Oscillation theory for Jacobi matrices and index pairings with applications to topological insulators

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Abstract and summary

This dissertation connects different topics and shows new results in the research field of mathematical physics. One of the main objects of consideration are certain self-adjoint operators on a Hilbert space, in particular, so-called discrete random Schrödinger operators because such operators are used in theoretical physics for describing disordered solid-state systems. They can characterise materials that are electrical conductors and others that are electrical insulators, often abbreviated to insulators. More than a decade ago, Charles L. Kane, Eugene J. Mele and others predicted a special physical system and material that is now known as a topological insulator. It links the mathematical topic of topology with the physics of the already mentioned insulator. This dissertation mainly considers mathematical aspects in this field and shows how functional analytic methods and index pairings can be used to classify operators with certain symmetry properties. This immediately leads to applications for physical systems like topological insulators since they can have fundamental symmetry relations, for example, a time-reversal symmetry, and these symmetries are described by corresponding operators.

The other main topic of this dissertation is the analysis of Jacobi operators and the related Jacobi matrices. They also are used to describe physical systems on lattices and, therefore, play an important role in theoretical solid-state physics. Here, the main object is not the application but rather the abstract oscillation theory of Jacobi matrices. It is shown how the spectrum of the Jacobi operator, which is a self-adjoint operator, can be described by a certain winding number. New results in a very general framework are developed and eventually applied to certain high-dimensional random Schrödinger operators.

Now, we briefly summarise the contents of the five chapters in this dissertation. We start with preliminaries in an introductory chapter that set the framework
for the rest of the work. In particular, we present the definition of Hilbert spaces with real structure and symmetry operators acting on them. After that we give an introduction to the theory of Krein spaces, which expands the geometry of Hilbert spaces by considering a not necessarily positive definite inner product. There, maximal isotropic subspaces known as Lagrangian subspaces are investigated. Then, we explain families of covariant operators that are the building blocks for describing random operators and, therefore, disordered systems. We close the chapter with a presentation of the Clifford groups and Clifford algebras that are necessary for describing symmetries contained in the Dirac operator, which in turn describes physical systems.

Chapter 2 focusses on some operator algebras and, in particular, on general C*-algebras. We present and emphasise the important properties of C*-algebras and von Neumann algebras. We close with a representation of the K-theory for C*-algebras and with the KR-theory for C*-algebras with symmetry, which have important applications for our treatment of topological insulators.

In Chapter 3, we give a short introduction to the field of Fredholm operators and the Noether index. Furthermore, we introduce another index, the so-called $\mathbb{Z}_2$-index. In certain situations it will be a second homotopy invariant. In addition, skew-adjoint real Fredholm operators that were considered by M. Atiyah and I. Singer are presented and connected to Fredholm operators on complex Hilbert spaces that fulfil certain symmetry relations.

Chapter 4 deals with index pairings and applications to topological insulators and is based on the already published joint work [GS16] with H. Schulz-Baldes. There, a Hilbert space with real structure is considered, and then projections and unitaries on this space that fulfil certain symmetry relations are chosen. Different pairings between these operators lead to Fredholm operators, whose Noether indices and $\mathbb{Z}_2$-indices are examined. These general index theorems with respect to the given symmetries can be applied to strong invariants of topological insulators. There, the symmetries are contained in the Fermi projection, coming from the Hamiltonian of the system, and in the Dirac operator, which is connected to the physical spatial dimension.

Chapter 5 finishes this dissertation with an extensive presentation of the oscillation theory for Jacobi matrices. In this chapter, we give a more generalised
and detailed presentation of the new results in the already published joint work [GSV17] with H. Schulz-Baldes and C. Villegas-Blas. Jacobi matrices are tridiagonal self-adjoint matrices of arbitrary size having invertible elements on the diagonal below and above the main diagonal. In classical considerations, the entries are complex numbers, and one can develop the Sturm-Liouville oscillation theory. Here, the entries are allowed to come from an arbitrary C*-algebra with unit element. This as well leads to some kind of oscillation theory and generalises known results from the finite-dimensional case presented in [Schu07]. For matrix-valued entries, this oscillation theory is connected to the eigenvalue counting function of the Jacobi matrix. In the general setting, the spectrum can be more complicated and another counting method is needed. For random Schrödinger operators that are given by covariant families of local operators, the integrated density of states is a natural substitute. Together with an approximation argument, a certain winding number can be interpreted as a spectral flow.
Zusammenfassung


Zusammenfassung

Windungszahl ausgedrückt werden kann. Dazu werden hier neue Ergebnisse in einem sehr allgemeinen Rahmen hergeleitet und schlussendlich auf zufällige hoch-dimensionale Schrödinger-Operatoren angewendet.


Kapitel 4 betrachtet Index-Paarungen und die Anwendungen zu topologi-

Contents

Acknowledgment 3

Abstract and summary 5

Zusammenfassung 9

Introduction and notations 17

1 Preliminaries 23

1.1 Basic notations 23

1.2 Krein spaces and symplectic methods 24

1.2.1 Definition of a Krein space 25

1.2.2 Lagrangian subspaces 28

1.2.3 Unitary groups in Krein spaces 32

1.3 Covariant families of operators and the density of states 35

1.3.1 Covariant families of operators 36

1.3.2 The density of states 39

1.3.3 Some properties related to Herglotz functions 46

1.3.4 The density of states for local operators 48

1.4 Symmetry operators 52

1.5 Some spectral properties 54

1.6 Clifford groups and algebras 58

1.6.1 Clifford algebras 59

1.6.2 Clifford groups 61

1.6.3 Representations of the Clifford group 64

1.6.4 Symmetry operators in the representation of the Clifford group 66
## 2 Operator algebras, $K$-theory and Symmetries

2.1 Some operator algebras ........................................ 73

2.1.1 Banach algebras ........................................... 73
2.1.2 C*-algebras and von Neumann algebras .................. 74
2.1.3 Building new C*-algebras ................................. 77
2.1.4 Exact sequences of C*-algebras .......................... 79
2.1.5 Unitisation of C*-algebras ............................... 80
2.1.6 Spectrum and the Gelfand–Naimark theorem ........... 83
2.1.7 Positive elements ........................................... 85
2.1.8 Traces on C*-algebras ...................................... 86
2.1.9 Factors of von Neumann algebras ......................... 87

2.2 Covariant families of operators as a C*-algebra ........ 90

2.3 C*-algebras with symmetry or Real C*-algebras ........... 93

2.3.1 C*-algebras with symmetry ............................... 94
2.3.2 Exact sequences of C*-algebras with symmetry .......... 97
2.3.3 Constructing lifts with symmetries ....................... 98

2.4 $K$-theory of C*-algebras .................................. 100

2.4.1 Complex $K$-groups ........................................ 100
2.4.2 The standard picture of the complex $K$-groups ....... 106
2.4.3 Real $KR$-groups ........................................ 107

2.5 Krein theory in C*-algebras ................................ 111

2.6 Some technical results ...................................... 113

## 3 Fredholm theory

3.1 Fredholm operators ........................................... 115

3.2 Skew-adjoint operators and Fredholm operators with symmetry 120

3.3 Classifying spaces ........................................... 131

## 4 Index pairings and topological insulators

4.1 Introduction .................................................. 137

4.2 Index pairings with symmetries ............................. 140

4.2.1 Index pairings ............................................. 140
4.2.2 Pairing projections and unitaries with symmetries ... 143
4.2.3 Pairing projections with projections ..................... 157
## 4.2.4 Pairing unitaries with unitaries ............................................ 158

## 4.3 Index pairings for topological insulators ............................... 160
  4.3.1 Topological insulators and their classification ...................... 160
  4.3.2 Invariants for zero-dimensional systems .............................. 163
  4.3.3 Reordering of the symmetries of the Hamiltonian .................... 165
  4.3.4 The Dirac operator and its symmetries .................................. 167
  4.3.5 Fredholm operators for topological insulators ....................... 169
  4.3.6 Examples .................................................................. 171

## 4.4 $KR$-cycles ................................................................. 175

## 5 Sturm-Liouville oscillation theory ............................................. 179
  5.1 Introduction and Sturm-Liouville problems .............................. 179
  5.2 Jacobi operators and Jacobi matrices ...................................... 181
  5.3 Analysis for Jacobi matrices ................................................ 182
    5.3.1 On transfer matrices ................................................ 189
    5.3.2 On Lagrangian frames and the Prüfer phase ....................... 193
    5.3.3 On estimates for the Prüfer phase ................................ 196
    5.3.4 Monotonicity of the Prüfer phase ................................ 198
    5.3.5 Asymptotics of the Prüfer phase ................................ 200
    5.3.6 Asymptotics for derivatives of Prüfer phases .................... 205
    5.3.7 Intersections of Lagrangian subspaces ............................ 210
  5.4 Oscillation theory in finite dimensions ................................. 212
    5.4.1 On the eigenphases of the Prüfer phase ............................ 214
    5.4.2 Counting function and rotation number ............................ 218
  5.5 Oscillation theory for high dimensional random Jacobi operators 221
    5.5.1 Overview ................................................................ 221
    5.5.2 Finite volume approximations ......................................... 225
    5.5.3 Proof of the main result .............................................. 229
  5.6 Some numerical analysis of the Anderson model ......................... 231
    5.6.1 Numerical setup ...................................................... 232
    5.6.2 Two-dimensional Anderson model for $\kappa = 0$ ................ 234
    5.6.3 Two-dimensional Anderson model for $\kappa = 0.5$ ............... 235
    5.6.4 Two-dimensional Anderson model for $\kappa = 1$ ................ 236
5.6.5 Two-dimensional Anderson model for $\kappa = 2$ . . . . . . . 236

Bibliography .................................................. 248
Introduction and notations

Quantum mechanics is often described as the fundamental theory of physics on small scales. Without doubt, the theoretical explanation of the strange behaviour of the building blocks of our world in the last century was one of the most important scientific contributions in modern times. In quantum mechanics, the physical system is described by a complex Hilbert space $\mathcal{H}$ and the observables are given by self-adjoint operators on this Hilbert space. This motivated mathematical research on such operators, and indeed the whole point of introducing unbounded linear operators by John von Neumann in 1929, see [Neu30a], was to explain the physical theory. In this dissertation, we are mostly interested in condensed matter physics, where one considers discrete Schrödinger operators on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. In the physics literature, such an operator is just called the Hamiltonian of the system and corresponds to the total energy in the system.

One typical example of a discrete Schrödinger operator $H : \mathcal{H} \to \mathcal{H}$ is given by the Anderson model in $d$ dimensions. It allows only nearest neighbour interactions and is defined by

$$H = \Delta_{\text{dis}} + V = \sum_{j=1}^d (S_j^* + S_j) + V,$$

where $S_j$ is the right shift operator on $\mathbb{Z}^d$ in the $j$th direction and $V$ is a multiplication operator with a real function on $\mathbb{Z}^d$. This operator describes a particle on a lattice within a potential given by $V$. One mostly finds two important physically motivated cases here. On the one hand, a periodic potential function describes a periodic system such as a periodic tight-binding model. On the other hand, disordered solid media correspond to random potentials,
meaning the function $V$ is replaced by a random variable.

We are especially interested in the last case since it describes a lot of physical systems that occur in condensed matter physics, in particular so-called topological insulators. These are materials that were predicted some years ago and invoked the interests of many physicists and mathematical physicists over the last decade. As the name suggests, these systems are electric insulators and are connected to the mathematical discipline of topology. The integer quantum Hall effect is one of the first discoveries that involve stable conducting channels, where, however, a strong external magnetic field is needed, see, e.g. [KDP80]. In 2005, Charles L. Kane and Eugene J. Mele predicted such a particular system, that is a topological insulator, where no external field is involved, see [KM05a] and [KM05b]. After that, this field of research attained an enormous acceleration and soon experimental confirmation, see [Kön+07] and [Geh+13]. By now, the field of topological insulators has grown even more and there are excellent surveys and monographs, namely [PS16a], [ORV15] and [BH13]. Here, we show how the classification of topological insulators is to be understood in the realm of Fredholm operators. These results have already been published in the author’s joint work [GS16] with Hermann Schulz-Baldes.

One approach to study the operator $H$ from above is to approximate it by finite dimensional matrices. For $d = 1$, one gets a so-called Jacobi-matrix, a matrix of the following form:

$$H_N = \begin{pmatrix} v_1 & 1 \\ 1 & v_2 & 1 \\ 1 & v_3 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & v_{N-1} & 1 \\ 1 & v_N \end{pmatrix}.$$ 

All self-adjoint matrices with invertible elements above and below the diagonal are called Jacobi-matrices. For analysing the spectrum of such matrices, there exists the Sturm-Liouville oscillation theory, which is named after Jacques Charles François Sturm and Joseph Liouville and known for almost two centuries.
However, to examine higher dimensional operators, one has to expand this classical oscillation theory. First generalisations, where the entries of the Jacobi matrices are finite dimensional matrices, seem due to Raoul Bott [Bot56]. A thorough examination of this finite dimensional case in the form of Jacobi matrices can be found in [Schu07]. Here, we go even further and allow entries coming from a unital C*-algebra. It is interesting to see that results from the classical oscillation theory hold even then. Again, the corresponding results have already been published in the author’s joint work [GSV17] with Hermann Schulz-Baldes and Carlos Villegas-Blas. However, in this dissertation, the material is presented in a slightly different way compared to [GSV17] since a more general point of view is considered. Also, a lot more details are provided here. The general framework can be applied to random Schrödinger operators since the covariant families of local operators build a unital C*-algebra. Here, **local operator** means an operator \( A \) on the Hilbert space \( \mathcal{H} = \ell^2(\mathbb{Z}^d) \) whose matrix entries \( \langle e_n, Ae_m \rangle \) decay sufficiently fast with the distance \( \|n - m\|_\infty \) between points \( n, m \in \mathbb{Z}^d \). The notion **covariant** describes an ergodic property for a whole family of operators on \( \mathcal{H} \), given by \( (A_\omega)_{\omega \in \Omega} \), where \( \Omega \) is a compact topological space together with commuting homeomorphisms \( T_1, \ldots, T_d \) on \( \Omega \) and \( A_\omega \) depends strongly continuously on \( \omega \). Then the covariance property means that

\[
S_j A_\omega S_j^* = A_{T_j(\omega)}, \quad j = 1, \ldots, d, \quad \omega \in \Omega,
\]

where \( S_1, \ldots, S_d \) denote the right shifts on \( \ell^2(\mathbb{Z}^d) \). It is well known, see, e.g. [Pas72] or Section 1.3 below, that, with respect to a given ergodic probability measure on \( \Omega \), each covariant family of local operators has a well-defined **integrated density of states**.

When we consider Jacobi matrices having entries that are covariant families of local operators, for example \( H_N \) above, we can approximate its integrated density of states by a generalised winding number of the associated **Prüfer phase**, which is a certain unitary element defined by the matrix entries of \( H_N \). Indeed, this result is a generalisation of the classical Sturm-Liouville oscillation theory and one of the main results in this work, cf. Theorem 5.37.
The following list is a short overview about the notation that is used in this dissertation.

**Symbols**

\[:=\] equality by definition

\[\subset\] subset symbol including equality

\[\mathbb{N}\] natural numbers without zero

\[\mathbb{N}_0\] natural numbers including zero

\[\mathbb{Z}\] integers

\[\mathbb{R}\] field of real numbers

\[\mathbb{C}\] field of complex numbers

\[\mathbb{K}\] field of real or complex numbers

\[i\] imaginary unit

\[\pi\] the ratio of a circle’s circumference to its diameter, may also be used for surjective maps or projection operators

\[\text{Im}\] imaginary part of a complex number

\[\text{Re}\] real part of a complex number

\[\mathbb{Z}^c\] complement of a set \(\mathbb{Z} \subset X\)

\[\delta_{jk}, \delta_{j,k}\] Kronecker delta that is 1 if \(j = k\) and 0 otherwise

\[L^p(M, \mu)\] Lebesgue space with respect to the measure \(\mu\) on \(M\)

\[\chi_A\] characteristic function for the set \(A\)

\[\mathcal{H}\] a Hilbert space (often complex and separable)

\[\mathcal{C}\] a complex conjugation on \(\mathcal{H}\)

\[\langle \cdot, \cdot \rangle\] scalar product in \(\mathcal{H}\), linear in the second entry

\[(\cdot | \cdot)\] standard scalar product in \(\mathbb{C}^n\), linear in the second entry

\[[\cdot, \cdot]\] inner product, not necessarily positive definite, linear in the second entry

\[\mathcal{B}(\mathcal{H})\] normed vector space of bounded linear maps from \(\mathcal{H}\) to \(\mathcal{H}\)

\[\mathcal{K}(\mathcal{H})\] normed vector space of compact linear maps from \(\mathcal{H}\) to \(\mathcal{H}\)

\[\mathcal{F}(\mathcal{H})\] topological space of Fredholm operators from \(\mathcal{H}\) to \(\mathcal{H}\)

\[\mathcal{U}(\mathcal{H})\] topological space of unitary operators from \(\mathcal{H}\) to \(\mathcal{H}\)

\[\mathcal{P}(\mathcal{H})\] topological space of (orthogonal) projections from \(\mathcal{H}\) to \(\mathcal{H}\)

\[E_A\] projection-valued spectral measure for the operator \(A\)
span $U$  linear hull of a set $U$ in a vector space
$f|_U$  restriction of a map $f$ to a subset $U$
$\oplus$  orthogonal sum in a Hilbert space or C*-algebras
$1_{\mathcal{H}}$  identity map $1_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$
$1, 1_A$  multiplicative unit element in an algebra $A$
spec$(A)$  spectrum of an operator $A$ or an element of a C*-algebra
spec$_{\text{ess}}(A)$  essential spectrum of an operator
spec$_{\text{dis}}(A)$  discrete spectrum of an operator
$\rho(A)$  resolvent set of an operator $A$
Ker $A$  null space of an operator $A$
Ran $A$  range of an operator $A$
$A$  an algebra, mostly a C*-algebra
$\mathcal{K}$  an inner product space, mostly a Krein space
$C$  Cayley transform
$\mathcal{F}$  Fourier transform
$\sigma_1, \sigma_2, \sigma_3$  the three Pauli matrices
$S^1$  unit circle in $\mathbb{C}$ given by $\{z \in \mathbb{C} \mid |z| = 1\}$
$\mathbb{C}_+$  open upper half plane $\{z \in \mathbb{C} \mid \text{Im} (z) > 0\}$
$C(\mathbb{R})$  set of continuous functions $f : \mathbb{R} \to \mathbb{C}$
$C^0(\mathbb{R})$  set of continuous functions $f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity
$\mathcal{C}_d$  real Clifford algebra with $d$ generators squaring to $1$
$\tilde{\mathcal{C}}_d$  real Clifford algebra with $d$ generators squaring to $-1$
Tr  usual trace on $\mathcal{B}(\mathcal{H})$
$\mathcal{T}$  a trace on a C*-algebra, mostly a normalised finite trace
$\mathbf{P}$  a probability measure
1 Preliminaries

This chapter serves as an introduction of the basic notations and concepts that are used in the present thesis. These are in particular symplectic or Krein space methods and covariant families of operators, which are extensively used in the last chapter when dealing with Sturm-Liouville oscillation theory. After that we present symmetry operators and Clifford algebras, which are used in Chapter 4 for giving topological insulators a mathematical treatment.

1.1 Basic notations

Throughout this work, \( \mathcal{H} \) will always denote a Hilbert space with a given scalar product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \), which is linear in the second argument, and with the induced norm \( \| \cdot \|_{\mathcal{H}} \), where the subscript \( \mathcal{H} \) is usually omitted. When not especially mentioned, all considered Hilbert spaces are always complex but not necessarily separable.

The normed space of bounded linear operators from a Hilbert space \( \mathcal{H} \) to a Hilbert space \( \mathcal{K} \) is denoted by \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \), and we simply write \( \mathcal{B}(\mathcal{H}) \) or \( \mathcal{B}(\mathcal{H}, \mathcal{H}) \). The space of Fredholm operators in \( \mathcal{H} \), i.e. the bounded linear operators with finite dimensional kernel and cokernel, is denoted by \( \mathcal{F}(\mathcal{H}) \), cf. Chapter 3 for more details. For the compact operators we use the notation \( \mathcal{K}(\mathcal{H}) \), and for the unitaries we use \( \mathcal{U}(\mathcal{H}) \). Moreover, we write \( 1_{\mathcal{H}} \) for the identity operator in \( \mathcal{H} \). If the corresponding Hilbert space is evident, we simply write \( 1 \) and even suppress the mentioning of the identity if there is no danger of confusion. However, it must be pointed out that the letter \( I \) is always reserved for a particular unitary and does never denote the unit in this work. Furthermore, we also use \( 1 \) to denote the unit in a group or the multiplicative unit in an algebra. An important part of this work is concerned with Hilbert spaces that have an
additional structure. Here, we use the following terminology.

**Definition 1.1.** A complex Hilbert space \( \mathcal{H} \) together with an anti-linear map \( C : \mathcal{H} \to \mathcal{H} \) with \( C^2 = 1 \) and \( \langle Cx, Cy \rangle = \langle x, y \rangle \) for all \( x, y \in \mathcal{H} \) is called a *Hilbert space with real structure*. The map \( C \) is then called a *complex conjugation* or a *real structure*.

Given a Hilbert space \( \mathcal{H} \) with real structure \( C \), for an operator \( T \in \mathcal{B}(\mathcal{H}) \), we usually abbreviate \( \overline{T} := CTC \) if there is no ambiguity about the real structure \( C \), and call \( \overline{T} \) the *complex conjugation of* \( T \). Moreover, \( T \) is said to be *real* if \( \overline{T} = T \) holds. Note that this notion heavily depends on the chosen real structure on the Hilbert space. The symbol \( T^* \) always denotes the usual adjoint of an operator \( T \in \mathcal{B}(\mathcal{H}) \), and \( T^t := \overline{T^*} \) denotes the *transpose of* \( T \). The last notation is also sometimes used for operators in real Hilbert spaces where it simply stands for the usual adjoint.

For the finite dimensional Hilbert space \( \mathbb{C}^n \) and the linear operators on it, which we can identify with the matrices \( \mathbb{C}^{n \times n} \), these notations all coincide with the common ones if we consider the usual complex conjugation on \( \mathbb{C}^n \). A standard infinite-dimensional example would be \( \ell^2(\mathbb{N}) \) where the real structure \( C \) is analogously given by componentwise complex conjugation.

There is yet another notion and structure we will sometimes use in connection with Hilbert spaces.

**Definition 1.2.** For a complex Hilbert space \( \mathcal{H} \) an anti-linear map \( Q : \mathcal{H} \to \mathcal{H} \) with \( Q^2 = -1 \) and \( \langle Qx, Qy \rangle = \langle x, y \rangle \) for all \( x, y \in \mathcal{H} \) is called a *quaternionic structure*.

Indeed, a quaternionic structure \( Q \) on a Hilbert space gives a representation of the algebra of quaternions by using the imaginary unit as a multiplication operator \( i \) and setting \( j := Q \) and \( k := ij \).

### 1.2 Krein spaces and symplectic methods

We give a short overview about infinite dimensional symplectic methods that are often studied in the analysis of Krein spaces or general inner product spaces. Good references for these topics are [Bog74], [EM04] and [Schu07].
1.2 Krein spaces and symplectic methods

1.2.1 Definition of a Krein space

In this section, we mainly examine Krein spaces, which are Banach spaces over \( \mathbb{K} \), meaning over the real or complex numbers, endowed with a suitable inner product that is potentially indefinite. This inner product is usually denoted by \([\cdot, \cdot]\) to distinguish it from an always positive definite scalar product. Explicitly, we make use of the following definition:

**Definition 1.3.** Let \( \mathcal{K} \) be a \( \mathbb{K} \)-vector space and \([\cdot, \cdot] : \mathcal{K} \times \mathcal{K} \to \mathbb{K} \) a map that is linear in the second argument and Hermitian, i.e. \([y, x] = [x, y]\) for all \( x, y \in \mathcal{K} \). Then \((\mathcal{K}, [\cdot, \cdot])\) is called an inner product space.

Note that by this definition an inner product space could also be a usual (pre-)Hilbert space. For an inner product space \( \mathcal{K} \) one calls an element \( x \in \mathcal{K} \) neutral if \([x, x] = 0\) and otherwise positive or negative depending on the sign of \([x, x]\). The space decomposes into a disjoint union of these three subsets but it is customary to include zero into all three sets and to define:

\[
P_{\mathcal{K}}^{++} := \{ x \in \mathcal{K} \mid [x, x] > 0 \} \cup \{ 0 \}, \quad P_{\mathcal{K}}^{--} := \{ x \in \mathcal{K} \mid [x, x] < 0 \} \cup \{ 0 \}.
\]

For an arbitrary subspace \( U \subset \mathcal{K} \), we define the orthogonal companion as

\[
U^{\perp} := \{ x \in \mathcal{K} \mid [y, x] = 0 \text{ for all } y \in U \}.
\]

This is obviously an analogue of the orthogonal complement in a (pre-)Hilbert space but one has to be very cautious when comparing these two concepts. Indeed, there might be non-trivial subspaces \( V \subset \mathcal{K} \) with \( V \subset V^{\perp} \), and we call them isotropic subspaces. Therefore, for each subspace \( U \subset \mathcal{K} \) the space \( U \cap U^{\perp} \) is called the isotropic part of \( U \). In the case where this intersection is non-trivial, we call \( U \) degenerate. For two subspaces \( U, V \subset \mathcal{K} \), we write \( U \oplus V \) for the usual inner direct sum of vector spaces, which implies that the intersection of the two subspaces is trivial. We write \( U \oplus^{\perp} V \) for the direct sum if, in addition, \([u, v] = 0\) for all \( u \in U \) and \( v \in V \). We simply call this the orthogonal sum of the two subspaces \( U \) and \( V \).

**Definition 1.4.** An inner product space \((\mathcal{K}, [\cdot, \cdot])\) is called a Krein space if one
of the two equivalent properties is fulfilled:

(a) There is a decomposition

$$\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_- \quad \text{with} \quad \mathcal{H}_+ \subset P_{\mathcal{K}}^{++}, \quad \mathcal{H}_- \subset P_{\mathcal{K}}^{--}$$

where \((\mathcal{H}_+,[\cdot,\cdot])\) and \((\mathcal{H}_-,-[\cdot,\cdot])\) are Hilbert spaces.

(b) There is a scalar product \(\langle \cdot, \cdot \rangle\) such that \((\mathcal{K},\langle \cdot, \cdot \rangle)\) is a Hilbert space. Moreover, there is a linear operator \(J : \mathcal{K} \rightarrow \mathcal{K}\) satisfying \(J^2 = 1_{\mathcal{K}}\) such that it is self-adjoint with respect to the scalar product and fulfils

$$[y, x] = \langle y, Jx \rangle \quad \text{for all} \quad x, y \in \mathcal{K}.$$ 

This operator \(J\) is called the fundamental symmetry.

It is not hard to prove that the two properties are equivalent, see, for example, [Lan82]. Because of the second property, one often calls a Hilbert space \(\mathcal{K}\) together with a fixed self-adjoint unitary \(J\) a Krein space without an explicit reference to the inner product \([\cdot, \cdot]\). In this work, we will also mainly follow this custom.

Note that by this definition a Krein space is also a topological space by using the topology of the underlying Hilbert space \((\mathcal{H}_+,[\cdot,\cdot]) \oplus (\mathcal{H}_-,-[\cdot,\cdot])\) or rather by using the topology of the Hilbert space in the second property of Definition 1.4. While this Hilbert space itself depends, of course, on the choice of the decomposition and the choice of the scalar product on \(\mathcal{K}\), respectively, the underlying normed space and, therefore, the topology is uniquely determined, see also [Lan82] for more details. This rises to the notion of isomorphisms:

**Definition 1.5.** Two Krein spaces \((\mathcal{K}_1,[\cdot,\cdot]_1)\) and \((\mathcal{K}_2,[\cdot,\cdot]_2)\) are called isometrically isomorphic if there is an invertible linear map \(T : \mathcal{K}_1 \rightarrow \mathcal{K}_2\) with \([Ty, Tx]_2 = [y, x]_1\) for all \(x, y \in \mathcal{K}_1\).

In the category of Krein spaces such invertible operators \(T : \mathcal{K}_1 \rightarrow \mathcal{K}_2\) are indeed the isomorphisms. Between the same Krein space an isomorphism is often called a unitary in \(\mathcal{K}\). Because of the likelihood of confusion with unitaries
in a Hilbert space we will rather speak of \( J \)-unitaries when the fundamental symmetry \( J \) is fixed.

By definition a Krein space consists of two Hilbert space \( \mathcal{H}_+ \) and \( \mathcal{H}_- \), and since a Hilbert space is uniquely determined by its dimension, two Krein spaces are isometrically isomorphic if and only if the dimensions of these Hilbert spaces coincide, respectively.

**Example 1.6.** For each complex Hilbert space \( \mathcal{H} \), the doubled Hilbert space \( \mathcal{K} = \mathcal{H} \oplus \mathcal{H} \) with scalar product \( \langle \cdot, \cdot \rangle_{\mathcal{H} \oplus \mathcal{H}} \) becomes a Krein space when equipped with either a self-adjoint unitary \( J \) or a skew-adjoint unitary \( I \), which can be chosen to be

\[
J = \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ 0 & -1_{\mathcal{H}} \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1_{\mathcal{H}} \\ 1_{\mathcal{H}} & 0 \end{pmatrix}.
\]

Here, the inner product is then defined by setting \([y, x]_J := \langle y, Jx \rangle_{\mathcal{H} \oplus \mathcal{H}}\) and \([y, x]_I := i \langle y, Ix \rangle_{\mathcal{H} \oplus \mathcal{H}}\), respectively. This means that the fundamental symmetry is here either \( J = J \) or \( J = iI \).

For Krein spaces with the same dimension for \( \mathcal{H}_+ \) and \( \mathcal{H}_- \), these are the common choices to describe the symplectic structure in a Hilbert space setting. Of course, both Krein spaces are isometrically isomorphic, and the fundamental symmetries are equivalent in the following sense:

\[
iI = C^*JC. \quad (1.1)
\]

Here, \( C \) is a unitary operator in the Hilbert space \( \mathcal{H} \oplus \mathcal{H} \), called the *Cayley transform* and given by

\[
C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{\mathcal{H}} & -i1_{\mathcal{H}} \\ 1_{\mathcal{H}} & i1_{\mathcal{H}} \end{pmatrix}.
\]

In later chapters, we will always work in the two Krein spaces from Example 1.6 above and use the Cayley transform to switch between these different representations. Accordingly, in the following, we will define a lot of objects for \( J = J \) and \( J = iI \) but often drop the imaginary unit \( i \) in the second case as we will seldom explicitly use the inner product.
1.2.2 Lagrangian subspaces

**Definition 1.7.** Let \((\mathcal{K}, \langle \cdot, \cdot \rangle)\) be a Krein space. A subspace \(L \subset \mathcal{K}\) satisfying \(L = L^{[\perp]}\) is called a **Lagrangian subspace**. The set of all Lagrangian subspaces is denoted by \(\mathbb{L}(\mathcal{K})\) and called the **Lagrangian Grassmannian**.

Lagrangian subspaces are found with different names in the literature. They are sometimes called *hypermaximal neutral subspaces* since, on the one hand, they have to be neutral spaces, meaning \([x, x] = 0\) for all \(x \in L\). On the other hand, they are maximal in the sense there is no neutral subspace \(V\) with \(V \supseteq L\), and, moreover, there is no proper extension of \(L\) in \(L \cup P_{\mathcal{K}}^{++}\) or \(L \cup P_{\mathcal{K}}^{--}\). This last property is often called *hypermaximal*, cf. [Bog74, Theorem 7.4].

Another term, one often uses, is *maximal isotropic subspaces* since Lagrangian subspaces are certainly isotropic, meaning \(L \subset L^{[\perp]}\), and there can be no proper isotropic extension. Consequently, Lagrangian subspaces also have to be closed.

**Lemma 1.8.** Let \(\mathcal{K}\) be a Krein space given by a Hilbert space and fundamental symmetry \(J\). Then \(L \subset \mathcal{K}\) is a Lagrangian subspace if and only if \(JL\) is a Lagrangian subspace.

**Proof.** Let \(L\) be a Lagrangian subspace. Then \(x \in L\) if and only if \(\langle y, Jx \rangle = 0\) for all \(y \in L\). But then we also have \(Jz \in L\) if and only if \(\langle Jw, J^2z \rangle = 0\) for each \(Jw \in L\). Since \(J\) is a self-adjoint unitary, this means that \(z \in JL\) if and only if \(\langle w, Jz \rangle = 0\) for each \(w \in JL\). Therefore \(JL\) is a Lagrangian subspace. The other implication immediately follows from \(J^2 = 1\). \(\square\)

One often writes \(\mathbb{L}(\mathcal{K}, J)\) for the Lagrangian Grassmannian when the Krein space is given by a Hilbert space \((\mathcal{K}, \langle \cdot, \cdot \rangle)\) and a fundamental symmetry \(J\). In particular, if we consider the Krein spaces in Example 1.6, then we write \(\mathbb{L}(\mathcal{K}, J)\) or \(\mathbb{L}(\mathcal{K}, I)\) for the Lagrangian Grassmannian, respectively. In the Krein spaces \((\mathcal{K}, J)\) and \((\mathcal{K}, I)\), all Lagrangian subspaces are half-dimensional, meaning the dimension is exactly the dimension of the Hilbert space \(\mathcal{H}\). On this account, one can use a technical simplification for describing such subspaces:

**Definition 1.9.** For a Hilbert space \(\mathcal{H}\) consider the Krein space \((\mathcal{K}, J)\) from Example 1.6. A linear operator \(\Psi : \mathcal{H} \rightarrow \mathcal{K}\) is called a **\(J\)-Lagrangian frame** if
\[ \Psi^* \Psi = 1_{\mathcal{H}} \text{ and } \text{Ran}(\Psi) \text{ is a Lagrangian subspace in } (\mathcal{K}, J). \] Analogously, one defines an \textit{I-Lagrangian frame} for the Krein space \((\mathcal{K}, I)\).

Obviously, a \(J\)-Lagrangian frame \(\Psi\) fulfils the equation \(\Psi^* J \Psi = 0\) since this condition is equivalent to \(\text{Ran}(\Psi)\) being isotropic. However, note that this is not a sufficient claim for being a \(J\)-Lagragian frame because the maximality of a Lagrangian subspace is missing here. A better characterisation will be presented in the next lemmata. In the following, we always use the notation \(\Psi = (a b)\) for given operators \(a, b \in \mathcal{B}(\mathcal{H})\) since it will be helpful when composing operators later.

**Lemma 1.10.** For every unitary \(u \in \mathcal{B}(\mathfrak{H})\), the operator \(\Psi = \frac{1}{\sqrt{2}} (u 1) : \mathfrak{H} \to \mathcal{K}\) is a \(J\)-Lagrangian frame.

**Proof.** Given a unitary \(u \in \mathcal{B}(\mathfrak{H})\), we can define \(L := \{u(x) \oplus x \mid x \in \mathfrak{H}\} \subset \mathcal{K}\), which is obviously isotropic. For an element \(y = y_1 \oplus y_2 \in \mathcal{K}\) with \(\langle y, Jz \rangle = 0\) for all \(z \in L\), we have

\[
0 = \langle y_1, u(x) \rangle_{\mathfrak{H}} - \langle y_2, x \rangle_{\mathfrak{H}} = \langle y_1 - u(y_2), u(x) \rangle_{\mathfrak{H}}
\]

for all \(x \in \mathfrak{H}\). Therefore, \(y_1 = u(y_2)\), which implies \(y \in L\). Hence, \(L\) is a Lagrangian subspace. \(\square\)

**Lemma 1.11.** For a \(J\)-Lagrangian frame \(\Psi = (a b)\) with operators \(a, b \in \mathcal{B}(\mathfrak{H})\), the scaled operators \(\sqrt{2}a, \sqrt{2}b\) are unitaries.

**Proof.** By definition of a \(J\)-Lagrangian frame, the operators \(a, b \in \mathcal{B}(\mathfrak{H})\) fulfil \(a^*a + b^*b = 1\) and \(a^*a - b^*b = 0\). Combining these equations, we conclude \(2a^*a = 2b^*b = 1\), and hence both operators \(\sqrt{2}a, \sqrt{2}b\) are isometries. Note that for isometries the range is always closed. Now, we will show that these operators are indeed surjective. Suppose there is a vector \(y \in \text{Ran}(a)\) that, hence, fulfils

\[
\langle y \oplus 0, Jw \rangle = 0 \text{ for all } w \in \text{Ran}(\Psi).
\]

Since \(\text{Ran}(\Psi)\) is maximal isotropic, \(y \oplus 0\) lies in \(\text{Ran}(\Psi)\). Consequently, there is an \(x \in \mathfrak{H}\) with \(\Psi(x) = a(x) \oplus b(x) = y \oplus 0\). Therefore, the vector \(y\) lies also
in the range of \( a \) and, thus, has to be zero. By the same argument for \( b \), we conclude \( \text{Ran}(a) = \text{Ran}(b) = \mathcal{H} \).

**Lemma 1.12.** For an \( I \)-Lagrangian frame \( \Phi = \begin{pmatrix} a \\ b \end{pmatrix} \) with operators \( a, b \in \mathcal{B}(\mathcal{H}) \), the operators \( a + ib \) and \( a - ib \) are unitaries. Moreover, \( aa^* + bb^* = 1 \) and \( ba^* = ab^* \).

**Proof.** The first part follows directly from Lemma 1.11 and the fact \( iI = C^*JC \). Then we get the second part from \( (a \pm ib)(a \pm ib)^* = 1 \).

At this point, we have not proven yet that there is always a \( J \)-Lagrangian frame for a given Lagrangian subspace in the Krein space \((\mathcal{K}, J)\). However, we will do this in the next lemma. Indeed, there might be many different choices for a \( J \)-Lagrangian frame, but we are merely interested in frames of the form already seen in Lemma 1.10. Now we generally construct such a \( J \)-Lagrangian frame.

**Lemma 1.13.** For a Lagrangian subspace in the Krein space \((\mathcal{K}, J)\), there is a unique \( J \)-Lagrangian frame of the form \( \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 1 \end{pmatrix} \) where \( v \in \mathcal{B}(\mathcal{H}) \) is unitary.

