Space versus energy oscillations for Sturm-Liouville and Jacobi operators

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There is a paper with a similar title in Elect. J. Diff. Eq. 2020

Sturm-Liouville operator and Prüfer phases

Self-adjoint acting densely on $L^2([0,1],\mathbb{R})$ as

$$H \;=\; -\partial_x\big(p\,\partial_x + q\big) \;+\; q\partial_x \;+\; v$$

with • real p, q, v bounded

- p, q differentiable
- $p \geqslant c_p > 0$ (not singular)
- Dirichlet boundary conditions (for sake of concreteness)

Solve Schrödinger equation with Dirichlet on left boundary:

$$H\phi^E \;=\; E\phi^E \qquad , \qquad \phi^E(0) = 0\,, \;\; \partial_x\phi^E(0) = 1$$

Define Prüfer phase

$$e^{i\theta^E(x)} \;=\; \frac{\phi^E(x) - i\partial_x\phi^E(x)}{\phi^E(x) + i\partial_x\phi^E(x)}$$

$(x, E) \mapsto \theta^E(x) \in \mathbb{R}$ differentiable in x and in E (lifted from $\mathbb{S}^1$ to $\mathbb{R}$)

Oscillations in space (Sturm) and energy

$$\begin{aligned}
&\#\Big\{\text{eigenvalues of } H \leqslant E\Big\} \\
&\quad = \ \#\Big\{\text{zeros of } x \in [0,1] \mapsto \phi^E(x)\Big\} \\
&\quad = \ \mathrm{SF}\Big(x \in [0,1] \mapsto \theta^E(x) \bmod 2\pi \text{ through } \pi\Big) \\
&\quad = \ \mathrm{SF}\Big(e \in (-\infty, E] \mapsto \theta^e(1) \bmod 2\pi \text{ through } \pi\Big)
\end{aligned}$$

Second equality is non-trivial! Indeed no going back through -1:

$$\partial_x \theta^E(x) \ > \ 0 \quad \text{whenever} \quad \theta^E(x) = \pi \bmod 2\pi$$

Further fact: global monotonicity in energy

$$\partial_E \theta^E(x) \ > \ 0$$

Homotopy property in the space-energy strip $(x, E) \in [0,1] \times \mathbb{R}$

Matrix-valued Sturm-Liouville operators

Self-adjoint acting densely on $L^2([0,1],\mathbb{C}^L)$ as

$$H \;=\; -\partial_x\big(p\,\partial_x + q\big) \;+\; q\partial_x \;+\; v$$

with • bounded functions $p, q, v = v^*$ matrix-valued $\mathbb{C}^{L\times L}$

- p, q differentiable
- $p \geqslant c_p\,\mathbf{1}_L > 0$ positive definite
- Dirichlet boundary conditions

Fundamental solution $x \mapsto \Phi^E(x) \in \mathbb{C}^{L\times L}$:

$$H\Phi^E \;=\; E\Phi^E \qquad , \qquad \Phi^E(0) = \mathbf{0}_L\;,\;\; \partial_x\Phi^E(0) = \mathbf{1}_L$$

Well-defined $\mathrm{U}(L)$-valued matrix Prüfer phase

$$U^E(x) \;=\; \big(\Phi^E(x) \,-\, i\,\partial_x\Phi^E(x)\big)\big(\Phi^E(x) \,+\, i\,\partial_x\Phi^E(x)\big)^{-1}$$

Fact: $(x,E) \mapsto U^E(x) \in \mathrm{U}(L)$ differentiable in x and in E

Oscillations in space and energy

$$\#\Big\{\text{eigenvalues of } H \leqslant E\Big\}$$
$$= \mathrm{SF}\Big(x \in [0,1] \mapsto \mathrm{Spec}(U^E(x)) \bmod 2\pi \text{ through } \pi\Big)$$
$$= \mathrm{SF}\Big(e \in (-\infty, E] \mapsto \mathrm{Spec}(U^e(1)) \bmod 2\pi \text{ through } \pi\Big)$$

Both intersection $\#$'s are Bott-Maslov index of $x, E \mapsto \mathrm{Ran}\binom{p(x)\partial_x \Phi^E(x)}{\Phi^E(x)}$

Still no going back through -1:

$$\frac{1}{i}(U(x))^* \partial_x U(x)\Big|_{\mathrm{Ker}(U(x)+\mathbf{1}_L)} > 0 \qquad \text{(Coppel 1965)}$$

