)}$. F is continuous:

$$\lim_{x\to d} F(x) = \lim_{x\to d} \frac{w(x)}{\exp(Q(x))} = \lim_{x\to d} \frac{\exp(Q(x))f(x)/p(x)}{\exp(Q(x))q(x)/p(x)} = \frac{f(d)}{q(d)} = F(d)$$

L'Hospitals rule can be applied for $x \downarrow d$, because the numerator and denominator converge both to 0 (see (4.7) and (4.9)) and for $x \uparrow d$ by Lemma 4.4 as the denominator converges to ∞.
Let $q_n := np' + q,\ f_0 := f$ and $f_n := f'_{n-1} - q'_{n-1}F^{(n-1)}$, especially $f_1 = f' - q'F$.
Then one shows by induction

$$pF^{(n+1)} + q_nF^{(n)} = f_n \tag{4.10}$$

Now we prove by induction that F is n times (continuously) differentiable in d for all $n \in \mathbb{N}$. First set $n = 1$.

$$\begin{aligned}\lim_{x\to d} F'(x) &= \lim_{x\to d} \frac{-q(x)w(x) + f(x)\exp(Q(x))}{\exp(Q(x))p(x)}\\ &= \lim_{x\to d} \frac{-q'(x)w(x) + f'(x)\exp(Q(x))}{\exp(Q(x))(p'(x) + q(x))} = \frac{-q'(d)F(d) + f'(d)}{p'(d) + q(d)}\end{aligned}$$

Again you can use L'Hospital for $x \downarrow d$ and $x \uparrow d$ for the same reasons as above. Thus F is continuously differentiable, even at $x = d$.

Now let $F^{(n)}$ be continuous. Furthermore let $Q_n := Q + n\ln|p|$ for $x \neq d$. According to (4.7) we have

$$\lim_{x\downarrow d} \exp(Q_n(x)) = \lim_{x\downarrow d} |p(x)|^n \exp(Q(x)) = 0, \quad \lim_{x\uparrow d} |p(x)| \exp(Q_n(x)) = \infty \tag{4.11}$$

If F is n-times continuously differentiable then f_1 is n times continuously differentiable, $f_2 = f_1' - b_1'F'$ is $n-1$ times continuously differentiable and so on. It follows by induction that f_n is continuously differentiable.
Let $w_n(x) := F^{(n)} \exp(Q_n(x))$. According to (4.11) we have $\lim_{x\downarrow d} w_n(x) = 0$ as $F^{(n)}$ is continuous. From (4.10) we get for $x \neq d$

$$F^{(n+1)} = \frac{f_n - q_nF^{(n)}}{p} = \frac{f_n \exp(Q_n) - q_nw_n}{\exp(Q_n)p}$$

Furthermore note that $Q_n' = Q' + n\frac{p'}{p} = \frac{q_n}{p}$ which leads to

$$\begin{aligned}q_n\, w_n' &= q_n\, F^{(n+1)} \exp(Q_n) + q_n\, F^{(n)}\, Q_n' \exp(Q_n)\\ &= Q_n'\, pF^{(n+1)} \exp(Q_n) + q_nF^{(n)}Q_n' \exp(Q_n) = f_n\, Q_n' \exp(Q_n)\end{aligned}$$

for $x \neq d$. Thus we get

$$\begin{aligned}\lim_{x\to d} F^{(n+1)}(x) &= \lim_{x\to d} \frac{-q_n(x)w_n(x) + f_n(x)\exp(Q_n(x))}{\exp(Q_n(x))p(x)} \\ &= \lim_{x\to d} \frac{-q_n'(x)w_n(x) + f_n'(x)\exp(Q_n(x))}{\exp(Q_n(x))(p'(x)+q_n(x))} = \frac{-q_n'(d)F^{(n)}(d) + f_n'(d)}{(n+1)p'(d)+q(d)}\end{aligned}$$

where L'Hospital can be used for $x \downarrow d$, because the numerator and denominator converge both to 0 according to (4.11) and for $x \uparrow d$ by Lemma 4.4 as the denominator converges to $\pm\infty$. Therefore $F^{(n)}$ is continuously differentiable in d.

Note that you can exchange '↑' and '↓' in equation (4.7) and you also get smooth solutions for (4.8). Then such a smooth solution is unique for $x < d$ and the proof goes analog. □

There is a similar situation as in the lemma above, where we get smooth solutions for a second order differential equation, although the coefficient before F'' is zero somewhere.