**Proof.** The ideas here are motivated by the proof in [EM04, Theorem 4.5]. We consider \( \mathcal{K} = \mathcal{H} \oplus \mathcal{H} \) as a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) and associated norm \( \|\cdot\| \). Let \( L \subseteq \mathcal{L}(\mathcal{K}, J) \) be a Lagrangian subspace. For each \( x = x_1 \oplus x_2 \in L \), we have \( \|x_1\|_J^2 = \|x_2\|_J^2 = \frac{1}{2} \|x\|^2 \). Now, we get two linear isometric maps by defining

\[
    v_i : L \to \mathcal{H}, \quad x_1 \oplus x_2 \mapsto \sqrt{2}x_i
\]

for \( i = 1, 2 \). By the same argument as in the proof of Lemma 1.11 above, we see that \( \text{Ran}(v_1) = \text{Ran}(v_2) = \mathcal{H} \).

Now we define the unitary operator \( v := v_1v_2^{-1} \in \mathcal{B}(\mathcal{H}) \). Then it only remains to show that \( L = \text{Ran}(\Psi) \) with \( \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 1 \end{pmatrix} \). For \( x = x_1 \oplus x_2 \in L \), we find \( v(x_2) = x_1 \) and, therefore, \( L \subseteq \text{Ran}(\Psi) \). For the other inclusion, we choose an arbitrary element \( v(x) \oplus x \in \text{Ran}(\Psi) \). Then, for an arbitrary vector \( z \in L \), we can write \( z = v(y) \oplus y \) by the first inclusion, and we get

\[
    \langle v(x) \oplus x, Jz \rangle = \langle v(x), v(y) \rangle - \langle x, y \rangle = 0
\]
since \( v \) is unitary. Hence, \( v(x) \oplus x \in L \) because \( L \) is maximal isotropic. The uniqueness is obvious since different unitaries have different graphs.

**Corollary 1.14.** For a Lagrangian subspace in the Krein space \((\mathcal{K}, I)\), there is a unique \( I \)-Lagrangian frame of the form \( \Psi = \frac{1}{2}(iv + i1) \) where \( v \in \mathcal{B}(\mathcal{J}) \) is unitary.

**Proof.** That immediately follows by Lemma 1.13 and using the Cayley transform \( C \).

The range of a \( J \)-Lagrangian frame \( \Psi \) does not change when considering the \( J \)-Lagrangian frame \( \Psi u \) where \( u : \mathcal{J} \to \mathcal{J} \) is a unitary operator. Indeed, the unitary group \( \text{U}(\mathcal{J}) \) defines an equivalence relation by this right action on the set of all \( J \)-Lagrangian frames, and we obtain a bijection between these equivalence classes and \( \mathbb{L}(\mathcal{K}, J) \). This means that the connection between Lagrangian subspaces and unitaries given in Lemma 1.13 is bijective. Of course, this immediately follows from the combination of Lemma 1.10 and Lemma 1.13. See also [EM04, Theorem 4.5] for an alternative representation.

**Proposition 1.15.** We call two \( J \)-Lagrangian frames \( \Psi_1 \) and \( \Psi_2 \) equivalent if there is a unitary \( u \in \mathcal{B}(\mathcal{J}) \) with \( \Psi_2 = \Psi_1 u \), analogously for \( I \)-Lagrangian frames. Then there is a bijection between the set of equivalence classes and Lagrangian subspaces, and we can identify:

\[
\mathbb{L}(\mathcal{K}, J) = \{[\Psi] \mid \Psi \text{ \( J \)-Lagrangian frame}\},
\]

\[
\mathbb{L}(\mathcal{K}, I) = \{[\Phi] \mid \Phi \text{ \( I \)-Lagrangian frame}\}.
\]

**Proof.** Firstly, we consider the set of \( J \)-Lagrangian frames, temporarily denoted by \( \mathcal{S} \) here. We define \( \Psi_1 \sim \Psi_2 \) if there is a unitary \( u \) with \( \Psi_2 = \Psi_1 u \) and, obviously, this is an equivalence relation. The map

\[
\mathcal{S}/\sim \to \mathbb{L}(\mathcal{K}, J), \quad [\Psi] \mapsto \text{Ran}(\Psi)
\]

is well-defined and bijective by Lemma 1.13 in combination with Lemma 1.11.
For the $I$-Lagrangian frames, we can use the Cayley transform and the fact $\mathbb{L}(\mathcal{K}, I) = C^*\mathbb{L}(\mathcal{K}, J)$. Since $\Phi$ is an $I$-Lagrangian if and only if $C\Phi$ is a $J$-Lagrangian frame, we immediately have the second bijection. 

Accordingly, we have already shown the most important connections between Lagrangian frames, Lagrangian subspaces and the unitary group. From now on, we will focus on a particular form of this bijection called the stereographic projection.

**Proposition 1.16.** The Lagrangian Grassmannians $\mathbb{L}(\mathcal{K}, J)$ and $\mathbb{L}(\mathcal{K}, I)$ are bijectively mapped onto the set of unitary operators on $\mathcal{H}$ by the two stereographic projections $\Pi_J : \mathbb{L}(\mathcal{K}, J) \to U(\mathcal{H})$ and $\Pi_I : \mathbb{L}(\mathcal{K}, I) \to U(\mathcal{H})$ defined by

$$
\Pi_J([\Psi]) = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^* \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1}, \quad \Pi_I([\Phi]) = \left[ \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right]^* \left[ \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \right]^{-1},
$$

where $\Psi$ and $\Phi$ are $J$-Lagrangian and $I$-Lagrangian frames, respectively.

**Proof.** Note that the maps are well-defined since the appearing inverses always exist and, indeed, are unitaries by Lemma 1.11 and Lemma 1.12. The one-to-one correspondence has been shown in Lemma 1.13 and Proposition 1.15 above.

**Remark 1.17.** Both stereographic projections are interconnected in the sense that

$$
\Pi_I([\Phi]) = \Pi_I([C^*C\Phi]) = \Pi_J([C\Phi]),
$$

for each $I$-Lagrangian frame $\Phi$.

### 1.2.3 Unitary groups in Krein spaces

We have already mentioned the isomorphisms in the category of Krein spaces and spoken of $J$-unitaries when there was an isomorphism between the same Krein spaces equipped with a fundamental symmetry $J$. Now, we define the set of $J$-unitaries for the fundamental symmetries $J = J$ and $J = iI$, where we again drop the imaginary unit in front of $I$ in all notations.
1.2 Krein spaces and symplectic methods

**Definition 1.18.** The sets of $J$-unitaries and $I$-unitaries are denoted by

\[ U(\mathcal{K}, J) = \{ T \in B(\mathcal{K}) \mid T \text{ invertible}, T^* J T = J \}, \]
\[ U(\mathcal{K}, I) = \{ T \in B(\mathcal{K}) \mid T \text{ invertible}, T^* I T = I \}. \]

These are groups under multiplication with neutral element $1_{\mathcal{K}}$.

In particular, note that when $T \in U(\mathcal{K}, J)$, then also the inverse $T^{-1}$ lies in $U(\mathcal{K}, J)$. Taking the inverse of the relation $T^* J T = J$, also shows that $T^*$ is in $U(\mathcal{K}, J)$ and analogously for $U(\mathcal{K}, I)$. It follows from the relation $i I = C^* J C$, see (1.1), that

\[ C^* U(\mathcal{K}, J) C = U(\mathcal{K}, I). \]

We will examine properties of these groups later. However, for us, the most important fact about these unitary groups is that they transform Lagrangian subspaces again into Lagrangian subspaces. This is a property that we have already shown for the fundamental symmetry $J$, cf. Lemma 1.8.

**Lemma 1.19.** If $L \in \mathbb{L}(\mathcal{K}, J)$ and $T \in U(\mathcal{K}, J)$, then $T(L) \in \mathbb{L}(\mathcal{K}, J)$. The same holds for $I$-Lagrangian subspaces.

**Proof.** Since $L$ is $J$-Lagrangian, we have for $x \in \mathcal{K}$ the equivalence:

\[ \langle y, Jx \rangle = 0 \text{ for all } y \in L \iff x \in L. \quad (1.2) \]

Now set $U := \text{Ran}(T|_L)$ and $z := Ty \in U$. Since $T$ is invertible, we can substitute $y = T^{-1}z$ in equation (1.2) above. Then by using $(T^{-1})^* J = J T$, we get

\[ \langle z, JTx \rangle = 0 \text{ for all } z \in U \iff Tx \in U, \]

and this shows that $U$ is $J$-Lagrangian as well. Of course, the same proof holds for $I$-Lagrangian subspaces. \qed

**Proposition 1.20.** The groups $U(\mathcal{K}, J)$ and $U(\mathcal{K}, I)$ act left on the Lagrangian Grassmannian $\mathbb{L}(\mathcal{K}, J)$ and $\mathbb{L}(\mathcal{K}, I)$, respectively, by

\[ U(\mathcal{K}, J) \times \mathbb{L}(\mathcal{K}, J) \to \mathbb{L}(\mathcal{K}, J), \quad (T, [\Psi]) \mapsto [T\Psi | T\Psi|^{-1}], \]
Analogously, one could also directly work with a Lagrangian subspace $L \subset K$ and write $(T, L) \mapsto T(L)$.

Above, we used the definition of the absolute value for operators on $\mathcal{H}$ given by $|T\Psi|^2 := (T\Psi)^*(T\Psi)$. Then, $|T\Psi|^{-1}$ is defined by the continuous functional calculus.

**Proof.** We prove just the second case and show how $U(K, I)$ acts on $I$-Lagrangian subspaces. Firstly, one should note that the map above is well-defined since, for each $I$-Lagrangian frame $\Phi$, the operator $(T\Phi)|T\Phi|^{-1}$ is also an $I$-Lagrangian frame by Lemma 1.19 above. Moreover, for a unitary $u \in B(\mathcal{H})$, we have

$$T(\Phi u)|T(\Phi u)|^{-1} = T\Phi uu^*|T\Phi|^{-1} u = T\Phi|T\Phi|^{-1} u$$

by functional calculus, cf. also Lemma 1.51 below. Therefore, the equivalence class is well-defined. Now, we consider $L \in \mathbb{L}(K, I)$ and set $T \cdot L := T(L)$. Then $1 \cdot L = L$ and $T_2 \cdot (T_1 \cdot L) = T_2(T_1(L)) = (T_2T_1) \cdot L$. The representation with $I$-Lagrangian frames is simply Proposition 1.15 and so both maps are left actions.

There is an interesting fact hidden in this claim above. The maps for the frames represent left actions and, hence, we have the following multiplication rule for the absolute value of the operators.

**Corollary 1.21.** If $\Phi$ is a $J$-Lagrangian frame and $T_1, T_2 \in U(K, J)$, then

$$\left| T_2T_1\Phi|T_1\Phi|^{-1} \right| |T_1\Phi| = |T_2T_1\Phi|.$$

The same holds for $I$-Lagrangian frames with operators from $U(K, I)$.

Another useful left action follows from those formulated in Proposition 1.20 above for the case of $I$-unitaries:

**Proposition 1.22.** The left action $U(K, I) \times \mathbb{L}(K, I) \to \mathbb{L}(K, I)$ given by $T \cdot [\Phi] = [T\Phi|T\Phi|^{-1}]$ is connected to another left action via the stereographic
1.3 Covariant families of operators and the density of states

projection

$$\Pi_I(T \cdot [\Phi]) = CTC^* \cdot \Pi_I([\Phi]),$$

where $\bullet: \mathbb{U}(\mathcal{K}, J) \times \mathbb{U}(\mathcal{H}) \to \mathbb{U}(\mathcal{H})$ is the so-called Möbius transform given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot u = (au + b)(cu + d)^{-1}.$$ 

Here, we represent the $J$-unitary $CTC^*$ as an operator block matrix with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$.

Proof. By using Remark 1.17 and the definitions of the stereographic projections from Proposition 1.16, we can calculate:

$$\Pi_I(T \cdot [\Phi]) = \Pi_J(C\Phi | T\Phi|^{-1}) = \Pi_J(CTC^*C\Phi | T\Phi|^{-1})$$

$$= [(a,b)C\Phi] [(c,d)C\Phi]^{-1}$$

$$= \left[ a \left( \begin{pmatrix} 1 \\ i \end{pmatrix} \right)^* \Phi + b \left( \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)^* \Phi \right] \left[ c \left( \begin{pmatrix} 1 \\ i \end{pmatrix} \right)^* \Phi + d \left( \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)^* \Phi \right]^{-1}. $$

By dividing by $\left( \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)^* \Phi$, one obtains the result. $\square$

1.3 Covariant families of operators and the density of states

The physical systems in condensed matter physics are in reality never ideal, and there are always small deviations to the theoretic model that is the Hamiltonian on $\ell^2(\mathbb{Z}^d)$, where $d$ is the spatial dimension. There may be, for example, impurities and perturbations of the perfect lattice. In order to deal with this disorder, one can see it as a random perturbation of the Hamiltonian and analyse a random family of operators. Here, we are merely interested in so-called covariant families of operators, fulfilling some translation invariance that results in the macroscopic homogeneity of the material. We use [PF92] and [Schu15a] in this section.
1.3.1 Covariant families of operators

Here, we start with a measure space, which will be a probability space in all applications later, and a discrete dynamical system.

**Definition 1.23.** Let $\Omega$ be a topological space, $\mu$ a Borel measure on the Borel $\sigma$-algebra and $T : \Omega \to \Omega$ a homeomorphism. This canonically defines a discrete dynamical system $\Phi : \mathbb{Z} \times \Omega \to \Omega$ by $\Phi(k, \omega) := T^k(\omega)$. Therefore, one also calls $(\Omega, T)$ a *discrete dynamical system*. The measure $\mu$ is called $T$-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for all Borel sets $A$. Moreover, the measure is called *ergodic* if it is $T$-invariant and for each set $A$ with $T^{-1}(A) = A$, one has $\mu(A) = 0$ or $\mu(\Omega \setminus A) = 0$.

For the Borel measure $\mu$, we will later often choose a probability measure $P$ on the Borel $\sigma$-algebra of $\Omega$ and want it to be ergodic with respect to a given dynamical system. However, since we deal with physical systems in $d$ dimensions, for $d \in \mathbb{N}$, we have dynamical systems in different directions or rather a group action $\mathbb{Z}^d \times \Omega \to \Omega$. This means that for each $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ there is a homeomorphism $T^n : \Omega \to \Omega$ and

$$T^n \circ T^m = T^{n+m} \quad \text{for all } n, m \in \mathbb{Z}^d.$$  \hfill (1.3)

Note that the superscript notation is convenient since it is compatible with taking powers.

Consequently, we adapt the notions from the one-dimensional discrete dynamical system of Definition 1.23 to finitely many dimensions.

**Definition 1.24.** Let $\Omega$ be a topological space and $\mathcal{T}$ a countable family of homeomorphisms $T : \Omega \to \Omega$. A Borel measure $\mu$ is called $\mathcal{T}$-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for all Borel sets $A$ and for all $T \in \mathcal{T}$. Moreover, the measure is called *ergodic* if it is $\mathcal{T}$-invariant and for each Borel set $A$ with $T^{-1}(A) = A$ for all $T \in \mathcal{T}$, one has $\mu(A) = 0$ or $\mu(\Omega \setminus A) = 0$.

For an element $n = (n_1, \ldots, n_d)$ in $\mathbb{Z}^d$, we always write $|n|_{\infty} = \max_i |n_i|$. Furthermore, $(e_n)_{n \in \mathbb{Z}^d} \subset \ell^2(\mathbb{Z}^d)$ denotes the canonical orthonormal basis. This
means that the map \( e_n : \mathbb{Z}^d \to \mathbb{C} \) fulfils:
\[
e_n(m) = \delta_{n_1,m_1} \cdots \delta_{n_d,m_d}.
\]

We also get a dynamical system \( \mathbb{Z}^d \times \Omega \to \Omega \) if there are given commuting homeomorphisms. Assume \( T_j : \Omega \to \Omega \) are homeomorphisms for all \( j = 1, \ldots, d \) with \( T_j \circ T_i = T_i \circ T_j \). Then the homeomorphisms \( T^n := T_1^{n_1} \cdots T_d^{n_d} \), for all \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \), fulfil (1.3). Accordingly, we will always choose \( d \) commuting homeomorphisms to fix the dynamical system.

**Definition 1.25.** Let \( \Omega \) be a topological space and \( T_1, \ldots, T_d \) commuting homeomorphisms. A family of operators \((A_{\omega})_{\omega \in \Omega}\), where \( A_\omega \in \mathcal{B}(\ell^2(\mathbb{Z}^d))\), is called **covariant** if \( \omega \mapsto A_\omega \) is strongly continuous and
\[
S_j A_\omega S_j^* = A_{T_j(\omega)}, \quad \omega \in \Omega, \tag{1.4}
\]
for all \( j = 1, \ldots, d \). Here \( S_j : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \) denotes the right shift on the \( j \)-th component.

Note that the notion of **covariance** always depends on the chosen dynamics \( \mathbb{Z}^d \times \Omega \to \Omega \) given by \( (n, \omega) \mapsto T^n(\omega) \) and that (1.4) is equivalent to
\[
S^n A_\omega (S^n)^* = A_{T^n(\omega)}, \quad \omega \in \Omega, \tag{1.5}
\]
for all \( n \in \mathbb{Z}^d \) and \( S^n := S_1^{n_1} \cdots S_d^{n_d} \).

**Example 1.26.** For a compact space \( K \) with probability measure \( p \) on the Borel \( \sigma \)-algebra, we define the compact space \( \Omega := K^{\mathbb{Z}^d} \) and denote by \( T_1, \ldots, T_d \) the right-shift actions on \( \Omega \) in analogy to the right shifts on \( \ell^2(\mathbb{Z}^d) \). Then, the probability measure \( P = p^{\mathbb{Z}^d} \) is ergodic.

**Example 1.27.** The **Anderson model** in dimension \( d \) for a coupling parameter \( \kappa \geq 0 \) is given by the compact space \( \Omega := K^{\mathbb{Z}^d} \) with \( K := [-\frac{1}{2}, \frac{1}{2}] \) and by the right shift actions \( T_1, \ldots, T_d \) as in Example 1.26 above. The associated random operator is then:
\[
H_\omega = \Delta + \kappa V_\omega, \quad \omega \in \Omega.
\]
Here, $\Delta := \sum_j (S_j^* + S_j)$ is the chosen discrete Laplacian with the right shifts $S_j \in B(\ell^2(\mathbb{Z}^d))$. Moreover, $V_\omega$ is a multiplication operator given by

$$(V_\omega x)_n = \tilde{v}_n(\omega) x_n, \quad x \in \ell^2(\mathbb{Z}^d), \quad n \in \mathbb{Z}^d,$$

where $\tilde{v}_n : \Omega \to K$ are independent and identically distributed random variables with probability distribution $p$. Then, $P = p^{\mathbb{Z}^d}$ is ergodic, and the the family $(H_\omega)_{\omega \in \Omega}$ is covariant if all $\tilde{v}_n$ are continuous functions with the property $\tilde{v}_n(T^m \omega) = \tilde{v}_{n-m}(\omega)$ for all $m \in \mathbb{Z}^d$.

**Definition 1.28.** Let $\Omega$ be a topological space, $T_1, \ldots, T_d$ commuting homeomorphisms. A covariant family $(A_\omega)_{\omega \in \Omega}$ is called of finite range if there is an $R \geq 0$ such that for all $\omega \in \Omega$ one has

$$\langle e_n, A_\omega e_m \rangle = 0, \quad \text{for all } |n - m|_\infty > R. \quad (1.6)$$

The Anderson model from Example 1.27 above is the typical example for a covariant family of finite range. Indeed, the range is here given by $R = 1$. Moreover, the Anderson model is an example of a covariant family that consists of self-adjoint elements. Our main interest lies in these families.

For convenience, we will now formulate the general setting in the following assertion:

**Hypothesis 1.29.** Let $\Omega$ be a compact topological space, $T_1, \ldots, T_d$ commuting homeomorphisms, $(A_\omega)_{\omega \in \Omega}$ a covariant family of self-adjoint operators and $P$ an ergodic probability measure on the Borel $\sigma$-algebra of $\Omega$.

An important property of a covariant family of self-adjoint operators on a compact space $\Omega$ is that there is a well-defined spectrum for the whole family in the following sense.

**Theorem 1.30.** Assume Hypothesis 1.29. Then there is bounded closed set $\Sigma \subset \mathbb{R}$ with $\text{spec}(A_\omega) = \Sigma$ $P$-almost surely.

**Proof.** See [PF92, Theorem 2.16] and [Pas72]. \qed

Note that the boundedness does not hold, in general, if $\Omega$ is not compact. This is why we will only consider compact topological spaces here. Another
advantage of this assertion is that the space of all covariant families of operators wears a well-defined norm:

**Proposition 1.31.** Let $\Omega$ be a compact topological space, $T_1, \ldots, T_d$ commuting homeomorphisms. Then the vector space of all covariant families $a = (A_\omega)_{\omega \in \Omega}$ has a norm given by

$$\|a\| := \sup_{\omega \in \Omega} \|A_\omega\|$$

and becomes a Banach space.

**Proof.** This a simple consequence of the uniform boundedness principle. Since by the strong continuity of the family and by the compactness of $\Omega$, one has $\sup_\omega \|A_\omega x\| < \infty$ for all $x \in \ell^2(\mathbb{Z}^d)$. Therefore, we also have $\sup_\omega \|A_\omega\| < \infty$. The norm properties and completeness naturally hold.

We will later examine covariant families of operators in more detail and show that the families of finite range build an algebra and are dense in the C*-algebra of families of local operators, see Section 2.2.

### 1.3.2 The density of states

The next object we will consider is the so-called density of states, which plays an important role in the characterisation of quantum mechanical systems. Roughly speaking, it measures the spectral values or energy values per unit volume and, therefore, induces a finite Borel measure on the real line. Its distribution function, called the integrated density of states, measures the part of the spectrum below each energy value. Accordingly, the integrated density of states is a counting function if there are only eigenvalues, for example, in finite dimensions or if there is only a discrete spectrum. Here, we will explicitly define this idea for covariant families of self-adjoint operators. By $C_0(\mathbb{R})$ we denote the Banach space of continuous functions on $\mathbb{R}$ that vanish at infinity. We use definitions and proofs from [Schu15a].

**Proposition 1.32.** Assume Hypothesis 1.29. Then there is a unique non-decreasing right-continuous function $N : \mathbb{R} \to \mathbb{R}$ with $N(\lambda) \xrightarrow{\lambda \to -\infty} 0$ such that
the associated Lebesgue-Stieltjes integral fulfills
\[ \int_{\mathbb{R}} f(\lambda) \, dN(\lambda) = \int_{\Omega} \langle e_0, f(A_\omega) e_0 \rangle \, dP(\omega) \]
for all \( f \in C_0(\mathbb{R}) \).

Because of Theorem 1.30, one can extend the functions considered above to all continuous functions \( C(\mathbb{R}) \). The uniqueness claim of Proposition 1.32 above gives rise to the following definition.

**Definition 1.33.** The real function \( N \) is called the *integrated density of states* (IDOS) and the associated Lebesgue-Stieltjes measure to \( N \) is called the *density of states* (DOS) of the covariant family of self-adjoint operators.

**Proof of Proposition 1.32.** For functions \( f \in C_0(\mathbb{R}) \), we set:
\[ \Psi(f) := \int_{\Omega} \langle e_0, f(A_\omega) e_0 \rangle \, dP(\omega). \]

Note that we always have \( |\Psi(f)| \leq ||f||_\infty \) by the spectral theorem. Indeed, we have for the operator norm \( ||\Psi|| = 1 \). This can be seen by choosing the function \( g = \chi_\Sigma \) where \( \Sigma \) is the almost surely spectrum from Theorem 1.30 above. Then \( \Psi(g) = 1 \). Obviously, \( \Psi : C_0(\mathbb{R}) \to \mathbb{R} \) is a positive linear functional. By the Riesz-Markov representation theorem, see, e.g. [Rud70, Theorem 2.14], we find a unique finite Borel measure \( \nu \) on \( \mathbb{R} \) that fulfills:
\[ \Psi(f) = \int_{\mathbb{R}} f \, d\nu, \quad f \in C_0(\mathbb{R}). \]

Since the operator norm of \( \Psi \) is 1, we get the same for the total variation norm of \( \nu \), and therefore \( \nu(\mathbb{R}) = 1 \). Consequently, \( \nu \) is a probability measure. Now, we can define the distribution function
\[ N(\lambda) := \nu((-\infty, \lambda]), \]
which is the unique non-decreasing right-continuous function that has \( \nu \) as its associated Lebesgue-Stieltjes measure. Therefore, the functional \( \Psi \) is expressible
as
\[ \Psi(f) = \int_{\mathbb{R}} f(\lambda) \, d\mathcal{N}(\lambda), \]
where this is to be understood as a Lebesgue-Stieltjes integral.

The integrated density of states \( \mathcal{N} \) is the unique distribution function for the whole family \( (A_\omega)_{\omega \in \Omega} \) and stays the same if we change the family on a \( \mathbb{P} \)-null set. In the next theorem, we connect \( \mathcal{N} \) to the eigenvalue counting function.

**Theorem 1.34.** Assume Hypothesis 1.29. Let \( \mathcal{N} \) be the IDOS of \( (A_\omega)_{\omega \in \Omega} \) and \( \Lambda_L := \{-L, -L + 1, \ldots, L - 1, L\}^d \) for \( L \in \mathbb{N} \) the centred cube in \( \mathbb{Z}^d \). Then, there is a Borel set \( \hat{\Omega} \) with \( \mathbb{P}(\hat{\Omega}) = 1 \) such that for all \( f \in C_0(\mathbb{R}) \) and for all \( \omega \in \hat{\Omega} \) the following holds:

\[ \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(\pi_L f(A_\omega)\pi_L^*) = \int_{\mathbb{R}} f(\lambda) \, d\mathcal{N}(\lambda). \]

Here, \( \text{Tr} \) is the usual trace on \( \ell^2(\mathbb{Z}^d) \), and we write \( \pi_L \) for the restriction operator on \( \ell^2(\mathbb{Z}^d) \) associated to \( \Lambda_L \) and defined by

\[ \pi_L : (x_n)_{n \in \mathbb{Z}^d} \mapsto (\chi_{\Lambda_L}(n)x_n)_{n \in \mathbb{Z}^d}. \]

The characteristic function for \( \Lambda_L \) is denoted by \( \chi_{\Lambda_L} \). This means that \( \pi_L \) restricts sequences to the cube \( \Lambda_L \) with Dirichlet boundary conditions.

**Proof.** Instead of \( C_0(\mathbb{R}) \), we use \( C(\Sigma) \) where \( \Sigma \) is the almost-sure spectrum of \( (A_\omega)_{\omega} \) given by Theorem 1.30. This is without loss of generality and has the advantage of using that the polynomials \( \mathcal{P}(\Sigma) \) on \( \Sigma \) lie dense in \( C(\Sigma) \). For all \( \omega \in \Omega \) and all polynomials \( p \in \mathcal{P}(\Sigma) \), we have by definition of the trace

\[ \frac{1}{|\Lambda_N|} \text{Tr}(\pi_L p(A_\omega)\pi_L^*) = \frac{1}{|\Lambda_L|} \sum_{n \in \Lambda_L} \langle e_n, p(A_\omega)e_n \rangle. \]

Using the right shift operators \( S_j \), we can transform \( e_n \) into \( e_0 \). Let us therefore use the notation \( S^n = S_1^{n_1} \cdots S_d^{n_d} \) for each \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \). Then, we simply get \( \langle e_n, p(A_\omega)e_n \rangle = \langle e_0, (S^n)^*p(A_\omega)S^n e_0 \rangle \) and can use the covariance
relation (1.5). Hence, we find

\[
\frac{1}{|\Lambda_L|} \text{Tr}(\chi_{\Lambda_L} p(A_\omega) \chi_{\Lambda_L}) = \frac{1}{|\Lambda_L|} \sum_{n \in \Lambda_L} \langle e_0, p(A_{T^n_\omega}) e_0 \rangle ,
\]

where we have used the notation \( T^n := T_1^{n_1} \cdots T_d^{n_d} \) as before. Now consider for each \( p \in \mathcal{P}(\Sigma) \) the continuous map

\[ \varphi_p : \Omega \to \mathbb{R}, \quad \omega \mapsto \langle e_0, p(A_\omega) e_0 \rangle , \]

which has the integral \( \int p \, d\mathcal{N} \) by Proposition 1.32. At this point, we can use Birkhoff’s ergodic theorem, see, e.g. [EW11, Theorem 2.30], for the dynamical system derived from the known group action \( \mathbb{Z}^d \times \Omega \to \Omega \) given by \( T^n \). Then, one gets that

\[
\lim_{L \to \infty} \frac{1}{|\Lambda_L|} \sum_{n \in \Lambda_L} \varphi_p(T^n(\omega))
\]

converges \( \mathbb{P} \)-almost everywhere to a map \( \varphi^T_p \). Since \( \mathbb{P} \) is ergodic, we also have by Birkhoff’s ergodic theorem that \( \varphi^T_p \) is \( \mathbb{P} \)-almost surely constant with

\[ \varphi^T_p(\omega) = \int \varphi_p \, d\mathbb{P} = \int_{\mathbb{R}} p(\lambda) \, d\mathcal{N}(\lambda) \]

for all \( \omega \in \Omega_p \), where \( \Omega_p \) is a chosen Borel set with \( \mathbb{P}(\Omega_p) = 1 \). This proof holds for all polynomials, and hence we define the set \( \hat{\Omega} := \bigcap_{p \in \mathcal{P}(\Sigma)} \Omega_p \). Then, \( \mathbb{P}(\hat{\Omega}) = 1 \) since this is a countable intersection. Accordingly, for all \( \omega \in \hat{\Omega} \) and polynomials \( p \in \mathcal{P}(\Sigma) \), we have

\[
\lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(\chi_{\Lambda_L} p(A_\omega) \chi_{\Lambda_L}) = \int_{\mathbb{R}} p(\lambda) \, d\mathcal{N}(\lambda) . \tag{1.7}
\]

In order to lift equation (1.7) to all continuous functions \( C(\Sigma) \), we use the Stone-Weierstraß theorem, see, e.g. [Wer05, Satz I.2.10]. Given an \( f \in C(\Sigma) \) and \( \varepsilon > 0 \), choose \( p \in \mathcal{P}(\Sigma) \) with \( \|f - p\|_\infty < \varepsilon \). Then for all \( \omega \in \hat{\Omega} \) and \( L \in \mathbb{N} \),
one has:

\[
\left| \frac{1}{|\Lambda_L|} \text{Tr}(\chi_{\Lambda_L} f(A_\omega) \chi_{\Lambda_L}) - \int_{\mathbb{R}} f(\lambda) d\mathcal{N}(\lambda) \right| \leq \left| \frac{1}{|\Lambda_L|} \text{Tr}(\chi_{\Lambda_L} (f - p)(A_\omega) \chi_{\Lambda_L}) \right| \\
+ \left| \frac{1}{|\Lambda_L|} \text{Tr}(\chi_{\Lambda_L} p(A_\omega) \chi_{\Lambda_L}) - \int_{\mathbb{R}} p(\lambda) d\mathcal{N}(\lambda) \right| + \int_{\mathbb{R}} (p - f)(\lambda) d\mathcal{N}(\lambda) \\
\leq 2\|f - p\|_\infty + \left| \frac{1}{|\Lambda_L|} \text{Tr}(\chi_{\Lambda_L} p(A_\omega) \chi_{\Lambda_L}) - \int_{\mathbb{R}} p(\lambda) d\mathcal{N}(\lambda) \right|.
\]

Taking the limit \( L \to \infty \) on both sides and using (1.7) shows

\[
\lim_{L \to \infty} \left| \frac{1}{|\Lambda_L|} \text{Tr}(\chi_{\Lambda_L} f(A_\omega) \chi_{\Lambda_L}) - \int_{\mathbb{R}} f(\lambda) d\mathcal{N}(\lambda) \right| \leq 2\varepsilon,
\]

and this concludes the proof.

\[\square\]

**Theorem 1.35.** Assume Hypothesis 1.29 where \((A_\omega)_{\omega \in \Omega}\) has finite range. Moreover, let \(\mathcal{N}\) be the IDOS of \((A_\omega)\) and \(\Lambda_L := \{-L, -L + 1, \ldots, L - 1, L\}^d\) for \(L \in \mathbb{N}\) the centred cube. Then, there is a Borel set \(\hat{\Omega}\) with \(P(\hat{\Omega}) = 1\) such that for all \(f \in C_0(\mathbb{R})\) and \(\omega \in \hat{\Omega}\) we have

\[
\int_{\mathbb{R}} f(\lambda) d\mathcal{N}(\lambda) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f(A_{\omega,L}))
\]

where \(A_{\omega,L} = \tilde{\pi}_L A_\omega \pi_{L}^*\) is the restriction of \(A_\omega\) to \(\ell^2(\Lambda_L)\), where the partial isometry \(\tilde{\pi}_L : \ell^2(\mathbb{Z}) \to \ell^2(\Lambda_L)\) is given by the restriction of the sequences. In particular, for all \(\lambda \in \mathbb{R}\) we have

\[
\mathcal{N}(\lambda) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \#\{\text{eigenvalues of } A_{\omega,L} \leq \lambda\} \quad P\text{-almost surely.}
\]

**Proof.** We fix \(\omega \in \hat{\Omega}\) where \(\hat{\Omega}\) is chosen from Theorem 1.34, which means the equality there holds. From now on, we write \(A = A_\omega\) for convenience. We also set \(A(L) = \pi_L A \pi_{L}^*\) for the restriction operator as in Theorem 1.34. Moreover, \(A(L^c) := \pi_{L^c} A \pi_{L^c}^*\) denotes the restriction to \(\Lambda_L^c\), where

\[
\pi_{L^c} : (x_n)_{n \in \mathbb{Z}^d} \mapsto (\chi_{\Lambda_L^c}(n)x_n)_{n \in \mathbb{Z}^d}
\]
restricts sequences to the complement of $\Lambda_L$. Of course, then we also have a boundary operator that describes the exchange between $\Lambda_L$ and $\Lambda_L^c$, and it will be denoted by

$$\Gamma_L = A - A(L) - A(L^c) : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d).$$

The finite range condition for $A$ implies that $\Gamma_L$ is a self-adjoint operator of finite rank. Now, we can restrain $A(L)$ to the invariant subspace $\pi_L \ell^2(\mathbb{Z}^d)$ and get the operator $A_L := \tilde{\pi}_L A \tilde{\pi}_L^*$. This means that the only difference between $A(L)$ and $A_L$ is the space they are acting on. Analogously, we define $A_{L^c}$.

Then, obviously, we have the orthogonal decomposition $A_L \oplus A_{L^c}$ with respect to the decomposition $\ell^2(\mathbb{Z}^d) = \ell^2(\Lambda_N) \oplus \ell^2(\Lambda_N^c)$.

Now, for arbitrary bounded self-adjoint operators $S, T$ and $z \in \mathbb{C} \setminus \mathbb{R}$, there is the, easily provable, resolvent identity

$$(S - z)(T + S - z)^{-1} = 1 - T(T + S - z)^{-1}.$$

Here, we use it for $A = \Gamma_L + A_L \oplus A_{L^c}$ and fixed $z \in \mathbb{C} \setminus \mathbb{R}$, so we get:

$$(A - z)^{-1} = (A_L \oplus A_{L^c} - z)^{-1} - (A_L \oplus A_{L^c} - z)^{-1} \Gamma_L (A - z)^{-1}.$$

Note that the resolvent of the operator above also has an orthogonal decomposition $(A_L \oplus A_{L^c} - z)^{-1} = (A_L - z)^{-1} \oplus (A_{L^c} - z)^{-1}$. Then, we choose $n \in \Lambda_L$ and get:

$$\langle e_n, (A - z)^{-1} e_n \rangle - \langle e_n, (A_L - z)^{-1} e_n \rangle = -\langle e_n, (A_L - z)^{-1} \Gamma_L (A - z)^{-1} e_n \rangle.$$

We will sum over all $n$, and then, on the right hand side, we can use Parvel’s identity several times. Therefore we obtain:

$$\sum_{n \in \Lambda_L} \langle e_n, (A_L - z)^{-1} \Gamma_L (A - z)^{-1} e_n \rangle = \sum_{n \in \Lambda_L} \sum_{j, m} \langle e_n, (A_L - z)^{-1} e_j \rangle \langle e_j, \Gamma_L e_m \rangle \langle e_m, (A - z)^{-1} e_n \rangle.$$
1.3 Covariant families of operators and the density of states

\[\sum_{j, m} \langle e_m, (A - z)^{-1}(A_L - z)^{-1}e_j \rangle \langle e_j, \Gamma_L e_m \rangle.\]

We know that \(A\) has finite range, and hence there are less than \(R|\partial\Lambda_L|\) non-vanishing summands in the last sum. Moreover, we can use the fact that the resolvent satisfies 
\[\| (A - z)^{-1} \| \leq |\text{Im} z|^{-1} \] and the rough estimate 
\[|\langle e_j, \Gamma_L e_m \rangle| \leq \|\Gamma_L\| \leq 3\|A\|.\] Therefore, we have a constant \(\alpha \geq 0\), which is independent of \(L\), such that

\[\left| \sum_{j, m} \langle e_m, (A - z)^{-1}(A_L - z)^{-1}e_j \rangle \langle e_j, \Gamma_L e_m \rangle \right| \leq \alpha |\partial\Lambda_L| |\text{Im} z|^{-2}.\]

Combining this with the equations above, we get the estimate:

\[\frac{1}{|\Lambda_L|} \left| \sum_{n \in \Lambda_L} \langle e_n, (A - z)^{-1}e_n \rangle - \langle e_n, (A_L - z)^{-1}e_n \rangle \right| \leq \alpha \frac{|\partial\Lambda_L|}{|\Lambda_L|} |\text{Im} z|^{-2}.\]

By assumption and Theorem 1.34, the limit of the first part on the left hand side exists for \(L \to \infty\). Then since \(|\partial\Lambda_L|/|\Lambda_L|\) goes to zero, we have the equality

\[\int_{\mathbb{R}} f_z(\lambda) dN(\lambda) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_L))\]

for \(f_z(t) = (t - z)^{-1}\). This equation can easily be lifted to all continuous functions since \(G = \text{span}\{f_z\}_{z \in \mathbb{C} \setminus \mathbb{R}}\) is dense in \((C_0(\mathbb{R}), \| \cdot \|_\infty)\), cf. Lemma 1.37 below. Note that then the limit on the right hand side also exists for \(f \in C_0(\mathbb{R})\) because

\[\left| \frac{1}{|\Lambda_L|} \text{Tr}(g(A_L)) - \frac{1}{|\Lambda_L|} \text{Tr}(f(A_L)) \right| \leq \|g - f\|_\infty \|A\|, \quad \text{for all } g \in G.\]

Accordingly, we get for all \(f \in C_0(\mathbb{R})\) and \(g \in G\), using the triangle inequality, the estimate:

\[\left| \int_{\mathbb{R}} f(\lambda) dN(\lambda) - \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f(A_L)) \right| \leq (\|A\| + 1)\|g - f\|_\infty.\]

This shows the first equation in the theorem. The second one follows by using
the function \( f = \chi_{\Sigma}(\chi_{(-\infty, \lambda]}) \), where \( \Sigma \) is the almost sure spectrum for the family \( (A_\omega)_{\omega \in \Omega} \).