Monotonicity in E

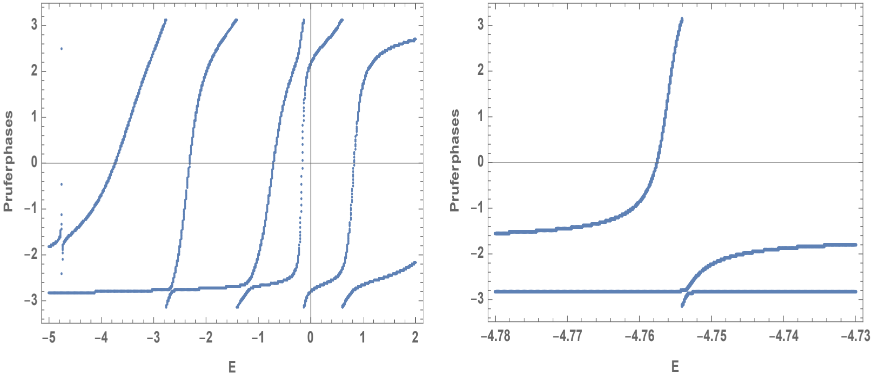
$$\frac{1}{i}(U^E(x))^* \partial_E U^E(x) \geqslant \mathbf{0}_L \qquad \text{(Bott 1956)}$$

Asymptotics $\lim_{E\to-\infty} U^E(x) = -\mathbf{1}_L$ clockwise

Numerical illustration for $L = 2$

$$p(x) = \begin{pmatrix} 2 + \cos(12\,x) & \sin(11.5\,x) \\ \sin(11.5\,x) & 3 - \sin(16\,x) \end{pmatrix} \quad , \quad q(x) = \ldots \quad , \quad v(x) = \ldots$$

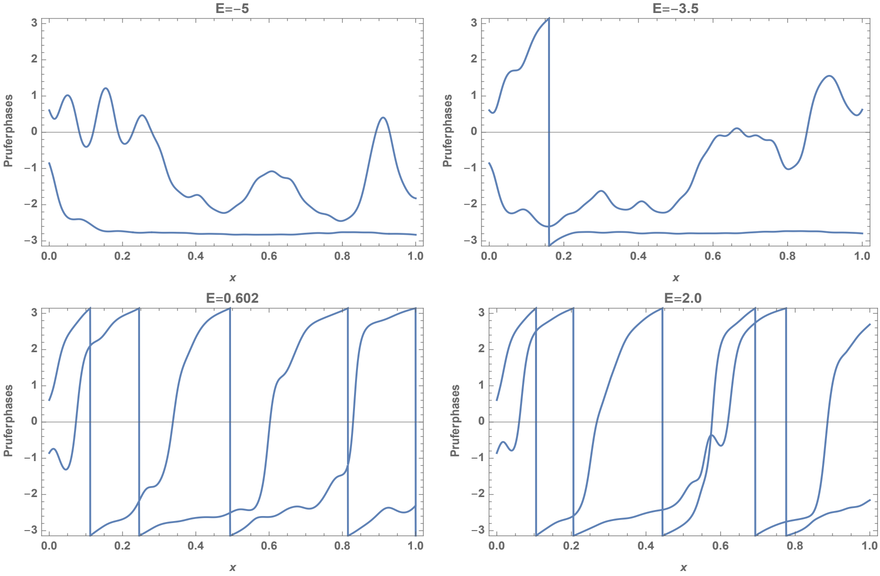
Plot of phases of spectrum of $U^E(1)$ as function of E



Avoided level crossings

Numerical illustration for $L = 2$

Plot of phases of spectrum of $U^E(x)$ as function of x for fixed E



Matrix Jacobi operators

Length $N \geqslant 3$ with $T_n, V_n = V_n^* \in \mathbb{C}^{L\times L}$ and $\mathrm{Ker}(T_n) = \{0\}$

$$H_N = \begin{pmatrix} V_1 & T_2 & & & & \\ T_2^* & V_2 & T_3 & & & \\ & T_3^* & V_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & V_{N-1} & T_N \\ & & & & T_N^* & V_N \end{pmatrix}$$

Suitable gauge transformation $G = \mathrm{diag}(G_1, \ldots, G_N) \Longrightarrow T_n > 0$

Fundamental solution Φ^E via three term recurrence (with $T_{N+1} = \mathbf{1}_L$)

$$T_{n+1}\Phi^E_{n+1} + V_n\Phi^E_n + T_n\Phi^E_{n-1} = E\Phi^E_n \qquad \Phi^E_0 = \mathbf{0}_L\,, \quad \Phi^E_1 = \mathbf{1}_L$$

Alternative construction: transfer matrices from Lorentz group $\mathrm{U}(L, L)$

Matrix Prüfer phases:

Produces eigenstate of H_N for energy E only if $\mathrm{Ker}(\Phi^E_{N+1}) \neq \{0\}$

Hence again intersection theory of Lagrangian subspaces

$$U^E_n \;=\; \big(T_{n+1}\Phi^E_{n+1} \,-\, i\,\Phi^E_n\big)\big(T_{n+1}\Phi^E_{n+1} \,+\, i\,\Phi^E_n\big)^{-1} \;\in\; \mathrm{U}(L)$$

Theorem

$$\textit{multiplicity of } E \textit{ as EV of } H_N \;=\; \textit{multiplicity of } -1 \textit{ as EV of } U^E_N$$

and

$$\#\{\textit{eigenvalues of } H_N \;\leqslant\; E\} \;=\; \mathrm{SF}\big(e \in (-\infty, E] \mapsto U^e_N \textit{ through } -1\big)$$

Moreover,

$$\frac{1}{i}\,(U^E_N)^*\partial_E U^E_N \;>\; 0$$

Question: how to define space oscillations from U^E_n to U^E_{n+1}?