Lemma 4.6. *Let $p, q, f \in C^\infty(\mathbb{R}), d \in \mathbb{R}$, such that*

$$p(d) = 0, \qquad \forall\, n \in \mathbb{N}_0 : \quad q(d) + np'(d) \neq 0 \tag{4.12}$$

and such that there exists a neighbourhood U of d, where $p(x) \neq 0\ \forall\, x \in U, x \neq d$. Let Q be an antiderivative of $\frac{q}{p}$ for $x \in U, x \neq d$ such that

$$\forall\, n \in \mathbb{N} : \quad \lim_{x\to d} |p|^n(x)\exp(Q(x)) = \infty\ . \tag{4.13}$$

Then any solution F in U for $x \neq d$ for the differential equation

$$pF' + qF = f\ . \tag{4.14}$$

can be smooth (C^∞) continued at d.
If you only need F to be m times continuously differentiable it is enough to postulate this for p, q and f (instead of C^∞) and equations (4.12) and (4.13) are only needed for $n = 0, 1, \ldots, m$.

Proof. The proof is very much the same as for Lemma 4.5, therefore we will not explain everything in detail. Let F be any solution of (4.14) for $x \in U, x \neq d$ and define $w(x) := F(x)\exp(Q(x))$. Furthermore let $F(d) := \frac{f(d)}{q(d)}$. Note from (4.14) that $w' = \exp(Q)f/p$. F is continuous in U as

$$\lim_{x\to d} F(x) = \lim_{x\to d} \frac{w(x)}{\exp(Q(x))} = \lim_{x\to d} \frac{\exp(Q(x))f(x)/p(x)}{\exp(Q(x))q(x)/p(x)} = \frac{f(d)}{q(d)} = F(d)$$

where L'Hospital's rule can be applied by Lemma 4.4 as the denominator converges to ∞ by (4.13). Let $q_n := np' + q$, $f_0 := f$ and $f_n := f_{n-1}' - q_{n-1}'F^{(n-1)}$, then one gets by induction

$$pF^{(n+1)} + q_nF^{(n)} = f_n\ . \tag{4.15}$$

Now we prove by induction that F is n times continuously differentiable in d for all $n \in \mathbb{N}$. First for $n = 1$ we get

$$\begin{aligned}\lim_{x\to d} F'(x) &= \lim_{x\to d} \frac{-q(x)w(x) + f(x)\exp(Q(x))}{\exp(Q(x))p(x)} \\ &= \lim_{x\to d} \frac{-q'(x)w(x) + f'(x)\exp(Q(x))}{\exp(Q(x))(p'(x)+q(x))} = \frac{-q'(d)F(d) + f'(d)}{p'(d)+q(d)}\end{aligned}$$

where L'Hospital can be used again due to Lemma 4.4.
Now let $F^{(n)}$ be continuous. Let $Q_n := Q + n\ln|p|$ for $x \neq d$. From (4.13) we get

$$\lim_{x\to d} |p(x)|\exp(Q_n(x)) = \infty . \tag{4.16}$$

As $f_1 = f' - q'F$ is n times continuously differentiable we get by induction that f_n is continuously differentiable. If we define $w_n := F^{(n)}\exp(Q_n)$ we can deduce

$$F^{(n+1)} = \frac{-q_n w_n + f_n \exp(Q_n)}{\exp(Q_n)p},\ Q_n' = \frac{q_n}{p} \quad \text{and} \quad q_n\, w_n' = f_n\, Q_n' \exp(Q_n)$$

for $x \neq d$. Using this we get

$$\begin{aligned}\lim_{x\to d} F^{(n+1)}(x) &= \lim_{x\to d} \frac{-q_n(x)w_n(x) + f_n(x)\exp(Q_n(x))}{\exp(Q_n(x))p(x)} \\ &= \lim_{x\to d} \frac{-q_n'(x)w_n(x) + f_n'(x)\exp(Q_n(x))}{\exp(Q_n(x))(p'(x)+q_n(x))} = \frac{-q_n'(d)F^{(n)}(d) + f_n'(d)}{(n+1)p'(d)+q(d)}\end{aligned}$$

where L'Hospital can be used due to Lemme 4.4. Therefore $F^{(n)}$ is continuously differentiable at d. □

4.6 Second degree anomaly

With the following Lemma we combine the pertubative formula for the invariant measure with the result in Lemma 4.2.