Theorem 1.35 expressively explains the name density of state. The Lebesgue-Stieltjes measure associated to \( \mathcal{N} \) gives the number of energy levels per unit volume. This means that it measures how much spectrum is detectable at a given point. The function \( \mathcal{N} \) is then a generalised counting function and measures the ratio of the spectrum below a given level.

### 1.3.3 Some properties related to Herglotz functions

To complete the proof of Theorem 1.35 above, we now show the following facts about Herglotz functions, which are by definition analytic functions that map the upper complex half-plane \( \mathbb{C}_+ := \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \} \) into itself. For more details, see [Schm14] and [Tes09], and for a generalisation, see [Gro16] and [GT00].

**Theorem 1.36 (Stieltjes inversion formula).** Let \( \omega \) be a positive finite Borel measure on \( \mathbb{R} \) and let \( m : \mathbb{C}_+ \to \mathbb{C}_+ \) be the Herglotz function given by

\[
m(z) = \int_{\mathbb{R}} \frac{1}{t - z} \, d\omega(t) .
\]

Then, the measure \( \omega \) is uniquely determined by the values of \( m \). Specifically, one has for \(-\infty \leq t_1 < t_2 \leq \infty\)

\[
\frac{1}{2} \omega(\{t_1\}) + \frac{1}{2} \omega(\{t_2\}) + \omega((t_1, t_2)) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{t_1}^{t_2} \text{Im} \, m(x + i\varepsilon) \, dx , \tag{1.8}
\]

where the convention \( \omega(\pm \infty) = 0 \) is used for unbounded intervals. Moreover, for \( x \in \mathbb{R} \), one has

\[
\lim_{\varepsilon \to 0^+} \varepsilon \, \text{Im} \, m(x + i\varepsilon) = \omega(\{x\}) . \tag{1.9}
\]

**Proof.** See also [Tes09, Appendix 3.4] for similar proofs. For the imaginary part of the Herglotz function, we can calculate:

\[
\text{Im} \, m(x + i\varepsilon) = \int_{\mathbb{R}} \text{Im} \left( \frac{1}{t - x - i\varepsilon} \right) \, d\omega(t) .
\]
Now, we use Fubini’s theorem and obtain:

\[
\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} \text{Im} \, m(x + i\varepsilon) \, dx = \lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} \left( \int_{t_1}^{t_2} \text{Im} \left( \frac{1}{s-x-i\varepsilon} \right) \, dx \right) \, d\omega(s)
\]

\[
= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \left[ \arctan \left( \frac{1}{\varepsilon} (t_2 - s) \right) - \arctan \left( \frac{1}{\varepsilon} (t_1 - s) \right) \right] \, d\omega(s).
\]

Here, we use the decomposition into maximal five parts

\[
\mathbb{R} = (-\infty, t_1) \cup \{t_1\} \cup (t_1, t_2) \cup \{t_2\} \cup (t_2, \infty),
\]

and separately apply Lebesgue’s Dominated Convergence Theorem, see, e.g. [Rud70, Theorem 1.34]. Note that we need here:

\[
\lim_{\varepsilon \to 0^+} \arctan \left( \frac{t-s}{\varepsilon} \right) = \begin{cases} 
\pi/2 & \text{, } t > s \\
0 & \text{, } t = s \\
-\pi/2 & \text{, } t < s
\end{cases}.
\]

This proves (1.8). For showing (1.9), choose \( \delta > 0 \) and calculate

\[
\lim_{\varepsilon \to 0^+} \varepsilon \text{Im} \, m(x + i\varepsilon) = \lim_{\varepsilon \to 0^+} \varepsilon \int_{x-\delta}^{x+\delta} \frac{1}{(t-x)^2 + \varepsilon^2} \, d\omega(t)
\]

\[
= \lim_{\varepsilon \to 0^+} \int_{(x-\delta, x) \cup (x, x+\delta)} \frac{1}{(t-x)^2 + 1} \, d\omega(t) + \int_{\{x\}} d\omega(t) = 0 + \omega(\{x\}),
\]

where in the first step, we have used that \( \omega(\mathbb{R}) < \infty \), and, in the last step, Lebesgue’s Dominated Convergence Theorem has been applied again. \( \square \)

**Lemma 1.37.** The vector space \( G = \text{span}\{f_z\}_{z \in \mathbb{C} \setminus \mathbb{R}} \) with \( f_z(t) = (t - z)^{-1} \) is dense in \( (C_0(\mathbb{R}), \|\cdot\|_\infty) \).

**Proof.** By the Riesz-Markov representation theorem, see e.g. [Rud70, Theorem 2.14], we know that each positive linear functional on \( (C_0(\mathbb{R}), \|\cdot\|_\infty) \) is given by a finite positive Borel measure \( \omega \) on \( \mathbb{R} \). However, an arbitrary linear functional \( y' \in C_0(\mathbb{R})' \) can be represented by positive functionals \( y_j' \) such that

\[
y' = y_1' - y_2' + iy_3' - iy_4'
\]
holds, where we denote the associated positive measures by $\omega_j$. Now, note that 
$\text{Re} f_z = \frac{1}{2}(f_z + f_{\overline{z}}) \in G$ and $\text{Im} f_z = \frac{1}{2i}(f_z - f_{\overline{z}}) \in G$. Hence, we get

$$\begin{align*}
\text{Re} y'(\text{Re} f_z) &= y'_1(\text{Re} f_z) - y'_2(\text{Re} f_z), \\
\text{Im} y'(\text{Re} f_z) &= y'_3(\text{Re} f_z) - y'_4(\text{Re} f_z).
\end{align*}$$

(1.10)

Choose now $y' \in C_0(\mathbb{R})'$ such that $y'(\text{Re} f_z) = 0$ and $y'(\text{Im} f_z) = 0$ for all $z \in \mathbb{C}$ with $\text{Im} z > 0$. Then by the two equations (1.10) above, we obtain

$$\begin{align*}
\text{Re} \int f_z \, d\omega_1 &= \text{Re} \int f_z \, d\omega_2, \\
\text{Re} \int f_z \, d\omega_3 &= \text{Re} \int f_z \, d\omega_4.
\end{align*}$$

By the same argument, we also get the result for the imaginary part. In summary, we have:

$$\begin{align*}
\int \frac{1}{t-z} \, d\omega_1(t) &= \int \frac{1}{t-z} \, d\omega_2(t), \\
\int \frac{1}{t-z} \, d\omega_3(t) &= \int \frac{1}{t-z} \, d\omega_4(t).
\end{align*}$$

Since the Herglotz functions coincide, the measures have to coincide as well by the inversion formula of Theorem 1.36. Hence, we get $y'_1 = y'_2$ and $y'_3 = y'_4$ and therefore $y' = 0$. In short, we have for all $y' \in C_0(\mathbb{R})'$:

$$y'(g) = 0 \text{ for all } g \in G \implies y' = 0.$$ 

Using the Hahn-Banach theorem, see, e.g. [Wer05, Korollar III.1.9], we conclude $G$ is dense in $C_0(\mathbb{R})$. 

\[\square\]

1.3.4 The density of states for local operators

Now, we show that the density of states with respect to such functions $f_z \in G$ continuously depends on the given family in the following sense:

**Lemma 1.38.** Assume Hypothesis 1.29 and let $a = (A_\omega)$ and $b = (B_\omega)$ be two covariant families of self-adjoint operators. Then for all $z \in \mathbb{C} \setminus \mathbb{R}$, the functions $f_z(t) = (t - z)^{-1}$ fulfil

$$\left| \int f_z(\lambda) \, dN_a(\lambda) - \int f_z(\lambda) \, dN_b(\lambda) \right| \leq \|f_z\|_\infty^2 \|a - b\|, \quad \text{for all } \lambda \in \mathbb{R},$$

where $\|f_z\|_\infty$ denotes the supremum norm of $f_z$. This result follows from the fact that the family of functions $f_z(t) = (t - z)^{-1}$ is dense in $C_0(\mathbb{R})$. 

Using the Hahn-Banach theorem, see, e.g. [Wer05, Korollar III.1.9], we conclude $G$ is dense in $C_0(\mathbb{R})$. 

\[\square\]
where $N_a$ and $N_b$ are the integrated density of states of $a$ and $b$, respectively.

**Proof.** By Proposition 1.32, we can directly calculate for a function $f \in C_0(\mathbb{R})$:

$$
\left| \int f(\lambda) dN_a(\lambda) - \int f(\lambda) dN_b(\lambda) \right| \leq \int_\Omega |\langle e_0, (f(A_\omega) - f(B_\omega))e_0 \rangle| \, d\mathcal{P}(\omega)
$$

$$
\leq \sup_{\omega \in \Omega} \| f(A_\omega) - f(B_\omega) \|.
$$

Now for a $z \in \mathbb{C} \setminus \mathbb{R}$, choose the function $f_z(t) = (t - z)^{-1}$, and, by using the resolvent identity, one gets:

$$
\left| \int f_z(\lambda) dN_a(\lambda) - \int f_z(\lambda) dN_b(\lambda) \right| \leq \sup_{\omega \in \Omega} \left( \| f_z(A_\omega) \| \| A_\omega - B_\omega \| \| f_z(B_\omega) \| \right)
$$

$$
\leq \| f_z \|_\infty \| a - b \|.
$$

Here, we used $(A_\omega - z)^{-1} - (B_\omega - z)^{-1} = (A_\omega - z)^{-1}(B_\omega - A_\omega)(B_\omega - z)^{-1}$ for the two self-adjoint operators $A_\omega, B_\omega$, and $\| f_z(A_\omega) \| \leq \| f_z \|_\infty$. \hfill \Box

**Lemma 1.39.** Assume Hypothesis 1.29 and that all families below consist of self-adjoint operators. Let $a = (A_\omega)_{\omega \in \Omega}$ be a covariant family such that there is a sequence $a_n = (A_n^\omega)_{\omega \in \Omega}$ of covariant families of finite range, for $n \in \mathbb{N}$, with $\| a_n - a \| \xrightarrow{n \to \infty} 0$. Then there is a Borel set $\hat{\Omega}$ with $\mathcal{P}(\hat{\Omega}) = 1$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ and for all $\omega \in \hat{\Omega}$ the limit

$$
\lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_\omega,L))
$$

exists, where $f_z(t) := (t - z)^{-1}$. Moreover, for all $z \in \mathbb{C} \setminus \mathbb{R}$, for all $\omega \in \hat{\Omega}$ and all $n \in \mathbb{N}$,

$$
\left| \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_\omega,L)) - \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr} \left( f_z(A_n^\omega,L) \right) \right| \leq \| f_z \|_\infty \| a_n - a \|,
$$

where $A_n^\omega,L = \pi_L A_n^\omega \pi_L^*$ is the restriction of $A_n^\omega$ to $\ell^2(\Lambda_L)$.

**Proof.** Here, we can perform a usual estimation and just calculate the traces
by using the orthonormal basis \((e_n)\):

\[
\left| \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_{\omega,L})) - \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A^{(n)}_{\omega,L})) \right| \\
\leq \frac{1}{|\Lambda_L|} \sum_{n \in \Lambda_L} \left| \langle e_n, \left( f_z(A_{\omega,L}) - f_z(A^{(n)}_{\omega,L}) \right) e_n \rangle \right| \\
\leq \|f_z(A_{\omega,L})\| \left\| A^{(n)}_{\omega,L} - A_{\omega,L} \right\| \|f_z(A^{(n)}_{\omega,L})\|.
\]

In the last inequality, we have applied, for two self-adjoint operators \(B, C\), the resolvent identity \((B - z)^{-1} - (C - z)^{-1} = (B - z)^{-1}(C - B)(C - z)^{-1}\). Now by using \(\|f_z(A_{\omega,L})\| \leq \|f_z\|_{\infty}\) by the functional calculus, we obtain the estimate. Combining this with Theorem 1.35 shows that the limit in (1.11) exists. \(\square\)

**Theorem 1.40.** Assume Hypothesis 1.29 and that all families below consist of self-adjoint operators. Let \(a = (A_{\omega})_{\omega \in \Omega}\) be a covariant family such that there is a sequence \(a_n = (A^{(n)}_{\omega})_{\omega \in \Omega}\) of covariant families of finite range with \(\|a_n - a\| \to 0\). Then, there is a Borel set \(\hat{\Omega}\) with \(P(\hat{\Omega}) = 1\) such that for all \(f \in C_0(\mathbb{R})\) and \(\omega \in \hat{\Omega}\) we have

\[
\int_{\mathbb{R}} f(\lambda) \, dN(\lambda) = \lim_{L \to \infty} \frac{1}{|\Lambda|} \text{Tr}(f(A_{\omega,L})),
\]

where \(N\) denotes the IDOS of \((A_{\omega})_{\omega}\). In particular, for all \(\lambda \in \mathbb{R}\),

\[
N(\lambda) = \lim_{L \to \infty} \frac{1}{|\Lambda|} \# \{ \text{eigenvalues of } A_{\omega,L} \leq \lambda \} \quad P\text{-almost surely},
\]

where \(A_{\omega,L}\) is the restriction of \(A_{\omega}\) to \(\ell^2(\Lambda_L)\).

In other words, the claim of Theorem 1.35 holds also for covariant families of operators that lie in the closure of the finite range families. These operator families are usually called *local* and will be considered in more detail later.

**Proof.** Choose \(z \in \mathbb{C} \setminus \mathbb{R}\) and \(f_z(t) = (t - z)^{-1}\) as always. Choose \(\omega \in \hat{\Omega}\) with \(\hat{\Omega}\) from Theorem 1.35. Let \(N_a\) and \(N_{a_n}\) denote the integrated density of states of \(a\) and \(a_n\), respectively. Then by using the existence of the limit in (1.11), we
can calculate:

\[
\left| \int f_z(\lambda) \, dN_a(\lambda) - \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_{\omega,L})) \right|
\leq \left| \int f_z(\lambda) \, dN_a(\lambda) - \int f_z(\lambda) \, dN_{a_n}(\lambda) \right| \\
+ \left| \int f_z dN_{a_n} - \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}\left(f_z(A_{\omega,L}^{(n)})\right) \right| \\
+ \left| \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}\left(f_z(A_{\omega,L}^{(n)})\right) - \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_{\omega,L})) \right| .
\]

The middle part is zero because of Theorem 1.35. The first and third part can be estimated by Lemma 1.38 and Lemma 1.39, respectively. Therefore, we have

\[
\left| \int f_z(\lambda) \, dN_a(\lambda) - \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_{\omega,L})) \right| \leq 2 \|f_z\|_\infty \|a_n - a\|
\]

for all \( n \in \mathbb{N} \). Since \( \|a_n - a\| \) converges to zero, we conclude:

\[
\int f_z(\lambda) \, dN_a(\lambda) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(f_z(A_{\omega,L}))
\]

for all \( z \in \mathbb{C} \setminus \mathbb{R} \). This equation can now again easily be lifted to all continuous functions since \( G = \text{span}\{f_z\}_{z \in \mathbb{C} \setminus \mathbb{R}} \) is dense in \((C_0(\mathbb{R}), \| \cdot \|_\infty)\), cf. Lemma 1.37 above. This is the same argument as in the proof of Theorem 1.35.

**Definition 1.41.** For a finite dimensional Hilbert space \( \mathcal{H} \) and self-adjoint \( A \in \mathcal{B}(\mathcal{H}) \), we call the normalised counting function

\[
N_A(\lambda) := \frac{1}{\dim(\mathcal{H})} \#\{\text{eigenvalues of } A \leq \lambda\} = \frac{1}{\dim(\mathcal{H})} \text{Tr}(\chi(-\infty, \lambda](A)),
\]

for \( \lambda \in \mathbb{R} \), also the **integrated density of states for** \( A \). Here, the eigenvalues are counted with multiplicities.

**Remark 1.42.** With the definition above, Theorem 1.40 implies for a covariant family \((A_{\omega})_{\omega \in \Omega}\) that lies in the closure of the finite range families, for all \( \lambda \in \mathbb{R} \):

\[
N(\lambda) = \lim_{L \to \infty} N_{A_{\omega,L}}(\lambda) \quad \text{P-almost surely.}
\]
1.4 Symmetry operators

In this section, we will consider operators on a Hilbert space with real structure, see Definition 1.1. In the following, we will explain what it means, on an operator theoretical level, when someone says that a physical system has a time-reversal symmetry, particle hole symmetry or a chiral symmetry. We formulate this for a bounded operator $H : \mathcal{H} \to \mathcal{H}$ since this is all we require in this thesis. However, this definition can easily be expanded to unbounded operators when we, in addition, consider suitable domains.

First we need a notion of a symmetry operator, which is often also just called a symmetry:

**Definition 1.43.** Let $\mathcal{H}$ be Hilbert space with real structure $\mathcal{C}$. An operator $S \in B(\mathcal{H})$ is called a symmetry operator if it is a real unitary, which means $S = \overline{S} = (S^*)^{-1}$, squares either to the identity or minus the identity and has eigenspaces of same dimension. In these respective cases $S^2 = 1$ or $S^2 = -1$, the symmetry operator is called even or odd.

Our notion of a symmetry operator is very general, and therefore the identity operator $1$ itself is also an even symmetry in this sense. Moreover, for an even symmetry operator $S$, the anti-linear operator $SC$ defines another real structure on the Hilbert space $\mathcal{H}$. On the other hand, if $S$ is an odd symmetry operator, then $SC$ gives a quaternionic structure on the Hilbert space in the sense of Definition 1.2. Note that a symmetry operator can only have $-1$ and $1$ in the spectrum, and in the case that both occur and if $\mathcal{H}$ is infinite dimensional, the two associated eigenspaces are both infinite dimensional.

**Definition 1.44.** Let $\mathcal{H}$ be Hilbert space with real structure $\mathcal{C}$. Fix a bounded self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$, called the Hamiltonian. Then we define the following symmetry types:

(1) The Hamiltonian is said to have a time reversal symmetry (TRS) if there is a symmetry operator $S_{tr} \in B(\mathcal{H})$ with

$$S_{tr}^* H S_{tr} = H .$$

The time reversal symmetry is called odd if $S_{tr}^2 = -1$ and even if $S_{tr}^2 = 1$. 
(2) The Hamiltonian is said to have a particle hole symmetry (PHS) if there is a symmetry operator $S_{ph} \in \mathcal{B}(\mathfrak{H})$ with

$$S_{ph}^*HS_{ph} = -H.$$ 

The particle hole symmetry is called odd if $S_{ph}^2 = -1$ and even if $S_{ph}^2 = 1$.

(3) The Hamiltonian is said to have a chiral symmetry (CHS) if there is a symmetry operator $S_{ch} \in \mathcal{B}(\mathfrak{H})$ with

$$S_{ch}^*HS_{ch} = -H.$$ 

This is also defined if $\mathfrak{H}$ does not carry a real structure.

If the Hamiltonian has got more than one symmetry, then we usually require, without loss of generality, that the operators $S_{tr}$ and $S_{ph}$ commute.

Note that for an unbounded Hamiltonian these definitions hold as well if we, in addition, demand that the domain of $H$ is invariant under the given symmetries and that the defining equations above hold for all vectors in the domain of $H$.

Now, it is often convenient, although it is slightly unconventional, to use the following terminology.

**Definition 1.45.** Let $\mathfrak{H}$ be Hilbert space with real structure $\mathcal{C}$. An operator $A \in \mathcal{B}(\mathfrak{H})$ is called even real or odd real if $S^*A^S = A$ for some even or odd symmetry operator $S$, respectively. An operator $A \in \mathcal{B}(\mathfrak{H})$ is called even symmetric or odd symmetric if $S^*A^S = A$ for an even or odd symmetry operator $S$, respectively. An orthogonal projection $P$ is called even Lagrangian or odd Lagrangian if $S^*PS = 1 - P$ with an even or odd symmetry $S$, respectively.

What we call here Lagrangian is sometimes called symplectic or rather symplectic Lagrangian, and it should be emphasised that only orthogonal projections can be Lagrangian.
1.5 Some spectral properties

In this section, we collect some properties like symmetry relations or special polar decompositions for operators that we will need in later chapters. We will use the spectral theorem for bounded and unbounded self-adjoint operators, see for example [Schm14, Theorem 5.1] and [Schm14, Theorem 5.7]. In this work, we often consider operators in the form $T = PFP + 1_{\mathcal{H}} - P$ where $F$ is a unitary and $P$ an orthogonal projection. They will ultimately be the object of research in Chapter 4. Here, we prove some important facts about these operators and always use the abbreviation $P^\perp = 1_{\mathcal{H}} - P$. Note that $P^\perp$ is the orthogonal projection onto the subspace $\text{Ran}(P)^\perp$.

Lemma 1.46. Let $F$ be a unitary operator on a Hilbert space $\mathcal{H}$ and $P$ be an orthogonal projection in $\mathcal{H}$. Then the restricted operator

$$\tilde{F} : \text{Ker}(PFP + P^\perp) \to \text{Ker}(P^\perp F^* P^\perp + P), \quad x \mapsto Fx$$

is a bijective isometric operator between the two Hilbert spaces $\text{Ker}(PFP + P^\perp)$ and $\text{Ker}(P^\perp F^* P^\perp + P)$.

Proof. We use ideas from [DS16]. Choose an element $x \in \text{Ker}(PFP + P^\perp)$. Note that $x$ necessarily lies in $\text{Ran}(P)$, and therefore we have $P^\perp x = 0$ and $PFPx = 0$. Considering $P^\perp Fx = Fx - PFPx = Fx$ shows $Fx \in \text{Ran}(P^\perp)$. Additionally, the calculation

$$P^\perp F^* P^\perp Fx = P^\perp F^* Fx = P^\perp x = 0$$

shows that $Fx \in \text{Ker}(P^\perp F^* P^\perp + P) \subset \text{Ran}(P^\perp)$, and hence $\tilde{F}$ is well-defined.

By substituting the operators $Q := P^\perp$ and $E := F^* = F^{-1}$, one can choose $y \in \text{Ker}(QE + Q^\perp) \subset \text{Ran}(Q)$ and repeat the argument above, showing that $E = \tilde{F}^{-1}$ is also well-defined. Since $F$ is unitary, the restriction $\tilde{F}$ is also an isometric operator. \hfill \Box

In the next Lemma, we show that a given symmetry for the Hamiltonian $H$ is conserved under a suitable function.
Lemma 1.47. Let $\mathcal{H}$ be a Hilbert space with real structure and $H \in \mathcal{B}(\mathcal{H})$ be self-adjoint with a PHS $S_{\text{ph}}$. Choose $f : \mathbb{R} \to \mathbb{R}$ as a Borel measurable function that is continuous on $\text{spec}(H) \subset \mathbb{R}$. If $f$ is an odd function, i.e. $f(-t) = -f(t)$ for all $t \in \mathbb{R}$, we have the symmetry relation

$$S_{\text{ph}}^* f(H) S_{\text{ph}} = -f(H).$$

If $f$ is an even function, i.e. $f(-t) = f(t)$ for all $t \in \mathbb{R}$, we have the symmetry relation

$$S_{\text{ph}}^* f(H) S_{\text{ph}} = f(H).$$

Proof. Since $S_{\text{ph}}$ is unitary, the equations above are obviously true for polynomials. The generalisation is shown by the Stone-Weierstraß approximation theorem, see, e.g. [Wer05, Satz I.2.10]. Given an odd function $f$, we choose $a > 0$ such that $[-a,a] \supset \text{spec}(H)$ and set $g : [-a,a] \to \mathbb{R}$ as a continuous extension of $f|_{\text{spec}(H)}$ that is still odd. Note that this is always possible. By Stone-Weierstraß, there is a sequence of odd polynomials $(p_n)_{n \in \mathbb{N}}$ on $[-a,a]$ such that $\|g - p_n\|_{\infty} \xrightarrow{n \to \infty} 0$. Therefore, we can calculate:

$$\|S_{\text{ph}}^* f(H) S_{\text{ph}} + f(H)\| =\]

$$= \|S_{\text{ph}}^* p_n(H) S_{\text{ph}} + p_n(H) - S_{\text{ph}}^* (p_n(H) - f(H)) S_{\text{ph}} - (p_n(H) - f(H))\|

\leq 2\|p_n(H) - f(H)\| = 2\|(p_n - g)(H)\| \leq 2\|p_n - g\|_{\infty} \xrightarrow{n \to \infty} 0.$$

Naturally, the same argument and an analogous calculation are true if the function $f$ is even. \qed

Since every function $f : \mathbb{R} \to \mathbb{R}$ can be split into an odd and an even part by writing

$$f(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) + f(-x)}{2},$$

one may use Lemma 1.47 for every function that is continuous on the spectrum of $H$. In particular, this is applicable for the characteristic function $\chi_{(-\infty,0]}$ if $0$ lies not in the spectrum of $H$.

Example 1.48. Let $H \in \mathcal{B}(\mathcal{H})$ a self-adjoint operator with $0 \notin \text{spec}(H)$ and a
PHS $S_{ph}$. Denote by $Q := \chi_{(-\infty,0]}(H)$ the spectral projection of the negative part of the spectrum. Then by using Lemma 1.47, we get $S_{ph}^*QS_{ph} = 1 - Q$. Therefore, $Q$ is Lagrangian with respect to $S_{ph}$.

For the sake of completeness, we now formulate the polar decomposition for a bounded or unbounded self-adjoint operator in a form we can use later.

**Proposition 1.49.** Let $A$ be a, possibly unbounded, self-adjoint operator on a complex Hilbert space. Then there is a unique polar decomposition

$$A = U |A| \quad (1.12)$$

where $|A| := \sqrt{A^*A}$ is given by the functional calculus and $U$ is a unitary with $U|\text{Ker}(A) = 1$. Moreover, $U$ commutes with $A$ and $|A|$.

**Proof.** In [Kat66, Chapter VI 2.7] and [Schm14, Theorem 7.2], it is shown that for each densely defined closed operator there is a unique polar decomposition $A = V |A|$ where $V$ is a partial isometry with initial space $\text{Ker}(A)^\perp$ and final space $\text{Ker}(A^*)^\perp$. If $A$ is self-adjoint, the two spaces coincide and thus $V$ is a unitary on $\text{Ker}(A)^\perp$. Therefore, we can set

$$Ux = \begin{cases} Vx, & \text{for } x \in \text{Ker}(A)^\perp \\ x, & \text{for } x \in \text{Ker}(A) \end{cases}$$

and get a unitary on the whole Hilbert space for which (1.12) still holds. The commutation property is due to the self-adjointness of $A$, cf. [Schm14, Example 7.1].

Some authors prefer not giving a name for this unitary and use the following convention, which we explain here and use in later chapters.

**Convention 1.50.** If $U$ is the unitary in the polar decomposition of $A$ from Proposition 1.49, we sometimes use the following notation:

$$\frac{A}{|A|} := U.$$
Note that reading this as an operator equation is only correct on the subspace \( \text{Ker}(A) \perp \).

The next claims are some important facts about using the functional calculus, especially in the context of Hilbert spaces with real structure. They are not hard to prove and, hence, often implicitly accepted. For the sake of completeness, we formulate the proofs here.

**Lemma 1.51.** Let \( A \) be a, possibly unbounded, self-adjoint operator on a Hilbert space \( \mathcal{H} \) with domain \( \mathcal{D}(A) \) and \( f : \mathbb{R} \to \mathbb{C} \) Borel measurable. Then \( f(A) \) is a normal operator with dense domain \( \mathcal{D}(f(A)) \) defined by the functional calculus. If \( U : \mathcal{K} \to \mathcal{H} \) is a unitary operator, where \( \mathcal{K} \) is another Hilbert space, then

\[
f(U^*AU) = U^*f(A)U \quad \text{and} \quad \mathcal{D}(f(U^*AU)) = U^*\mathcal{D}(f(A)).
\] (1.13)

**Proof.** For the functional calculus of unbounded self-adjoint operators, see, for example, [Schm14, Theorem 5.9]. Using Stone’s formula for the spectral measures, see [Schm14, Theorem 5.14], we find

\[
E_{U^*AU} = U^*E_AU,
\] (1.14)

where \( E_B \) denotes the unique spectral measure of a self-adjoint operator \( B \) by the spectral theorem. Applying this formula to the domain

\[
\mathcal{D}(f(A)) = \left\{ x \in \mathcal{H} \mid \int |f(t)|^2 \, d\langle x, E_A(t)x \rangle < \infty \right\},
\]

cf. [Schm14, Formula (5.11)], this immediately yields the second part of (1.13). For obtaining the first part, we choose \( x \in \mathcal{D}(f(U^*AU)) \), \( y \in \mathcal{K} \) and calculate by using equation (1.14):

\[
\langle y, f(U^*AU)x \rangle = \int f(t) \, d\langle y, E_{U^*AU}(t)x \rangle
\]
\[
= \int f(t) \, d\langle Uy, E_AUx \rangle = \langle y, U^*f(A)Ux \rangle.
\]

Therefore \( f(U^*AU)x = U^*f(A)Ux \) for all \( x \in \mathcal{D}(f(U^*AU)) \) and the operators
Now we can use Lemma 1.51 above to show that the absolute value for self-adjoint operators is invariant under unitary transformations in the following sense.

**Corollary 1.52.** For a, possibly unbounded, self-adjoint operator $A$ and a unitary $U$, one always has $|U^*AU| = U^*|A|U$.

In a Hilbert space with real structure, one reckons that the same claim holds for the complex conjugation $\mathcal{C}$. However, since $\mathcal{C}$ is not a linear operator we cannot use the result of Corollary 1.52 above. Nevertheless, we can indeed show the invariance of the absolute value under complex conjugations.

**Lemma 1.53.** Let $A$ be a, possibly unbounded, self-adjoint operator on a Hilbert space with real structure $\mathcal{C}$. Then

$$\mathcal{C}|A|\mathcal{C} = |\mathcal{C}A\mathcal{C}|.$$

**Proof.** We write $\overline{A} := \mathcal{C}A\mathcal{C}$ and calculate

$$(\mathcal{C}|A|\mathcal{C})^2 = \mathcal{C}|A|^2|A|\mathcal{C} = |\overline{A}|^2 = \overline{A}\overline{A}.$$ 

Since the square root of a positive operator is unique, one gets $|\overline{A}| = \mathcal{C}|A|\mathcal{C}$. Note that $\mathcal{C}|A|\mathcal{C}$ is also a positive operator because it has the same spectrum as $|A|$.

### 1.6 Clifford groups and algebras

For analysing physical systems like topological insulators, one needs the so-called *Dirac operator* since it describes the propagation of Fermions, for example electrons. This section is based on the appendix of the publication [GS16], see also Chapter 4 for more details. The Dirac operator is an unbounded operator, and in its definition we always need anti-commuting elements, known as $\Gamma$-matrices, that ultimately come from a *Clifford algebra* $\mathcal{Cl}_d$ for a given dimension $d \in \mathbb{N}$. We explain some facts here but refer to the monograph...
[Por95] for a more detailed presentation. One possible definition of the Dirac operator on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^{d'}$ for some $d, N, d' \in \mathbb{N}$ is

$$D = \sum_{j=1}^{d} X_j \otimes 1 \otimes \Gamma_j,$$

where $X_j$ stands for the, unbounded, position operator on $\ell^2(\mathbb{Z}^d)$ in the $j$th direction, and $\{\Gamma_j\}_{j=1,...,d}$ are some anti-commuting matrices. The physical interested reader may be more familiar with the continuous representation of the Dirac operator that one gets when using a discrete Fourier transform, see Chapter 4 for details.

The construction of the $\Gamma$-matrices from (1.15) as well as their symmetries are reviewed in this section. The real Clifford algebra $\mathcal{O}_d$ for $d \in \mathbb{N}$ is the real algebra generated by some elements $\gamma_1, \ldots, \gamma_d$ satisfying the relation $\gamma_n \gamma_m + \gamma_m \gamma_n = 2\delta_{n,m}$. It will be convenient to first look at the finite subgroup of the multiplicative group of $\mathcal{O}_d$ generated by $\gamma_1, \ldots, \gamma_d$, called the Clifford group, see Section 1.6.2. Then irreducible representations of this group are constructed, see Section 1.6.3, and symmetry operators in these representations are studied, see Section 1.6.4. Useful references containing a lot of the material below are [ABS64], [Roe99], [Kar78], [LM89], [Por95], [GVF01] and [Por95]. However, it was not possible to locate all of the facts we need, there. For this reason, we present a short self-contained description adapted to the needs of Chapter 4.

1.6.1 Clifford algebras

Clifford algebras can be defined in a general setting where we only need a vector space with some inner product. However, we have to emphasise that in the construction of Clifford algebras a bilinear form is needed rather than a sesquilinear form, so that we cannot use the definition of the inner product space given in Definition 1.3. Accordingly, we just speak of a vector space equipped with a bilinear form.

**Definition 1.54.** Let $V$ vector space over $\mathbb{K}$ and $b : V \times V \rightarrow \mathbb{K}$ a bilinear form. A pair $(\mathcal{A}, \phi)$ consisting of a $\mathbb{K}$-algebra $\mathcal{A}$ with unit $1$ and a linear map
φ : V → A satisfying
\[ \phi(v)^2 = b(v,v)1 \]
is called a Clifford algebra for \((V,b)\) if for each other such pair \((A',\phi')\) there is a unique algebra homomorphism \(A \to A'\) such that the diagram
\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & A \\
\downarrow{\phi'} & & \downarrow \\
A' & & 
\end{array}
\]
is commutative.

By this universal property, it is obvious that a Clifford algebra for \(V\) is uniquely determined. Indeed, it is easy to show that, for each finite dimensional vector space with bilinear form, there exists this Clifford algebra.

Furthermore, each real Hilbert space gives a bilinear form by its scalar product, and each complex Hilbert space \(\mathcal{H}\) with real structure \(\mathcal{C}\) defines a bilinear form by \(b(y,x) := \langle \mathcal{C}y, x \rangle_{\mathcal{H}}\). Consequently, we can meaningfully speak about Clifford algebras for inner product spaces when there is a real structure involved. Since we are merely interested in the Hilbert spaces \(\mathbb{R}^d\) and \(\mathbb{C}^d\) equipped with the standard scalar product, denoted by \((\cdot|\cdot)\), we give the following claim.

**Proposition 1.55.** For the Hilbert space \((\mathbb{K}^d, (\cdot|\cdot))\), there exists the Clifford algebra and is denoted by \(\mathcal{C}_d^\mathbb{K}\). For the inner product space \((\mathbb{K}^d, -(\cdot|\cdot))\), there exists the Clifford algebra and is denoted by \(\mathcal{C}_d^\mathbb{K}\).

**Proof.** Here, we give the proof for \((\mathbb{K}^d, (\cdot|\cdot))\). Choose the standard basis of \(\mathbb{K}^d\) and denote it by \((\gamma_j)_{j=1,\ldots,d}\). We consider the \(2^d\) elements given by a multiplicative identity \(1\) and further elements \(\gamma_{j_1} \cdots \gamma_{j_k}\) for increasing indices \(1 \leq j_1 < \cdots < j_k \leq n\) and \(k \in \{1,\ldots,d\}\). The multiplication of these elements is completely given by the two formulas
\[
\gamma_j \gamma_m + \gamma_m \gamma_j = 0 \quad \text{and} \quad \gamma_j^2 = 1 \quad \text{for all} \quad j \neq m.
\]

This defines a unital algebra \(\mathcal{C}_d^\mathbb{K}\) and also the linear map \(\phi : \mathbb{K}^d \to \mathcal{C}_d^\mathbb{K}\) by setting \(\phi(\gamma_j) = \gamma_j\). This map fulfils \(\phi(v)^2 = (v|v)1\) for all \(v \in \mathbb{K}^d\). Moreover, the pair \((\mathcal{C}_d^\mathbb{K}, \phi)\) has also the universal property since each linear map \(\phi' : \mathbb{K}^d \to A'\)
into a unital algebra $\mathcal{A}'$ with $\phi'(\gamma_j)^2 = 1$ extends naturally to an algebra homomorphism $\mathcal{A}_d^K \to \mathcal{A}'$. The same proof holds for $\widetilde{\mathcal{A}}_d^K$ with only one sign changed. \hfill $\square$

The Clifford algebra can be fully understood if one observes these anti-commuting elements generating the algebra. Therefore, we first consider the underlying group structure of these elements. For the real Clifford algebras we simply write $\mathcal{C}_d$ and $\widetilde{\mathcal{C}}_d$. Before turning to the technical details of the Clifford group, we want to give an important result:

**Proposition 1.56** (Bott periodicity, cf. [ABS64]). The real Clifford algebras are eightfold periodic, $\mathcal{C}_{d+8} \cong \mathcal{C}_d$, and the complex Clifford algebras are twofold periodic, $\mathcal{C}_{d+2}^C \cong \mathcal{C}_d^C$.

### 1.6.2 Clifford groups

In this section, we survey the **Clifford group** which is a group generated from some anti-commuting elements, and related to the well-known Clifford algebra from above. We define it in the following way:

**Definition 1.57.** For $d \in \mathbb{N}$ a group $F_d$ is called a $d$-th Clifford group if it is generated by $(d+2)$ different elements, namely the neutral element $1$, $d$ elements $\gamma_1, \ldots, \gamma_d$ and an extra element $-1$ commuting with $\gamma_n$ for all $n$, such that the relations $(-1)^2 = \gamma_n^2 = 1$ for all $n$ and $\gamma_n \gamma_m = (-1) \gamma_m \gamma_n$ for $n \neq m$ hold.

**Example 1.58.** For $F_1$ we can choose the following $2 \times 2$-matrix group that is generated by these three elements:

$$
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Obviously, the group $F_1$ consists of four elements since only $\gamma(-1) = (-1)\gamma$ is a new element that can be generated, and the group is commutative. Therefore $F_1$ coincides with the Klein four-group.