Discrete Sturm oscillations

To measure $\#$ of sign changes of solution from n to $n+1$ consider

$$S_n^E = (\Phi_n^E)^* T_{n+1} \Phi_{n+1}^E = (S_n^E)^*$$

Morse indices $\imath_<(S_n^E) = \mathrm{Tr}(\chi(S_n^E < 0))$ and $\imath_>(S_n^E) = \mathrm{Tr}(\chi(S_n^E > 0))$

Theorem (inspired by book of Dosly-Elyseeva-Hilscher)

For E not in the finite singular set

$$\mathcal{S} = \bigcup_{n=1,\ldots,N-1} Spec(H_n) \qquad , \qquad H_n = H_N|_{[1,n]}$$

one has

$$\#\{eigenvalues\ of\ H_N \leqslant E\} = \sum_{n=1}^{N} \imath_>(S_n^E) = NL - \sum_{n=1}^{N} \imath_<(S_n^E)$$

Proof: short, but intricate geometric argument

Interpolation between U^E_{n-1} and U^E_n

Using principal branch Log of logarithm

$$Q^E_n \;=\; -i\,\mathrm{Log}(U^E_n)$$

Set

$$W^E(x) \;=\; \begin{cases} e^{-i\,3(x-n+\frac{2}{3})\,Q^E_{n-1}}\,, & x\in[n-1,n-\frac{2}{3}] \\ e^{i\,3(x-n+\frac{2}{3})\,2\pi\,\chi(S^E_n\geqslant 0)}\,, & x\in[n-\frac{2}{3},n-\frac{1}{3}] \\ e^{i\,3(x-n+\frac{1}{3})\,Q^E_n}\,, & x\in[n-\frac{1}{3},n] \end{cases}$$

Corollary

Suppose that $E\notin\mathcal{S}$. Then

$$\#\{\textit{eigenvalues of } H_N \;\leqslant\; E\} \;=\; \mathrm{SF}\big(x\in[0,N]\mapsto W^E(x)\;\; \textit{through} \;-1\big)$$

Short-coming: excludes $\mathcal{S}$! Progress by Hilscher-Sepitka 2021

Alternative: work with a suitable ODE on each interval $[n-1,n]$

Elements of geometric proof (disc. Sturm osc.):

Claim: $\mathrm{Tr}(\chi(H_N \leqslant E)) = N^E = \sum_{n=1}^{N} \imath_>(S_n^E)$ with $S_n^E = (\Phi_n^E)^* T_{n+1} \Phi_{n+1}^E$

Aim 1: exhibit $(H_N - E)$-non-pos. def. subs. $\mathcal{E}_{\leqslant}^E \subset \mathbb{C}^{NL}$ with $\dim = N^E$

Aim 2: exhibit $(H_N - E)$-pos. def. subs. $\mathcal{E}_{>}^E \subset \mathbb{C}^{NL}$ with $\dim = NL - N^E$

For $* = \leqslant, >$ iterative construction $\mathcal{E}_*^E = \bigoplus_{n=1}^N \mathcal{E}_*^{E,n}$ with $\mathcal{E}_*^{E,n} \subset \mathcal{E}^n$ and

$\mathcal{E}^n = \Big\{\psi = (\psi_1, \ldots, \psi_N) \in \mathbb{C}^{NL} \,:\, \psi_n \neq 0 \text{ and } \psi_{n+1} = \ldots = \psi_N = 0\Big\} \cup \{0\}$

By construction, $\mathcal{E}_*^{E,n} \cap \mathcal{E}_*^{E,m} = \{0\}$ for $n \neq m$

For $\mathcal{E}_>^{E,n}$, choose $v \in \mathbb{C}^L$ such that $v^* S_n^E v < 0$, so $\phi_n^E v \neq 0$. Set

$$\psi_v^{E,n} = \Big(\phi_1^E v, \ldots, \phi_n^E v, 0, \ldots, 0\Big) \in \mathcal{E}^n$$

Then

$$(\psi_v^{E,n})^* (H_N - E) \psi_v^{E,n} = - v^* (\phi_n^E)^* T_{n+1} \phi_{n+1}^E v = - v^* S_n^E v > 0$$

Holds for all v satisfying $v^* S_n^E v < 0$. Therefore

$$\dim(\mathcal{E}_>^{E,n}) \geqslant \imath_<(S_n^E)$$

Further extensions of oscillation theory:

- unitary scattering zippers (generalization of CMV matrices)
- Jacobi operators with blocks of infinite dimension
- Oscillation theory for surface states in $2D$ systems