Spectral flow applied to solid state systems

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Clerment-Ferrand November, 2016

Plan

- Review of classical spectral flow
- Laughlin arguments
- η-invariants and finite-volume calculation of indices
- $\mathbb{Z}_2$-valued spectral flow
- Application to a topological insulator (Kitaev chain)

Review of spectral flow

$\mathcal{H}$ separable Hilbert space and $\mathbb{B}(\mathcal{H})$ bounded operators

$T \in \mathbb{B}(\mathcal{H})$ Fredolm $\iff$ $\mathrm{Ker}(T)$, $\mathrm{Ker}(T^*)$ finite dimensional

$T = T^*$ Fredholm $\iff$ $0 \notin \sigma_{\mathrm{ess}}(T)$

$\mathbb{F}_{\mathrm{sa}} = \{ T = T^* \text{ Fredholm } \}$ has 3 components which contract to

$$\begin{aligned}
\mathbb{F}^*_{\mathrm{sa}} &= \{T \in \mathbb{F}_{\mathrm{sa}} \,|\, \sigma_{\mathrm{ess}}(T) = \{-1, 1\}\} \\
\mathbb{F}^+_{\mathrm{sa}} &= \{T \in \mathbb{F}_{\mathrm{sa}} \,|\, \sigma_{\mathrm{ess}}(T) = \{1\}\} \\
\mathbb{F}^-_{\mathrm{sa}} &= \{T \in \mathbb{F}_{\mathrm{sa}} \,|\, \sigma_{\mathrm{ess}}(T) = \{-1\}\}
\end{aligned}$$

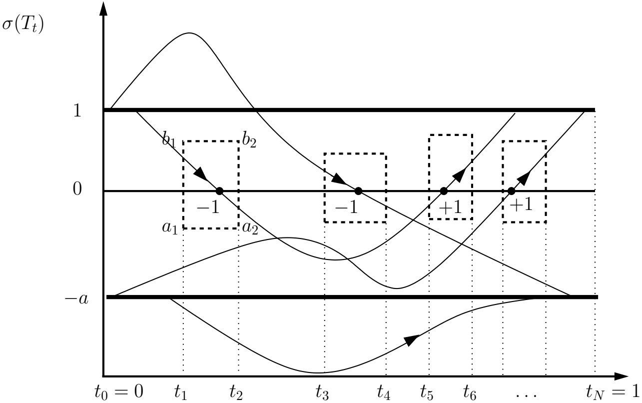
Theorem (Atiyah-Singer 1969)

Homotopy groups of $\mathbb{F}^*_{\mathrm{sa}}$ *are* $\pi_{2n}(\mathbb{F}^*_{\mathrm{sa}}) = 0$ *and* $\pi_{2n+1}(\mathbb{F}^*_{\mathrm{sa}}) = \mathbb{Z}$

Aim: spectral flow calculates $\pi_1(\mathbb{F}^*_{\mathrm{sa}})$

Intuitive notion of spectral flow

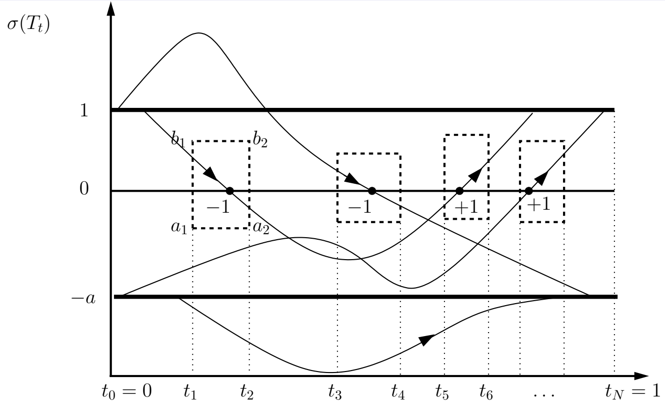
Given path $t \in [0,1] \mapsto T_t = (T_t)^*$ of self-adjoint Fredholms on $\mathcal{H}$



Counting of eigenvalues passing 0 works if path analytic (APS)

For continuous paths need to go to "generic position", or:

Phillips' analytic approach (1996)