Clearly, one can choose representations of the groups in such a way that $F_{d-1} \subset F_d$ is a subgroup. Actually, one can represent these groups such that
one has \( F_d = F_{d-1} \cup \gamma_d F_{d-1} \). From this, one can conclude that \( F_d \) has \( 2^{d+1} \) elements. In Chapter 4 of this work, we will need the following three important automorphisms of the Clifford groups:

**Definition 1.59.** Let \( F_d \) be the \( d \)-th Clifford group. Then we define the element \( \kappa = (-1)^{\left\lfloor \frac{d}{2} \right\rfloor} \in \{-1, 1\} \) where \( \lfloor \cdot \rfloor \) stands for the common floor function. Moreover, the following automorphisms \( \alpha, \beta, \hat{\alpha} : F_d \to F_d \) are given by

\[
\alpha(\gamma_n) = \kappa(-1)^{n+1}\gamma_n, \quad \beta(\gamma_n) = (-1)^{n}\gamma_n, \quad \hat{\alpha}(\gamma_n) = \kappa(-1)^{n}\gamma_n,
\]

together with the natural formulas \( \alpha(\pm 1) = \pm 1, \beta(\pm 1) = \pm 1 \) and \( \hat{\alpha}(\pm 1) = \pm 1 \).

Note that all these objects depend on \( d \), but in most cases, we will not place an index if there is no risk of confusion. Clearly, one has \( \alpha \circ \beta = \beta \circ \alpha = \hat{\alpha} \). The peculiar choice of the element \( \kappa \) is crucial because it ensures that \( \alpha \) is an inner automorphism, namely that the automorphism is given by an element \( \sigma \in F_d \) with \( \alpha(\gamma) = \sigma \gamma \sigma^{-1} \) for all \( \gamma \in F_d \). We call the property of being an inner automorphism adjoinable. See also Proposition 1.61 below for more details.

The involutions \( \beta \) and \( \hat{\alpha} \) turn out to be adjoinable only for even \( d \). In many prior works starting with [ABS64], the involution \( \beta \) plays a prominent role and is often called the main involution.

Since the generators \( \{\gamma_n\} \) of a Clifford group anti-commute, we are getting a special element when considering the product of these generators.

**Definition 1.60.** For a Clifford group \( F_d \) we define the following different elements:

\[
\sigma = \gamma_2 \gamma_4 \gamma_6 \cdots \gamma_{2 \left\lfloor \frac{d}{2} \right\rfloor}, \quad \omega = \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_d, \quad \hat{\sigma} = \gamma_1 \gamma_3 \gamma_5 \cdots \gamma_{2 \left\lfloor \frac{d}{2} \right\rfloor + 1}.
\]

Here, \( \omega \) is called the volume element of the Clifford group.

Now by using the anti-commutativity of the generators, we find the following important relations:

\[
\sigma^2 = \begin{cases} 
1, & d = 1, \\
(-1)^{\kappa}, & d \neq 1,
\end{cases}
\]
and
\[ \omega^2 = (-1)^{d-1}, \quad \hat{\sigma}^2 = (-1)^{\lfloor \frac{d-1}{2} \rfloor}, \quad \omega \sigma = \kappa^d \hat{\sigma}. \]

Using \( \omega^{-1} = \gamma_d \gamma_{d-1} \cdots \gamma_1 \), one readily checks
\[ \omega \gamma_n \omega^{-1} = (-1)^{d-1} \gamma_n, \quad n = 1, \ldots, d. \]

Hence for odd \( d \), the volume element \( \omega \) lies in the centre of the group, and actually the centre is always \( \{1, -1, \omega, (-1)\omega\} \) for odd \( d \). On the other hand, for even \( d \) the centre is only \( \{1, -1\} \), and one can then decompose the Clifford group into
\[ F_d = F_{d,+} \cup F_{d,-}, \quad \text{with} \quad F_{d,\pm} = \{ \gamma \in F_d \mid \omega \gamma \omega^{-1} = (\pm 1)\gamma \}. \]

This means that we have either elements having an even number of the generators \( \gamma_1, \ldots, \gamma_d \) or elements having an odd number of these generators.

**Proposition 1.61.** The automorphism \( \alpha \) is always adjoinable, and

\[ \alpha(\gamma) = \sigma \gamma \sigma^{-1}, \quad \text{for all} \quad \gamma \in F_d. \quad (1.16) \]

The automorphisms \( \beta \) and \( \hat{\alpha} \) are not adjoinable for odd \( d \), and for even \( d \) one has

\[ \beta(\gamma) = \omega \gamma \omega^{-1}, \quad \hat{\alpha}(\gamma) = \hat{\sigma} \gamma \hat{\sigma}^{-1}, \quad \text{for all} \quad \gamma \in F_d. \quad (1.17) \]

**Proof.** For the first claim, it is sufficient to check that \( \sigma \gamma_n \sigma^{-1} = \kappa (-1)^{d+1} \gamma_n \). This is just a long elementary calculation. Let now \( d \) be odd and let us first show that \( \beta \) is not adjoinable. Suppose that there is an \( \eta \in F_d \) with \( \eta \gamma_n \eta^{-1} = (-1)\gamma_n \) for all \( n = 1, \ldots, d \). Then one would have \( \eta \omega \eta^{-1} = (-1)^d \omega = (-1)\omega \), which is in contradiction to the fact that \( \omega \) is in the centre for odd \( d \). For \( \hat{\alpha} \) one can argue similarly, or use that its adjoinability would imply the adjoinability of \( \beta \). The formulas in (1.17) can again be readily checked. \( \square \)

Furthermore, let us point out that \( \sigma \) is not uniquely determined by the adjoinable equation (1.16) for \( \alpha \). In fact, for any \( \eta \) in the centre of \( F_d \), one has, of course, \( \alpha(\gamma) = (\eta \sigma) \gamma (\eta \sigma)^{-1} \). For even \( d \), this only allows for a sign
change of \( \sigma \), but for odd \( d \) one can choose \( \eta = \omega \) such that \( \alpha(\gamma) = \tilde{\sigma}\gamma\tilde{\sigma}^{-1} \).
Indeed, \( \omega\sigma \) is either equal to \( \tilde{\sigma} \) or to \( (-1)\tilde{\sigma} \). Moreover, note that \( \omega \) and \( \tilde{\sigma} \) are the uniquely determined elements satisfying the adjoinable equation (1.17) up to a multiplication with \(-1\) for even \( d \).

### 1.6.3 Representations of the Clifford group

By a complex representation of the group \( F_d \) we understand a group homomorphism \( \pi_d : F_d \to \mathbb{C}^{d' \times d'} \) for some \( d' \in \mathbb{N} \). If this map is injective, the representation is called faithful, and it is called irreducible if there is no non-trivial subrepresentation. We have already seen such a representation of \( F_2 \) in Example 1.58.

Here, the task is to construct a particular faithful and irreducible complex representation such that \( \pi_d(\gamma_{2n-1}) \) are real and \( \pi_d(\gamma_{2n}) \) are purely imaginary matrices for all fitting \( n \), and, moreover, \( \pi_d(-1) = -\pi_d(1) \). This alternation is feasible in an irreducible complex representation, and it turns out that the dimension of the representation is given by \( d' = 2\lfloor \frac{d}{2} \rfloor \), except for the special case of \( d = 1 \), for which \( d' = 2 \), as we have seen in Example 1.58. The dimension of a real representation, which means choosing real matrices \( \pi_d(\gamma_n) \), would be larger, in general. The construction presented here is done iteratively using the \( 2 \times 2 \) Pauli matrices

\[
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.18}
\]

Also let \( \mathbb{1}_n \in \mathbb{C}^{n \times n} \) denote the identity matrix. Then \( \pi_d(-1) = -\mathbb{1}_{d'} \) and

\[
\pi_d(\gamma_n) = \sigma_3 \otimes \pi_{d-2}(\gamma_{n-2}), \quad \pi_d(\gamma_1) = \sigma_1 \otimes \mathbb{1}_{\frac{d'}{2}}, \quad \pi_d(\gamma_2) = \sigma_2 \otimes \mathbb{1}_{\frac{d'}{2}}. \tag{1.19}
\]

In the following, we will use the short notation \( \Gamma_n := \pi_d(\gamma_n) \). Note that for even \( d \), the representations \( \pi_d \) are given by the restriction of \( \pi_{d+1} \), meaning just rejecting \( \Gamma_{d+1} \) as an element of the group. For sake of concreteness, let us write out the non-trivial generators of the first few representations explicitly:

\[
F_1 : \quad \Gamma_1 = \sigma_1.
\]
$F_2$ and $F_3$ : $\Gamma_1 = \sigma_1$, $\Gamma_2 = \sigma_2$, $\Gamma_3 = \sigma_3$.

$F_4$ and $F_5$ : $\Gamma_1 = \sigma_1 \otimes 1_2$, $\Gamma_2 = \sigma_2 \otimes 1_2$, $\Gamma_3 = \sigma_3 \otimes \sigma_1$, $\Gamma_4 = \sigma_3 \otimes \sigma_2$, $\Gamma_5 = \sigma_3 \otimes \sigma_3$.

$F_6$ and $F_7$ : $\Gamma_1 = \sigma_1 \otimes 1_2 \otimes 1_2$, $\Gamma_2 = \sigma_2 \otimes 1_2 \otimes 1_2$, $\Gamma_3 = \sigma_3 \otimes \sigma_1 \otimes 1_2$, $\Gamma_4 = \sigma_3 \otimes \sigma_2 \otimes 1_2$, $\Gamma_5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1$, $\Gamma_6 = \sigma_3 \otimes \sigma_3 \otimes \sigma_2$, $\Gamma_7 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3$.

The above representation $\pi_d$ naturally extends to a complex irreducible representation of the Clifford algebra $\mathcal{C}_d$, which we will call by the same name. This means that we represent the real Clifford algebra $\mathcal{C}_d$ as a real algebra embedded into a complex matrix algebra. Clearly, $\pi_d(\mathcal{C}_d)$ is a subalgebra of $\pi_{d+1}(\mathcal{C}_{d+1})$ for even $d$. Furthermore, $1_2 \otimes \pi_{d-1}(\mathcal{C}_{d-1})$ is a subalgebra of $\pi_d(\mathcal{C}_d)$, still for even $d$. The representation (1.19) is not unique as faithful irreducible representation with real $\Gamma_{2n-1}$ and purely imaginary $\Gamma_{2n}$. There are unitarily equivalent representations, but arguments similar as in the proof of Proposition 1.63 below allow to reduce to the above for even $d$. On the other hand, it is well known that for odd $d$ there are two inequivalent representations. A second representation is obtained by changing the sign of just one of the $\Gamma$-matrices, say $\Gamma_d$, so this changes the sign of $\Gamma_1 \cdots \Gamma_d$ that is, moreover, proportional to the identity. One particular unitarily equivalent representation is obtained by exchanging the roles of $\sigma_1$ and $\sigma_3$, and simultaneously adding a sign to each $\sigma_2$. The real unitary basis change is given by

$$f = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = f^* = f^{-1},$$

which satisfies $f^* \sigma_1 f = \sigma_3$ and $f^* \sigma_2 f = -\sigma_2$ and can be extended as $f^{\otimes \lfloor \frac{d}{2} \rfloor}$ to the representation of $F_d$. It is also possible to cyclically exchange the roles of the Pauli matrices using the Cayley transform in $\mathbb{C}^2$ defined by

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$  \hspace{1cm} (1.20)
It satisfies $C^3 = e^{-\frac{1\pi}{4}} \mathbb{1}_2$ as well as

$$C\sigma_1 C^* = \sigma_2, \quad C\sigma_2 C^* = \sigma_3, \quad C\sigma_3 C^* = \sigma_1.$$ 

Conjugating with $C^\otimes \lfloor \frac{d}{2} \rfloor$ leads to a unitarily equivalent representation $\Gamma'_n$, for $n = 1, \ldots, d$. This representation has real $\Gamma'_{2n+1}$ and imaginary $\Gamma'_{2n}$, except for $\Gamma'_d$ in the case of odd $d$, which is always real.

**1.6.4 Symmetry operators in the representation of the Clifford group**

Next, let us consider the representations of the elements $\sigma$, $\omega$ and $\hat{\sigma}$ introduced in Definition 1.60 above. They will be decorated by phase factors so that they become symmetry operators in the sense of Definition 1.43:

$$\Sigma = i \lfloor \frac{d}{2} \rfloor \pi_d(\sigma), \quad \Omega = (-i)^{\lfloor \frac{d}{2} \rfloor} \pi_d(\omega), \quad \hat{\Sigma} = \kappa^d \pi_d(\hat{\sigma}).$$

Here, the imaginary factors are important in order to produce real matrices, but the additional signs are not established and merely added to produce nice formulas in (1.21), (1.22) and (1.23) below. Note again that all these matrices depend on $d$ but we suppress this for sake of notational simplicity as long as there is no ambiguity. The added factors are chosen such that $\Sigma$, $\Omega$ and $\hat{\Sigma}$ are real and one has

$$\hat{\Sigma} = \Omega \Sigma, \quad \Omega = \left\{ \begin{array}{ll} \sigma_3^\otimes \frac{d}{2}, & d \text{ even}, \\ \mathbb{1}_{d'}, & d \text{ odd} \end{array} \right.$$ 

(1.21)

Note that $\Omega = \Gamma_{d+1}$ for even $d$. Let us also point out that there is an alternative construction of $\Sigma$ that, using the supplementary variable on $\Sigma(d)$, is iteratively given by

$$\Sigma(d) = -\Sigma(d-4) \otimes \sigma_1 \otimes i\sigma_2,$$

$$\Sigma(0) = \Sigma(1) = \mathbb{1}_{d'}, \quad \Sigma(2) = \Sigma(3) = i\sigma_2.$$ 

(1.22)
Furthermore, one has for some sign $\eta \in \{-1, 1\}$:

\[
\hat{\Sigma}(d) = \eta \Sigma(d - 1) \otimes \sigma_1, \quad \text{for } d \text{ even}.
\]  

For sake of concreteness, let us write out these symmetry operators for low dimensions:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\Omega$</th>
<th>$\Sigma$</th>
<th>$\hat{\Sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\sigma_3$</td>
<td>$i\sigma_2$</td>
<td>$\sigma_1$</td>
</tr>
<tr>
<td>3</td>
<td>$i\sigma_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\sigma_3 \otimes \sigma_3$</td>
<td>$-\sigma_1 \otimes i\sigma_2$</td>
<td>$i\sigma_2 \otimes \sigma_1$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$-\sigma_1 \otimes i\sigma_2$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\sigma_3 \otimes \sigma_3 \otimes \sigma_3$</td>
<td>$-i\sigma_2 \otimes \sigma_1 \otimes i\sigma_2$</td>
<td>$\sigma_1 \otimes i\sigma_2 \otimes \sigma_1$</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>$-i\sigma_2 \otimes \sigma_1 \otimes i\sigma_2$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3$</td>
<td>$\sigma_1 \otimes i\sigma_2 \otimes \sigma_1 \otimes i\sigma_2$</td>
<td>$i\sigma_2 \otimes \sigma_1 \otimes i\sigma_2 \otimes \sigma_1$</td>
</tr>
</tbody>
</table>

The main properties of $\Omega$ and $\Sigma$ are recollected in the next Proposition 1.62 and Proposition 1.63 below and can easily be checked with some patience. Moreover, we now show that these symmetry operators can be uniquely characterised.

**Proposition 1.62.** Consider $\pi_d(\mathcal{O}_d) \subset \mathbb{C}^{d' \times d'}$ with dimension $d' = 2^{\left\lfloor \frac{d}{2} \right\rfloor}$. Set $\kappa = (-1)^{\left\lfloor \frac{d}{2} \right\rfloor}$ as a sign, analogously as above.

(i) Up to a sign there is a unique real unitary $\Sigma \in \mathbb{C}^{d' \times d'}$, given by the construction above, satisfying

\[
\Sigma^* \Gamma_n \Sigma = -\kappa (-1)^n \Gamma_n, \quad n = 1, \ldots, d.
\]

This can be restated as $\Sigma^* \overline{\Gamma_n} \Sigma = \kappa \Gamma_n$. 


(ii) For even \( d \) and up to a sign, there is a unique real unitary \( \Omega \in \mathbb{C}^{d' \times d'} \), given by (1.21), satisfying
\[
\Omega^* \Gamma_n \Omega = -\Gamma_n, \quad n = 1, \ldots, d.
\]

(iii) For even \( d \) and up to a sign, there is a unique real unitary \( \Sigma \) characterised by
\[
\hat{\Sigma}^* \Gamma_n \hat{\Sigma} = \kappa (-1)^n \Gamma_n, \quad n = 1, \ldots, d,
\]
which can be restated as \( \hat{\Sigma}^* \Gamma_n \hat{\Sigma} = -\kappa \Gamma_n \). For odd \( d \) there exists no \( \hat{\Sigma} \) satisfying (1.24).

(iv) The following relations in Table 1.1 below hold with \( \mathbb{1} = \mathbb{1}_{d'} \).

<table>
<thead>
<tr>
<th>( d \mod 8 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma^2 = )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \kappa = )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \Omega^2 = )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \Omega \Sigma = )</td>
<td>-( \Sigma \Omega )</td>
<td>( \Sigma \Omega )</td>
<td>-( \Sigma \Omega )</td>
<td>( \Sigma \Omega )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \hat{\Sigma}^2 = )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \hat{\Sigma} \Sigma = )</td>
<td>-( \Sigma \hat{\Sigma} )</td>
<td>( \Sigma \hat{\Sigma} )</td>
<td>-( \Sigma \hat{\Sigma} )</td>
<td>( \Sigma \hat{\Sigma} )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( \Omega \hat{\Sigma} = )</td>
<td>-( \Sigma \Omega )</td>
<td>( \Sigma \Omega )</td>
<td>-( \Sigma \Omega )</td>
<td>( \Sigma \Omega )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1.1: Properties of symmetry operators

**Proof.** For (i): For \( d = 1 \) the claim is obvious. Hence, let us suppose \( d \geq 2 \) and first consider the case of even \( d \). The proof will be done by induction over \( d \) and, therefore, the variable \( d \) will be added on \( \Gamma_j(d) \) and \( \Sigma(d) \) for sake of clarity. Let us start by writing \( \Gamma_1(d) \), \( \Gamma_2(d) \) and \( \Sigma(d) \) in the grading of the first fiber of \( \mathbb{C}^{d'} = \mathbb{C}^2 \otimes \mathbb{C}^{d'_2} \):
\[
\Gamma_1(d) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2(d) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma(d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
1.6 Clifford groups and algebras

The imposed commutation relations can be written as \( \Gamma_1(d)^* \Sigma(d) \Gamma_1(d) = \kappa \Sigma(d) \) and \( \Gamma_2(d)^* \Sigma(d) \Gamma_2(d) = -\kappa \Sigma(d) \). A short calculation shows that with a real unitary \( b = \Sigma' \) acting on \( \mathbb{C}^{d'} \) one has

\[
\Sigma(d) = \begin{pmatrix} 0 & \Sigma' \\ \kappa \Sigma' & 0 \end{pmatrix} = \begin{cases} 
\sigma_1 \otimes \Sigma', & \kappa = 1, \\
\kappa \Sigma' & \sigma_2 \otimes \Sigma', & \kappa = -1.
\end{cases}
\]

Now for \( j \geq 3 \) one has \( \Gamma_j(d) = \sigma_3 \otimes \Gamma_{j-2}(d-2) \) by definition and hence, due to the anti-commutation property in the first factor, the other required relations become

\[
(\Sigma')^* \Gamma_{j-2}(d-2) \Sigma' = \kappa(-1)^{j-2} \Gamma_{j-2}(d-2), \quad j = 3, \ldots, d.
\]

But this determines \( \Sigma' \) uniquely by the induction hypothesis, just up to a sign. Hence, the uniqueness of \( \Sigma(d) \) holds for even \( d \). For odd \( d \), \( \Sigma(d) = \Sigma(d-1) \) is also unique and has the right commutation relation with \( \Gamma_d(d) \).

For (ii): The proof is analogous to the argument in (i), but more simple.

For (iii): The proof for \( \Sigma \) is identical to (i), except that for odd \( d \) the required commutation relation with \( \Gamma_d(d) \) does not hold.

For (iv): This follows from a straightforward calculation.

The following general result for operators in a Hilbert space allows to bring the two commuting or anti-commuting symmetry operators into a normal form, which means into a suitable block form. It will then be applied to the representations of the Clifford group.

**Proposition 1.63.** Let \( \Omega \) and \( \Sigma \) be symmetry operators on a Hilbert space with real structure. Suppose that \( \text{spec}(\Omega) = \{-1, 1\} \) and let \( \eta, \tau \in \{-1, 1\} \) be such that \( \Sigma^2 = \eta 1 \) and \( \Omega \Sigma = \tau \Sigma \Omega \). Then, in each of the following cases, there is a real unitary \( O \) with

(i) \( O^* \Omega O = \sigma_3 \otimes 1 \) and \( O^* \Sigma O = \sigma_1 \otimes 1 \) for \( \eta = 1 \) and \( \tau = -1 \).

(ii) \( O^* \Omega O = \sigma_3 \otimes 1 \) and \( O^* \Sigma O = i \sigma_2 \otimes 1 \) for \( \eta = -1 \) and \( \tau = -1 \).

(iii) \( O^* \Omega O = \sigma_3 \otimes 1 \) and \( O^* \Sigma O = 1 \otimes i \sigma_2 \) for \( \eta = -1 \) and \( \tau = 1 \).
(iv) \( O^\ast \Omega O = \sigma_3 \otimes 1 \otimes 1 \) and \( O^\ast \Sigma O = 1 \otimes \sigma_3 \otimes 1 \) for \( \eta = 1 \) and \( \tau = 1 \) and provided that \( \Omega \) and \( \Sigma \) have 4 common eigenspaces of the same dimension.

Here, we have used the Pauli matrices \( \sigma_1, \sigma_2 \) and \( \sigma_3 \), defined in (1.18).

**Proof.** The eigenspaces of \( \Omega \) are invariant under complex conjugation since \( \overline{\Omega} = \Omega \) by definition of a symmetry operator. Hence, it is possible to choose a real basis for both eigenspaces. These basis are by hypothesis of equal dimension because this is also part of Definition 1.45, and constitute \( O' \) satisfying \( (O')^\ast \Omega O' = \sigma_3 \otimes 1 \). If now \( \tau = -1 \), one has \( (O')^\ast \Sigma O' = \begin{pmatrix} 0 & a \\ \eta a^\ast & 0 \end{pmatrix} \) with some real unitary \( a \). Setting \( O = O' \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \) concludes the proof of (i) and (ii). If \( \tau = 1 \), then \( (O')^\ast \Sigma O' = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) with real unitaries \( a \) and \( b \). If \( \eta = -1 \) both of these can be diagonalised to \( i \sigma_2 \) by a real unitary. Indeed, the spectrum of \( a \) is \( \{i, -i\} \) and the associated eigenspaces \( \mathcal{E}_{-i} \) and \( \mathcal{E}_i \) are complex conjugates of each other and are, in particular, of the same dimension. Hence, there is a unitary \( V = (\overline{v}, v) \) built from the basis \( v = (v_1, v_2, \ldots) \) of \( \mathcal{E}_i \) such that \( V^\ast a V = -i\sigma_3 \otimes 1 \). Now the operator block Cayley transform \( C' = C \otimes 1 \), with \( C \) defined in (1.20), leads to a real unitary \( VC' \), which satisfies \( (VC')^\ast a VC' = i\sigma_2 \otimes 1 \). Similarly, \( (WC')^\ast b WC' = i\sigma_2 \otimes 1 \) and then \( O = O' \begin{pmatrix} VC' & 0 \\ 0 & WC' \end{pmatrix} \) has the desired properties. The proof of (iv) is done by diagonalising \( a \) and \( b \) and using the supplementary hypothesis. \( \square \)

**Proposition 1.64.** The symmetries \( \Omega \) and \( \Sigma \) defined by (1.21) and (1.22) have common eigenspaces of same dimension. Hence, also item (iv) of Proposition 1.63 applies.

**Proof.** The following argument also allows to construct the basis changes \( O \) in Proposition 1.63 explicitly for all four cases when \( \Omega \) and \( \Sigma \) are the symmetry operators of the Clifford algebra. Let us introduce the vectors

\[
e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_{-1} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

in \( \mathbb{C}^2 \). Furthermore, for a multi-index \( i = (i_1, \ldots, i_d) \in \{-1, 1\}^d \) let us introduce the complementary index \( i^c \) obtained by flipping all signs. Now the vectors \( v_i = e_{i_1} \otimes \ldots \otimes e_{i_d} \), with \( i \) running through \( \{-1, 1\}^d \), constitute an orthonormal
basis of \( \mathbb{C}^{d'} \). Moreover, this is an eigenbasis of \( \Omega \) and we have the eigenvalue equation \( \Omega v_i = i_1 \cdots i_d v_i \). Now one checks that for some sign \( \kappa \in \{-1, 1\} \) we have

\[
(v_j)^{\ast} \Sigma v_i = \kappa \delta_{j,i}, \quad (v_i)^{\ast} \Sigma v_j = \kappa \delta_{j,i}.
\]

Consequently, \( \Sigma \) is block diagonal in this basis with \( 2 \times 2 \) blocks given by

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

Both blocks have eigenvalues 1 and \(-1\) of equal multiplicity.

\[
\begin{array}{cccc}
\hline
 d \mod 8 & O^{\ast} \Sigma O & \kappa & O^{\ast} \Omega O \\
\hline
 1 & \sigma_1 \otimes 1 & 1 & \\
 2 & i\sigma_2 \otimes 1 & -1 & \sigma_3 \otimes 1 \\
 3 & i\sigma_2 \otimes 1 & -1 & \\
 4 & 1 \otimes i\sigma_2 & 1 & \sigma_3 \otimes 1 \\
 5 & i\sigma_2 \otimes 1 & 1 & \\
 6 & \sigma_1 \otimes 1 & -1 & \sigma_3 \otimes 1 \\
 7 & \sigma_1 \otimes 1 & -1 & \\
 8 & 1 \otimes \sigma_1 \otimes 1 & 1 & \sigma_3 \otimes 1 \otimes 1 \\
\hline
\end{array}
\]

Table 1.2: Basis change with real unitary \( O \)

Combining Proposition 1.63 and Proposition 1.64, one finds that there is always an adequate orthogonal basis change \( O \) that brings the two symmetries in the normal form given in the Table 1.2 above.
2 Operator algebras, $K$-theory and Symmetries

This chapter serves as a preparation for the presented results about topological insulators in Chapter 4 and about the oscillation theory in Chapter 5. Mainly, we introduce $C^*$-algebras as objects of research, traces on them and the associated $K$-theory, which we will use for classification of topological insulators later.

2.1 Some operator algebras

2.1.1 Banach algebras

By a real or complex algebra, we understand a vector space $\mathcal{A}$ over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ together with a bilinear map $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ that is associative. We simply call such a map a multiplication. If there is a multiplicative unit $1 \in \mathcal{A}$, we call the algebra unital, and if the multiplication is commutative, we call the algebra commutative.

Definition 2.1. A Banach space $(\mathcal{A}, \| \cdot \|)$ together with a multiplication is called a Banach algebra if $\|ab\| \leq \|a\|\|b\|$ holds for all $a, b \in \mathcal{A}$.

Here, a Banach algebra can be a complex or a real algebra, but note that some authors use the term Banach algebra only for complex ones.

Definition 2.2. Let $\mathcal{A}$ be a real or complex algebra.

(a) A map $^* : \mathcal{A} \rightarrow \mathcal{A}$, $a \mapsto a^*$ is called a *-involution if

$$
(a^*)^* = a, \quad (ab)^* = b^*a^*, \quad (a + \lambda b)^* = a^* + \bar{\lambda}b^*,
$$
for all \(a, b \in \mathcal{A}\) and \(\lambda \in \mathbb{K}\).

(b) \(\mathcal{A}\) together with a \(*\)-involution is called a \(*\)-algebra.

(c) A Banach algebra \(\mathcal{A}\) together with a chosen \(*\)-involution is called a Banach \(*\)-algebra.

### 2.1.2 C*-algebras and von Neumann algebras

One of the most important objects of research in this work are so-called operator algebras, which always mean \(C^*\)-algebras and von Neumann algebras here. The standard example is \(\mathcal{B}(\mathcal{H})\), and, moreover, all operator algebras can be seen as subalgebras of \(\mathcal{B}(\mathcal{H})\) by the Gelfand-Naimark theorem, cf. [Dix77, Theorem 2.6.1]. We mostly use lower-case letters for elements of our algebras, but when they act as linear maps, we also sometimes use capitals.

**Definition 2.3.** A complex Banach space \((\mathcal{A}, \| \cdot \|)\) together with a multiplication and a \(*\)-involution is called a \(C^*\)-algebra if

\[
\|ab\| \leq \|a\| \|b\|, \quad \|a^*a\| = \|a\|^2
\]

for all \(a, b \in \mathcal{A}\) and \(\lambda \in \mathbb{C}\).

The morphisms in the category of \(C^*\)-algebras are algebra-homomorphisms that preserve the involution and are, therefore, called \(*\)-homomorphisms. Note that by definition \(C^*\)-algebras are complex algebras, and this is in some sense essential. However, we will also describe how to transfer the essential properties to real algebras since these also play an important role in Chapter 4 when describing symmetry relations.

The equation \(\|a^*a\| = \|a\|^2\) is often called the \(C^*\)-property or \(C^*\)-equation since it is the most important ingredient of \(C^*\)-algebras.

**Remark 2.4.** The \(C^*\)-property can be weakened to \(\|a^*a\| \geq \|a\|^2\) for all \(a \in \mathcal{A}\). Combining this with the submultiplication, one gets \(\|a^*\| \leq \|a\|\). Then using the involution, one immediately gets \(\|a^*\| = \|a\|\). From this, we can conclude the other inequality \(\|a^*a\| \leq \|a\|^2\) for \(a \in \mathcal{A}\).
2.1 Some operator algebras

We call a C*-algebra **unital** if the algebra has an multiplicative unit. We always denote the unit by \( 1 \) or by \( 1_A \) if necessary. Note that we will have cases where there is no unit in the C*-algebra.

Example 2.5. (a) If \( \Omega \) is a locally compact Hausdorff space, then \( A = C_0(\Omega) \) is a commutative C*-algebra. Here, \( C_0(\Omega) \) denotes the set of complex functions \( f \) that satisfy that for all \( \varepsilon > 0 \) there is a compact set \( K \) such that \( |f(x)| < \varepsilon \) for all \( x \in \Omega \setminus K \). The C*-algebra \( A \) is unital if and only if \( \Omega \) is compact.

(b) Given a Hilbert space \( \mathcal{H} \), the space \( B(\mathcal{H}) \) is a unital C*-algebra. If the dimension \( \dim \mathcal{H} > 1 \), the algebra is non-commutative.

(c) For an infinite-dimensional Hilbert space \( \mathcal{H} \), the space of compact operators \( K(\mathcal{H}) \) is a non-unital C*-algebra.

(d) Let \( A \in B(\mathcal{H}) \) for an arbitrary Hilbert space \( \mathcal{H} \). The norm closure of the set of all polynomials in the three variables \( A, A^* \) and \( 1 \) is a unital C*-algebra and denoted by \( C^*(A, 1) \).

Definition 2.6. For a C*-algebra \( A \), a \( * \)-homomorphism \( \varphi : A \to B(\mathcal{H}) \) is called a representation of \( A \). If \( \varphi \) is injective, the representation is called faithful.

All notions for operators in Hilbert spaces that can be formulated in an algebraic manner are also used in C*-algebras. This means, for example, that an element \( a \in A \) is called normal if \( a^*a = aa^* \) and self-adjoint if \( a^* = a \). Other properties will be discussed later.

Although we will mostly work with general C*-algebras here, it is sometimes convenient to restrict to some special operator algebras that have sufficiently many projections to work with. In particular, this is important when one needs so-called semi-finite traces, see, e.g. [LSZ12]. Here, we want to give a short survey of these operator algebras that were primarily introduced by John von Neumann in 1929, see [Neu30b].

Definition 2.7. A unital C*-algebra \( A \) is called a von Neumann algebra if it fulfils one of the following equivalent properties:
(a) $\mathcal{A} \subset \mathcal{B}($H$)$, and $\mathcal{A}'' = \mathcal{A}$ holds. Here, $\mathcal{A}''$ denotes the bicommutant $(\mathcal{A}')'$ where $\mathcal{S}' := \{T \in \mathcal{B}($H$) \mid ST = TS \text{ for all } S \in \mathcal{S}\}$ for a subset $\mathcal{S} \subset \mathcal{B}($H$)$.

(b) $\mathcal{A} \subset \mathcal{B}($H$)$, and it is closed in the weak operator topology.

(c) $\mathcal{A} \subset \mathcal{B}($H$)$, and it is closed in the strong operator topology.

(d) $\mathcal{A} \subset \mathcal{B}($H$)$, and it is closed in the weak* topology obtained from the predual of $\mathcal{B}($H$)$, the trace class operators.

(e) $\mathcal{A}$ is the topological dual space of a Banach space.

Obviously, we have hidden some deep mathematical results in the definition above. Firstly, one should note that von Neumann algebras are mostly defined and given as concrete operator algebras. Only item (e) shows how one can define a von Neumann algebra as a completely abstract C*-algebra that carries besides a norm topology a chosen weak* topology. However, one mostly does not work with this definition. The equivalence of (b) and (c) holds for every convex set $\mathcal{S} \subset \mathcal{B}($H$)$. This means that the closure of $\mathcal{S}$ in the weak operator topology coincides with the closure of $\mathcal{S}$ in the strong operator topology. The equivalence of (c) and (a) is known as the von Neumann bicommutant theorem and combines the topological nature with the algebraic structure, cf. [Neu30b]. The last equivalence (a) and (e) is due to Shōichirō Sakai, see [Sak62].

Example 2.8. For a $\sigma$-finite measure space $(\Omega, \mathcal{A}, \mu)$, we denote the Hilbert space of equivalence classes of square integrable functions on $\Omega$ by $\mathcal{H} = L^2(\Omega, \mu)$. Moreover, we write $L^\infty(\Omega, \mu)$ for the equivalence classes of essentially bounded functions on $\Omega$. Since we can identify each $\phi \in L^\infty(\Omega, \mu)$ with the multiplication operator

$$M_\phi : \mathcal{H} \rightarrow \mathcal{H}, \quad (M_\phi f)(t) = \phi(t) f(t) \quad \mu\text{-almost everywhere,}$$

we read $L^\infty(\Omega, \mu) \subset \mathcal{B}($H$)$, which is a C*-algebra. It is easy to show that, with this identification, we have $(L^\infty(\Omega, \mu))' = L^\infty(\Omega, \mu)$. Since all multiplication operators commute, $L^\infty(\Omega, \mu)$ is a commutative von Neumann algebra.
2.1.3 Building new C*-algebras

Recall that for rings one has the notion of an ideal. This is a subgroup with respect to the addition and closed under the multiplication by arbitrary ring elements. If the ring is non-commutative, one has to distinguish left ideals, right ideals and two-sided ideals. For simplicity, we will only use two-sided closed ideals in the theory of C*-algebras.

**Proposition 2.9.** Let \( J \) be a two-sided and closed ideal in the C*-algebra \( A \). The quotient \( A/J \) is a C*-algebra with norm given by

\[
\|a + J\| := \inf\{\|a + j\| \mid j \in J\}.
\]

**Proof.** See, for example, [Weg93, Chapter 1.4]. \( \square \)

**Definition 2.10.** For two C*-algebras \( A \) and \( B \), we define \( A \oplus B \) as the C*-algebra given by the set \( A \times B \) with componentwise addition and multiplication. We call it the direct sum of \( A \) and \( B \).

**Definition 2.11.** For two C*-algebras \( A \) and \( B \), the completion of the algebraic tensor product \( A \odot B \) with respect to the spatial C*-norm, which is the smallest possible C*-norm on \( A \odot B \), is called the spatial tensor product of \( A \) and \( B \) and denoted by \( A \otimes B \).

For the definition and an elaborate presentation of the spatial C*-norm and the spatial tensor product, see [Weg93, Appendix T].

**Definition 2.12** (Matrix algebras). For a C*-algebra \( A \), we denote the matrix algebra of dimension \( n \in \mathbb{N} \) by

\[
M_n(A) := \left\{ (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in A, \ i, j = 1, \ldots n \right\},
\]

where addition is componentwise defined and the multiplication is the formal matrix multiplication. By choosing a faithful representation for \( A \) by an injective \(*\)-homomorphism \( \varphi : A \to B(\mathcal{H}) \), the above matrix algebra \( M_n(A) \) is
naturally also a C*-algebra, where the involution is given by \((a_{ij})^* := (a_{ji}^*)\) and the norm defined by:

\[
\| (a_{ij}) \| := \left\| \begin{pmatrix} \varphi(a_{11}) & \cdots & \varphi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{n1}) & \cdots & \varphi(a_{nn}) \end{pmatrix} \right\|_{\mathcal{H}^n \to \mathcal{H}^n}
\]

Here, the latter norm denotes the operator norm in \(\mathcal{B}(\mathcal{H}^n)\) with \(\mathcal{H}^n = \mathcal{H} \oplus \cdots \oplus \mathcal{H}\) as the usual orthogonal sum.

Note that the Hilbert space \(\mathcal{H}^n\) has a well-defined scalar product, and therefore the operator norm of \(\mathcal{B}(\mathcal{H}^n)\) above is well-defined. Choosing a representation is indeed always possible by the Gelfand-Naimark theorem, see Theorem 2.23 below. Moreover, the definition of the norm in \(M_n(\mathcal{A})\) is independent of the choice of the representation \(\varphi\), see, e.g. [RLL00, Chapter 1.3].