Lemma 4.7. *Let $T_{\lambda,\sigma}$ have an anomaly of second degree (and K-th kind). Let f be a twice continuously differentiable, π-periodic function. We assume that there is a constant c and a differentiable π-periodic solution ρ for*

$$\frac{1}{2}\partial_\theta(\mathbb{E}(p_{1,\sigma}^2)\rho) - \mathbb{E}(q_\sigma)\rho = c \tag{4.17}$$

such that $\int_0^\pi \rho(\theta)\mathrm{d}\theta = 1$. Furthermore we assume that $c \neq 0$ or that $\left(\mathbb{E}(p_{1,\sigma}^2)\rho\right)^{-1}$ is twice continuously differentiable. If there is a twice continuously differentiable, π-periodic function G such that

$$f - \int_0^\pi f(\theta)\rho(\theta)\mathrm{d}\theta = \frac{1}{2}\mathbb{E}(p_{1,\sigma}^2)G' + \mathbb{E}(q_\sigma)G \tag{4.18}$$

then one has

$$\mathbb{E}\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \int_0^\pi f(\theta)\rho(\theta)\mathrm{d}\theta \;+\; \mathcal{O}\left(\frac{\lambda^{-2\alpha_1}}{N}, \lambda^{\beta-2\alpha_1}\right)$$

Proof. If $c \neq 0$ let $F(\theta) = \int_0^\theta G(\tilde{\theta})\mathrm{d}\tilde{\theta}$. Then F is 3 times continuously differentiable. The equations (4.17) and (4.18) lead to

$$F(\pi) \;=\; \frac{1}{c}\int_0^\pi c\,G(\theta)\mathrm{d}\theta \;=\; \frac{1}{c}\int_0^\pi c\,G(\theta) - \partial_\theta \frac{1}{2}[\mathbb{E}(p_{1,\sigma}^2(\theta))\rho(\theta)\,G(\theta)]\,\mathrm{d}\theta \;=$$

$$=\; \frac{1}{c}\int_0^\pi \left[-\frac{1}{2}\mathbb{E}(p_{1,\sigma}^2(\theta))G'(\theta) - \mathbb{E}(q_\sigma(\theta))G(\theta)\right]\rho(\theta)\,\mathrm{d}\theta \;=\; 0\;.$$

If $c = 0$ we deduce from (4.17) that

$$\frac{1}{2}\mathbb{E}(p_{1,\sigma}^2)\partial_\theta\left(\frac{1}{\mathbb{E}(p_{1,\sigma}^2)\rho}\right) \;+\; \mathbb{E}(q_\sigma)\frac{1}{\mathbb{E}(p_{1,\sigma}^2)\rho} \;=\; 0\;.$$

Thus $\hat{G} := G - \frac{\hat{C}}{\mathbb{E}(p_{1,\sigma}^2)\rho}$ still solves (4.18) for any constant $\hat{C}$. We choose $\hat{C}$ in such a way, that $\int_0^\pi \hat{G}(\theta)\,\mathrm{d}\theta = 0$ and define $F(\theta) := \int_0^\theta \hat{G}(\tilde{\theta})\,\mathrm{d}\tilde{\theta}$.
Therefore in both cases we get a π-periodic, 3 times continuously differentiable function F satisfying

$$f \;-\; \int_0^\pi f(\theta)\rho(\theta)\mathrm{d}\theta \;=\; \frac{1}{2}\mathbb{E}(p_{1,\sigma}^2)F'' \;+\; \mathbb{E}(q_\sigma)F'$$

and Lemma 4.2 completes the proof. □

The easiest case that can occur is $\mathbb{E}(p_{1,\sigma}^2) > 0$ for all θ which was already treated by Schulz-Baldes [Sch1]. Indeed for this case with f being continuously differentiable one can easily solve the differential equations (4.17) and (4.18) with π-periodic functions and sufficient regularity. For the first one this was already done with equation (3.12). To solve (4.18) set $g := f - \int_0^\pi f(\theta)\rho(\theta)\,\mathrm{d}\theta$ and