$\exists$ finite partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ of $[0,1]$ and $a_n < 0 < b_n$ with $t \in [t_{n-1}, t_n] \mapsto \chi(T_t \in [a_n, b_n])$ continuous. Set:

$$\mathrm{SF}(t \in [0,1] \mapsto T_t) = \sum_{n=1}^{N} \mathrm{Tr}_{\mathcal{H}} \left(\chi(T_{t_{n-1}} \in [a_n, 0]) - \chi(T_{t_n} \in [a_n, 0])\right)$$

Theorem (Phillips 1996)

$\mathrm{SF}(t \in [0,1] \mapsto T_t)$ *independent of partition and* $a_n < 0 < b_n$.

It is a homotopy invariant when end points are kept fixed.

It satisfies concatenation and normalization:

$\mathrm{SF}(t \in [0,1] \mapsto T + (1-2t)P) = -\dim(P)$ *for* $TP = P$

Theorem (Lesch 2004)

Homotopy invariance, concatenation, normalization characterize SF

Theorem (Perera 1993, Phillips 1996)

SF *on loops establishes isomorphism* $\pi_1(\mathbb{F}^*_{\mathrm{sa}}) = \mathbb{Z}$

Theorem (Phillips 1996, based on Avron-Seiler-Simon 1994)

Let $T_1 = U^* T_0 U$ *invertible with* U *unitary and* $[U, T_0]$ *compact*

$\mathrm{SF}(t \in [0,1] \mapsto (1-t)T_0 + tT_1) = -\mathrm{Ind}(PUP|_{P\mathcal{H}})\,,\;\; P = \chi(T_0 > 0)$

Example: Laughlin argument 1981

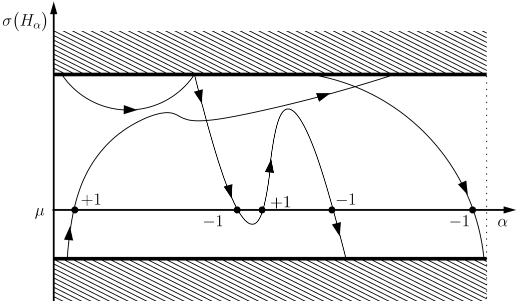
Theorem (Macris 2002, De Nittis, S-B 2016)

H disordered Harper-like operator on $\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^L$ with $\mu\in$ gap

H_α Hamiltonian with extra flux $\alpha\in[0,1]$ through 1 cell of $\mathbb{Z}^2$

Then $T_\alpha = H_\alpha - \mu \in \mathbb{F}^*_{\mathrm{sa}}$ and with $P=\chi(H_\alpha\leq\mu)$, $U=\frac{X_1+iX_2}{|X_1+iX_2|}$

$$\mathrm{SF}\Big(\alpha\in[0,1]\mapsto H_\alpha \text{ through } \mu\Big) \;=\; -\,\mathrm{Ind}(PUP) \;=\; -\mathrm{Ch}(P)$$



$\mathbb{Z}_2$ invariant for QSH by spectral flow

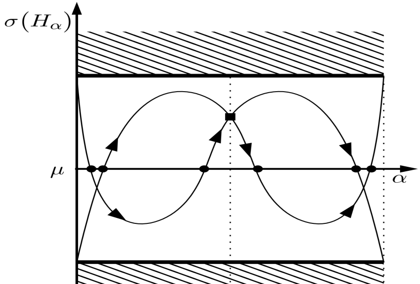
TRS implemented by a real unitary $S_{\rm tr}$ with $S_{\rm tr}^2 = -\mathbf{1}$

$$S_{\rm tr}^* \, \overline{H_\alpha} \, S_{\rm tr} \;=\; H_{-\alpha} \;=\; U^* \, H_{1-\alpha} \, U$$

Both for $\alpha = 0$ (no flux) and $\alpha = \frac{1}{2}$ (half flux) one has TRS

Theorem (De Nittis, S-B 2016)

$\mathrm{Ind}_2(PUP) = \dim(\ker(PUP)) \bmod 2 = 1$, *namely non-trivial QSH*
$\Longrightarrow \quad H_{\frac{1}{2}}$ *has Kramers pair bound state in gap*