**Lemma 2.13.** If \(\mathcal{A}\) is a C*-algebra, then we define for each pair \(a, b \in \mathcal{A}\) the element

\[
\text{diag}(a, b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A}).
\]

The norm is then given by

\[
\| \text{diag}(a, b) \| = \max\{\|a\|, \|b\|\}.
\]

**Proof.** By definition, the norm of \(\text{diag}(a, b)\) is given via a Hilbert space \(\mathcal{H}\) and an injective morphism \(\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})\). Then for each diagonal operator \(T = \text{diag}(A, B) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})\), one has

\[
\|T\|_{\mathcal{H}^2 \to \mathcal{H}^2} = \max\{\|A\|_{\mathcal{H} \to \mathcal{H}}, \|B\|_{\mathcal{H} \to \mathcal{H}}\}
\]

by usual matrix calculation. Therefore, we conclude

\[
\| \text{diag}(a, b) \| = \| \text{diag}(\varphi(a), \varphi(b)) \|_{\mathcal{H}^n \to \mathcal{H}^n} = \max\{\|a\|, \|b\|\},
\]

where we used that each injective map \(\varphi\) is isometric. \(\square\)
One can show that forming matrices is a functor in the category of C*-algebras as stated below. Here, we use the following inclusion definitions: For numbers $n, m \in \mathbb{N}$ there is a naturally given map $\oplus : M_n(A) \times M_m(A) \to M_{n+m}(A)$ by identifying the diagonal block matrix $a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with the matrix in $M_{n+m}(A)$. In this sense, we also obtain the inclusion $M_n(A) \subset M_{n+m}(A)$ by identifying $a = a \oplus 0$.

**Proposition 2.14.** Let $A$ and $B$ be C*-algebras and $\varphi : A \to B$ a *-homomorphism. Then for all $n \in \mathbb{N}$, the map $\varphi_n : M_n(A) \to M_n(B)$ given by $\varphi_n((a_{ij}))_{ij} = (\varphi(a_{ij}))_{ij}$ is a *-homomorphism. In particular, the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow i & & \downarrow i \\
M_n(A) & \xrightarrow{\varphi_n} & M_n(B)
\end{array}
$$

Here, $i$ denotes the natural inclusion map from above.

**Proof.** This directly follows from a short calculation. \qed

### 2.1.4 Exact sequences of C*-algebras

We will often consider so-called *short exact sequences of C*-algebras*. These are sequences given by three C*-algebras $A_1, A_2, A_3$ and four *-homomorphisms

$$
0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} 0
$$

(2.1)

where $\text{Ran}(f_j) = \text{Ker}(f_{j+1})$ holds for all $j = 0, \ldots, 2$. Since $f_1(A_1) = \text{Ker}(f_2)$ is a closed two-sided ideal in $A_2$, we can directly identify $A_1$ with $J := f_1(A_1)$. Of course, one could shorten the diagram by using arrows for injection and surjection, and then the exact sequence above can be written as

$$
\begin{array}{c}
J \\
\xleftarrow{\iota} A \\
\xrightarrow{p} A/J
\end{array}
$$

where we use $A := A_2$, the inclusion map $\iota$ and $p$ as the projection $a \mapsto [a]$.

A short exact sequence of C*-algebras given by equation (2.1) is called a *split exact sequence at $A_2$* if there is a right inverse for $f_2$, which means a
*-homomorphism \(g_2 : A_3 \to A_2\). We denote a split exact sequence by the following diagram:

\[
0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} 0.
\]

### 2.1.5 Unitisation of C*-algebras

Since we often discover C*-algebras that do not have a unit but always want to talk about invertibility and the spectrum, we are forced to introduce a method of ‘adding a unit’. In the following, we will define for each C*-algebra \(A\) a slightly larger C*-algebra \(A^+\) that is unital and contains \(A\) as an ideal such that \(A^+/A \cong \mathbb{C}\). We will call \(A^+\) the *unitisation of \(A\) even if \(A\) already is a unital algebra.

**Definition 2.15.** For each C*-algebra \(A\), we define the set

\[
A^+ := A \times \mathbb{C} = \{(a, \lambda) \mid a \in A, \lambda \in \mathbb{C}\}
\]

and equip it with pointwise summation and mixed multiplication:

\[
(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda \mu).
\]

We have the embedding \(i : A \to A^+\) by \(i(a) = (a, 0)\), and, in this sense, we can just identify \(A \subset A^+\). We denote the projection onto the second argument by

\[
\pi : A^+ \to \mathbb{C}, \ (a, \lambda) \mapsto \lambda.
\]

For an \(x \in A^+\), we define the norm \(\|x\|_{A^+}\) as the maximum of \(\pi(x)\) and \(\sup\{\|ax\|_A \mid a \in A, \|a\|_A \leq 1\}\). Together with this norm, \(A^+\) is a C*-algebra with unit \(1_{A^+} = (0, 1)\) and called the *unitisation of \(A\). In accordance with the multiplication and the embedding \(i\), we will write \(a + \lambda 1_{A^+} := (a, \lambda)\) for elements in \(A^+\).

Definition 2.15 above contains some statements that are not clear and should be proven. We will show that \(\| \cdot \|_{A^+}\) is indeed a norm and that \(A^+\), with this definition, is a C*-algebra.
2.1 Some operator algebras

Proof. Of course, we have $\|b\|_{A^+} = \|b\|_A$ for all $b \in A$ because the C*-property holds for $\| \cdot \|_A$, namely $\|a^* a\|_A = \|a\|^2_A$ for all $a \in A$. To show that $\| \cdot \|_{A^+}$ is positive definite, choose an $x \in A^+$ with $\|x\|_{A^+} = 0$. Then, we have $x = (b, 0)$ with $b \in A$ since $\pi(x)$ has to vanish. Hence, we obtain $\|ax\|_A = 0$ for all $a \in A$, and therefore

$$0 = \|b^* b\|_A = \|b\|^2_A.$$  

In the next step, we show the triangular inequality. We will use the short notation

$$\|x\|' := \sup\{\|ax\|_A \mid a \in A, \|a\| \leq 1\}.$$  

For this function $\| \cdot \|'$, we can prove the desired inequality by using the norm properties of $\| \cdot \|_A$:

$$\|x + y\|' \leq \sup\{\|ax\|_A + \|ay\|_A \mid a \in A, \|a\| \leq 1\} \leq \|x\|' + \|y\|'.$$

The triangular inequality is also true for $|\pi(\cdot)|$, and by combining these results, we get that $\| \cdot \|_{A^+}$ is a norm on the algebra $A^+$.

In the same manner, we can show that the norm is submultiplicative: We choose $x, y \in A^+$ with $x \neq 0$ and obtain

$$\frac{1}{\|x\|_{A^+}} \|xy\|' = \sup\left\{\left\| \left( a \frac{x}{\|x\|_{A^+}} \right) y \right\|_A \mid a \in A, \|a\| \leq 1 \right\}$$  

$$\leq \sup\{\|by\|_A \mid b \in A, \|b\| \leq 1\} \leq \|y\|_{A^+}.$$  

In combination with the submultiplicativity for $|\pi(\cdot)|$, this shows the submultiplicativity for $\| \cdot \|_{A^+}$.

It is straightforward to show that we have a Banach space. Thus, all that remains to show is the C*-property for $\| \cdot \|_{A^+}$. Choose $x \in A^+$ and $a \in A$ with $\|a\| \leq 1$, then we obtain by using the properties of $\| \cdot \|_A$:

$$\|a (x^* x)\|_A \geq \|(ax^*) x\|_A \|a^*\|_A \geq \|(ax^*) xa^*\|_A = \|ax^*\|_A \|xa^*\|_A.$$
From this, we can conclude:

\[ \|x^*x\|' \geq \|x\|^2. \]

Again, combining this with \(|\pi(\cdot)|\) shows the C*-property. \(\square\)

**Remark 2.16.** Note that the proof above also works for a real Banach *-algebra \(A\) that has the additional property \(\|a^*a\| = \|a\|^2\) for all \(a\). One just has to substitute \(\mathbb{C}\) by \(\mathbb{R}\) and gets a unital real Banach *-algebra \(A^+\).

We note that in the case that \(A\) is already unital, \(A^+\) can be identified with the direct sum \(A \oplus \mathbb{C}\), which means we have an isomorphism of C*-algebras given by \(A^+ \ni (a, \lambda) \mapsto (a + \lambda 1_A) \oplus \lambda \in A \oplus \mathbb{C}\). However, this is not true for non-unital C*-algebras, although this map is still an isomorphism of vector spaces. Therefore, we get a similar result if one embeds the non-unital \(A\) into \(B(\mathcal{H})\) for a chosen Hilbert space \(\mathcal{H}\). Then, one can define the smallest unital C*-algebra \(A^\sim \subset B(\mathcal{H})\) that contains \(A\). In this case \(A^\sim\) is isomorphic to \(A^+\) as C*-algebras.

The unitisation of a C*-algebra is part of a split exact sequence of C*-algebras of the following form

\[ 0 \longrightarrow A \xrightarrow{i} A^+ \xrightarrow{\pi} \mathbb{C} \longrightarrow 0, \]

where there exists an embedding \(i' : \mathbb{C} \to A^+\) given by \(i'(\lambda) = (0, \lambda)\), which is a right inverse to \(\pi\). Then \(s = i' \circ \pi : A^+ \to A^+\) extracts the scalar part and given by \(s(a + \lambda 1_{A^+}) = \lambda 1_{A^+}\).

**Proposition 2.17.** If \(\varphi : A \to B\) is a *-homomorphism between C*-algebras, then there is a unique extension to the unitisations such that \(\varphi^+ : A^+ \to B^+\) is a *-homomorphism with \(\varphi^+(1_{A^+}) = 1_{B^+}\).

**Proof.** Simply set \(\varphi^+(a + \lambda) := \varphi(a) + \lambda\) for all \(a + \lambda \in A^+\). This so defined \(\varphi^+\) is unique since it is the only *-homomorphismus that makes the following diagram commutative.
2.1 Some operator algebras

\[ \begin{align*}
0 & \xrightarrow{i} \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{\varphi^+} \mathbb{C} \xrightarrow{0}, \\
\mathcal{A} & \xrightarrow{i} \mathcal{A}' \xleftarrow{\varphi} \mathbb{C} \xrightarrow{0}, \\
\mathcal{B} & \xrightarrow{i} \mathcal{B}' \xleftarrow{\varphi^+} \mathbb{C} \xrightarrow{0},
\end{align*} \]

where \( i \) stands for two different embeddings. \( \square \)

2.1.6 Spectrum and the Gelfand–Naimark theorem

The notion of the spectrum for operators on Hilbert spaces or Banach spaces can easily be extended to general C*-algebras if one distinguishes unital and non-unital cases:

**Definition 2.18.** For a unital C*-algebra \( \mathcal{A} \), the spectrum of \( a \in \mathcal{A} \) is given by

\[ \text{spec}_\mathcal{A}(a) = \{ \lambda \in \mathbb{C} \mid (a - \lambda 1_\mathcal{A}) \in \mathcal{A} \text{ is not invertible} \} . \]

For a, not necessarily unital, C*-algebra \( \mathcal{A} \), we can analogously define the spectrum of \( a \in \mathcal{A} \) when considering the unitisation by

\[ \text{spec}_{\mathcal{A}^+}(a) = \{ \lambda \in \mathbb{C} \mid (a - \lambda 1_{\mathcal{A}^+}) \in \mathcal{A}^+ \text{ is not invertible} \} . \]

The complements of these sets are called *resolvent sets* and denoted by \( \rho_\mathcal{A}(a) \) and \( \rho_{\mathcal{A}^+}(a) \), respectively.

Here, it is important to note that for a unital C*-algebra the notions \( \text{spec}_\mathcal{A}(a) \) and \( \text{spec}_{\mathcal{A}^+}(a) \) do not coincide. Indeed, by the definition of \( \mathcal{A}^+ \) one always has \( 0 \in \text{spec}_{\mathcal{A}^+}(a) \), which obviously is not correct for the usual spectrum, in general. More concretely, we get:

**Proposition 2.19.** For a unital C*-algebra \( \mathcal{A} \) and for all \( a \in \mathcal{A} \), one has

\[ \text{spec}_\mathcal{A}(a) \cup \{0\} = \text{spec}_{\mathcal{A}^+}(a) . \]

**Proof.** Let \( a \in \mathcal{A} \) and \( \lambda \neq 0 \) such that \( a - \lambda 1_\mathcal{A} \) is invertible with inverse \( b \). Then set \( c = b + \frac{1}{\lambda} 1_\mathcal{A} \) and observe \( (a - \lambda 1_\mathcal{A})c = \frac{a}{\lambda} \). Therefore

\[ (a - \lambda 1_{\mathcal{A}^+})(c - \frac{1}{\lambda} 1_{\mathcal{A}^+}) = ((a - \lambda 1_\mathcal{A})c - \frac{1}{\lambda}a) + 1_{\mathcal{A}^+} = 1_{\mathcal{A}^+} , \]
and the same calculation naturally holds for the left inverse. Hence, this implies $ho_{A^+}(a) \supset \rho_A(a) \setminus \{0\}$.

To show the other inclusion, we take an element $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ such that $a - \lambda 1_{A^+}$ is invertible. This means that $\lambda \neq 0$ and that the inverse is given by an element $c - \frac{1}{\lambda} 1_{A^+}$ with the property $ac - \lambda c - \lambda^{-1} a = 0$ in $\mathcal{A}$. From this, we conclude

$$(a - \lambda 1_{\mathcal{A}})(c - \lambda 1_{\mathcal{A}}) = ac - \lambda c - \frac{1}{\lambda} a + 1_{\mathcal{A}}.$$ 

Again, the same can be done for the left inverse. This shows $\rho_{A^+}(a) \subset \rho_A(a) \setminus \{0\}$ and so the claim. \hfill \Box

The spectrum of an element is always non-empty by Liouville’s theorem and compact by the Neumann series, cf. e.g. [Wer05, Satz VI.1.3]. Moreover, we have the following:

**Proposition 2.20** ([Bla06], Corollary II.1.6.7). Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{B} \subset \mathcal{A}$ be a unital sub-$C^*$-algebra with same unit. For all $b \in \mathcal{B}$, the following holds:

$$\text{spec}_\mathcal{B}(b) = \text{spec}_\mathcal{A}(b).$$

This is an important fact: The spectrum of an element does not depend on the $C^*$-algebra around it. It is an intrinsic property and it sufficient to consider the smallest unital $C^*$-algebra containing the element $a \in \mathcal{A}$, which is $C^*(a, 1)$, cf. Example 2.5 (d). Therefore, we can simply write $\text{spec}(a)$ without loss of information.

We collect some facts about the spectrum:

**Proposition 2.21** ([Weg93], Section 1.3). Let $\mathcal{A}$ be a $C^*$-algebra. Then:

(a) If $a \in \mathcal{A}$ is self-adjoint, that is $a^* = a$, then $\text{spec}(a) \subset \mathbb{R}$.

(b) If $a \in \mathcal{A}$ is such that $a = b^* b$ for some $b \in \mathcal{A}$, then $\text{spec}(a) \subset [0, \infty)$.

(c) Suppose $\mathcal{A}$ is unital. If $a \in \mathcal{A}$ is unitary, that is $a^* a = a a^* = 1$, then $\text{spec}(a) \subset S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

In the case that $a \in \mathcal{A}$ is normal, that is $a^* a = a a^*$, the implications formulated in (a), (b) and (c) are in fact equivalences.
For completeness, we cite the important generalisation of the continuous functional calculus for C*-algebras and the related Gelfand–Naimark theorem, which, roughly speaking, just states that each C*-algebra is indeed a closed subalgebra of $B(\mathcal{H})$.

**Theorem 2.22.** Let $\mathcal{A}$ be a unital C*-algebra and $a \in \mathcal{A}$ normal. Then there is a unique injective $*$-homomorphism,

$$\Phi : C(\text{spec}(a)) \to C^*(a, 1),$$

between the continuous functions and the C*-algebra generated by $a, a^*, 1$ with the property:

$$\Phi(t \mapsto t) = a, \quad \Phi(t \mapsto 1) = 1.$$

**Proof.** See e.g. [Dix77, Theorem 1.5.1].

**Theorem 2.23.** For every C*-algebra $\mathcal{A}$, there is a Hilbert space $\mathcal{H}$ and a bijective $*$-homomorphism $\varphi : \mathcal{A} \to B(\mathcal{H})$.

**Proof.** See e.g. [Dix77, Theorem 2.6.1].

### 2.1.7 Positive elements

After defining the spectrum of an element in a C*-algebra, we can rightfully speak of positive elements in analogy of positive operators in Hilbert spaces.

**Definition 2.24.** Let $\mathcal{A}$ be a C*-algebra. An element $a \in \mathcal{A}$ is called **positive** if $a$ is normal and $\text{spec}_{\mathcal{A}^+}(a) \subset [0, \infty)$. We will write $\mathcal{A}_{\text{pos}}$ for the set of positive elements and $\mathcal{A}_{\text{sa}}$ for the self-adjoint elements. The latter one becomes an ordered set with $a \leq b : \iff b - a$ positive.

Moreover, if $\mathcal{A}$ is unital, we call $a \in \mathcal{A}$ **strictly positive** if $a$ is positive and invertible, which means $\text{spec}_{\mathcal{A}}(a) \subset (0, \infty)$. We write in this case $a > 0$.

Indeed, this a standard notation that is also used for linear operators in Banach spaces, although it is in some sense counter-intuitive since we do not use the better fitting term **non-negative**. However, a few authors in fact do
this, but we will stay here by the traditional term. For the C*-algebra \( \mathcal{B}(\mathcal{H}) \), we obviously have that \( T \in \mathcal{B}(\mathcal{H}) \) is positive if and only if \( T \) is self-adjoint and fulfills \( \langle x, Tx \rangle \geq 0 \) for all \( x \in \mathcal{H} \).

Let us also note that \( a^*a \) is a positive element for all \( a \in \mathcal{A} \), but this is, however, not trivial in an abstract C*-algebra, cf. [Bla06, Proposition II.3.1.3].

### 2.1.8 Traces on C*-algebras

Here, we will explain some facts about traces for C*-algebras and von Neumann algebras. For a detailed survey, we refer to [Bla06]. Recall that for a finite dimensional Hilbert space \( \mathcal{H} \) the C*-algebra \( \mathcal{B}(\mathcal{H}) \) owns the well-known trace for matrices. It can be extended to an infinite dimensional Hilbert space \( \mathcal{H} \). Indeed, this trace is then not defined on the whole C*-algebra \( \mathcal{B}(\mathcal{H}) \) but rather on the positive elements, denoted by \( \mathcal{B}(\mathcal{H})_{\text{pos}} \), and we obtain the following:

**Proposition 2.25.** For the C*-algebra \( \mathcal{A} = \mathcal{B}(\mathcal{H}) \), where \( \mathcal{H} \) is a Hilbert space, there is a map \( T : \mathcal{A}_{\text{pos}} \to [0, \infty] \) with the following properties:

(a) \( T(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 T(a_1) + \lambda_2 T(a_2) \) for all \( \lambda_1, \lambda_2 \geq 0 \) and \( a_1, a_2 \in \mathcal{A}_{\text{pos}} \).

(b) \( T(a^*a) = T(aa^*) \) for all \( a \in \mathcal{A} \).

(c) \( T(a) = 0 \) implies \( a = 0 \).

(d) \( T(a) = \sup_i T(a_i) \) for every increasing net \( (a_i) \subset \mathcal{A}_{\text{pos}} \) with \( \lim a_i = a \).

(e) For all \( a \in \mathcal{A}_{\text{pos}} \), we have \( T(a) = \sup \{ T(b) \mid b \leq a, T(b) < \infty \} \).

**Definition 2.26.** Let \( \mathcal{A} \) be a C*-algebra and \( T : \mathcal{A}_{\text{pos}} \to [0, \infty] \) be a map. Then \( T \) is called a **weight** if (a) holds. It is called a **trace** if (a) and (b) holds. A weight or a trace with property (c) is called **faithful**, and it is called **normal** if (d) holds. A trace \( T \) is called **finite** if the range lies in \([0, \infty)\) and **semi-finite** if (e) holds.

Note that if \( \mathcal{A} \) is a unital C*-algebra and \( T \) is a trace, then property (b) implies \( T(u^*au) = T(a) \) for all unitaries \( u \) and \( a \in \mathcal{A}_{\text{pos}} \), so that each trace is also unitarily invariant.
Remark 2.27. A finite trace $\mathcal{T}: A_{\text{pos}} \to [0, \infty)$ can be extended to the whole algebra $A$ by polarisation and is then a continuous linear functional. Moreover, one can also show that $\mathcal{T}(ab) = \mathcal{T}(ba)$ for all $a, b \in A$. If $A$ is a unital C*-algebra, then one usually normalises the finite trace by setting $\mathcal{T}(1) = 1$. It is then called a normalised trace.

Definition 2.28. A von Neumann algebra $A$ is called semi-finite von Neumann algebra if there is a faithful normal semi-finite trace.

Moreover, a semi-finite trace on a von Neumann algebra always gives a dimension function, that generalises the usual notion.

Definition 2.29. Each semi-finite von Neumann algebra $A \subset \mathcal{B}(\mathcal{H})$ with a faithful normal semi-finite trace $\mathcal{T}$ has a dimension function for closed subspaces $U \subset \mathcal{H}$ by defining

$$\dim_{\mathcal{T}}(U) := \mathcal{T}(P_U),$$

where $P_U$ is the orthogonal projection onto $U$.

2.1.9 Factors of von Neumann algebras

The traces defined above are very important in special von Neumann algebras, so-called factors, which we will define in this section. For the sake of completeness, we also explain the notion of Hilbert bundles and Hilbert spaces that are given by a direct integral. Here, all contemplated Hilbert spaces are separable.

Definition 2.30. Let $(\Omega, \mathcal{A}, \mu)$ be $\sigma$-finite measure space, and for each $\omega \in \Omega$ let $(\mathcal{H}_{\omega}, \langle \cdot , \cdot \rangle_{\omega})$ be a separable Hilbert space. A family of maps $\mathcal{F}$ of the form

$$f : \Omega \to \bigcup_{\omega \in \Omega} \mathcal{H}_{\omega}, \quad f(\omega) \in \mathcal{H}_{\omega},$$

where $f \in \mathcal{F}$, is called a measurable Hilbert bundle if it has the following properties:

(1) For all $f, g \in \mathcal{F}$, the map $\omega \mapsto \langle f(\omega), g(\omega) \rangle_{\omega}$ is measurable.

(2) If for all $f \in \mathcal{F}$ the map $\omega \mapsto \langle f(\omega), h(\omega) \rangle_{\omega}$ is measurable, then $h \in \mathcal{F}$. 
(3) There is a countable subset \( \{f_j\}_j \) in \( \mathcal{F} \) such that for all \( \omega \in \Omega \) the set \( \{f_j(\omega)\}_j \) is dense in \( \mathcal{H}_\omega \).

We define the seminorm of a function \( f \in \mathcal{F} \) by

\[
\|f\| := \left( \int_{\Omega} \langle f(\omega), f(\omega) \rangle \mu(\omega) \right)^{1/2}.
\]

Constructing a normed space by standard methods gives us the Hilbert space

\[
\int_{\Omega} \mathcal{H}_\omega \, d\mu(\omega) := \{ f \in \mathcal{F} \mid \|f\| < \infty \}/\{ f \in \mathcal{F} \mid \|f\| = 0 \},
\]

which is called the \textit{direct integral of the Hilbert bundle}. Moreover, an operator \( T \) acting on this space is called \textit{decomposable} if there is a uniformly bounded set of operators \( T_\omega \in \mathcal{B}(\mathcal{H}_\omega) \) such that \( (Tf)(\omega) = T_\omega(f(\omega)) \) for all \( \omega \in \Omega \). In this case, one writes

\[
T = \int_{\Omega} T_\omega \, d\mu(\omega).
\]

The direct integral of Hilbert bundles is very useful to describe a lot of known spaces in a general setting, as one can see in the following.

**Example 2.31.** (a) \( L^2(\Omega, \mu) \) is given as a direct integral if \( (\Omega, \mu) \) is a separable measure space:

\[
L^2(\Omega, \mu) = \int_{\Omega} \mathcal{H}_\omega \, dx(\omega), \text{ with } \mathcal{H}_\omega = \mathbb{C}.
\]

(b) If \( \mu \) is a sum of \( n \) point measures on \( \mathbb{R} \), we get:

\[
\mathbb{C}^n = \bigoplus_n \mathbb{C} = \int_{\mathbb{R}} \mathcal{H}_\omega \, d\mu(\omega), \text{ with } \mathcal{H}_\omega = \mathbb{C}.
\]

**Definition 2.32.** For a von Neummann algebra \( \mathcal{A} \), the \textit{centre} is defined by

\[
Z(\mathcal{A}) := \{ a \in \mathcal{A} \mid ab = ba \text{ for all } b \in \mathcal{A} \}.
\]

If the centre is isomorphic to \( \mathbb{C} \), meaning that it only consists of scalar multiples of the identity, then \( \mathcal{A} \) is called a \textit{factor}. 
2.1 Some operator algebras

Example 2.33. (a) The von Neumann algebra $\mathcal{B}(\mathcal{H})$ is a factor. One easily finds that each element $T$ in the centre is normal and has all vectors as eigenvectors. Therefore, it can only have one eigenvalue $\lambda$, and by the spectral theorem $T = \lambda 1$.

(b) Let $(\Omega, \mu)$ be a $\sigma$-finite measure space. Then the von Neumann algebra $L^\infty(\Omega, \mu)$ from Example 2.8 is not a factor since it is commutative. However, one can write the Hilbert space $\mathcal{H} = L^2(\Omega, \mu)$ as a direct integral, see Example 2.31, and then each multiplication operator is decomposable:

$$M_\phi = \int_\Omega (M_{\phi, \omega}) d\mu(\omega), \quad \text{with } M_{\phi, \omega} = \phi(\omega).$$

Therefore, $L^\infty(\Omega, \mu)$ is, in this sense, decomposable into factors, and one writes then:

$$L^\infty(\Omega, \mu) = \int_\Omega \mathbb{C} d\mu(\omega).$$

In fact, it is true that one can write each von Neumann algebra given as an operator algebra on a separable Hilbert space as such a direct integral over factors, see [Bla06, Section III.1.6.4]. Therefore, it is often sufficient to study different factors of von Neumann algebras.

For the classification of factors, it is helpful to look at projections in the von Neumann algebra, which are as always orthogonal projections, meaning $p \in \mathcal{A}$ is a projection if $p^* = p = p^2$. Moreover, two projections $p, q$ are called orthogonal if $pq = 0$. Two projections $p, q$ are called (Murray-von Neumann) equivalent if there is a partial isometry $v$ with $vv^* = p$ and $v^*v = q$. We write then $p \sim q$. Moreover, we can order the set of projections by writing $p \leq q$ if $pq = p$.

Note here that the ordering of projections is compatible with the definition of positive operators in $\mathcal{B}(\mathcal{H})$, that is $q \geq p$ if and only if $\langle x, (q-p)x \rangle \geq 0$ for all $x \in \mathcal{H}$, cf. also Section 2.1.7. We give a short proof of this equivalence: If $pq = qp = p$, then $(q-p)^2 = q-p$ and $q-p$ is as a projection also a positive operator. On the other hand, if $\langle x, (q-p)x \rangle \geq 0$, then for $x \in \text{Ran}(p)$ we find $\langle x, qx \rangle = \langle x, x \rangle$ and therefore $\langle x - qx, x - qx \rangle = 0$. This means $x \in \text{Ran}(q)$, and we have $pq = p$. 
Definition 2.34. A projection \( q \) is called finite if \( p \leq q \) and \( p \sim q \) implies \( p = q \). Otherwise we call it infinite. Moreover, we call \( q \neq 0 \) minimal if \( p \leq q \) implies \( p = 0 \).

At this point, it is good to consider the von Neumann algebra \( \mathcal{A} = \mathcal{B}(\mathcal{H}) \), where the finite projections are the projections onto finite dimensional subspaces of \( \mathcal{H} \) and the minimal projections are the rank one projections. This seems like an easy understandable von Neumann algebra since every finite projection can be built from the minimal projections and the equivalence classes of finite projections are classified by the rank. Therefore, one can essentially distinguish four possibilities:

Definition 2.35. A factor \( \mathcal{A} \) is called of type I, II\(_1\), II\(_\infty\) or III if:

- For I: \( \mathcal{A} \) has a minimal projection.
- For II\(_1\): \( \mathcal{A} \) has no minimal projections and every projection is finite.
- For II\(_\infty\): \( \mathcal{A} \) has no minimal projections but both a finite and an infinite projection.
- For III: \( \mathcal{A} \) all non-vanishing projections are infinite.

This classification can also be reformulated by traces. A type I factor has a unique trace \( \mathcal{T} \) with \( \mathcal{T}(p) = 1 \) for each minimal projection \( p \). A type II\(_1\) factor has a finite normalised trace \( \mathcal{T} \) with range \([0, 1]\) for all projections. On the other hand, II\(_\infty\) has a semi-finite trace that is infinite for infinite projections and finite for finite projections. Finally, a factor of type III can only have traces with values in \( \{0, \infty\} \). For more details about this, see e.g. [Bla06, Theorem III.1.7.9] and [Con95, Chapter V].

2.2 Covariant families of operators as a C*-algebra

In this section, we will reconsider the covariant families of operators from Section 1.3 and show that the families with finite range induce an important C*-algebra.
Proposition 2.36. Let $\Omega$ be a compact topological space and $T_1, \ldots, T_d$ be commuting homeomorphisms on $\Omega$. The vector space of all covariant families $a = (A_\omega)_{\omega \in \Omega} \subset B(\ell^2(\mathbb{Z}^d))$ with finite range, together with pointwise operations, becomes a complex $*$-algebra. Moreover,

$$\|a\| = \sup_{\omega \in \Omega} \|A_\omega\|$$

defines a norm that fulfils $\|a^*a\| = \|a\|^2$. We will denote this algebra together with this norm by $A_{\text{cov},0}$.

Proof. Obviously, $A_{\text{cov},0}$ is complex vector space by pointwise operations. The multiplication is also pointwise defined, but we have to check the covariance relation (1.4) and the finite range condition (1.6). The covariance relation is fulfilled since the right shift operator $S_j$ is unitary on $\ell^2(\mathbb{Z}^d)$, and the finite range condition can be verified by using Parseval’s identity: If $a \in A_{\text{cov},0}$ has the finite range $R_a$ and $b \in A_{\text{cov},0}$ has $R_b$, chosen as in equation (1.6), then for $ab$ one can set $R_{ab} = R_a + R_b$ since

$$\langle e_n, A_\omega B_\omega e_m \rangle = \sum_{k} \langle e_n, A_\omega e_k \rangle \langle e_k, B_\omega e_m \rangle .$$

Then, one of the factors in the sum always has to be zero, and so $A_{\text{cov},0}$ is a complex $*$-algebra. Since by Proposition 1.31 $\| \cdot \|$ is a norm, it only remains to show that $\| \cdot \|$ has the C*-property $\|a^*a\| = \|a\|^2$. However, this is obvious since the space $\Omega$ is compact, and at each point $\omega \in \Omega$ we have the C*-algebra $B(\ell^2(\mathbb{Z}^d))$. □

Definition 2.37. Let $\Omega$ be a compact space and $T_1, \ldots, T_d$ be commuting homeomorphisms on $\Omega$. We call the norm-closure of $A_{\text{cov},0}$ in the Banach space of all covariant families the algebra of covariant families of local operators and denote this unital C*-algebra by $A_{\text{cov},c}$.

The space $A_{\text{cov},c}$ still consists of families of operators $(A_\omega)_{\omega \in \Omega}$ that are still covariant but they do not have finite range anymore. However, they fulfil some decaying properties in the off-diagonal part such that we call them local.
Remark 2.38. The $C^*$-algebra $A_{\text{cov},c}$ is isomorphic to the reduced crossed product $C(\Omega) \rtimes_{\alpha} \mathbb{Z}^d$ with respect to the homeomorphism $\alpha : \mathbb{Z}^d \to \text{Aut}(C(\Omega))$ with

$$\alpha(n)(f) : \omega \mapsto f(T^{-n}\omega).$$

See for the definition of reduced crossed products [Dav96] and [PS16a].

From now on, we will fix a probability measure $P$ on $\Omega$ that has to be invariant and ergodic with respect to all the dynamical systems given by $T_1, \ldots, T_d$, see Section 1.3 for the explicit definition. Then, we define for $a = (A_\omega)_{\omega \in \Omega} \in A_{\text{cov},c}$ the map

$$\mathcal{T}(a) := E(e_0, A_\omega e_0) = \int_{\Omega} \langle e_0, A_\omega e_0 \rangle \, dP(\omega),$$

where $e_0 \in \ell^2(\mathbb{Z}^d)$ is given by the map $e_0 : \mathbb{Z}^d \to \mathbb{C}$ with $e_0(n) = \delta_{0,n_1} \cdots \delta_{0,n_d}$ for all $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. Then, we summarise the following easily provable properties:

**Proposition 2.39.** The map $\mathcal{T} : A_{\text{cov},c} \to \mathbb{C}$ is linear and fulfils:

(a) $\mathcal{T}(a) \geq 0$ for all positive $a \in A_{\text{cov},c}$.

(b) $\mathcal{T}(a^*a) = \mathcal{T}(aa^*)$ for all $a \in A_{\text{cov},c}$.

(c) $\mathcal{T}(u^*au) = \mathcal{T}(a)$ for all unitaries $u \in A_{\text{cov},c}$ and positive $a \in A_{\text{cov},c}$.

(d) $\mathcal{T}(1) = 1$.

This proposition implies that $\mathcal{T}$ defines a normalised finite trace on the unital $C^*$-algebra $A_{\text{cov},c}$. We will extensively use this trace and the next remark in Chapter 5. From the definitions of Section 1.3 and, in particular, Proposition 1.32 and Theorem 1.40, we obtain the next result.

**Remark 2.40.** For a covariant family of local operators $a = (A_\omega)_{\omega \in \Omega} \in A_{\text{cov},c}$ the integrated density of states is given by

$$\mathcal{N}(\lambda) = \mathcal{T}(\chi_{(-\infty, \lambda]}(a))$$
2.3 C*-algebras with symmetry or Real C*-algebras

for all \( \lambda \in \mathbb{R} \). Moreover, we have almost surely

\[
\mathcal{T}(a) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \text{Tr}(A_{\omega,L}),
\]

where \( A_{\omega,L} \) is the finite-dimensional restriction of \( A_{\omega} \) to \( \ell^2(\Lambda_L) \). This means that \( \mathcal{T} \) is almost surely given by the trace per unit volume.

2.3 C*-algebras with symmetry or Real C*-algebras

By the definition of a C*-algebra, it is claimed that the underlying Banach space is a complex one. This seems arbitrary, but, in fact, the fruitful results one gets from the C*-property do not hold in general for a real Banach algebra. We refer to [Kas81], [Pal70] and [Schr93] for overviews about real Banach spaces with an algebra structure.

That the analogue definition of a real C*-algebra would not be powerful enough, one can see in the following:

**Example 2.41.** Consider the real Banach space \( \mathcal{A} = \{(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}) \mid x, y \in \mathbb{R}\} \) with the usual operator norm \( \| \cdot \| \) and the identity as involution, that is \( a^* = a \) for all \( a \in \mathcal{A} \). By the common matrix multiplication, this defines a unital real algebra that fulfills \( \|ab\| \leq \|a\|\|b\| \) and \( \|a^*a\| = \|a\|^2 \). However, not all elements of the form \( a^*a \) are positive, for example, \( I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) gives \( I^*I = -\mathbb{1}_2 \).

This example shows the crucial difference that occurs by defining complex and real C*-algebras since it was an important fact that all elements of the form \( a^*a \) have a positive spectrum, as known for operators in a Hilbert space. This means that we have to characterise real C*-algebras in another way:

**Definition 2.42.** A real Banach space \((\mathcal{A}, \| \cdot \|)\) is called a real Banach *-algebra if there is a multiplication that turns \( \mathcal{A} \) into a real algebra, there is an involution \( a \mapsto a^* \) with

\[
(a^*)^* = a, \quad (ab)^* = b^*a^*, \quad (a + \lambda b)^* = a^* + \lambda b^*, \quad \text{for } a, b \in \mathcal{A}, \lambda \in \mathbb{R}
\]
and \( \|ab\| \leq \|a\|\|b\| \) for all \( a, b \in A \). It is called a \textit{real C*-algebra} if, in addition, the complexification \( A_C = A \oplus iA \) becomes a C*-algebra with a norm that restricted to \( A \) is the original norm \( \| \cdot \| \).

**Remark 2.43.** The complexification \( A_C = A \oplus iA \) is just defined as the algebra on the set \( A \times A \) with componentwise addition and scalar multiplication by \( \mathbb{C} \) given by \((\lambda + i\mu)(a, b) = (\lambda a - \mu b, \lambda b + \mu a)\). For this reason, one can just denote \( (a, b) = a + ib \) and explain the multiplication and involution in the standard way.

Definition 2.42 above is one possible definition that gives all the properties from (complex) C*-algebras to real C*-algebras. Of course, it may seem peculiar to use the definition of complex C*-algebras to define some related objects, and therefore other authors use different approaches, which we present here:

**Proposition 2.44 ([Goo82]).** For a real Banach \(*\)-algebra \( A \), the following claims are equivalent:

(a) \( A \) is a real C*-algebra.

(b) There is a real Hilbert space \( S_r \) and a norm-closed subalgebra \( B \subset B(S_r) \) such that there is an isometric \(*\)-isomorphism \( A \to B \).

(c) For all \( a \in A \) one has \( \|a^*a\| = \|a\|^2 \) and \( 1_{A^+} + a^*a \in A^+ \) is invertible, where \( A^+ \) is the unitisation of \( A \), cf. Remark 2.16.

(d) One has \( \|a\|^2 \leq \|a^*a + b^*b\| \) for all \( a, b \in A \).

Beside these equivalent notions there is another object related to real C*-algebras that gives us indeed a special complex C*-algebra. Often these are called Real C*-algebras or “Real” C*-algebras. This terminology is not ideal as such algebras are derived from complex C*-algebras and is therefore not used here. We solely use the term explained in the next section.

### 2.3.1 C*-algebras with symmetry

**Definition 2.45.** Let \( A \) be a (complex) C*-algebra. An anti-linear isometric map \( \tau : A \to A \) is called a \textit{symmetry} or also a \textit{real structure} for \( A \) if it fulfils...
\( \tau(a^*) = \tau(a)^* \) and \( \tau(ab) = \tau(a)\tau(b) \) for all \( a, b \in A \). The pair \((A, \tau)\) is called a \( C^* \)-algebra with symmetry.