$$\kappa(\theta) \;:=\; \int_0^\theta \frac{2\mathbb{E}(q_\sigma(\tilde{\theta}))}{\mathbb{E}(p_{1,\sigma}^2(\tilde{\theta}))}\,\mathrm{d}\tilde{\theta}, \quad H_g(\theta) \;:=\; \int_0^\theta \frac{2g(\tilde{\theta})}{\mathbb{E}(p_{1,\sigma}^2(\tilde{\theta}))}\,e^{\kappa(\tilde{\theta})}\,\mathrm{d}\tilde{\theta}, \quad C_g \;=\; \frac{e^{-\kappa(\pi)}H_g(\pi)}{1-e^{-\kappa(\pi)}}$$

if $\kappa(\pi) \neq 0$, otherwise define $C_g = 0$. Notice that $\kappa(\pi) = 0$ if and only if $c = 0$ in (4.17) which corresponds to $C = 0$ in (3.12) and one has $H_g(\pi) = \frac{1}{\hat{c}}\int_0^\pi g(\theta)\rho(\theta)\,\mathrm{d}\theta = 0$. Thus we always find a twice continuously differentiable, π-periodic function G by

$$G(\theta) \;=\; e^{-\kappa(\theta)}[H_g(\theta) + C_g] \tag{4.19}$$

and Lemma 4.7 leads to the following Corollary. Note that if $c = 0$ the solution ρ from equation (3.12) is always positive, thus $(\mathbb{E}(p_{1,\sigma}^2)\rho)^{-1}$ is a smooth function.

Corollary 4.8. *Let $T_{\lambda,\sigma}$ have an anomaly of second degree, let $\mathbb{E}(p_{1,\sigma}^2) > 0$ for all θ and let ρ be defined as in (3.12). Then for any π-periodic, continuously differentiable function f one has*

$$\mathbb{E}\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \int_0^\pi f(\theta)\,\rho(\theta)\,\mathrm{d}\theta \,+\, \mathcal{O}((\lambda^{2\alpha_1}N)^{-1}, \lambda^{\beta-2\alpha_1})$$

However there are situations, for example after the transformation of a parabolic first degree anomaly to a second degree anomaly, where $\mathbb{E}(p_{1,\sigma}^2(\hat{\theta})) = 0$ for some $\hat{\theta}$. By equation (4.5) this leads to a linear dependence of the random variables $a_{1,\sigma}, b_{1,\sigma}$ and $c_{1,\sigma}$. Thus if this holds for three different angles on $[0,\pi)$ we can conclude $P_{1,\sigma} = 0$ for almost all σ, which is not the case. Therefore this happens for at most two different angles.
We will now consider the situation, where we have $\mathbb{E}(p_{1,\sigma}^2(\hat{\theta})) = 0$ for one angle $\hat{\theta}$ on $\mathbb{R}\mathrm{P}(1)$ and where $\mathbb{E}(p_{K+1,\sigma}(\hat{\theta})) \neq 0$. W.l.o.g. we can assume that $\hat{\theta} = \pi/2$ (otherwise we get this after a rotation basis change). To solve (4.17) and (4.18) we define $g(\theta)$ and $\kappa(\theta)$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ as above. Furthermore we define

$$H(\theta) \;:=\; \int_\xi^\theta e^{-\kappa(\tilde{\theta})}\,\mathrm{d}\tilde{\theta}\,, \quad H_g(\theta) \;:=\; \int_{-\xi}^\theta \frac{2g(\tilde{\theta})}{\mathbb{E}(p_{1,\sigma}^2(\tilde{\theta}))}\,e^{\kappa(\tilde{\theta})}\,\mathrm{d}\tilde{\theta}$$

where $\xi = \frac{\pi}{2}$ if $\mathbb{E}(b_{K+1,\sigma}) < 0$ and $\xi = -\frac{\pi}{2}$ if $\mathbb{E}(b_{K+1,\sigma}) > 0$. For $|\theta| < \frac{\pi}{2}$ let

$$\rho(\theta) \;:=\; \frac{2ce^{\kappa(\theta)}}{\mathbb{E}(p_{1,\sigma}^2)}H(\theta), \quad G(\theta) = e^{-\kappa(\theta)}\,H_g(\theta) \tag{4.20}$$

where $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho(\theta)\,\mathrm{d}\theta = 1 \;\Rightarrow\; c \neq 0$ (this solution was already given with equation (3.13)). G solves equation (4.18). Together with Corollary 4.8 the following shows Theorem 3.10.