Spectral flow in a BdG-Hamiltonian

Flux tube in two-dimensional BdG Hamiltonian

$$S_{\rm ph}^* \, \overline{H_\alpha} \, S_{\rm ph} \;=\; -\,H_{-\alpha} \qquad , \qquad S_{\rm ph}^2 = \pm \mathbf{1}$$

Then $S_{\rm ph}^* \, \overline{H_\alpha} \, S_{\rm ph} = -U^* H_{1-\alpha} U$ so that

$$\sigma(H_\alpha) \;=\; -\,\sigma(H_{-\alpha}) \;=\; -\,\sigma(H_{1-\alpha})$$

PHS only for $\alpha = 0, \frac{1}{2}, 1$ and thus $\mathrm{Ind}_2(H_{\frac{1}{2}})$ wel-defined

Theorem (De Nittis, S-B 2016)

$\mathrm{Ind}(PUP) \bmod 2 = \mathrm{Ind}_2(H_{\frac{1}{2}})$

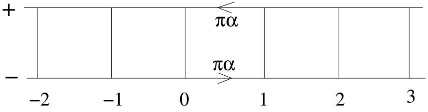
or: odd Chern number implies existence of zero mode at defect

These zero modes are Majorana fermions (Read-Green 2000)

Worth noting: $S_{\rm ph}^2 = -\mathbf{1} \implies \mathrm{Ind}(PUP)$ even $\implies$ no zero mode

Laughlin arguments in other dimensions (in preparation with Carey)

$d = 1$: chiral spectral flow in SSH leads to bound state of $H_{\frac{1}{2}}$



On $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L$ with $d \geq 3$: insert non-abelian Wu-Yang monopol

$$A \;=\; \frac{i}{2}\,\frac{[D,\gamma]}{D^2} \qquad , \qquad D = \sum_{j=1}^{d} \gamma_j X_j$$

into non-abilian translations (say without magnetic field):

$$S_k^\alpha \;=\; e^{i\nabla_k^\alpha} \;=\; U^\alpha(X) S_k \qquad , \qquad \nabla_k^\alpha \;=\; i\partial_k + \alpha A_k$$

Then study (chiral) spectral flow for $H_\alpha = P(S_1^\alpha, \ldots, S_d^\alpha)$

Spectral flow and η-invariant

$B = B^*$ invertible operator on $\mathcal{H}$ with compact resolvent. Then

$$\eta(B) \;=\; \mathrm{Tr}(B|B|^{-s-1})|_{s=0} \;=\; \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty dt\; t^{\frac{s-1}{2}}\;\mathrm{Tr}(B\,e^{-tB^2})\Big|_{s=0}$$

provided it exists! If $\dim(\mathcal{H}) < \infty$, then $\eta(B) = 2\,\mathrm{Sig}(B)$

Proposition

Given $D > 0$ and A bounded, and B with η-invariant

$$B \;=\; \begin{pmatrix} D & A \\ A^* & -D \end{pmatrix} \qquad \Longrightarrow \qquad \eta(B) \;=\; 0$$

Proposition (Corollary of result in Getzler, 1993)

$\lambda \in [0,1] \mapsto B(\lambda) - B(0) \in \mathcal{L}^1$, $B(0)$ *and* $B(1)$ *have* η*-invariants*

$$\eta(B(1)) - \eta(B(0)) \;=\; 2\,\mathrm{SF}\big(\lambda \in [0,1] \mapsto B(\lambda)\big)$$

η-Invariant for chiral Hamiltonian

Chiral Hamiltonian $H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ in $d = 1$ on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}$

Introduce operator with compact resolvent:

$$B \;=\; (\kappa X + iH)J \;=\; \begin{pmatrix} \kappa X & -iA \\ iA^* & -\kappa X \end{pmatrix} \quad , \quad J \;=\; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

Proposition (in preparation with Loring)

If $[H, X]$ *bounded,* $\eta(B)$ *exists*

Use path $\lambda \in [0,1] \mapsto H(\lambda)$ splitting H on $\ell^2(\mathbb{N}_+) \oplus \ell^2(\mathbb{N}_-)$

Leads to path $\lambda \in [0,1] \mapsto B(\lambda)$ with $B(0) = B$ and

$$B(1) \;=\; \begin{pmatrix} \kappa X_+ & -iA_+ \\ iA_+^* & -\kappa X_+ \end{pmatrix} \oplus \begin{pmatrix} \kappa X_- & -iA_- \\ iA_-^* & -\kappa X_- \end{pmatrix}$$

Thus $\eta(B(1)) = 0$ by above proposition!