The term real structure is in fact related to Hilbert spaces with real structures from Definition 1.1. If \( A = B(H) \), where \( H \) is a Hilbert space with real structure, then \( \tau(A) := A^* \) for \( A \in B(H) \) turns \((A, \tau)\) into a \( C^* \)-algebra with symmetry. Furthermore, an element \( a \in A \) in a \( C^* \)-algebra with symmetry \((A, \tau)\) is called real if \( \tau(a) = a \) and it is called symmetric if \( \tau(a) = a^* \). Again, this coincides with the common terms used for ordinary matrices.

**Example 2.46.** The complex conjugation of complex numbers transforms \( M_n(\mathbb{C}) \) via

\[
\tau(A) = \tau \left( \begin{array}{ccc}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nn}
\end{array} \right) := \left( \begin{array}{ccc}
    \bar{a}_{11} & \cdots & \bar{a}_{1n} \\
    \vdots & \ddots & \vdots \\
    \bar{a}_{n1} & \cdots & \bar{a}_{nn}
\end{array} \right)
\]

into a \( C^* \)-algebra with symmetry. A real element \( A \in M_n(\mathbb{C}) \) is simply a real matrix and a symmetric element is a symmetric matrix in the usual manner.

**Proposition 2.47** ([Li03], Proposition 5.1.3). Given a \( C^* \)-algebra with symmetry \((A, \tau)\), its fixed point set

\[
A_\tau = \{ a \in A \mid \tau(a) = a \}
\]

becomes a real \( C^* \)-algebra. Inversely, every real \( C^* \)-algebra can be represented by the fixed point set of a \( C^* \)-algebra with symmetry.

**Proof.** The fix point set \( A_\tau \) is closed in the Banach space \( A \) and by definition of the map \( \tau \) it is a Banach *-algebra over \( \mathbb{R} \). Since \( A_\tau \oplus iA_\tau = A \) holds, the required \( C^* \)-norm is naturally given and so \( A_\tau \) is a real \( C^* \)-algebra. On the other hand, if \( B \) is a real \( C^* \)-algebra, one can define an anti-linear involutive *-automorphism \( \tau \) on the complexification \( B_\mathbb{C} = B \oplus iB \) by \( \tau(a + ib) = a - ib \) for \( a, b \in B \). This turns \( B \oplus iB \) into a \( C^* \)-algebra with symmetry and given fix point set \( B \). \( \square \)
Note that by this proof, the fixed point set and the complexification are connected: \((A_r)_C = A\). Now we will give some definitions in relation to other well-known constructions.

**Definition 2.48.** Let \((A, \tau)\) be a C*-algebra with symmetry.

(a) For the unitisation \(A^+\), there is always a symmetry given by

\[ \tau^+(A + \lambda) := \tau(A) + \overline{\lambda}. \]

Therefore, \((A^+, \tau^+)\) is a unital C*-algebra with symmetry.

(b) For \(M_n(A)\), one sets the symmetry:

\[ \tau_n((a_{ij})) := (\tau(a_{ij})) \quad \text{for} \quad 1 \leq i, j \leq n. \]

Therefore, \((M_n(A), \tau_n)\) is always a C*-algebra with symmetry.

In the category of C*-algebras the morphisms are given by *-homomorphisms. Now in the category of C*-algebras with symmetry, morphisms preserve the given real structure \(\tau\) as well. Therefore, we set the following definition.

**Definition 2.49.** Let \((A, \tau)\) and \((B, \tau')\) two C*-algebras with symmetry. A *-homomorphism \(\varphi : A \to B\) is called a symmetry preserving *-homomorphism if \(\varphi(\tau(a)) = \tau'(\varphi(a))\) for all \(a \in A\). This means the diagram

\[
\begin{array}{ccc}
A & \overset{\varphi}{\longrightarrow} & B \\
\downarrow{\tau} & & \downarrow{\tau'} \\
A & \overset{\varphi}{\longrightarrow} & B
\end{array}
\]

commutes for symmetry preserving *-homomorphisms.

**Remark 2.50.** The spectrum of an element \(a \in A\) in a unital C*-algebra with symmetry fulfils:

\[ \sigma(\tau(a)) = \overline{\sigma(a)} := \{ \overline{\lambda} \mid \lambda \in \sigma(a) \}. \]

This follows from the fact that the element \(a - \lambda 1\) is invertible if and only if \(\tau(a - \lambda 1) = \tau(a) - \overline{\lambda} 1\) is invertible. In particular, the spectrum of a real element
2.3 $C^*$-algebras with symmetry or Real $C^*$-algebras

\[ a \in A, \text{ i.e. } \tau(a) = a, \text{ is symmetric to the real line:} \]
\[ \sigma(a) = \overline{\sigma(a)}. \]

2.3.2 Exact sequences of $C^*$-algebras with symmetry

For a $C^*$-algebra with symmetry $(A, \tau)$, the relevant ideals are the two-sided closed ideals that are, in addition, also invariant under $\tau$. We shortly call them $\tau$-ideals. In this case, we get the following exact sequence of $C^*$-algebras with symmetry

\[ 0 \to (J, \tau) \xrightarrow{\iota} (A, \tau) \xrightarrow{p} (A/J, \tau_{A/J}) \to 0, \]

where $A/J$ carries the natural symmetry $\tau_{A/J}$ given by:

\[ \tau_{A/J}(a + J) := \tau(a) + J. \]

Example 2.51 (Real Toeplitz extension). Here, we consider the $C^*$-algebra $B(\ell^2(\mathbb{N}))$. Let $S \in B(\ell^2(\mathbb{N}))$ be the right shift and $C^*(S)$ the complex Toeplitz algebra, i.e. the $C^*$-algebra generated by $S$ in $B(\ell^2(\mathbb{N}))$. Then, one has the exact sequence

\[ 0 \to K(\ell^2(\mathbb{N})) \to C^*(S) \to C(S^1) \to 0. \]

Here, $K(\ell^2(\mathbb{N}))$ denotes the compact operators and $S^1$ the unit sphere in $\mathbb{C}$. Now $\ell^2(\mathbb{N})$ is a Hilbert space with real structure by the componentwise complex conjugation. This leads to $C^*$-algebras with symmetry and the following exact sequence of real $C^*$-algebras, which is usually called the real Toeplitz extension:

\[ 0 \to K(\ell^2(\mathbb{N}))_{\tau} \to C^*(S)_{\tau} \to C(S^1)_{\tau} \to 0. \]

These are the well-known fixed point sets. Let us write out the last algebra a bit more explicitly:

\[ C(S^1)_{\tau} = \left\{ S^1 \ni z \mapsto f(z) \mid f(z) = \overline{f(z)} \right\}. \]

The spectrum of an operator in a real $C^*$-algebra $A$ is always defined as the spectrum of this operator seen as an element in the complexification $A_{\mathbb{C}}$. This
is then in harmony with seeing the real C*-algebra as the fixed point set of a C*-algebra with symmetry.

2.3.3 Constructing lifts with symmetries

When there are two algebraic structures, e.g. groups or C*-algebras, $A$ and $B$ together with a morphism $\varphi : A \to B$, we call $a \in A$ a lift of $b \in B$ if $\varphi(a) = b$. Moreover, we want to get lifts that preserve certain properties of $b$. In [RLL00, Chapter 2], we find the following results in the case of C*-algebras $A$, $B$ and a surjective $*$-homomorphism $\varphi : A \to B$:

(i) For each element in $B$ there is a norm-preserving lift.

(ii) For each self-adjoint element in $B$ there is a norm-preserving self-adjoint lift.

(iii) For each positive element in $B$ there is a norm-preserving positive lift.

(iv) For normal elements, projections or unitaries, there, in general, is not such a preserving lift.

Now we are interested in lifts when we have C*-algebras with symmetries $(A, \tau)$ and $(B, \tau')$. This is especially interesting when transferring results about $K$-theory, see next the section below, from the complex case to the real case. Now, we prove that there is always a symmetry-preserving lift.

**Proposition 2.52.** Let $(A, \tau)$ and $(B, \tau')$ be C*-algebras with symmetries and $\varphi : A \to B$ a symmetry-preserving surjective $*$-homomorphism. Then:

(i) Every self-adjoint real element $b \in B$ lifts to a real self-adjoint $a \in A$.

(ii) For each real element $b \in B$, there is a real lift $a \in A$ with the same norm as $b$.

(iii) Every positive real element $b \in B$ lifts to a positive real element $a \in A$.

**Proof.** This proof is a suitable modification of the proof of [RLL00, Section 2.2.10].
For (i): For an arbitrary lift $x \in \mathcal{A}$ of $b$, we can set
\[
a_0 := \frac{1}{4} \left( x + x^* + \tau(x) + \tau(x^*) \right).
\]
Then $a_0$ is a self-adjoint and real lift of $b$. For getting the same norm, we choose a real continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(t) = t$ for $t \in [-\|b\|, \|b\|]$ and constant elsewhere. By the spectral mapping theorem, we immediately obtain $\sigma(f(a_0)) = f(\sigma(a_0)) \subset [-\|b\|, \|b\|]$. Therefore the norm of $a := f(a_0)$ is not greater than that of $b$. The element $a$ remains self-adjoint and real. The last part can be shown by using
\[
\|\tau(a) - a\| \leq \|\tau(f(a_0) - p(a_0))\| + \|p(a_0) - f(a_0)\|
\]
for each polynomial $p$ and the Stone-Weierstraß theorem. To specify this: Note that the norm is preserved by $\tau$ and that by the Stone-Weierstraß theorem there is a sequence of polynomial $(p_n)$ on $\sigma(a_0)$ such that $\|f - p_n\| \to 0$. Therefore, we have:
\[
\|\tau(a) - a\| \leq 2\|p_n(a_0) - f(a_0)\| \xrightarrow{n \to \infty} 0.
\]
Since also by Stone-Weierstraß $\varphi(a) = \varphi(f(a_0)) = f(\varphi(a_0)) = f(b) = b$ holds, it remains only to show that $\|a\| \geq \|b\|$. However, this follows immediately from the operator norm of $\varphi$, which can not be greater than 1.

For (ii): For an element $b \in \mathcal{B}$, we can define a self-adjoint element
\[
v := \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix},
\]
whose norm is
\[
\|v\|^2 = \|v^*v\| = \max\{\|bb^*\|, \|b^*b\|\} = \|b\|^2
\]
by Lemma 2.13. Using (i), we know that there is a real self-adjoint lift $x \in M_2(\mathcal{A})$ of $v$ having the same norm. The upper right matrix entry of $x$ gives a real lift of $b$ and is called $a$. Of course, we have $\|a\| \leq \|x\| = \|b\|$ by definition and $\|a\| \geq \|b\|$ since the operator norm of $\varphi$ does not exceed 1. Overall, $a$ is a real
lift of $b$ with same norm.

For (iii): For a lift $x \in \mathcal{A}$ of $b$, we define the positive element $a_0 := (x^*x)^{1/2}$, which is a lift of $b$. As before, we set $f(a_0) = a$ using the function $f$ from (i). Then we can do the exact same proof as in (i). Note that by the spectral mapping theorem $\sigma(a) = f(\sigma(a_0)) \subset [0, \|b\|]$ holds and that the element $a$ is also positive.

\[\square\]

2.4 $K$-theory of C*-algebras

The $K$-theory of C*-algebras is a vast area with interlockings with many other mathematical topics and cannot be stated here in full generality. There are some excellent monographs about this theory in the realm of C*-algebras, see, for example, [Weg93] and [RLL00].

We will use a different approach for defining the $K$-groups than the literature above and are inspired by [HL11]. This approach seems reasonable since it helps to define analogue $K$-groups for C*-algebras with symmetry. The following sections are based on the publication [GS16].

2.4.1 Complex $K$-groups

In the classical treatment, there are two important groups connected to a C*-algebra, which we will call the complex $K$-groups to emphasise the notion of a complex C*-algebra instead of a C*-algebra with symmetry, which is considered in the next chapter. We start with the first $K$-group, which will be denoted by $K_0$. The classical representation of the $K_0$-group of a C*-algebra $\mathcal{A}$ is given by projections in the matrix algebra of $\mathcal{A}$ or rather in the unitisation $\mathcal{A}^+$. See [RLL00] for more details. However, here inspired by [HL11] and [Dae88], self-adjoint elements squaring to the identity will be used for the definition of the $K_0$-group instead of projections.

For a C*-algebra $\mathcal{A}$, which does not have to be unital, we will consider the matrix algebras $M_n(\mathcal{A}^+)$ over the unitisation $\mathcal{A}^+$ together with the function $s(a + \lambda 1_{\mathcal{A}^+}) = \lambda 1_{\mathcal{A}^+}$ extracting the scalar part for $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. By the functorial property of building matrices, see Proposition 2.14, this extends to a
*-homomorphism $s_n : M_n(A^+) \rightarrow M_n(A^+)$, where we may drop the index when there is no confusion. Moreover, we will use capital letters for matrices over a C*-algebra to emphasise the construction from the underlying C*-algebra $A$.

Note that each subset $U \subset M_n(A^+)$ is a topological space with respect to the unique C*-norm on it and hence there is a natural equivalence relation for $A, B \in U$ given by

$$A \sim_h B \quad \text{in} \quad U : \iff \text{There is a continuous path in } U \text{ from } A \text{ to } B. \quad (2.2)$$

If the two elements $A$ and $B$ are in the same path-connected component of $U$, we will say that $A$ and $B$ are homotopic in $U$. This is just the common notion of $\pi_0(U)$ in a topological space but note that it is crucial for having a path completely inside the set $U$. Otherwise the notion would not be very useful since each C*-algebra, as a linear space, is path-connected.

We will need a very special subset of $M_n(A^+)$ and define the following self-adjoint elements

$$V_{0,2n}(A) = \left\{ A \in M_{2n}(A^+) \mid A^* = A, \ A^2 = 1_{A^+} \right\}.$$  

Note that $E_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of $V_{0,2}(A)$, and analogously we get that $E_{2n} := E_2^{\oplus n} = E_2 \oplus \cdots \oplus E_2$ is an element of $V_{0,2n}(A)$. Now, define the set

$$V_0(A) = \bigcup_{n=1}^{\infty} \left\{ A \in V_{0,2n}(A) \mid s(A) \sim_h E_{2n} \text{ in } s(V_{0,2n}(A)) \right\},$$

which contains matrices of arbitrary size. Note that the homotopy is to be understood in space of scalar matrices.

Now on this set $V_0(A)$, we define an equivalence relation $\sim_0$ by distinguishing two different cases. For $A, B \in V_{0,2n}(A)$ we set $A \sim_0 B$ if they are simply homotopic, this means $A \sim_h B$ in $V_{0,2n}(A)$. On the other hand, we define for each $A \in V_{0,2n}(A)$ the equivalence

$$A \sim_0 \begin{pmatrix} A & 0 \\ 0 & E_2 \end{pmatrix} \in V_{0,2n+2}(A). \quad (2.3)$$
Then recursively $\sim_0$ is well-defined on $V_0(A)$, and obviously this defines an equivalence relation with the following properties for the orthogonal sum $A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$:

**Proposition 2.53.**

(i) $E_2 \sim_0 E_2 \oplus E_2 \sim_0 1_n \oplus -1_n$.

(ii) $A \oplus B \sim_0 B \oplus A$.

(iii) If $A \sim_0 B$ and $C \sim_0 D$, then also $A \oplus C \sim_0 B \oplus D$.

(iv) $A \oplus -A \sim_0 E_2$.

**Proof.** For (i): We can explicitly write down the path from $E_2 \oplus E_2$ to $1_n \oplus -1_n$ by setting

$$[0, \pi] \ni t \mapsto \begin{pmatrix} 1 & -\cos(t) & \sin(t) \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For (ii): Here, we just have to rotate the block matrix in the following form:

$$f(t) := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}^* \begin{pmatrix} A & \cos(t) - \sin(t) \\ B & \sin(t) \cos(t) \end{pmatrix}.$$

We obtain $f(0) = A \oplus B$ and $f(\pi/2) = B \oplus A$.

For (iii): This is shown by writing the two homotopies as a $2 \times 2$ matrix and by using equation (2.3) and part (ii).

For (iv): Let $A \in V_0(A)$ be of size $2n$. By using the functional calculus, see Theorem 2.22, and the characteristic functions $\chi_{\{1\}}$ and $\chi_{\{-1\}}$, one can consider the following continuous path:

$$f(t) := \begin{pmatrix} \chi_{\{1\}}(A) - \chi_{\{-1\}}(A) \cos(t) \\ \chi_{\{-1\}}(A) \sin(t) \end{pmatrix} \begin{pmatrix} \chi_{\{-1\}}(A) \sin(t) \\ -\chi_{\{1\}}(A) + \chi_{\{-1\}}(A) \cos(t) \end{pmatrix}.$$

Here, we have $f(0) = A \oplus -A$ and $f(\pi) = 1_n \oplus -1_n \sim_0 E_2$ by part (i).

This proposition shows that the orthogonal sum $\oplus$ fulfils all properties of a commutative addition when considering equivalent classes with respect to $\sim_0$. Therefore, we can define the following group.
Definition 2.54. The set $V_0(\mathcal{A})/\sim_0$ becomes an abelian group with respect to the addition $[A] + [B] := [A \oplus B]$ and the neutral element $0 := [E_2]$. This group is called the complex $K_0$-group of the C*-algebra $\mathcal{A}$ and denoted by $K_0(\mathcal{A})$.

Example 2.55. (a) If $\mathcal{A} = \mathbb{C}$, then each equivalence class $[A]_{\sim_0}$ is uniquely determined by the difference of the multiplicity of the eigenvalue 1 of $A$ and the multiplicity of the eigenvalue $-1$ of $A$. Therefore we have $K_0(\mathbb{C}) = \mathbb{Z}$.

(b) If $\mathcal{H}$ is a finite dimensional Hilbert space, then by the same argument as above $K_0(\mathcal{B}(\mathcal{H})) = \mathbb{Z}$.

(c) If $\mathcal{H}$ is an infinite dimensional Hilbert space, each matrix $A$ has non-empty essential spectrum so that all eigenvalues can be changed within the equivalence class and we get $K_0(\mathcal{B}(\mathcal{H})) = 0$.

In the complex case, there is second important group, denoted by $K_1(\mathcal{A})$. It can be constructed by, moreover, imposing a symmetry. In fact this is a chiral symmetry in the terminology of Section 1.4. Hence, let us introduce the even symmetry operator $R_{2n} = R_{2n}^\otimes$ with $R_2 := \begin{pmatrix} 0 & 1_A^+ \\ 1_A^+ & 0 \end{pmatrix}$. We extend it naturally to the set $\cup_{n \geq 1} M_{2n}(\mathcal{A}^+) \bigcup \{ A \in M_{2n}(\mathcal{A}^+) \mid A^* = A, \ A^2 = \mathbf{1}_{2n}, \ R_{2n}^* AR_{2n} = -A \}$.

The imposed symmetry implies the following helpful property.

Lemma 2.56. For all $A \in V_{1,2n}(\mathcal{A})$, we have that $s(A)$ and $E_{2n}$ are homotopic in $s(V_{1,2n}(\mathcal{A}))$.

Proof. Consider an element $A \in V_{1,2}(\mathcal{A})$. Note that we then have for the scalar part $s(A)^2 = \mathbf{1}_{A^+}$, $s(A)^* = s(A)$ and $R_2 s(A) R_2 = -s(A)$. Using our knowledge of complex matrices, we conclude that there is a $t_0 \in \mathbb{R}$ such that

$$s(A) = \begin{pmatrix} \cos(t_0) \mathbf{1}_{A^+} & \mathbf{i} \sin(t_0) \mathbf{1}_{A^+} \\ -\mathbf{i} \sin(t_0) \mathbf{1}_{A^+} & -\cos(t_0) \mathbf{1}_{A^+} \end{pmatrix}.$$
By writing $t$ instead of $t_0$ and varying $t$ from $t_0$ to 0, one has the wanted continuous path in $s(V_{1,2}(\mathcal{A}))$. \hfill \Box

Using Lemma 2.56, we can define $V_1(\mathcal{A})$ in two different ways:

$$V_1(\mathcal{A}) := \{ Q \in V_0(\mathcal{A}) | R^*QR = -Q \} = \bigcup_{n=1}^{\infty} V_{1,2n}(\mathcal{A}).$$

On the set $V_1(\mathcal{A})$ we will again define an equivalence relation $\sim_1$ in an analogue manner as before: For $A, B \in V_{1,2n}(\mathcal{A})$, we set $A \sim_1 B$ if they are homotopic, meaning $A \sim_h B$ in the set $V_{1,2n}(\mathcal{A})$. Also in the same way as before, we define for $A \in V_{1,2n}(\mathcal{A})$ the equivalence

$$A \sim_1 \begin{pmatrix} A & 0 \\ 0 & E_2 \end{pmatrix} \in V_{1,2n+2}(\mathcal{A}).$$

Hence, $\sim_1$ is a well-defined equivalence relation on $V_1(\mathcal{A})$, and we can summarise the properties analogous to Proposition 2.53. Of course, here one just has to check that all homotopies are in fact in $V_{1,2n}(\mathcal{A})$.

**Proposition 2.57.** Let $A, B, C, D \in V_1(\mathcal{A})$ and $\sim_1$ defined as above. Then the following holds:

(i) If $A \sim_1 B$ and $C \sim_1 D$, then also $A \oplus C \sim_1 B \oplus D$.

(ii) $A \oplus B \sim_1 B \oplus A$.

(iii) $A \sim_1 -A$ for $A \in V_{1,2}(\mathcal{A})$. In particular, $E_2 \sim_1 -E_2$.

(iv) $A \oplus (-E_{2n}AE_{2n}) \sim_1 E_2$.

**Proof.** The parts (i) and (ii) have the same proofs as the analogous claims from Proposition 2.53 since all homotopies are in fact in $V_{1,2n}(\mathcal{A})$.

For (iii): If $A \in M_2(\mathcal{A}^+)$, we get from the symmetry relation $R_2^*AR_2 = -A$ the following:

$$A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \text{ with } a, b \in \mathcal{A}^+.$$
2.4 K-theory of C*-algebras

In accordance with the other properties, we have $a = a^*$, $b = -b^*$ and $a^2 - b^2 = 1$. If we define the matrix

$$J(t) := \begin{pmatrix} \cos(t) & i \sin(t) \\ i \sin(t) & \cos(t) \end{pmatrix},$$

then we can easily show $R_2 J(t) = J(t) R_2$ for all $t \in \mathbb{R}$. Therefore, the continuous path

$$f(t) := J(t)^* A J(t) = \begin{pmatrix} a \cos(2t) + ib \sin(2t) & b \cos(2t) + ia \sin(2t) \\ -b \cos(2t) - ia \sin(2t) & -a \cos(2t) - ib \sin(2t) \end{pmatrix}$$

lies in $V_{1,2}(A)$. This path fulfils $f(0) = A$ and $f(\pi/2) = -A$. For (iv): Here we use ideas from [Dae88, Proposition 2.11]. For a chosen $A \in V_{1,2n}(A)$, we set $B := -E_{2n} A E_{2n}$ and consider the following path in $V_1(A)$:

$$g(t) := \begin{pmatrix} A \cos(t) & E_{2n} \sin(t) \\ E_{2n} \sin(t) & B \cos(t) \end{pmatrix}.$$ 

Then we obtain $g(0) = A \oplus B$ and $g(\pi/2) = (0, E_{2n})$. This last element is via a similar path homotopic to $E_{2n} \oplus -E_{2n}$. By using part (iii), we get $A \oplus B \sim_1 E_{4n}$. 

Hence, the quotient $V_1(A)/\sim_1$ becomes again a group, see also [Dae88] for more details about this construction.

**Definition 2.58.** The set $V_1(A)/\sim$ is an abelian group with respect to the addition $[A] + [B] = [A \oplus B]$ and the neutral element $0 = [E_2]$. The inverse of $[A]$ is given by $[-E_{2n} A E_{2n}]$. This group is called the complex $K_1$-group of the C*-algebra $A$ and denoted by $K_1(A)$.

We refer to [Weg93] for a lot of examples of $K_1$-groups, and in fact every abelian group can be a $K_1$-group for a suitable C*-algebra, cf. [Weg93, Exercise 9.H]. We present some important examples here without proofs:

**Example 2.59.** (a) $K_1(\mathbb{C}) = 0$.

(b) $K_1(\mathbb{B}(\mathcal{H})) = 0$ for each Hilbert space $\mathcal{H}$. 

(c) $K_1(A) = 0$ for every von Neumann algebra $A$.

(d) $K_1(B(\mathcal{H})/K(\mathcal{H})) = \mathbb{Z}$ for infinite dimensional $\mathcal{H}$, where $K(\mathcal{H})$ denotes the compact operators on $\mathcal{H}$.

### 2.4.2 The standard picture of the complex $K$-groups

Here, we want to explain how the $K$-groups are usually defined and that these definitions coincide with ours, given in Section 2.4.1. The usual definition are found, for example, in [RLL00], [Weg93] and [GVF01], and we simply call it the standard picture, here.

**Definition 2.60.** For a C*-algebra $A$, we set the following:

1. The standard picture of the first complex $K$-group is given by
   \[ \hat{K}_0(A) = \{ [P]_0 - [s(P)]_0 \mid P = P^2 = P^* \in \bigcup_{n \geq 1} M_n(A^+) \} . \]
   Here, $s(P)$ is the scalar part of $P \in M_n(A^+)$ and the difference is formal in the sense of Grothendieck. The equivalence relation is taken on the set of projections in matrix algebras over $A^+$, by homotopy in the sense of (2.2) and $P \sim_0 \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$ similar as in (2.3). The addition in $\hat{K}_0(A)$ is also given by $\oplus$.

2. The standard picture of the second $K$-group is given by
   \[ \hat{K}_1(A) = W_1(A)/\sim_1 \]
   where
   \[ W_1(A) = \{ U \in \bigcup_{n \geq 1} M_n(A^+) \mid U^{-1} = U^* \} , \]
   on which the equivalence relation $\sim_1$ is defined by homotopy using (2.2) and the definition $U \sim_1 U \oplus 1_{A^+}$. The addition is again given by $\oplus$.

**Proposition 2.61.** The definitions of the complex $K$-groups in Section 2.4.1 coincide with the standard picture. This means, there are group isomorphisms $\varphi_0 : \hat{K}_0(A) \to K_0(A)$ and $\varphi_1 : \hat{K}_1(A) \to K_1(A)$.

**Proof.** We define the map $\varphi_0 : \hat{K}_0(A) \to K_0(A)$ by
   \[ \varphi_0([P]_0 - [s(P)]_0) = \begin{pmatrix} 1_n - 2P & 0 \\ 0 & 2s(P) - 1_n \end{pmatrix}_0 , \]
   $P \in M_n(A^+)$.
It can easily be checked that $\varphi_0$ is a group homomorphism with inverse given by

$$\varphi_0^{-1}([Q]_0) = [\chi(-\infty, 0)(Q)]_0 - [\chi(-\infty, 0)(s(Q))]_0.$$ 

Now we consider the second $K$-group. A standard homotopy argument shows that the addition also fulfills $[U]_1 + [U']_1 = [UU']_1$ and that $[U]_1 = [Us(U)^*]_1$ holds. Now we set $f_{2n} = f_{2n}^{\otimes n}$ with the matrix $f_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then we see $f_2 R_2 f_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence, there is a permutation $g_{2n}$ such that we have $g_{2n} f_{2n} R_2 f_{2n} g_{2n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Now we define $O = \bigoplus_{n \geq 1} f_{2n} g_{2n}$ and obtain $\hat{R} = O^* R O = \bigoplus_{n \geq 1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover, we define $\hat{Q} = O^* Q O$ and obtain the relations $\hat{R}^* \hat{Q} \hat{R} = -\hat{Q}$ and $\hat{Q}^2 = 1$. This implies that there is a unitary $U$ such that

$$\hat{Q} = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} U + U^* & -U + U^* \\ U - U^* & -U - U^* \end{pmatrix}. \quad (2.4)$$

Now $\varphi_1 : \hat{K}_1(A) \to K_1(A)$ will be given by $\varphi_1([U]_1) = [O \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix} O^*]_1$, and it can be checked to be a group isomorphism. Note that, in particular, one has the following homotopy $s(O \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix} O^*) \sim E_{2n}$. The inverse is then given by

$$\varphi_1^{-1}([Q]_1) = \left[ \frac{1}{2} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}^* Q \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \right]_1.$$ 

Just as $K_1(A)$ was introduced above as classes of self-adjoints squaring to the identity satisfying a supplementary symmetry relation, one can also view $K_0(A)$ as classes of even-dimensional unitaries $Q$ satisfying the relations $Q = Q^*$ and $s(Q) \sim_0 E_{2n}$, in analogy to the definition of the set $V_0(A)$.

### 2.4.3 Real $KR$-groups

For a C*-algebra with symmetry, we can also define groups that we will call the $KR$-groups. Since the C*-algebras have a real structure, we sometimes call them the *real K-groups* and always denote them by $KR_j(A, \tau)$, where $\tau$ is the symmetry on the C*-algebra $A$. We will discover that there are eight different groups rather than just two as in the complex case stated before. This fact is related to the Bott periodicity like for complex and real Clifford algebras. This connection is also named by Wood-Karoubi, see [Schr93, Theorem 1.4.7].

The groups $KR_j(A, \tau)$ will be defined as the sets of equivalence classes of
projections that satisfy some special symmetry relations. We will eventually need these symmetries in Chapter 4, for applications, which are based on the work [GS16]. Unfortunately, we were unable to localise a reference where the $KR$-groups are defined in this manner, except for the cases $j = 0, 1, 2$, which are treated in [HL11], and a unitary picture, which is discussed in the independent and parallel work [BL16]. The possible connections to [Kar78], [Kas81] and [Schr93] can be the object of further investigations. Other contributions in this direction can be found in [Dae88] and in the recent work [Kel17], which was published after the work [GS16] that this section is based on. Again, we will use self-adjoint elements squaring to the identity rather than projections. Also, we will use the extension of $\tau$ to $A^+$ as $\tau^+(A + \lambda 1_{A^+}) = \tau(A) + \overline{\lambda} 1_{A^+}$ such that $(A^+, \tau)$ is also a C*-algebra with symmetry. For shortening the notation, we will write simply $\tau^+(a) = \overline{a}$ for every $a \in A^+$, and, as always, we extend this notion to the matrices $M_n(A)$ as well, cf. Definition 2.48. We will distinguish the $KR$-groups with even index and odd index and it will be clear later why this is a good arrangement. Let us start with the even ones and set:

\[
\begin{align*}
V_0(A, \tau) &:= \{ Q \in V_0(A) \mid S_0^* Q S_0 = Q, \ s(Q) \sim_h E_{2n} \} , \\
V_2(A, \tau) &:= \{ Q \in V_0(A) \mid S_2^* Q S_2 = -Q, \ s(Q) \sim_h E_{2n} \} , \\
V_4(A, \tau) &:= \{ Q \in V_{0, ev}(A) \mid S_4^* Q S_4 = Q, \ s(Q) \sim_h E_{2n} \otimes 1_2 \} , \\
V_6(A, \tau) &:= \{ Q \in V_0(A) \mid S_6^* Q S_6 = -Q, \ s(Q) \sim_h E_{2n} \} .
\end{align*}
\]

The homotopies $s(Q) \sim_h E_{2n}$ and $s(Q) \sim_h E_{4n}$ have to respect the required symmetries, and in $V_{0, ev}(A)$ matrices have doubled dimension, which means:

\[
V_{0, ev}(A) := \{ Q \in \bigcup_{n \geq 1} M_{4n}(A^+) \mid Q^* = Q, \ Q^2 = 1, \ s(Q) \sim E_{2n} \otimes 1_2 \} .
\]

Finally, the symmetry operators are given by $S_0 := 1$ and

\[
S_2 := \bigcup_{n \geq 1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus^n , \quad S_4 := \bigcup_{n \geq 1} \begin{pmatrix} \Sigma_2 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \oplus^n , \quad S_6 := \bigcup_{n \geq 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus^n ,
\]

where $\Sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the notation $\bigcup_{n \geq 1}$ is again a useful abbreviation for the
extension of considered operators. Moreover, we have the following terminology:

| \(V_0(A, \tau)\) | real self-adjoint |
| \(V_2(A, \tau)\) | even symplectic self-adjoint |
| \(V_4(A, \tau)\) | quaternionic self-adjoint |
| \(V_6(A, \tau)\) | odd symplectic self-adjoint |

Above, the term *symplectic self-adjoint* for a symmetry of type \(S^* Q S = -Q\) reflects that the associated projection \(P = \frac{1}{2}(Q + 1)\) has a symplectic symmetry \(S^* P S = 1 - P\), and this symplectic or Lagrangian symmetry can be either even or odd pending on the sign of \(S^2\). In the terminology of Section 1.4, these symmetries are also called even or odd particle-hole symmetries, while the real and quaternionic symmetries are called even and odd time-reversal symmetries.

Next let us note that \(E_2 \in V_j(A, \tau)\) for \(j = 0, 2, 6\) and \(E_2 \otimes 1_2 \in V_4(A, \tau)\). On each set \(V_{2i}(A, \tau)\) an equivalence relation \(\sim\) is defined by homotopy and equation (2.3). As the symmetries of \(V_{2i}(A, \tau)\) are conserved under the addition of \(V_0(A)\), it provides a semi-group structure on the quotient \(V_{2i}(A, \tau)/ \sim\) with neutral element \(E_2\) for \(j = 0, 2, 6\) and \(E_2 \otimes 1_2\) for \(j = 4\). The associated Grothendieck group is by definition \(KR_{2i}(A, \tau)\). Furthermore, let us stress that while the same symbol \(\sim\) is used, homotopies are always only constructed within \(V_{2i}(A, \tau)\) so that, in particular, it is wrong to conclude that \(KR_{2i}(A, \tau)\) is a subgroup of \(K_0(A) = V_0(A)/ \sim_0\).

For odd \(j = 2i + 1\), one now imposes conditions in \(V_1(A)\) that correspond to relations that will be need later in Chapter 4, cf. e.g. Theorem 4.5 and Table 4.2. This will be explained later in more detail. Again, we need \(V_1(A)\) with doubled dimensions:

\[
V_{1, ev}(A) = \{ Q \in V_{0, ev}(A) \mid R^*_\text{ev} Q R_{\text{ev}} = -Q \},
\]

where \(R_{\text{ev}} = \cup_{n \geq 1} (R_2 \otimes 1_2)^\oplus n\).

\[
V_1(A, \tau) := \{ Q \in V_1(A) \mid S_1^* Q S_1 = Q, \ s(Q) \sim E_{2n} \},
\]

\[
V_3(A, \tau) := \{ Q \in V_{1, ev}(A) \mid S_3^* Q S_3 = -Q, \ s(Q) \sim E_{2n} \otimes 1_2 \},
\]
\[ V_5(A, \tau) := \{ Q \in V_{1, ev}(A) \mid S_5^* \overline{Q} S_5 = Q, \quad s(Q) \sim E_{2n} \otimes 1_2 \}, \]
\[ V_7(A, \tau) := \{ Q \in V_1(A) \mid S_7^* \overline{Q} S_7 = -Q, \quad s(Q) \sim E_{2n} \}, \]

where the symmetries are given by \( S_1 = 1 \) and

\[ S_3 := \bigcup_{n \geq 1} \left( \begin{array}{cc} 0 & -\Sigma_2 \\ \Sigma_2 & 0 \end{array} \right)^{\oplus_n}, \quad S_5 := \bigcup_{n \geq 1} \left( \begin{array}{cc} \Sigma_2 & 0 \\ 0 & \Sigma_2 \end{array} \right)^{\oplus_n}, \quad S_7 := \bigcup_{n \geq 1} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)^{\oplus_n} \]

with the same notations as before. Furthermore, we have the following useful terminology:

<table>
<thead>
<tr>
<th>( V_j(A, \tau) )</th>
<th>real chiral self-adjoint</th>
<th>even symplectic chiral self-adjoint</th>
<th>quaternionic chiral self-adjoint</th>
<th>odd symplectic chiral self-adjoint</th>
</tr>
</thead>
</table>

Again, \( E_2 \in V_j(A, \tau) \) for \( j = 1, 7 \) and \( E_2 \otimes 1_2 \in V_j(A, \tau) \) for \( j = 3, 5 \). Furthermore, (2.3) defines an equivalence relation \( \sim \) on \( V_j(A, \tau) \), and using the usual addition coming from \( \oplus \), we get a monoid structure on the quotient \( V_j(A, \tau)/\sim \). The associated Grothendieck groups are now by definition \( KR_j(A, \tau) \) for odd \( j \).

Let us point out that the odd \( V_{2n+1}(A, \tau) \) are obtained from \( V_{2n}(A, \tau) \) by imposing the chirality constraint while keeping \( S_{2n} = S_{2n+1} \), except for the pair \( j = 2, 3 \) because in \( j = 3 \) one requires the chiral symmetry \( R \) to anti-commute with the even symplectic symmetry, and this requires to pass to \( 2 \times 2 \) matrices so that \( S_2 \) and \( S_3 \) are different. However, both still commute to \( 1 \) and the symmetry relation is in both cases \( S^* \overline{Q} S = -Q \).

Note that we have the following relations for \( j = 3, 7 \):

\[ RS_1 = S_1 R, \quad R_{ev} S_3 = -S_3 R_{ev}, \quad R_{ev} S_5 = S_5 R_{ev}, \quad RS_7 = -S_7 R. \]

Like in the case \( K_1(A) \), it is also possible to view the odd \( KR \)-groups as classes of unitaries. For this set \( \Sigma = \bigcup_{n \geq 1} \Sigma_2^{\oplus_n} \). By reducing out the definitions
for $V_{2i+1}(A, \tau)$ one finds that with

\[
W_1(A, \tau) = \{ U \in W_1(A) \mid \overline{U} = U \},
\]

\[
W_3(A, \tau) = \{ U \in W_{1, ev}(A) \mid \Sigma^* \overline{U} \Sigma = U^* \},
\]

\[
W_5(A, \tau) = \{ U \in W_{1, ev}(A) \mid \Sigma^* \overline{U} \Sigma = U \},
\]

\[
W_7(A, \tau) = \{ U \in W_1(A) \mid \overline{U} = U^* \},
\]

where

\[
W_{1, ev}(A) = \{ U \in \bigcup_{n \geq 1} M_{2n}(A^+) \mid U^{-1} = U^* \}.
\]

Furthermore, we have the following terminology:

<table>
<thead>
<tr>
<th>$W_1(A, \tau)$</th>
<th>real unitary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_3(A, \tau)$</td>
<td>odd-symmetric unitary</td>
</tr>
<tr>
<td>$W_5(A, \tau)$</td>
<td>quaternionic unitary</td>
</tr>
<tr>
<td>$W_7(A, \tau)$</td>
<td>symmetric unitary</td>
</tr>
</tbody>
</table>

With these definitions, one can check $KR_{2i+1}(A, \tau) = W_{2i+1}(A, \tau)/\sim_1$ in a similar manner as in Proposition 2.61 using that $Q \in V_1(A)$ is given in terms of a unitary by (2.4). It is also possible to express the even $KR$-groups by unitaries, but this is not written out here, see [BL16]. The relations to the different representations can be found in [Kel17] and [BL16].