Theorem 4.9. *Let $T_{\lambda,\sigma}$ have an anomaly of second degree and let $\mathbb{E}(p_{1,\sigma}^2) > 0$ for $\theta \in \mathbb{R}\mathrm{P}(1), \theta \neq \frac{\pi}{2}$ and $\mathbb{E}(p_{1,\sigma}^2(\frac{\pi}{2})) = 0$. Furthermore let ρ be defined as in (4.20) (π-periodically continued) and let f be π-periodic and twice continuously differentiable. Then one has*

$$\mathbb{E}_\omega\,\frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; \int_0^\pi f(\theta)\,\rho(\theta)\,\mathrm{d}\theta \,+\, \mathcal{O}((\lambda^{2\alpha_1}N)^{-1}, \lambda^{\beta-2\alpha_1})$$

To prove this theorem we need the following lemma.

Lemma 4.10. *We define*

$$Q(\theta) \;:=\; \int_0^\theta \frac{2\mathbb{E}(p_{K+1,\sigma}(\tilde{\theta}))}{\mathbb{E}(p_{1,\sigma}^2(\tilde{\theta}))}\,\mathrm{d}\tilde{\theta}, \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \;. \tag{4.21}$$

If $p_{K+1,\sigma}(\frac{\pi}{2}) > 0$ then for any $j \in \mathbb{R}$ one has

$$\lim_{\theta\uparrow\frac{\pi}{2}} [\mathbb{E}(p_{1,\sigma}^2(\theta))]^j \exp(Q(\theta)) \;=\; \lim_{\theta\downarrow-\frac{\pi}{2}} [\mathbb{E}(p_{1,\sigma}^2(\theta))]^j \exp(-Q(\theta)) \;=\; \infty$$

and

$$\lim_{\theta\uparrow\frac{\pi}{2}} [\mathbb{E}(p_{1,\sigma}^2(\theta))]^j \exp(-Q(\theta)) \;=\; \lim_{\theta\downarrow-\frac{\pi}{2}} [\mathbb{E}(p_{1,\sigma}^2(\theta))]^j \exp(Q(\theta)) \;=\; 0 \;.$$

If $\mathbb{E}(p_{K+1,\sigma}(\frac{\pi}{2})) < 0$ *we get the same results with* 0 *and* ∞ *being exchanged.*

Proof. As $\mathbb{E}(p_{1,\sigma}^2(\frac{\pi}{2})) = 0$ according to (4.5) we get that $\mathbb{E}(b_{1,\sigma}^2) = 0$ and we have

$$\mathbb{E}(p_{1,\sigma}^2) \;=\; \cos^2(\theta)\,\tilde{p}(\theta), \quad \tilde{p}(\theta) := \mathbb{E}([c_{1,\sigma}\cos(\theta) - 2a_{1,\sigma}\sin(\theta)]^2) \;. \tag{4.22}$$

If $\tilde{p} > 0$ for all θ then one has two constants $0 < C_1 < C_2$ such that $C_1 \leq \tilde{p} \leq C_2$ as $\tilde{p}$ is continuous and π-periodic. For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we define

$$\tilde{Q}(\theta) \;:=\; \int_0^\theta \frac{2p_{K+1,\sigma}(\tilde{\theta})}{\cos^2(\tilde{\theta})}\,\mathrm{d}\tilde{\theta} \;.$$

Using (4.5) we realize that $\mathbb{E}(b_{K+1,\sigma}) < 0$ iff $\mathbb{E}(p_{K+1,\sigma}(\frac{\pi}{2})) > 0$ and

$$\tilde{Q}(\theta) \;=\; -\mathbb{E}(b_{K+1,\sigma})\tan(\theta) + [\mathbb{E}(c_{K+1,\sigma}) + \mathbb{E}(b_{K+1,\sigma})]\theta + 2\mathbb{E}(a_{K+1,\sigma})\ln|\cos(\theta)|\,. \tag{4.23}$$

We know that $Q(\theta)$ lies in between $\frac{1}{C_1}\tilde{Q}(\theta)+C_3$ and $\frac{1}{C_2}\tilde{Q}(\theta)+C_4$ for suitable constants C_3 and C_4 (which may be different for the cases $\theta > 0$ and $\theta < 0$), because in some interval around $\pm\frac{\pi}{2}$ the sign of $\mathbb{E}(p_{K+1,\sigma})$ does not change and the integrand in the definition of Q lies between $\frac{1}{C_1}$ and $\frac{1}{C_2}$ times the integrand of $\tilde{Q}$. As one can write $p_{1,\sigma}^2$ as a rational function in $\tan(\theta)$ equation (4.23) gives the desired result.