Finite-volume calculation of index

$$\eta(B) \;=\; 2\,\mathrm{SF}(\lambda\in[0,1]\mapsto B(\lambda))$$

Calculation: spectral flow equal to $\mathrm{Ind}(\Pi A \Pi)$ where Π Hardy

Moreover, signature stabilizes for finite volume approximation of B

Theorem (with Loring, in preparation)

Let B_Λ restriction of B to $\ell^2([-\Lambda,\Lambda])\otimes\mathbb{C}^{2N}$

If $\frac{1}{\Lambda}$ and $\kappa>0$ sufficiently small,

$$\tfrac{1}{2}\,\mathrm{Sig}(B_\Lambda) \;=\; \mathrm{Ind}(\Pi A \Pi)$$

Similar results for all d, but K-theoretic proofs (fuzzy spheres)

Symmetries can be implemented and lead to Pfaffians

Basics on skew-adjoint Fredholm operators

$\mathcal{H}_{\mathbb{R}}$ real Hilbert space with complexification $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \oplus i\mathcal{H}_{\mathbb{R}}$

$T \in \mathbb{B}(\mathcal{H}_{\mathbb{R}})$ extends to complex linear operator (e.g. for spectrum)

$T^* = -T$ skew-adjoint $\implies \sigma(T) = \overline{\sigma(T)} \subset i\,\mathbb{R}$

$T^* = -T$ Fredholm $\iff 0 \notin \sigma_{\mathrm{ess}}(T)$

Theorem (Atiyah Singer 1969)

$\mathbb{F}_{\mathrm{sk}} = \{T = -T^* \text{ Fredholm } \}$ *has two connected components distinguished by:* $\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2$

Homotopy groups satisfy $\pi_n(\mathbb{F}_{\mathrm{sk}}) = \pi_{n+8}(\mathbb{F}_{\mathrm{sk}})$ *and are given by*

n	0	1	2	3	4	5	6	7
$\pi_n(\mathbb{F}_{\mathrm{sk}})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\,\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Aim: define $\mathbb{Z}_2$-valued spectral flow calculating $\pi_1(\mathbb{F}^*_{\mathrm{sa}})$

Note: $\mathrm{SF}(t \in [0,1] \mapsto T_t \in \mathbb{F}_{\mathrm{sk}}) = 0$

Start with example in $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$

$$T_t \;=\; (2t-1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \qquad \widetilde{T}_t \;=\; |2t-1|\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Spectra identical $\sigma(T_t) = \sigma(\widetilde{T}_t) = \{(1-2t)i, (2t-1)i\}$, but

$$\widetilde{T}_t(s) \;=\; |2ts-1|\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;\in\; \mathbb{F}_{\mathrm{sk}}$$

homotopy of paths with $\widetilde{T}_t(1) = \widetilde{T}_t$ and $\widetilde{T}_t(0)$ constant

No such homotopy for T_t !

Obstruction is change of orientation of eigenfunctions:

$$T_1 \;=\; A^* T_0 A \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\mathrm{sgn}(\det(A)) < 0$

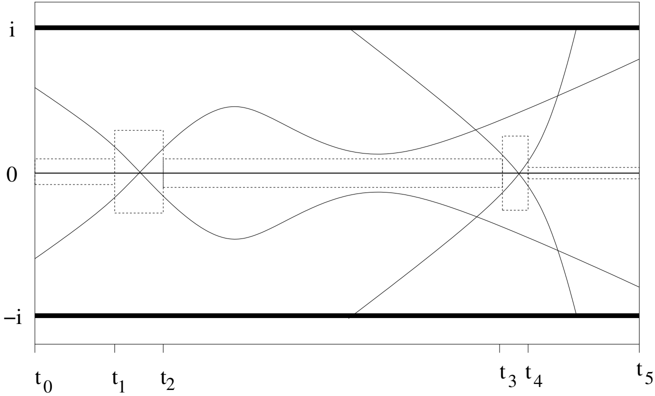
Definition

$\dim(\mathcal{H}_{\mathbb{R}}) < \infty$ and $T_0, T_1 \in \mathbb{F}_{\rm sk}$ with nullity $\dim(\mathcal{H}_{\mathbb{R}}) \bmod 2$