### 2.5 Krein theory in C*-algebras

In Section 1.2, we discussed generalisations of Hilbert spaces for general inner products, so-called Krein spaces. Sometimes it is suitable to consider the operators on Krein spaces using C*-algebraic techniques. For a unital C*-algebra $A$ with unit $1$, we consider the matrix algebra $M_2(A)$ and define the well-known elements:

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i1 \\ 1 & i1 \end{pmatrix}.
\]
Then we can also define the Krein space unitary groups as in Definition 1.18:

**Definition 2.62.** Let \( \mathcal{A} \) be a unital C*-algebra. Then the set of \( J \)-unitaries and \( I \)-unitaries are given by

\[
\mathcal{U}(\mathcal{A}, J) = \{ T \in M_2(\mathcal{A}) \mid T \text{ invertible, } T^*JT = J \},
\]

\[
\mathcal{U}(\mathcal{A}, I) = \{ T \in M_2(\mathcal{A}) \mid T \text{ invertible, } T^*IT = I \}.
\]

These are groups under multiplication with neutral element \( 1_2 \).

We can also transfer the notion of Lagrangian frames to the C*-algebra setting, where we extend vector and matrix multiplication to the objects with entries in \( \mathcal{A} \) in the natural way.

**Definition 2.63.** Let \( \mathcal{A} \) be a unital C*-algebra. A combination of two unitaries \( a, b \in \mathcal{U}(\mathcal{A}) \) written as \( \Psi = (a \ b) \) is called a \( J \)-Lagrangian frame if \( \Psi^*\Psi = 1 \) and \( \Psi^*J\Psi = 0 \). A combination of two elements \( a, b \in \mathcal{A} \) written as \( \Phi = (a \ b) \) is called \( I \)-Lagrangian frame if \( C\Psi \) is a \( J \)-Lagrangian frame.

So without having the notion of Lagrangian subspaces like in Krein spaces, we can use the frames defined above since they, in fact, generalise the usual definition by Lemma 1.10. This means that if \( \mathcal{A} = \mathcal{B}(\mathcal{H}) \), then the definition here and the one of Section 1.2 do coincide. Therefore, we can use the equivalence classes of the frames to generalise the set of Lagrangian subspaces since they are to be identified by Proposition 1.15.

**Definition 2.64.** We call two \( J \)-Lagrangian frames \( \Psi_1 \) and \( \Psi_2 \) equivalent if there is a unitary \( u \in \mathcal{U}(\mathcal{A}) \) with \( \Psi_2 = \Psi_1 u \), analogously for \( I \)-Lagrangian frames. Then we define:

\[
\mathbb{L}(\mathcal{A}, J) = \{ [\Psi] \mid \Psi \text{ \( J \)-Lagrangian frame} \},
\]

\[
\mathbb{L}(\mathcal{A}, I) = \{ [\Phi] \mid \Phi \text{ \( I \)-Lagrangian frame} \}.
\]

As already mentioned, by Proposition 1.15 this is a indeed a generalisation of the usual notion of Lagrangian subspaces. In this sense, we can also use
the stereographic projections of Proposition 1.16 and the action of the group $\mathbb{U}(\mathcal{A}, J)$ onto $\mathbb{L}(\mathcal{A}, J)$ by Proposition 1.20. We summarise this:

**Proposition 2.65.** There are two bijections, called stereographic projections, and given by $\Pi_J : \mathbb{L}(\mathcal{A}, J) \to \mathbb{U}(\mathcal{A})$ and $\Pi_I : \mathbb{L}(\mathcal{A}, I) \to \mathbb{U}(\mathcal{A})$ with

$$\Pi_J([\Psi]) = \begin{pmatrix} 1^* & 0 \\ 0 & 1 \\ \end{pmatrix} \begin{pmatrix} 1^* & \Psi \\ 0 & 1 \\ \end{pmatrix}^{-1}, \quad \Pi_I([\Phi]) = \begin{pmatrix} 1^* & \Phi \\ 1 & -1 \\ \end{pmatrix} \begin{pmatrix} 1^* & \Phi \\ -1 & 1 \\ \end{pmatrix}^{-1},$$

where $\Psi$ and $\Phi$ are $J$-Lagrangian and $I$-Lagrangian frames, respectively.

**Proposition 2.66.** The groups $\mathbb{U}(\mathcal{A}, I)$ and $\mathbb{U}(\mathcal{A}, J)$ act left on the sets $\mathbb{L}(\mathcal{A}, I)$ and $\mathbb{L}(\mathcal{A}, J)$, respectively, by

$$\mathbb{U}(\mathcal{A}, I) \times \mathbb{L}(\mathcal{A}, I) \to \mathbb{L}(\mathcal{A}, I), \quad (T, [\Phi]) \mapsto [T\Phi | T\Phi|^{-1}]$$

$$\mathbb{U}(\mathcal{A}, J) \times \mathbb{L}(\mathcal{A}, J) \to \mathbb{L}(\mathcal{A}, J), \quad (T, [\Psi]) \mapsto [T\Psi | T\Psi|^{-1}]$$

where $|T\Phi|^2 := (T\Phi)^*(T\Phi)$.

The groups of $J$-unitaries and $I$-unitaries can be written out more explicitly using elements $a, b, c, d \in \mathcal{A}$, for example:

$$\mathbb{U}(\mathcal{A}, J) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ invertible} \mid a^*a - c^*c = 1, \; d^*d - b^*b = 1, \; a^*b = c^*d \right\}$$

$$\mathbb{U}(\mathcal{A}, I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ invertible} \mid a^*c = c^*a, \; b^*d = d^*b, \; a^*d - c^*b = 1 \right\}$$

**Remark 2.67.** Using the representation of a $J$-unitary above, we immediately get $d^*d = 1 + b^*b \geq 1$ and thus $\|d^{-1}\| \leq 1$. As the adjoint also lies in $\mathbb{U}(\mathcal{K}, J)$, one further has $dd^* = 1 + cc^*$ so that $d^{-1}c(d^{-1}c)^* = 1 - d^{-1}(d^{-1})^* < 1$. In particular, $\|d^{-1}c\| < 1$.

### 2.6 Some technical results

In this short section, we will state some properties of C*-algebras that are not conceptual interesting but will be helpful for proofs in later chapters.
Proposition 2.68. Let $\mathcal{A}$ be a $C^*$-algebra and let $a_0, \ldots, a_n \in \mathcal{A}$ such that $p(\lambda) = a_n \lambda^n + \cdots + a_1 \lambda + a_0$ is a polynomial. Assume that $p(\lambda)$ is invertible for every $\lambda \in \mathbb{R}$. Then $\lambda \mapsto p(\lambda)^{-1}$ is real analytic.

Proof. Choose $\lambda_0 \in \mathbb{R}$ and $r > 0$ such that $\|p(\lambda) - p(\lambda_0)\| < 1$ for all $\lambda$ with $|\lambda - \lambda_0| < r$. We write $T_\lambda := p(\lambda_0) - p(\lambda)$. Then

$$p(\lambda)^{-1} = (1 - T_\lambda)^{-1}p(\lambda_0)^{-1} = \left( \sum_{j=0}^{\infty} T_\lambda^j \right) p(\lambda_0)^{-1}$$

by the Neumann series. Since $T_\lambda$ is a polynomial in $\lambda$ this gives a power series $p(\lambda)^{-1} = \sum_j b_j (\lambda - \lambda_0)^j$ with $b_j \in \mathcal{A}$. \hfill $\Box$

Lemma 2.69. Let $\mathcal{A}$ be a unital $C^*$-algebra and $a \in \mathcal{A}$. If $\|a\| \leq \frac{1}{2}$, then $1 - a$ is invertible and one has

$$(1 - a)^{-1} = 1 + a(1 - a)^{-1} \text{ and } \|(1 - a)^{-1}\| \leq 2.$$  

Proof. This is just a simple consequence of the Neumann series and the equation $1 = (1 - a) + a$. \hfill $\Box$
3 Fredholm theory

This chapter serves as a short introduction for Fredholm operators in Banach spaces, and then we mainly examine Fredholm operators on Hilbert spaces with real structure and show index theorems that will be needed in Chapter 4. We firstly introduce the classical definition of a Fredholm-operator that was considered for the first time in [Noe21].

**Definition 3.1.** Let $X$ and $Y$ be two Banach spaces. A bounded linear operator $T : X \to Y$ is called a Fredholm operator if the kernel $\ker(T)$ and the cokernel $Y/\text{Ran}(T)$ is finite dimensional. The number

$$\text{ind}(T) := \dim \ker(T) - \dim Y/\text{Ran}(T)$$

is called the *Noether index* of $T$. The set of all these Fredholm operators is denoted by $\mathcal{F}(X,Y)$.

Note that two things are uncommon here. Firstly, one usually finds the name *Fredholm index* for the number $\text{ind}(T)$. However, since Fritz Noether was the first that examined the now-called Fredholm operators with non-vanishing indices in his work [Noe21], in 1921, we should honour him with this. Secondly, in the definition of a Fredholm operator, one usually demands that the range of $T$ is closed. However, this is redundant and already follows from the finite dimensional cokernel. A proof can be found, for example, in [HS91], but we will explain it in more detail below.

### 3.1 Fredholm operators

In this section, we collect some well-known facts about Fredholm operators, especially Fredholm operators on Hilbert spaces. We start with the proof that
the range of a Fredholm operator is necessarily closed.

**Proposition 3.2.** Let $X$ and $Y$ be Banach spaces and $T \in B(X,Y)$ with finite dimensional cokernel $Y/\text{Ran}(T)$. Then $\text{Ran}(T)$ is closed in $Y$.

**Proof.** We use the ideas from [HS91, Lemma 25.7] here. First, we note that for each vector space $V$, independently of a norm or even of the underlying field, and for each subspace $U \subset V$, we can split the space in the following way:

$$V = U \oplus \tilde{U} \quad \text{with} \quad \tilde{U} \cong V/U.$$

(3.1)

Here, $\oplus$ is the inner direct sum for subspaces, i.e. the whole space is generated by the two subspaces and the intersection is just the zero vector. The proof of the claim (3.1) is not hard: One chooses a basis for $U$ and extends it to a basis for $V$. This extension immediately provides a subspace $\tilde{U}$ together with a basis for $\tilde{U}$, which is denoted by $(e_j)$. Accordingly, $([e_j])$ is a basis for $V/U$, and this gives us the wanted isomorphism in (3.1).

For proving that $\text{Ran}(T)$ is closed, we use the following simplification: If we have a linear map $T : X \to Y$, it is always possible to transform it into an injective linear map:

$$\tilde{T} : X/\text{Ker}(T) \to Y, \quad \tilde{T}([x]) = T(x).$$

The range of $\tilde{T}$ and $T$ are the same, and hence we can reduce our problem for the injective operator $\tilde{T}$. Using the decomposition from above, we can write

$$Y = \text{Ran}(T) \oplus \tilde{U} \quad \text{with} \quad \tilde{U} \cong Y/\text{Ran}(T),$$

and there is also a finite basis $(e_j)$ for $\tilde{U}$ such that $([e_j])$ is a basis for $Y/\text{Ran}(T)$. Now, we can transform $\tilde{T}$ into another operator that is even surjective. Let $n$ denote the dimension of the cokernel $Y/\text{Ran}(T)$, and the we define:

$$\widehat{T} : X/\text{Ker}(T) \times \mathbb{K}^n \to Y, \quad ([x], (\lambda_1, \ldots, \lambda_n)) \mapsto Tx + \sum \lambda_j e_j.$$

This operator is obviously still continuous and, by definition, also bijective. Consequently, we have a continuous bijective operator between two Banach
spaces and can apply the open mapping theorem. This implies that the inverse
operator $\hat{T}^{-1}$ is also continuous, and so $\hat{T}$ sends closed set to closed sets. Thus
$\hat{T}(X/\text{Ker}(T)) = \text{Ran}(T)$ is closed. \hfill \Box

**Proposition 3.3.** Let $X$, $Y$ and $Z$ be Banach spaces. For all $T \in \mathcal{F}(X,Y)$
and $S \in \mathcal{F}(Y,Z)$, the operator $ST$ is also Fredholm.

**Proof.** Obviously, $\text{Ker}(ST) = \{x \in X \mid STx = 0\} \cup \text{Ker}(T)$ and therefore
$\dim(\text{Ker}(ST)) \leq \dim \text{Ker}(S) + \dim \text{Ker}(T)$. The same holds for the dimensions
of the cokernel. \hfill \Box

When considering Fredholm operators on Hilbert spaces, one does not need
to calculate the cokernel because one can simply use the kernel of the adjoint
by $\text{Ker}(T^*) = \text{Ran}(T)^\perp$. We use the notation $\mathcal{F}(\mathcal{H}) := \mathcal{F}(\mathcal{H},\mathcal{H})$.

**Remark 3.4.** Let $\mathcal{H}$ be a real or complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then:

$$\dim \left( Y/\overline{\text{Ran}(T)} \right) = \dim \text{Ker}(T^*) .$$

*In particular, if $T \in \mathcal{F}(\mathcal{H})$, one has $\text{ind}(T) = \dim \text{Ker}(T) - \dim \text{Ker}(T^*)$.*

However, one has to notice that the closure above is necessary. Specifically,
knowing that both kernels, $\text{Ker}(T)$ and $\text{Ker}(T^*)$, are finite dimensional is not
sufficient for $T$ being a Fredholm operator. We present some properties of
Fredholm operators in the following here.

**Lemma 3.5.** Let $T \in \mathcal{F}(\mathcal{H})$ with a real or complex Hilbert space $\mathcal{H}$. Then $T^*$
is a Fredholm operator with $\text{ind}(T^*) = -\text{ind}(T)$.

**Proof.** One can show, for example, using the open mapping theorem and Hahn-
Banach theorem, see [DS58, Chapter VI.6], that the range of $T$ is closed if and
only if the range of $T^*$ is closed. Combining this with Remark 3.4 above, we
know that $T^*$ is Fredholm if and only if $T$ is Fredholm. \hfill \Box

An important notion that is connected to Fredholm operators is the essential
spectrum. We focus now on complex Hilbert spaces $\mathcal{H}$ again. One possible
definition is the following:
**Definition 3.6.** Let $T \in B(\mathcal{H})$. The set of all $\lambda \in \mathbb{C}$ for which $T - \lambda$ is not a Fredholm operator is called the *essential spectrum of $T$* and denoted by $\text{spec}_{\text{ess}}(T)$.

For normal operators $T \in B(\mathcal{H})$, the essential spectrum has a very useful description. It consists of all complex numbers in the spectrum that are not isolated eigenvalues of finite multiplicity. The later eigenvalues build by definition the *discrete spectrum*, denoted by $\text{spec}_{\text{dis}}(T)$. Accordingly, for a normal operator $T$, we have the disjoint union

$$\text{spec}(T) = \text{spec}_{\text{dis}}(T) \cup \text{spec}_{\text{ess}}(T).$$

In particular, a self-adjoint $T \in B(\mathcal{H})$ is a Fredholm operator if and only if there is an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \cap \text{spec}(T) \subset \{0\}$ and $0$ is at most an eigenvalue of finite multiplicity. A good presentation of the essential spectrum can be found in [EE87].

**Lemma 3.7.** For every operator $T \in B(\mathcal{H})$, we have $\text{Ker}(T^*T) = \text{Ker}(T)$ and $\text{Ker}(TT^*) = \text{Ker}(T^*)$. In particular, we also have $\overline{\text{Ran}(T^*T)} = \overline{\text{Ran}(T^*)}$ and $\overline{\text{Ran}(TT^*)} = \overline{\text{Ran}(T)}$.

**Proof.** We just have to show one equality since the other follows by switching the role of $T$ and $T^*$. Obviously, $\text{Ker}(T^*T) \supset \text{Ker}(T)$, so we just show the other inclusion. Choose $y \in \text{Ker}(T^*T)$. Then $\langle T^*Ty, z \rangle = 0$ for all $z \in \mathcal{H}$. Especially for $z = y$, we have $0 = \langle Ty, Ty \rangle = \|Ty\|^2$ and therefore $y \in \text{Ker}(T)$. For showing the range claims, one just has to notice:

$$\overline{\text{Ran}(T)} = \text{Ker}(T^*)^\perp = \text{Ker}(TT^*)^\perp = \overline{\text{Ran}(TT^*)}.$$ 

The same reasoning applies for the last equality. \qed

**Lemma 3.8.** An operator $T \in B(\mathcal{H})$ is a Fredholm operator if and only if $\text{spec}_{\text{ess}}(T^*T) \subset (0, \infty)$ and $\text{spec}_{\text{ess}}(TT^*) \subset (0, \infty)$.

**Proof.** First, let the essential spectrum of $T^*T$ and $TT^*$ be strictly positive, meaning both operators are Fredholm operators. By Lemma 3.7 above, we
know then that \( \ker(T) \) and \( \ker(T^*) \) are finite dimensional and both \( \operatorname{ran}(T) \) and \( \operatorname{ran}(T^*) \) are closed. Therefore, also \( T \) and \( T^* \) are Fredholm operators. On the other hand, if \( T \) is a Fredholm operator, then obviously also \( T^*T \) and \( TT^* \) are Fredholm operators by Proposition 3.3.

\[ \square \]

**Corollary 3.9.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be a self-adjoint operator. Then \( T \) is a Fredholm operator if and only if \( \operatorname{spec}_{\text{ess}}(T^2) \subset (0, \infty) \).

**Lemma 3.10.** Let \( T \in \mathcal{F}(\mathcal{H}) \) and \( B \) be a bounded operator on \( \mathcal{H} \oplus \mathcal{H} \) given by

\[
B = \begin{pmatrix}
TT^* + T^*T & i(TT^* - T^*T) \\
-i(TT^* - T^*T) & TT^* + T^*T
\end{pmatrix}.
\]

Then \( B \) is essentially strictly positive, that is, \( \operatorname{spec}_{\text{ess}}(B) \subset (0, \infty) \).

**Proof.** It is easy to see that this block operator is always positive: \( B \geq 0 \). However, for being essentially strictly positive, \( B \) has to be invertible except on a subspace of finite dimension. Firstly, we write \( K := \ker(T) \) and \( L := \ker(T^*) \) and define the invertible operators \( T_1 = T + P_K \) and \( T_2 = T^* + P_L \), where \( P_U \) denotes the orthogonal projection onto a given subspace \( U \). Note that the product of the operators is given by \( T_1T_2 = TT^* + P_KP_L \). Moreover, for \( B \) we have the decomposition:

\[
B = \begin{pmatrix}
TT^* + T^*T & i(TT^* - T^*T) \\
-i(TT^* - T^*T) & TT^* + T^*T
\end{pmatrix} = \begin{pmatrix}
T & -iT^*
-iT & T
\end{pmatrix} \begin{pmatrix}
T & iT^*
\end{pmatrix}.
\]

Both block operators on the right-hand side are invertible with the exception of a finite dimensional subspace. To see this, we consider the first one and calculate:

\[
\begin{pmatrix}
T & -iT^*
-iT & T
\end{pmatrix} \begin{pmatrix}
T_1^{-1} & iT_1^{-1}
-iT_2^{-1} & T_2^{-1}
\end{pmatrix} = \begin{pmatrix}
P_K & iP_{K^\perp} - iP_{L^\perp}
P_{L^\perp} - iP_{K^\perp} & P_{K^\perp} + P_{L^\perp}
\end{pmatrix},
\]

\[
\begin{pmatrix}
T_1^{-1} & iT_1^{-1}
-iT_2^{-1} & T_2^{-1}
\end{pmatrix} \begin{pmatrix}
T & -iT^*
-iT & T
\end{pmatrix} = \begin{pmatrix}
P_K & P_{L^\perp} & 0
0 & P_{K^\perp} + P_{L^\perp}
\end{pmatrix}.
\]

Naturally, we find an analogous result for the second matrix. Therefore,
both block matrices are invertible on the Hilbert space $\mathfrak{H}_0 \oplus \mathfrak{H}_0$ where we set $\mathfrak{H}_0 = K^\perp \cap L^\perp = \ker(T)^\perp \cap \ker(T^*)^\perp$. In conclusion, $B$ restricted to the the Hilbert space $\mathfrak{H}_0 \oplus \mathfrak{H}_0$ is invertible. Since the space $\ker(T) + \ker(T^*) = \mathfrak{H}_0^\perp$ is finite dimensional by assumption, the essential spectrum of $B$ does not include zero.

By definition the Fredholm operators are almost invertible, meaning they are invertible with the exception of a finite dimensional subspace. It can be shown that $T \in \mathcal{F}(\mathfrak{H})$ if and only if there is an operator $S \in \mathcal{B}(\mathfrak{H})$ such that

$$ST - 1 \quad \text{and} \quad TS - 1$$

have finite dimensional range, also called finite rank operators. This characterisation is known as Atkinson’s theorem, see, e.g. [AM10].

We close this introduction about Fredholm operators with an important fact about the Noether index that we will use in this work.

**Theorem 3.11.** Let $X$, $Y$ and $Z$ be Banach spaces. Then, the map given by the Noether index $\text{ind} : \mathcal{F}(X, Y) \to Z$ is norm-continuous. In particular, it is locally constant. Moreover, for $T \in \mathcal{F}(X, Y)$ and $S \in \mathcal{F}(Y, Z)$ one has

$$\text{ind}(ST) = \text{ind}(S) + \text{ind}(T).$$

**Proof.** See [Arv06, Chapter 3.4].

### 3.2 Skew-adjoint Fredholm operators and Fredholm operators with symmetry

In this section, we will consider Fredholm operators on real Hilbert spaces that anti-commute with a chosen representation of the real Clifford algebra $\tilde{\mathcal{C}}_d$. See Proposition 1.55 for the definition and properties of the real Clifford algebra $\tilde{\mathcal{C}}_d$. We will show that there is a natural connection between these Fredholm operators in a real Hilbert space and some Fredholm operators in a complex Hilbert space fulfilling some symmetry relations.

If we look into the classical paper of Atiyah and Singer [AS69], we find spaces consisting of Fredholm operators in a real Hilbert space that are anti-commuting
with some chosen operators $J_i$. These operators $J_i$ should represent the real Clifford algebras. We show that these spaces, explained in Proposition 3.12 below, are in some sense isomorphic to operators in a complex Hilbert space. In the next proposition, this is formulated in a more detailed sense.

For a better understanding we will sometimes equip real Hilbert spaces with an index $r$ and complex Hilbert spaces with an index $c$. Moreover, all complex Hilbert spaces carry a real structure, here called a complex conjugation $C$, which is just an isometric anti-linear map with $C^2 = 1$, cf. Definition 1.1. We write $T^t := (CTC)^*$ for the transpose of $T \in \mathcal{B}(\mathfrak{f}_r)$ and also $A^t$ for the adjoint of an operator in real Hilbert spaces, i.e. $A \in \mathcal{B}(\mathfrak{f}_r)$. This helps to distinguish the different spaces.

**Proposition 3.12.** For $k \in \mathbb{N}$ let $\mathfrak{f}_r$ be an infinite dimensional real Hilbert space and $J_1, \ldots, J_{k-1} \in \mathcal{B}(\mathfrak{f}_r)$ skew-adjoint operators, i.e. $J_i^t = -J_i$, with $J_i^2 = -1$ for $i \in \{1, \ldots, k-1\}$ such that $J_1, \ldots, J_{k-1}$ define a faithful representation of the Clifford algebra $\tilde{\mathcal{C}}_{k-1}$ and such that this representation restricted to simple subalgebras of $\tilde{\mathcal{C}}_{k-1}$ is still infinite. Define then the real vector spaces

\[
S_0 := \mathcal{B}(\mathfrak{f}_r),
S_k := \{ A \in \mathcal{B}(\mathfrak{f}_r) \mid A = -A^t, AJ_m = -J_mA \text{ for } m = 1, \ldots, k-1 \}.
\]

Then for each $k \in \mathbb{N}$, there is an infinite complex Hilbert space $\mathfrak{f}_c$ with real structure and an $\mathbb{R}$-linear isometric isomorphism $\Phi_k$ between the following spaces:

\[
\begin{align*}
S_0 & \xrightarrow{\Phi_0} \{ T \in \mathcal{B}(\mathfrak{f}_c) \mid T = T^t \}, \\
S_1 & \xrightarrow{\Phi_1} \{ T \in \mathcal{B}(\mathfrak{f}_c) \mid T = T^*, T = -T^t = -\bar{T} \}, \\
S_2 & \xrightarrow{\Phi_2} \{ T \in \mathcal{B}(\mathfrak{f}_c) \mid T = -T^t \}, \\
S_3 & \xrightarrow{\Phi_3} \{ T \in \mathcal{B}(\mathfrak{f}_c) \mid T = T^*, I^*\bar{T}I = T \}, \\
S_4 & \xrightarrow{\Phi_4} \{ T \in \mathcal{B}(\mathfrak{f}_c) \mid I^*\bar{T}I = T \}, \\
S_5 & \xrightarrow{\Phi_5} \{ T \in \mathcal{B}(\mathfrak{f}_c) \mid T = T^*, I^*\bar{T}I = -T \}, \\
S_6 & \xrightarrow{\Phi_6} \{ T \in \mathcal{B}(\mathfrak{f}_c) \mid T = T^t \}.
\end{align*}
\]
Here, \( I \in \mathcal{B}(\mathcal{H}_c) \) is a skew-adjoint operator, i.e. \( I^* = -I \), satisfying \( I^2 = -\mathbb{1} \) and \( \overline{I} = I \). Moreover, for \( k \geq 8 \) we use the eightfold Bott periodicity of the Clifford algebra to get analogue isomorphisms.

Note that in the proposition above, we have to choose for each \( k \) a suitable complex Hilbert space \( \mathcal{H}_c \) and an operator \( I \). Using this proposition, we are able to switch between real Hilbert spaces and complex Hilbert spaces with a complex conjugation. For this reason, we will often use the following fact.

**Lemma 3.13.** Each complex Hilbert space \( \mathcal{H}_c \) can be turned into a real Hilbert space \( \mathcal{H}_r \) by using the forgetful functor that dismisses the complex scalar multiplication.

**Proof.** As a vector space \( \mathcal{H}_c \) consists of an underlying additive group, which we will call \( G \). On \( G \) there is a scalar multiplication \( s : \mathbb{C} \times G \to G \) given with the usual compatible rules. By using the inclusion \( j : \mathbb{R} \to \mathbb{C} \) with \( j(\lambda) = \lambda + i0 \), one gets a real scalar multiplication \( s_{\mathbb{R}}(\lambda, g) := s(j(\lambda), g) \) and therefore a real Hilbert space \( \mathcal{H}_r \). The scalar product is then defined via the polarisation formula: \( \langle x, y \rangle_{\mathcal{H}_r} := \frac{1}{4} (\langle x + y, x + y \rangle_{\mathcal{H}_c} - \langle x - y, x - y \rangle_{\mathcal{H}_c}) \).

**Lemma 3.14** (cf. [Wei80], Ex. 5.32). For each real Hilbert space \( \mathcal{H} \) the Hilbert space \( \mathcal{H} \oplus \mathcal{H} \) can be turned into a complex Hilbert space \( \mathcal{H}_C \) with real structure, called the complexification of \( \mathcal{H} \). Moreover, we have:

(a) For \( \lambda, \mu \in \mathbb{R} \) and \( x, y \in \mathcal{H} \), the scalar multiplication in \( \mathcal{H}_C \) is given by \( (\lambda + i\mu) \cdot x \oplus y = (\lambda x - \mu y) \oplus (\lambda y + \mu x) \), and the scalar product is defined by \( \langle x_1 \oplus y_1, x_2 \oplus y_2 \rangle_{\mathcal{H}_C} = \langle x_1, x_2 \rangle + i\langle y_1, x_2 \rangle - i\langle y_1, x_2 \rangle + \langle y_1, y_2 \rangle \).

(b) Every operator \( A \in \mathcal{B}(\mathcal{H}) \) has a unique complexification \( A_C \in \mathcal{B}(\mathcal{H}_C) \) with same norm and \( (A_C)^* = (A^*)_C \) by setting \( A_C(x \oplus y) = Ax \oplus Ay \). The map \( A \mapsto A_C \) is \( \mathbb{R} \)-linear and injective.

(c) Each operator \( T \in \mathcal{B}(\mathcal{H}_C) \) can be bijectively and norm-preserving mapped to an operator \( \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \) with \( a, b \in \mathcal{B}(\mathcal{H}) \). In this case, \( T = a_C + ib_C \).
Proof. By defining the scalar multiplication as described in (a), \( \mathcal{H} \oplus \mathcal{H} \) becomes a complex vector space. The specified scalar product turns it into a Hilbert space. In conformity with these definitions, one can write elements of \( \mathcal{H}_C \) as \( x + iy \) for \( x, y \in \mathcal{H} \). Therefore, we see that a real structure is given by \( \mathcal{C} : \mathcal{H}_C \rightarrow \mathcal{H}_C \) with \( \mathcal{C}(x + iy) = x - iy \).

For (b): An operator \( A \in \mathcal{B}(\mathcal{H}) \) on the real Hilbert space can be extended by simply defining \( A_C(x + iy) := Ax + iAy \), and this operator is then \( \mathbb{C} \)-linear, bounded and has the same norm as \( A \). The equation \( (A_C)^* = (A^*)_C \) directly follows from the definition of the scalar product of \( \mathcal{H}_C \). The \( \mathbb{R} \)-linearity is obvious.

For (c): Each operator in \( \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \) can be represented by an operator block matrix \( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \). This represents a \( \mathbb{C} \)-linear operator on \( \mathcal{H}_C \) if and only if \( c = -b \) and \( d = a \). Then, by the definition of the complexification, we can conclude \( T = a_C + ib_C \). \( \square \)

Lemma 3.15. Let \( \mathcal{H} \) be a complex Hilbert space with real structure \( \mathcal{C} \). Then there is a unique decomposition \( \mathcal{H} = \mathcal{H}_0 \oplus i\mathcal{H}_0 \), where \( \mathcal{H}_0 = \{ x \in \mathcal{H} \mid \mathcal{C}x = x \} \) is closed. Furthermore:

(a) By restricting the scalar multiplication, \( \mathcal{H}_0 \) becomes a real Hilbert space \( \mathcal{H}_R \), called the real part of \( \mathcal{H} \). We have then \( (\mathcal{H}_R)_C = \mathcal{H} \).

(b) Each operator \( T \in \mathcal{B}(\mathcal{H}) \) can be bijectively and norm-preserving mapped to an operator \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathcal{B}(\mathcal{H}_R \oplus i\mathcal{H}_R) \) with \( a, b \in \mathcal{B}(\mathcal{H}_R) \). In this case, \( T = a_C + ib_C \).

Proof. We have \( \mathcal{C}^2 = 1 \) and can define \( \mathcal{H}_0 \) as the set of all \( x \in \mathcal{H} \) with \( \mathcal{C}x = x \). Then we find for each \( z \in \mathcal{H} \) the following:

\[
    z = \frac{1}{2}(z + \mathcal{C}z) + i \frac{1}{2i}(z - \mathcal{C}z).
\]

Since the first part lies in \( \mathcal{H}_0 \) and the second in \( i\mathcal{H}_0 \), we have a direct decomposition of \( \mathcal{H} \). Moreover, \( \mathcal{H}_0 \) and \( i\mathcal{H}_0 \) are closed and orthogonal since \( \mathcal{C} \) is isometric, and therefore \( \langle x, iy \rangle = \langle \mathcal{C}x, \mathcal{C}iy \rangle = -\langle x, iy \rangle \) for all \( x, y \in \mathcal{H}_0 \). Accordingly, we obtain the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_0 \oplus i\mathcal{H}_0 \).
For (a): The complexification of $\mathcal{H}_R$ is given by $\mathcal{H}_R \oplus \mathcal{H}_R$ together with the rule $i \cdot (x \oplus y) = -y \oplus x$. This is the same as in $\mathcal{H} = \mathcal{H}_0 \oplus i\mathcal{H}_0$, and also the scalar products coincide.

For (b): Since $\mathcal{H}$ is just the complexification of $\mathcal{H}_R$, we can use part (c) of Lemma 3.14.

Lemma 3.16. Let $\mathcal{H}$ be a real Hilbert space and $I \in \mathcal{B}(\mathcal{H})$ be a skew-adjoint operator satisfying $I^2 = -1$. Then $\mathcal{H}$ becomes a complex Hilbert space $\mathcal{H}'$ with real structure by setting

$$(\lambda + i\mu) \cdot x := \lambda x + \mu Ix, \quad \lambda, \mu \in \mathbb{R}, \ x \in \mathcal{H}. \quad (3.2)$$

Proof. For all $x \in \mathcal{H}$, we have $\langle x, Ix \rangle = \langle -Ix, x \rangle$ since $I^* = -I$. However, the scalar product is symmetric, and therefore $Ix \perp x$ for all $x$. Hence, the Hilbert space can be decomposed, for example with the Gram–Schmidt process, into an orthogonal sum $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $I(\mathcal{H}^+) = \mathcal{H}^-$ and $\dim(\mathcal{H}^+) = \dim(\mathcal{H}^-)$ hold. By setting the scalar multiplication as given by (3.2), one gets a complex vector space, here denoted by $\mathcal{H}'$, where one defines the scalar product as before by $\langle x^+ \oplus x^-, y^+ \oplus y^- \rangle_{\mathcal{H}'} := \langle x^+, y^+ \rangle + i\langle x^+, y^- \rangle - i\langle y^+, x^- \rangle + \langle x^-, y^- \rangle$. There is a natural real structure on this Hilbert space that is defined by $C : \mathcal{H}^+ \oplus \mathcal{H}^- \ni x \oplus y \mapsto x \oplus -y$.

Now we have set the suitable ground work and can prove the claims about isomorphic spaces of operators.

Proof of Proposition 3.12. We start the proof by observing the largest considered Clifford algebra. Since the Clifford algebra $\widetilde{\mathcal{C}}_6$ is isomorphic to the algebra of $8 \times 8$ real matrices, the Hilbert space can be decomposed to $\mathcal{H}_r = \mathbb{R}^8 \otimes \mathcal{H}'_r$. We will write $1$ for the identity operator in $\mathcal{H}'_r$ and use the following real $2 \times 2$ matrices, given by the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operators $J_1, \ldots, J_6$ in $\mathcal{B}(\mathcal{H}_r)$ have representations as operator block
matrices that can be shortly written as

\[
J_1 = (\sigma_3 \otimes i\sigma_2 \otimes 1_2) \otimes 1, \quad J_2 = (\sigma_3 \otimes \sigma_3 \otimes i\sigma_2) \otimes 1, \\
J_3 = (\sigma_3 \otimes \sigma_1 \otimes i\sigma_2) \otimes 1, \quad J_4 = (i\sigma_2 \otimes 1_2 \otimes 1_2) \otimes 1, \\
J_5 = (\sigma_1 \otimes 1_2 \otimes i\sigma_2) \otimes 1, \quad J_6 = (\sigma_1 \otimes i\sigma_2 \otimes \sigma_3) \otimes 1. \tag{3.3}
\]

This representation of the Clifford Algebra \(\widetilde{\mathcal{C}}^{\ell}_6\) is unique up to equivalence, and, on that account, we will explicitly use the operators above. Note that \(\widetilde{\mathcal{C}}^{\ell}_3\) and \(\widetilde{\mathcal{C}}^{\ell}_7\) are the only Clifford algebras that are not simple and that we need representations having infinite multiplicity on each simple subalgebra. In this case, every such representation of \(\widetilde{\mathcal{C}}^{\ell}_m\) is equivalent to the given operators \(J_1, \ldots, J_m\) above for each \(m = 1, \ldots, 6\).

For \(S_0\): By Lemma 3.15 part (a), we can choose \(\mathfrak{H}_c = (\mathfrak{H}_r)_C\) as the complexification of \(\mathfrak{H}_r\) without loss of generality. Then we can use Lemma 3.14 and set \(\Phi_0 : A \mapsto A_C\), which is \(\mathbb{R}\)-linear, norm-preserving and injective. On the other hand, each operator \(T \in \mathcal{B}(\mathfrak{H}_c)\) is always of the form \(T = A_C + iB_C\) for uniquely determined \(A, B \in \mathcal{B}(\mathfrak{H}_r)\). The condition \(T = \overline{T}\) is equivalent to \(B = 0\) and so \(\Phi_0\) is also surjective.

For \(S_1\): Just like above, we choose \(\mathfrak{H}_c = (\mathfrak{H}_r)_C\) as the complexification of \(\mathfrak{H}_r\). For \(A \in S_1\) one can define \(T = iA_C \in \mathcal{B}(\mathfrak{H}_c)\), which satisfies \(T^* = -iA_C^t = T\), and \(T^t = -T^* = -T\). Hence, \(\Phi_1 : A \mapsto iA_C\) is well-defined, \(\mathbb{R}\)-linear, injective and norm-preserving. On the other hand, for each \(T \in \mathcal{B}(\mathfrak{H}_c)\) with \(T = T^*\) and \(T = -T^t\), the operator \(-iT\) is a real operator, i.e. \(-iT = -iT\), and is consequently given as \(-iT = A_C\) for a unique operator \(A \in \mathcal{B}(\mathfrak{H}_r)\). The condition \(T = T^*\) implies \(A = -A^t\) by Lemma 3.14 part (b). We conclude that \(\Phi_1\) is surjective and the wanted isomorphism.

For \(S_2\): Since \(J_1\) is a skew-adjoint operator with \(J_1^2 = -1\) on the real Hilbert space, we can use Lemma 3.16 and choose the complex Hilbert space \(\mathfrak{H}_c = \mathfrak{H}_r^{J_1}\) with real structure \(\mathcal{C}\). We will write \(q : \mathfrak{H}_r \to \mathfrak{H}_c\) for the \(\mathbb{R}\)-linear inclusion map. For each \(A \in S_2\) the operator \(qAq^{-1} : \mathfrak{H}_c \to \mathfrak{H}_c\) is anti-linear because, for \(\lambda, \mu \in \mathbb{R}\) and \(x \in \mathfrak{H}_r\), one has

\[
(\lambda + i\mu) \cdot Ax = \lambda Ax + \mu J_1 Ax = A(\lambda x - \mu J_1 x) = A((\lambda - i\mu) \cdot x).
\]
Therefore, we obtain a linear operator $\Phi_2(A) = CqAq^{-1}$, which fulfils the condition $\Phi_2(A)^t = -\Phi_2(A)$. Accordingly, the map $\Phi_2(A)$ is $\mathbb{R}$-linear, injective, norm-preserving and well-defined.