Now if $\tilde{p} = 0$ for some $\theta \in \mathbb{R}\mathrm{P}(1)$, then this must be for $\hat{\theta} = \frac{\pi}{2}$ (as this is the only angle where $\mathbb{E}(p_{1,\sigma}^2)$ is zero). That means that $\mathbb{E}(a_{1,\sigma}^2) = 0$ and equation (4.22) reduces to

$$\mathbb{E}(p_{1,\sigma}^2) \;=\; \mathbb{E}(c_{1,\sigma}^2)\,\cos^4(\theta)$$

and we have

$$Q(\theta) \;=\; \frac{1}{\mathbb{E}(c_{1,\sigma}^2)}\left[-\mathbb{E}(b_{K+1,\sigma})\frac{\tan^3(\theta)}{3} - \mathbb{E}(a_{K+1,\sigma})\tan^2(\theta) + \mathbb{E}(c_{K+1,\sigma})\tan(\theta)\right] \tag{4.24}$$

which also leads to the mentioned results.

□

Proof of Theorem 4.9. We continue the functions ρ and G and κ from equation (4.20) π-periodically for $\theta \notin \frac{\pi}{2} + \pi\mathbb{Z}$. Then for $\theta \in [0,\pi], \theta \neq \frac{\pi}{2}$ one has

$$\frac{1}{2}\mathbb{E}(p_{1,\sigma}^2)\rho' + \mathbb{E}(p_{1,\sigma}\partial_\theta p_{1,\sigma} - q_\sigma)\rho \;=\; c$$

where $c \neq 0$ and

$$\frac{1}{2}\mathbb{E}(p_{1,\sigma}^2)G' + \mathbb{E}(q_\sigma)G \;=\; g$$

where $g := f - \int_0^\pi f(\tilde\theta)\rho(\tilde\theta)\,\mathrm{d}\tilde\theta$. For $\theta \neq \frac{\pi}{2}$ let $\hat{Q}$ be an antiderivative of $\frac{2\mathbb{E}(p_{1,\sigma}\partial_\theta p_{1,\sigma} - q_\sigma)}{\mathbb{E}(p_{1,\sigma}^2)}$ such that $\hat{Q}(\pi) = \hat{Q}(0) = 0$ and remember that κ is an antiderivative of $\frac{2\mathbb{E}(q_\sigma)}{\mathbb{E}(p_{1,\sigma}^2)}$. Comparing these functions with Q (π-periodically continued) as defined in (4.21) one has

$$\kappa(\theta) \;=\; Q(\theta) + \frac{1}{2}\ln(\mathbb{E}(p_{1,\sigma}^2)), \quad \hat{Q} \;=\; -Q + \frac{1}{2}\ln(\mathbb{E}(p_{1,\sigma}^2))\,.$$

Thus for $\mathbb{E}(p_{K+1,\sigma}(\frac{\pi}{2})) > 0$ Lemma 4.10 yields

$$\lim_{\theta\uparrow\frac{\pi}{2}} \frac{\exp(\hat{Q}(\theta))}{\mathbb{E}(p_{1,\sigma}^2(\theta))} \;=\; 0, \quad \forall\, n\in\mathbb{N}:\; \lim_{\theta\downarrow\frac{\pi}{2}} (\mathbb{E}(p_{1,\sigma}^2(\theta)))^n \exp(\hat{Q}(\theta)) \;=\; \infty$$

and

$$\lim_{\theta\downarrow\frac{\pi}{2}} \frac{\exp(\kappa(\theta))}{\mathbb{E}(p_{1,\sigma}^2(\theta))} \;=\; 0, \quad \forall\, n\in\mathbb{N}:\; \lim_{\theta\uparrow\frac{\pi}{2}} (\mathbb{E}(p_{1,\sigma}^2(\theta)))^n \exp(\kappa(\theta)) \;=\; \infty\,,$$