If $T_1 = A^* T_0 A$ for some invertible A, then

$$\mathrm{SF}_2(T_0, T_1) \;=\; \mathrm{sgn}(\det(A)) \;\in\; \mathbb{Z}_2$$

Now similar as Phillips: path $t \mapsto T_t \in \mathbb{F}_{\rm sk}$ with $\mathrm{Ind}_2(T_t) = 0$



Definition of $\mathbb{Z}_2$-valued spectral flow

For $a > 0$ set $Q_a(t) = \chi(T_t \in (-ia, ia))$

$\exists$ finite partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ and $a_n > 0$

- $t \in [t_{n-1}, t_n] \mapsto Q_{a_n}(t)$ continuous and constant finite rank
- $\|Q_{a_n}(t) - Q_{a_n}(t')\| \; < \; \epsilon \;\; \forall \;\; t, t' \in [t_{n-1}, t_n]$
- $\|\pi(T_t) - \pi(T_{t'})\|_{\mathcal{Q}} \; < \; \epsilon \;\; \forall \;\; t, t' \in [t_{n-1}, t_n]$

for some $\epsilon \leq \frac{1}{5}$

$V_n : \mathrm{Ran}(Q_{a_n}(t_{n-1})) \to \mathrm{Ran}(Q_{a_n}(t_n))$ orthogonal projection,

namely $V_n v = Q_{a_n}(t_n)v$. Check: V_n is a bijection.

Define $T_t^{(a)} \; = \; Q_a(t)\, T_t\, Q_a(t) + R_t$ with R_t lifting kernel

Definition

$$\mathrm{SF}_2(t \in [0,1] \mapsto T_t) \; = \; \sum_{n=1}^{N} \mathrm{SF}_2\big(T^{(a_n)}_{t_{n-1}}, V_n^* T^{(a_n)}_{t_n} V_n\big) \bmod 2$$

Basic properties

Theorem

$\mathrm{SF}_2(t \in [0,1] \mapsto T_t \in \mathbb{F}_{\mathrm{sk}})$ *independent of partition and* $a_n > 0$.

It is a homotopy invariant when end points are kept fixed.

It satisfies concatenation.

It satisfies a normalization (later).

SF_2 has characterizing properties of SF, but *no* "spectral flowing"

Theorem

SF_2 *on loops establishes isomorphism* $\pi_1(\mathbb{F}_{\mathrm{sk}}) = \mathbb{Z}_2$

Reformulation

$J \in \mathbb{B}(\mathcal{H}_\mathbb{R})$ complex structure $\iff$ $J^* = -J$ and $J^2 = -\mathbf{1}$

Theorem

J_0, J_1 complex structures with $\|\pi(J_0) - \pi(J_1)\|_\mathcal{Q} < 1$. Then

$$\mathrm{SF}_2(t \in [0,1] \mapsto tJ_0 + (1-t)J_1 \in \mathbb{F}_{\mathrm{sk}}) = \tfrac{1}{2}\,\mathrm{dim}(\mathrm{Ker}(J_0 + J_1)) \bmod 2$$

Proof: Both sides are homotopy invariants... □

Theorem

For above partition of path $t \in [0,1] \mapsto T_t$, set $J_n = T_{t_n}|T_{t_n}|^{-1}$

Then

$$\mathrm{SF}_2(t \in [0,1] \mapsto T_t) = \left(\sum_{n=1}^{N} \tfrac{1}{2}\,\mathrm{dim}(\mathrm{Ker}(J_{n-1} + J_n)) \right) \bmod 2$$

For classical spectral flow similar with index of pairs of projections

An index formula and canonical example

Theorem

J complex structure, $O = (O^*)^{-1}$ orthogonal with $[O, J]$ compact

$$\mathrm{SF}_2(t \in [0,1] \mapsto (1-t)J + tO^*JO) \; = \; \dim \mathrm{Ker}(POP|_{P\mathcal{H}}) \bmod 2$$

where $P = \chi(iJ > 0)$ Hardy

Example:

$\mathcal{H}_{\mathbb{R}} = L^2_{\mathbb{R}}(\mathbb{S}^1) \otimes \mathbb{R}^2$ and $\mathcal{H}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{S}^1) \otimes \mathbb{C}^2$

Fourier $\mathcal{F} : \mathcal{H}_{\mathbb{C}} \to \ell^2_{\mathbb{C}}(\mathbb{Z}) \otimes \mathbb{C}^2$

$J = \mathcal{F}^* \widehat{J} \mathcal{F}$ where $\widehat{J} = i\,\mathrm{sgn}(X) \otimes \mathbf{1}_2 + |0\rangle\langle 0| \otimes i\sigma_2$

$O = (O(k))_{k \in \mathbb{S}^1}$ fibered with 2×2 rotation matrix $O(k)$ by k

$\mathrm{SF}_2(t \in [0,1] \mapsto (1-t)J + tO^*JO) = 1 = \dim \mathrm{Ker}(POP|_{P\mathcal{H}_{\mathbb{C}}})$

Skew-adjoint Fredholm = gapped BdG

Fermionic quadratic Hamiltonian $\mathbf{H} = (a\ a^*)H\binom{a}{a^*}$ on $\mathcal{F}_-(\mathcal{H})$

BdG Hamiltonian $H \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ satisfies even PHS

$$\sigma_1^* \overline{H} \sigma_1 \;=\; -H \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then Majorana representation:

$$H_{\rm Maj} \;=\; C^* H C \;=\; -\overline{H_{\rm Maj}} \;=\; i\,T\,, \qquad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

Then: $\overline{T} = T$ and $T^* = -T$ and

$T \in \mathbb{F}_{\rm sk} \iff$ 0 in gap of H

Thus: paths of BdG's have a $\mathbb{Z}_2$-valued spectral flow

Kitaev chain with flux (disorder suppressed)

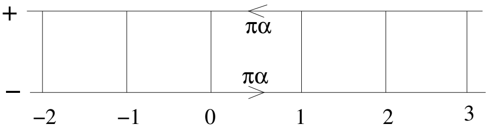
Here $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ and with shift S and $\mu \in \mathbb{R}$:

$$H = \frac{1}{2} \begin{pmatrix} S + S^* + 2\mu & i(S - S^*) \\ i(S - S^*) & -(S + S^* + 2\mu) \end{pmatrix}$$

$$= S_0 + S_0^* + \mu\, \mathbf{1} \otimes \sigma_3 \,, \qquad S_0 = S \otimes \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

Insert flux: $H_\alpha = S_\alpha + S_\alpha^* + \mu\, \mathbf{1} \otimes \sigma_3$

$$S_\alpha = S_0 + |1\rangle\langle 0| \otimes \frac{1}{2} \begin{pmatrix} e^{-i\pi\alpha} - 1 & i(e^{-i\pi\alpha} - 1) \\ i(e^{i\pi\alpha} - 1) & -(e^{i\pi\alpha} - 1) \end{pmatrix}$$



Spectral flow and bound states at defect

Proposition

For $|\mu| < 1$,

$$\mathrm{SF}_2(\alpha \in [0,1] \mapsto H_\alpha) \;=\; 1$$

Time-reversal symmetry $\sigma_3 \overline{H} \sigma_3 = H$, hence in CAZ Class BDI

Also holds for half flux: $\sigma_3 \overline{H_{\frac{1}{2}}} \sigma_3 = H_{\frac{1}{2}}$

Proposition

For $|\mu| < 1$, $H_{\frac{1}{2}}$ *has odd number of evenly degenerate zero modes:*

$$\tfrac{1}{2}\,\dim_{\mathbb{C}}(\mathrm{Ker}_{\mathbb{C}}(H_{\frac{1}{2}})) \bmod 2 \;=\; 1$$

Proof: Symmetry $\sigma(H_\alpha) = \sigma(H_{1-\alpha})$ and above Proposition