with $\uparrow$ and $\downarrow$ being exchanged if $\mathbb{E}(p_{K+1,\sigma}(\frac{\pi}{2})) < 0$. As we also have $\partial_\theta\mathbb{E}(p_{1,\sigma}^2(\frac{\pi}{2})) = 0$ and $\mathbb{E}(p_{K+1,\sigma}(\frac{\pi}{2})) \neq 0$ all conditions for Lemma 4.5 are satisfied and we can apply it to get that ρ and G can be continued to twice continuously differentiable functions at $\frac{\pi}{2}$. (This also shows that ρ really is normalizable, i.e. one finds a constant c such that $\int_0^\pi \rho(\theta)\,\mathrm{d}\theta = 1$.) Therefore we can apply Lemma 4.7 which now proves the theorem. □

Finally let us consider the case stated in Theorem 3.12, i.e. let

$$\mathbb{E}(p_{1,\sigma}^2(\hat\theta)) \;=\; \mathbb{E}((\partial_\theta\, p_{1,\sigma}(\hat\theta))^2) \;=\; 0,\; \mathbb{E}(q_\sigma(\hat\theta)) \;=\; 0,\; \partial_\theta\mathbb{E}(q_\sigma(\hat\theta)) \;<\; 0\,.$$

We may again assume that $\hat\theta = \frac{\pi}{2}$. Then from equation (4.5) one deduces that

$$\mathbb{E}(a_{1,\sigma}^2) \;=\; \mathbb{E}(b_{1,\sigma}^2) \;=\; 0 \;\Rightarrow\; \mathbb{E}(p_{1,\sigma}^2) \;=\; \mathbb{E}(c_{1,\sigma}^2)\cos^4(\theta)$$

as well as $\mathbb{E}(b_{K+1,\sigma}) = 0$ and $\mathbb{E}(a_{K+1,\sigma}) < 0$. The conditions on $p_{1,\sigma}$ say that there is almost no (random) movement in a small neighbourhood of $\hat\theta$ to the lowest order α_1. The conditions on q_σ say that $\hat\theta$ is a stable fix point of $S_{\lambda,\sigma}$ to order $2\alpha_1$. All together we have a dynamical System θ_n that is very close concentrated at this fix point. The following theorem is equivalent to Theorem 3.12.

Theorem 4.11. *Let $T_{\lambda,\sigma}$ have a second degree anomaly. Furthermore let $a_{1,\sigma} = b_{1,\sigma} = 0$ $\mathbf{p}$-almost surely (then of course one has $\mathbb{E}(c_{1,\sigma}^2) > 0$) and*

$$\mathbb{E}(b_{K+1,\sigma}) \;=\; 0 \,\wedge\, \mathbb{E}(a_{K+1,\sigma}) \;<\; 0 \;\Leftrightarrow\; \mathbb{E}(q_\sigma(\pi/2)) \;=\; 0 \,\wedge\, \partial_\theta\mathbb{E}(q_\sigma(\pi/2)) \;<\; 0\,.$$

If $f \in C^3(\mathbb{R}\mathrm{P}(1))$ then one has

$$\mathbb{E}_\omega\, \frac{1}{N}\sum_{n=0}^{N-1} f(\theta_n) \;=\; f\left(\frac{\pi}{2}\right) + \mathcal{O}((\lambda^{2\alpha_1}N)^{-1}, \lambda^{\beta-2\alpha_1})\,.$$

Proof. For $|\theta| < \frac{\pi}{2}$ we get

$$\begin{aligned}\kappa(\theta) &= \int_0^\theta \frac{2\mathbb{E}(q_\sigma(\tilde\theta))}{\mathbb{E}(p_{1,\sigma}^2(\tilde\theta))}\,\mathrm{d}\tilde\theta = \\ &= \frac{-\mathbb{E}(a_{K+1,\sigma})}{\mathbb{E}(c_{1,\sigma}^2)}\tan^2(\theta) + \frac{\mathbb{E}(c_{K+1,\sigma})}{\mathbb{E}(c_{1,\sigma}^2)}\tan(\theta) + \ln(\cos^2(\theta))\end{aligned}$$

As $\mathbb{E}(a_{K+1,\sigma}) < 0$ and κ' is a rational function in $\tan(\theta)$ one has

$$\lim_{\theta\uparrow\frac{\pi}{2}} \partial_\theta^n e^{-\kappa(\theta)} = \lim_{\theta\downarrow-\frac{\pi}{2}} \partial_\theta^n e^{-\kappa(\theta)} = 0$$

where ∂_θ^n denotes the n-th derivative. Therefore the function $e^{-\kappa(\theta)}$ can be continued to a π-periodic and smooth (C^∞) function. Note that $\int_0^\pi e^{-\kappa(\theta)}\,\mathrm{d}\theta > 0$ and we have

$$\frac{1}{2}\mathbb{E}(p_{1,\sigma}^2(\theta))\,\partial_\theta\, e^{-\kappa(\theta)} + \mathbb{E}(q_\sigma(\theta))\, e^{-\kappa(\theta)} = 0\,. \tag{4.25}$$

Let $g(\theta) := f(\theta) - f(\frac{\pi}{2})$. For $|\theta| < \frac{\pi}{2}$ define similar as above

$$H_g(\theta) := \int_0^\theta \frac{2g(\tilde\theta)}{\mathbb{E}(p_{1,\sigma}^2(\tilde\theta))}\, e^{\kappa(\tilde\theta)}\,\mathrm{d}\tilde\theta, \quad G(\theta) := e^{-\kappa(\theta)} H_g(\theta)$$

and continue these functions π-periodically for $\theta \notin \frac{\pi}{2} + \pi\mathbb{Z}$. Then we have

$$\frac{1}{2}\,\mathbb{E}(p_{1,\sigma}^2)\, G' + \mathbb{E}(q_\sigma)\, G = g \tag{4.26}$$

We want to show that G can be continued to a twice continuously differentiable function, therefore it is enough to show this for $\theta = \frac{\pi}{2}$.
As $g(\frac{\pi}{2}) = 0$ and g is 3 times continuously differentiable we get that $\hat g(\theta) := \frac{g(\theta)}{\theta-\pi/2}$ is twice continuously differentiable, even at $\frac{\pi}{2}$ (suitable continued), as well as $\hat p(\theta) := \frac{\mathbb{E}(p_{1,\sigma}^2(\theta))}{2(\theta-\pi/2)}$ and $\hat q(\theta) := \frac{\mathbb{E}(q_\sigma(\theta))}{\theta-\pi/2}$. Then we get from (4.26)

$$\hat p\, G' + \hat q\, G = \hat g\,.$$

We can deduce $\hat p(\frac{\pi}{2}) = 0$, $n\hat p'(\frac{\pi}{2}) + \hat q(\frac{\pi}{2}) = \hat q(\frac{\pi}{2}) \neq 0$, furthermore κ is an antiderivative of $\frac{\hat q}{\hat p}$ and one has

$$\lim_{\theta\uparrow\frac{\pi}{2}} |\hat p(\theta)|^n \exp(\kappa(\theta)) = \lim_{\theta\downarrow\frac{\pi}{2}} |\hat p(\theta)|^n \exp(\kappa(\theta)) = \infty\,.$$

Therefore we can apply Lemma 4.6 to get that G (suitable continued) is twice continuously differentiable, even at $\frac{\pi}{2}$. Now define

$$\hat G(\theta) := G(\theta) - \frac{\int_0^\pi G(\tilde\theta)\,\mathrm{d}\tilde\theta}{\int_0^\pi e^{-\kappa(\tilde\theta)}\,\mathrm{d}\tilde\theta}\, e^{-\kappa(\theta)}\,, \quad F(\theta) := \int_0^\theta \hat G(\tilde\theta)\,\mathrm{d}\tilde\theta\,.$$

Then one has $F(\pi) = 0 = F(0)$. Thus F is a π-periodic, 3 times continuously differentiable function and equations (4.25) and (4.26) lead to

$$g \;=\; \frac{1}{2}\mathbb{E}(p_{1,\sigma}^2)\, F'' \;+\; \mathbb{E}(q_\sigma)\, F' \; .$$

Now Lemma 4.2 proves the theorem. □

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Declaration

I hereby declare that this document has been composed by me and is based on my own work, unless otherwise acknowledged in the text.

Erlangen, 17 March, 2006

Selbständigkeitserklärung

Hiermit erkläre ich, diese Arbeit selbständig und nur unter der Verwendung der angegebenen Literatur verfasst zu haben.

Erlangen, den 17. März 2006