For showing surjectivity, we consider a complex Hilbert space $\mathcal{H}_c$ with real structure $\mathcal{C}$ and define $\mathcal{H}_r$ as the embedded real Hilbert space, cf. Lemma 3.13. We set $q : \mathcal{H}_r \to \mathcal{H}_c$ as before. Then, we can define $J_1 \in \mathcal{B}(\mathcal{H}_r)$ by setting $J_1(q^{-1}x) = q^{-1}(ix)$. For each operator $T \in \mathcal{B}(\mathcal{H}_c)$ with $T^t = -T$, the operator $q^{-1}CTq \in \mathcal{B}(\mathcal{H}_r)$ anti-commutes with $J_1$ and is skew-adjoint. This shows that the map $\Phi_2 : A \mapsto CqAq^{-1}$ is an isomorphism and preserves the norm.

For $S_3$: Now we have the two operators $J_1$, $J_2$, but, in this case, it will be more suitable to use the unitarily equivalent operators on a decomposition $\mathcal{H}_r = \mathcal{H}_0 \oplus \mathcal{H}_0$ given by

$$\tilde{J}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{J}_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $I \in \mathcal{B}(\mathcal{H}_0)$ with the properties $I^2 = -1$ and $I^{-1} = I^* = -I$. The first two equations for an operator $A \in S_3$, i.e. $A^t = -A$ and $\tilde{J}_1A = -A\tilde{J}_1$, show the following block form

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{with} \quad a^t = -a \quad \text{and} \quad b^t = -b.$$

The third equation $\tilde{J}_2A = -A\tilde{J}_2$ implies $a = IaI$ and $b = -IbI$. Then, we define $\mathcal{H}_c = \mathcal{H}_0^I$ by Lemma 3.16 and $q$ as before. Moreover, we define the $\mathbb{R}$-linear map $\varphi : \mathcal{B}(\mathcal{H}_c) \to \mathcal{B}(\mathcal{H}_r)$ by $c \mathcal{C} + id \mathcal{C} \mapsto cI + id$, where we used Lemma 3.15 by representing each operator in $\mathcal{B}(\mathcal{H}_c)$ with operators $c, d \in \mathcal{B}(\mathcal{H}_0)$. Now, consider the operator

$$T := \varphi q \tilde{J}_1CAq^{-1} = -ia + bI$$

and observe that $T^* = ia^t - Ib^t = T$ holds. Moreover, $I^*\overline{T}I = I^*(ia + bI)I = T$.

Eventually, this defines the $\mathbb{R}$-linear map $\Phi_3$, which is again norm-preserving and injective.
3.2 Skew-adjoint operators and Fredholm operators with symmetry

On the other hand, if $\mathfrak{H}_c$ is a complex Hilbert space with real structure, then use Lemma 3.15 and write $\mathfrak{H}_c = \mathfrak{H}_0 \oplus i\mathfrak{H}_0$ and also $\mathfrak{H}_r = \mathfrak{H}_0 \oplus \mathfrak{H}_0$ as a real Hilbert space. Since $I \in B(\mathfrak{H}_c)$ fulfills $I = I' \oplus I'$, one has a block decomposition with $I = I' \oplus I'$. Note again that each operator $T \in B(\mathfrak{H}_c)$ can be represented by two operators $c, d \in B(\mathfrak{H}_0)$ as $T = cC \oplus dC$. If $T$ is a self-adjoint operator with $T = I^*TI$, one obtains

$$A := C^{-1}\tilde{J}_1^{-1}q^{-1}\varphi^{-1}Tq = \begin{pmatrix} -d & -cI' \\ -cI' & d \end{pmatrix} \text{ with } d^t = -d \text{ and } (-cI')^t = cI'. $$

In fact, this defines an element of $S_3$, and therefore $\Phi_3 : A \mapsto \varphi q\tilde{J}_1CAq^{-1}$ is the wanted isomorphism.

For $S_4$: Choose the operators $J_1, J_2, J_3$ as in equation (3.3) and the decomposition $\mathfrak{H}_r = \mathbb{R}^8 \otimes \mathfrak{H}_r'$. Then, these operators define a faithful representation of $\mathcal{C} \ell_3$. An operator $A \in B(\mathfrak{H}_r)$ that anti-commutes with $J_1, J_2$ and $J_3$ is off-diagonal with respect to the given decomposition, i.e.

$$A = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \text{ with } X\tilde{J}_i = \tilde{J}_iX \text{ and } Y\tilde{J}_i = \tilde{J}_iY, \text{ for } i = 1, 2, 3.$$

Here, $\tilde{J}_1 := i\sigma_2 \otimes 1_2 \otimes 1, \tilde{J}_2 := \sigma_3 \otimes i\sigma_2 \otimes 1$ and $\tilde{J}_3 = \sigma_1 \otimes i\sigma_2 \otimes 1$. Because $A$ is skew-adjoint, we know $Y = -X^t$. If we set $I := i\sigma_2 \otimes 1$, we find the block representation for $X$ and so analogously for $Y$:

$$X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ with } a = -IaI \text{ and } b = IbI. \quad (3.4)$$

The definition for $\mathfrak{H}_c$ is the same as in the case of $S_3$, and, consequently, the operator $T = b_C + ia_C \in B(\mathfrak{H}_c)$ suffices $ITI = T$.

For the surjectivity part, we do the same as in the case $S_3$ for a complex Hilbert space $\mathfrak{H}_c$ with real structure. Each operator $T$ can be written as
$T = c_C + i d_C$, and, by using $ITI = T$, we can define

$$A := \begin{pmatrix} 0 & 0 & d & c \\ 0 & 0 & -c & d \\ -d^t & c^t & 0 & 0 \\ -c^t & -d^t & 0 & 0 \end{pmatrix} \quad \text{with } d = -I'd'I' \text{ and } c = I'cI',$$

and so $A \in \mathcal{S}_4$. This defines the $\mathbb{R}$-linear isomorphism $\Phi_4$.

**For $\mathcal{S}_5$:** The operators $J_1, J_2, J_3, J_4$ from equation (3.3) define a faithful representation of $\widetilde{\mathcal{C}}_4$. We find as shown in the case for $\mathcal{S}_4$ that $A \in \mathcal{S}_5$ can be written as

$$A = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \quad \text{with } X \tilde{J}_i = \tilde{J}_iX, \ Y \tilde{J}_i = \tilde{J}_iY, \ X^t = -Y \quad \text{for } i = 1, 2, 3,$$

where $\tilde{J}_1 := i \sigma_2 \otimes 1_2 \otimes 1, \tilde{J}_2 := \sigma_3 \otimes i \sigma_2 \otimes 1$ and $\tilde{J}_3 = \sigma_1 \otimes i \sigma_2 \otimes 1$. The fourth equation $AJ_4 = -J_4A$ implies $X = Y = -X^t$. By doing the same as before, using equation (3.4) and setting $T := b_C + ia_C$, one gets $T^* = T$ and $I^*TI = -T$, where $I := i \sigma_2 \otimes 1$. For showing the surjectivity, the proof is the same as above for $\mathcal{S}_4$, so that $\Phi_5$ is also well-defined.

**For $\mathcal{S}_6$:** The operators $J_1, \ldots, J_5$ from (3.3) define a faithful representation of $\widetilde{\mathcal{C}}_5$. The proof of $\mathcal{S}_5$ above implies

$$A = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \quad \text{with } X^t = -X,$$

$$X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with } a = -IaI, \ b = IbI.$$

The remaining equation $J_5A = -AJ_5$ results in $bI = -Ib$ and $aI = -Ia$, so we conclude $a = 0$. Hence, $b$ acting on $\mathcal{S}_c$ is an antilinear operator that is related to a linear operator $T \in \mathbb{B}(\mathcal{S}_c)$ with $T^t = T$, cf. proof for $\mathcal{S}_2$. The definition of $\Phi_6$ is almost the same as before.

**For $\mathcal{S}_7$:** We already know that the operator $A \in \mathcal{S}_7$ has the block representa-
3.2 Skew-adjoint operators and Fredholm operators with symmetry

\[ A = \sigma_1 \otimes i\sigma_2 \otimes -b \text{ with } b^t = b \text{ and } b = IbI. \]

Now, we have one more equation, namely \( AJ_6 = -J_6A \), and derive \( Jb = -bJ \) with \( J := \sigma_3 \otimes 1 \). This implies
\[ b = \begin{pmatrix} y \\ y^t \end{pmatrix} \text{ with } y^t = y. \]

Accordingly, we find an operator \( T \in \mathcal{B}(\mathfrak{H}) \) satisfying \( T = T^* = T^\ast \) by the same procedure as before. The same holds for the definition of \( \Phi_7 \).

The result of Proposition 3.12 shows that one can equivalently consider real Hilbert spaces with anti-commuting operators or complex Hilbert spaces with real structure where the operators fulfils some symmetry relations. It is this last viewpoint that we will use in the rest of this work, but originally Atiyah and Singer worked with the first one, see [AS69]. There, a real Hilbert space \( \mathfrak{H}_r \) and the Fredholm operators \( F(\mathfrak{H}_r) \) are considered. As previously arranged, \( J_1, \ldots, J_6 \) are skew-adjoint operators on \( \mathfrak{H}_r \) with \( J_k^2 = -1 \) such that \( J_1, \ldots, J_m \) define a faithful representation of the Clifford Algebra \( \tilde{\mathfrak{C}}_{m} \).

Now, we are able to reformulate Proposition 3.12 such that we can identify the Fredholm operators in a real Hilbert space with the Fredholm operators in a complex Hilbert space. Of course, we do not have linear isomorphisms since the Fredholm operators do not form a linear space, but they are still homeomorphisms.

**Theorem 3.17.** For \( k \in \mathbb{N} \), let \( \mathfrak{H}_r \) be an infinite dimensional real Hilbert space and \( J_1, \ldots, J_{k-1} \) be skew-adjoint operators on \( \mathfrak{H}_r \) satisfying \( J_i^2 = -1 \) for all \( i \in \{1, \ldots, k-1\} \) such that \( J_1, \ldots, J_{k-1} \) define a faithful representation of the Clifford algebra \( \tilde{\mathfrak{C}}_{k-1} \) and such that this representation restricted to simple subalgebras of \( \tilde{\mathfrak{C}}_{k-1} \) is still infinite. Define the topological spaces
\[
F^0_r := F(\mathfrak{H}_r), \\
F^k_r := \{ A \in F(\mathfrak{H}_r) \mid A = -A^t, AJ_m = -J_mA \text{ for } m = 1, \ldots, k-1 \}.
\]
Then for each \( F^k_r \), there is a complex Hilbert space \( \mathcal{H}_c \) and a topological embedding \( \Phi_k : F^k_r \to F(\mathcal{H}_c) \) with

\[
F^0_c := \Phi_0(F^0_r) = \{ T \in F(\mathcal{H}_c) \mid T = \overline{T} \},
\]

\[
F^1_c := \Phi_1(F^1_r) = \{ T \in F(\mathcal{H}_c) \mid T = T^*, \ T = -T^t = -\overline{T} \},
\]

\[
F^2_c := \Phi_2(F^2_r) = \{ T \in F(\mathcal{H}_c) \mid T = -T^t \},
\]

\[
F^3_c := \Phi_3(F^3_r) = \{ T \in F(\mathcal{H}_c) \mid T = T^*, \ I^*\overline{TI} = T \},
\]

\[
F^4_c := \Phi_4(F^4_r) = \{ T \in F(\mathcal{H}_c) \mid I^*\overline{TI} = T \},
\]

\[
F^5_c := \Phi_5(F^5_r) = \{ T \in F(\mathcal{H}_c) \mid T = T^*, \ I^*\overline{TI} = -T \},
\]

\[
F^6_c := \Phi_6(F^6_r) = \{ T \in F(\mathcal{H}_c) \mid T = T^t \},
\]

\[
F^7_c := \Phi_7(F^7_r) = \{ T \in F(\mathcal{H}_c) \mid T = T^*, \ \overline{T} = T \}.
\]

Here, \( I \in \mathcal{B}(\mathcal{H}_c) \) is a skew-adjoint operator satisfying \( I^2 = -1 \) and \( \overline{I} = I \). By the Bott periodicity of the Clifford algebra, one has analogue embeddings for \( k \geq 8 \).

Proof. By using of Proposition 3.12, we just have to show that \( \Phi_k(A) \) is a Fredholm operator if and only if \( A \in F^k_r \). We do this for all the different cases.

For \( F^0_r \) and \( F^1_r \): Note that the complexification of an operator \( A \) just doubles the real dimension of the kernel and range of \( A \). Therefore, \( A_C \) and \( iA_C \) are Fredholm operators in \( \mathcal{H}_c \) if and only if \( A \) is a Fredholm operator in \( \mathcal{H}_r \).

For \( F^2_r \) and \( F^3_r \): Since we have \( \Phi_k(A) = \psi_k^{-1}A\varphi_k \) for some \( \mathbb{R} \)-linear isomorphisms \( \varphi_k, \psi_k : \mathcal{H}_c \to \mathcal{H}_r \) for \( k = 2, 3 \), the real dimensions of the kernel and cokernel of \( A \) and \( \Phi_k(A) \) are the same.

For \( F^4_r, F^5_r, F^6_r \) and \( F^7_r \): The operators \( A \in F^k_r \) can be represented as operator block matrices, cf. the proof of Proposition 3.12:

\[
A = \begin{pmatrix}
a & b \\
-b & a \\
-a^t & b^t \\
-b^t & -a^t \\
\end{pmatrix} \xrightarrow{\Phi_4} b_C + i a_C = T,
\]

\[
A = \sigma_1 \otimes \begin{pmatrix}
a & b \\
-b & a \\
\end{pmatrix} \xrightarrow{\Phi_5} b_C + i a_C = T,
\]
\begin{equation*}
  A = \sigma_1 \otimes i\sigma_2 \otimes -b \xrightarrow{\Phi_6} Cq\bar{q}^{-1} = T, \\
  A = \sigma_1 \otimes i\sigma_2 \otimes \sigma_1 \otimes y \xrightarrow{\Phi_7} y_C = T.
\end{equation*}

Obviously, the dimension of the kernel of \( A \) is an integral multiple of the dimension of the kernel of \( T \), so if one of them is finite, then this is true for the other. The same argumentation holds for the cokernel.

Note that the real Clifford algebra satisfies the eightfold Bott periodicity \( \widetilde{\mathcal{A}}_{m+8} \simeq \mathbb{R}^{16 \times 16} \otimes_{\mathbb{R}} \widetilde{\mathcal{A}}_{m} \), and therefore the above spaces of Fredholm operators are homeomorphic \( F_{r}^{k+8} \approx F_{r}^{k} \). For example, an operator \( A \in F_{r}^{8} \) has the form

\begin{equation*}
  A = \sigma_1 \otimes i\sigma_2 \otimes \sigma_1 \otimes \left( x^t \ x \right)
\end{equation*}

with a Fredholm operator \( x \in F_{r}^{0} \). This gives the homeomorphism \( F_{r}^{8} \approx F_{r}^{0} \).

**Remark 3.18.** For \( k \in \mathbb{N} \), an operator \( A \in F_{r}^{k} \) fulfils \( \ker(A) = \ker(A^t) \) and therefore has index 0.

### 3.3 Classifying spaces

Theorem 3.17 above claims that the topological spaces \( F_{r}^{k} \) and \( F_{c}^{k} \) are homeomorphic for all \( k \in \mathbb{N} \). Not all of the spaces are connected and there might be trivial components that one should exclude. For example, the topological space of self-adjoint Fredholm operators decomposes into threee components, either the essential spectrum is positive, negative or both. The first two components are contractible and not interesting, so that we will concentrate on the component where the essential spectrum is positive and negative. We have this distinction only in the case where the Clifford algebra is not simple, that is \( \widetilde{\mathcal{A}}_3 \) and \( \widetilde{\mathcal{A}}_7 \). We define:

\[
  F_{r}^{k*} := \\
  \begin{cases} 
  F_{r}^{k}, & k \neq 3 \mod 4, \\
  \{ A \in F_{r}^{k} \mid \sigma_{ess}(J_1 \cdots J_{k-1}A) \cap \mathbb{R}^{\pm} \neq \emptyset \}, & k = 3 \mod 4,
  \end{cases}
\]
and

\[ F^k_{c*} := \begin{cases} F^k_c, & k \neq 3 \mod 4, \\ \{ T \in F^k_c \mid \sigma_{ess}(T) \cap \mathbb{R}^\pm \neq \emptyset \}, & k = 3 \mod 4. \end{cases} \]

Note that operators in \( F^1_c \) and \( F^5_c \) are always self-adjoint and must have both positive and negative essential spectrum. When asking about the homotopy groups of these spaces it does not matter if we consider the real or complex spaces since the spaces are homeomorphic. Therefore, we may drop the index \( r \) and \( c \). Also one may omit the star in the following, and, for sake of concreteness, we define

\[ F^k := F^k_{c*} \text{ for all } k. \]

Now one can ask which elements are connected by a continuous path, meaning what are the connected components of each space. These can be classified by suitable topological invariants. One of them is the Noether index, and the other is a \( \mathbb{Z}_2 \)-index denoted by \( \text{ind}_2 \).

**Definition 3.19.** Let \( T \) be a Fredholm operator with Noether index \( \text{ind}(T) = 0 \). Then

\[ \text{ind}_2(T) := \dim(\text{Ker}(T)) \mod 2 \in \mathbb{Z}_2 \]

is called the \( \mathbb{Z}_2 \)-index of \( T \).

Of course in general, the \( \mathbb{Z}_2 \)-index is not a homotopy invariant on the space of Fredholm operators, but it is one on special subsets. For example, given odd symmetric Fredholm operators from \( F^1 \), the \( \mathbb{Z}_2 \)-index is a well-defined homotopy invariant, cf. [Schu15b].

More generally, it is proved in [AS69] that \( F^1 \) has the same zeroth homotopy group as the stabilised orthogonal group \( O \) and the higher homotopy groups are related by \( \pi_{k-1}(O) = \pi_0(F^k) \), and it is also proved that:

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_0(F^k) )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
<td>0</td>
<td>( 2\mathbb{Z} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{Z} )</td>
</tr>
</tbody>
</table>

The observing reader immediately notes the factor 2 in the fourth entry, which seems unnecessary since it does not change the group. However, the connected
3.3 Classifying spaces

components of $F^k$ are labelled by $\text{ind}$ and $\text{ind}_2$, and this explains the entry $2\mathbb{Z}$ for $F^4$ in the table above because kernel and cokernel of $T \in F^4$ clearly carry a quaternionic structure.

Now we close this chapter with some explicit connections between the homotopy groups of these Fredholm operators. In [AS69], the isomorphisms $\pi_0(F^{k+1}) \rightarrow \pi_1(F^k)$, for all $k \in \mathbb{N}$, are proved, and it may alternatively be possible to establish these isomorphisms by the loops of the following proposition. However, this is not done here and remains an object of further research.

**Proposition 3.20.** For each $T \in F^{k+1}$, we have a loop $[0, 2] \ni t \mapsto \alpha^k_t(T)$ in the Fredholm operators $F^k$ defined by

$$
\alpha^0_t(T) = \cos(\pi t) - iT \sin(\pi t)
$$

$$
\alpha^1_t(T) = \begin{pmatrix}
-i & \cos(\pi t) + \frac{1}{2} \left( \begin{array}{cc}
i(T-T^*) & T+T^* \\
T+T^* & -i(T-T^*)
\end{array} \right)
\end{pmatrix} \sin(\pi t)
$$

$$
\alpha^2_t(T) = I \cos(\pi t) + \frac{1}{2} ((T + \overline{T}) I + (T - \overline{T})) \sin(\pi t)
$$

$$
\alpha^3_t(T) = \begin{pmatrix}
-1 & \cos(\pi t) + T
\end{pmatrix} \sin(\pi t)
$$

$$
\alpha^4_t(T) = -i \cos(\pi t) + T \sin(\pi t)
$$

$$
\alpha^5_t(T) = \begin{pmatrix}
i & \cos(\pi t) + \frac{1}{2} \left( \begin{array}{cc}
(T+T^*) & -i(T-T^*) \\
-i(T-T^*) & -(T+T^*)
\end{array} \right)
\end{pmatrix} \sin(\pi t)
$$

$$
\alpha^6_t(T) = -\cos(\pi t) + iT \sin(\pi t)
$$

$$
\alpha^7_t(T) = \begin{pmatrix}
1 & \cos(\pi t) + T
\end{pmatrix} \sin(\pi t).
$$

**Proof.** It is easy to check that $\alpha^k_t(T) \in \Phi_k(S_k)$ for all $t \in \mathbb{R}$ and $T \in F^{k+1}$. Therefore, only the Fredholm property remains to show for each case.

For $\alpha^0$: If we consider the numerical range of the operator $\alpha^0_t(T)$, we can show that it is invertible for suitable $t \in [0, 2]$. For fixed $t$, the real part is given by $\text{Re} \langle x, \alpha^0_t(T)x \rangle = \cos(\pi t)$ for $\|x\| = 1$. Hence, the operator is invertible for $t \neq \frac{1}{2}$ and $t \neq \frac{3}{2}$. However, at these two exception points, we clearly have a Fredholm operator by definition.
For $\alpha^1$: Here, we can just square the operator and get:

$$(\alpha^1_t(T))^2 = \cos(\pi t)^2 \mathbf{1} + \frac{1}{2} \sin(\pi t)^2 \begin{pmatrix} TT^* + T^*T & i(TT^* - T^*T) \\ -i(TT^* - T^*T) & TT^* + T^*T \end{pmatrix}.$$ 

Therefore, one needs to check the essentially positivity of the block operator

$$B = \begin{pmatrix} TT^* + T^*T & i(TT^* - T^*T) \\ -i(TT^* - T^*T) & TT^* + T^*T \end{pmatrix},$$

which is given by Lemma 3.10.

For $\alpha^2$: Here again, we observe the square of the operator, but now with consideration of the adjoint. We obtain

$$(\alpha^2_t(T))^*\alpha^2_t(T) = \cos^2(\pi t)\mathbf{1} + \frac{1}{2} \left[ T^2 + \overline{T}^2 + I(\overline{T}^2 - T^2) \right] \sin^2(\pi t)$$

and

$$(\alpha^2_t(T))(\alpha^2_t(T))^* = \cos^2(\pi t)\mathbf{1} + \frac{1}{2} \left[ T^2 + \overline{T}^2 + (\overline{T}^2 - T^2)I \right] \sin^2(\pi t).$$

Now, we consider the numerical range as in the case for $\alpha^0$ and get that both operators are invertible for $t \neq 1/2$ and $t \neq 3/2$. Therefore, $\alpha^2_t(T)$ is a Fredholm operator for $t \neq 1/2$ and $t \neq 3/2$ by Lemma 3.8. For the two exception points, we set $B_\pm = T^2 + \overline{T}^2 \pm I(\overline{T}^2 - T^2)$ and can calculate:

$$\langle x, B_\pm x \rangle = \|Tx \pm TIx\|^2.$$

This is positive if and only if $x \notin \text{Ker}(T(\mathbf{1} \pm I))$. This kernel is by assumption finite dimensional since $T$ is Fredholm and $(\mathbf{1} \pm I)$ is invertible. Accordingly, the operators $B_\pm$ are essentially positive and $\alpha^2_{1/2}(T)$ and $\alpha^2_{3/2}(T)$ are also Fredholm operators by Lemma 3.8.

For $\alpha^3$: A straightforward calculation shows

$$(\alpha^3_t(T))^2 = \cos(\pi t)^2 \mathbf{1} + \sin(\pi t)^2 \begin{pmatrix} TT^* \\ T^*T \end{pmatrix},$$
which obviously is essentially positive for all $t$.

For $\alpha^4$: Here, we can use the same idea as for $\alpha^0$ and observe the numerical range. We find $\text{Im} \langle x, \alpha^4_t(T)x \rangle = -\cos(\pi t)$ for $\|x\| = 1$. Hence, the operator is again invertible for $t \neq \frac{1}{2}$ and $t \neq \frac{3}{2}$. However, for the exception points, we clearly have a Fredholm operator.

For $\alpha^5$: Here, we repeat the argumentation for the case $\alpha^1$ using the operator:

$$(\alpha^5_C(T)(t))^2 = \cos(\pi t)^2 1 + \frac{1}{2} \sin(\pi t)^2 \begin{pmatrix} TT^* + T^*T & i(TT^* - T^*T) \\ -i(TT^* - T^*T) & TT^* + T^*T \end{pmatrix}.$$

For $\alpha^6$: The proof for $\alpha^0$ carries over when changing one sign.

For $\alpha^7$: Again we can use a previous proof, namely that for $\alpha^3$. □
4 Index pairings and topological insulators

This chapter is based on the work [GS16] and considers a Hilbert space with real structure that is, in addition, equipped with a symmetry operator. In the sense of Definition 1.45, we can then have real, quaternionic, symmetric and anti-symmetric linear operators and so-called Lagrangian orthogonal projections. Here, we investigate index pairings of projections and unitaries when they fulfil some symmetry relations. We will find different scenarios for this index pairing: The Noether indices of the contemplated Fredholm operators can take either arbitrary integer values or only even integer values, or they can vanish and then possibly have secondary $\mathbb{Z}_2$-invariants.

With these general results, we are able to prove index theorems for strong invariants in disordered physical systems, so-called topological insulators. From the physical point of view, we have the Fermi projection that comes from the Hamiltonian of the system and the Dirac operator. In mathematical terms, the Fermi projection matches with the $K$-theoretical part and the Dirac operator with the $K$-homological part.

4.1 Introduction

In recent years, the topic *topological insulators* has been getting very popular in the community of mathematical physics and especially in solid state physics. The first question one could ask is where the name comes from. Generally speaking, a topological insulator is a physical system, namely a solid state system and a special material, that is indeed an insulator in the usual meaning. This means that there is a band gap in the energy spectrum between the valence
band and the conducting band. However, if we have a boundary in this system, then it is only an insulator in the bulk, meaning away from the boundary it is a usual insulator. On the other hand, on the boundary we find indeed a metallic property: The material is conducting on the boundary. For this physical system, we find so-called boundary states where the electrons can move freely.

This special property is interesting enough but that is not all. One often hears that these conducting boundary state are topological protected by continuous deformation of the system, like the hole in a torus is stable under continuous transformations. Indeed, this is in some sense exactly the case since they both have the mathematical description as topological invariants as we will see later. We refer here to [PS16a] and [Sch+08] for a good introduction to this topic.

In summary, this all explains the name of this solid state system since we have an insulator and the mathematical field of topology, which explains the strange behaviour on the boundary.

For sake of concreteness, we give a mathematical definition of the term described above:

**Definition 4.1.** Let $\mathcal{H}$ be a Hilbert space with real structure. An operator $H \in \mathcal{B}(\mathcal{H})$ is called the Hamiltonian of a topological insulator if it has a spectral gap around zero, satisfies one or a combination of time-reversal, particle-hole or chiral symmetry, and the Fermi projection $\chi_{(-\infty,0]}(H)$ is topologically distinguishable from the trivial projection.

The last part means that there is no norm-continuous path inside the Hamiltonians with the given symmetries that transforms the Fermi projection into the Fermi projection of a normal insulator. Of course, this is the essence of a topological insulator, but we will make this clear later. However, the definition above is not the general one, since often disordered systems are considered and then the spectral gap condition can be weakened by the fact that zero lies in a region of dynamical Anderson localisation. In this chapter, all mentioned proofs work with this weaker assumption that there is just a so-called mobility gap.

The main object of this chapter and the joint work [GS16], which this whole presentation is based on, is to establish the following:
**Goal.** Find a systematic construction of index pairings for strong invariants of topological insulators.

For the topological insulators that do not satisfy a symmetry relation or just a chiral symmetry, the so-called complex classes, this construction was already achieved in [PLB13] and [PS16b]. Furthermore, one finds such index theorems in [Schu15b] and [DS16], where they are proved for all cases in dimension $d = 2$. For these cases, the index pairing is well known from the theory of the integer quantum Hall effect, see [ASS94] and [BES94].

The basic idea here will be to consider the index pairings of the complex classes, which are known for arbitrary even dimensions [PLB13] and arbitrary odd dimensions [PS16b], and to implement the symmetries of the Fermi projection in order to deduce $\mathbb{Z}$-, $\mathbb{Z}_2$- and $2\mathbb{Z}$-valued index pairings. While in dimension $d = 2$, this essentially leads to Fredholm operators lying in the classifying spaces of Atiyah and Singer [AS69], new types of $\mathbb{Z}_2$- and $2\mathbb{Z}$-valued index pairings emerge in other dimensions, and it was not possible to understand them in terms of the classifying spaces, but this remains a possibility for further investigations in future. The existence and well-definedness of these pairings is rooted in Kramers degeneracy arguments, presented in Section 4.2.2, which are also new and interesting on a level of linear algebra. Section 4.2 presents and proves all these index pairings and constitutes the mathematical core of this chapter. These mathematical results need only a basic functional analytic framework, such that the main results should be more accessible to a wider audience. However, it is possible to understand and apply these index pairings in the realm of non-commutative geometry [Con94] as the result of a pairing of $KR$-groups, cf. Section 2.4.3, [Kar78] and [Schr93], with $KR$-cycles, see [Kas81] and [GVF01]. This is explained in Section 4.4, which could be used for further applications of the functional analytic results of Section 4.2.

The remainder of the chapter, Section 4.3 on topological insulators, can be seen as an example where all types of index pairings with symmetries appear. In connection with topological insulators, a major advantage of this index approach here is that it allows to deal with systems with a so-called broken translation invariance. Most prior works as [KM05b], [FKM07], [Ryu+10], [Kit09], [ASV13], [DG14] and [KZ16] were restricted to periodic systems, except
for [HL11], [GP13], [FM13] and [Thi16], which do not develop an index theory approach though. The indices constructed in this chapter allow to distinguish topological ground states, often also called quantum phases, of disordered systems or of local perturbations of periodic systems, and thereby prove a much stronger stability result on these systems. Moreover, the same strategy to implement symmetries also works for the boundary index pairings, see [KRS02], [EG11] and [QHZ08], an issue that also could be further developed. Recent important papers on very related matters are [Kel17] and [BCR16].

4.2 Index pairings with symmetries

4.2.1 Index pairings

Here, we will mainly consider a complex Hilbert space $\mathcal{H}$ with real structure and a Fredholm operator $T \in F(\mathcal{H})$ given in the form

$$T = PFP + P^\perp,$$  

(4.1)

where $P$ is an orthogonal projection in $\mathcal{H}$, $F$ is a unitary operator on $\mathcal{H}$ and $P^\perp := 1 - P$ is the orthogonal projection onto $\text{Ran}(P)^\perp$. Naturally, $P$ and $F$ have to be given in such a way that the operator $T$ above is indeed a Fredholm operator. For the set of orthogonal projections on $\mathcal{H}$, we will write $\mathcal{P}(\mathcal{H})$, and the unitaries are denoted by $\mathcal{U}(\mathcal{H})$.

Since the complex Hilbert space $\mathcal{H}$ is furnished with a complex conjugation $C$, the operators $P$ and $F$ can then have special symmetries, i.e. $P$ can be real or quaternionic or Lagrangian and $F$ can be real or quaternionic or symmetric or anti-symmetric. We will explain these symmetry relations later in detail.

When fixing a symmetry relation and consider all operators $P$ and $F$ that fulfil this chosen condition, one can look at the Noether index $\text{ind}(T) \in \mathbb{Z}$ of the associated operator $T$ from (4.1). One possibility is that these symmetries do not restrict the value of the Noether index, another one that it is always even, and yet another one is that $\text{ind}(T) = 0$. In this last case, one may find a
4.2 Index pairings with symmetries

secondary invariant using the definition

\[ \text{ind}_2(T) = \dim(\text{Ker}(T)) \mod 2 \in \mathbb{Z}_2. \]

This then means that \( \text{ind}_2(PFP + P^\perp) \) is invariant under norm-continuous homotopies of the two operators \( P \) and \( F \) in the chosen symmetry class, and, on the other hand, the Fredholm property of this operator \( T \) has to be preserved.

Above we have defined an index for a combination of a projection and a unitary with the help of the Fredholm operator in \( (4.1) \). Now, we expand this index pairing for two projections and also for two unitaries. Note that for two given projections \( P \) and \( E \), the operators \( 1 - 2P \) and \( 1 - 2E \) are unitaries, and we can define a Fredholm operator like in \( (4.1) \) by combining one unitary and one projection, see \( (4.2) \) below. Of course, we still assume the assumption that \( P \) and \( E \) are chosen such that \( T \) and \( T' \) are Fredholm operators. Firstly, we will show that it does not matter which of the two operators we define as the pairing for two projections:

**Proposition 4.2.** Let \( P, E \in \mathcal{P}(\mathfrak{H}) \) be orthogonal projections such that

\[
T = P(1 - 2E)P + P^\perp, \quad T' = E(1 - 2P)E + E^\perp \quad \text{(4.2)}
\]

are Fredholm operators. Then \( \text{ind}(T) = \text{ind}(T') = 0 \) and \( \text{ind}_2(T) = \text{ind}_2(T') \).

**Proof.** By definition \( T \) and \( T' \) are self-adjoint, and therefore the Noether indices vanish. Next we will show that the kernels are of same dimension. We need the fact

\[
T = 1 - 2PEP, \quad T' = 1 - 2EPE. \quad \text{(4.3)}
\]

Note also that \( \text{Ker}(T) \subset \text{Ran}(P) \) and \( \text{Ker}(T') \subset \text{Ran}(E) \). It is easy to see that for each \( x \in \text{Ker}(T) \) we have \( Ex \in \text{Ker}(T') \), and in the same way for each \( y \in \text{Ker}(T') \) we have \( Py \in \text{Ker}(T) \). Then by equation \( (4.3) \) each \( x \in \text{Ker}(T) \) and \( y \in \text{Ker}(T') \) fulfills

\[
PEx = PEPx = \frac{1}{2}x, \quad EPy = EPEy = \frac{1}{2}y.
\]

This means, in summary, that we have the commutative diagram
and therefore all maps have to be isomorphisms. In particular, $\ker(T)$ and $\ker(T')$ have the same dimension. \hfill \square

In the same manner, we can now define a pairing between two unitary operators $U$ and $F$ on $\mathcal{F}$. They can again transform into two projections on the Hilbert space $\mathcal{F} \oplus \mathcal{F}$ by setting $P = \frac{1}{2} \left( \frac{1}{U^*} U \right)$ and $E = \frac{1}{2} \left( \frac{1}{F^*} F \right)$. Consequently, we have again two index pairings in the form of Fredholm operators deduced from equation (4.1):

**Proposition 4.3.** Let $U, F \in \mathbb{U}(\mathcal{F})$ be unitaries such that

$$
T = P \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} P + P^\perp, \quad T' = E \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} E + E^\perp \quad (4.4)
$$

are Fredholm operators in $\mathbb{B}(\mathcal{F} \oplus \mathcal{F})$ where $P = \frac{1}{2} \left( \frac{1}{U^*} U \right)$ and $E = \frac{1}{2} \left( \frac{1}{F^*} F \right)$. Then one has $\text{ind}(T) = \text{ind}(T') = 0$ and

$$
\text{ind}_2(T) = \text{ind}_2(T') = \dim(\ker(UF + FU)) \mod 2.
$$

Of course, there are other possibilities than choosing the unitaries block diagonal, but (4.4) is the only acceptable choice in connection with commuting symmetry operators as we will see later.

**Proof.** Again, $T$ and $T'$ are self-adjoint by definition and have therefore vanishing Noether indices. For examining the kernel of $T$, one can write $P$ explicitly in (4.4), and one sees that $x \oplus y \in \mathcal{F} \oplus \mathcal{F}$ lies in kernel of $T$ if and only if

$$
(F + UFU^* + 2)x + (FU + UF - 2U)y = 0, \\
(U^* F + FU^* - 2U^*)x + (U^* FU + F + 2)y = 0.
$$
By multiplying the second equation with $U$ and combining with the first, we conclude that vectors in the kernel of $T$ are of the form $Uy \oplus y$ with $y \in \ker(UF + FU)$. By switching the role of $U$ and $F$, we get that the kernel of $T'$ consists of elements of the form $Fy \oplus y$ with $y \in \ker(UF + FU)$. This implies the equality and thus also $\text{ind}_2(T) = \text{ind}_2(T')$. Furthermore, by the same argument the kernel of $T^*$ consists of the vectors $F^*y \oplus y$ with $y \in \ker(UF^* + F^*U)$. Since $F^*\ker(UF^* + F^*U) = \ker(UF + FU)$, the dimensions of $\ker(T)$ and $\ker(T^*)$ coincide. Thus $\text{ind}(T) = 0$. \hfill $\square$

The Proposition 4.3 above shows that the suitable pairing for two unitaries $U, F \in U(\mathcal{F})$ is given by the operator $UF + FU$. Now, let us summarise and arrange all the possible pairings between two projections $P$ and $E$ and two unitaries $U$ and $F$ in a table:

<table>
<thead>
<tr>
<th>$F \in U(\mathcal{F})$</th>
<th>$E \in P(\mathcal{F})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P \in P(\mathcal{F})$</td>
<td>$PFP + P \perp$</td>
</tr>
<tr>
<td>$U \in U(\mathcal{F})$</td>
<td>$UF + FU$</td>
</tr>
</tbody>
</table>

Table 4.1: Pairings between projections and unitaries.

As already noted, we will always assume that the unitaries and projections are chosen such that the appearing operators are Fredholm. The pairings on the diagonal are analysed in Section 4.2.2, those in the upper right and lower left can only be $\mathbb{Z}_2$-valued due to Propositions 4.2 and 4.3 and are studied in Sections 4.2.3 and 4.2.4, respectively.

### 4.2.2 Pairing projections and unitaries with symmetries

First of all, we will use the notion of symmetries and symmetry operators as introduced in Section 1.4. As we will consider different real and quaternionic structures on the Hilbert space $\mathcal{F}$ involved, we choose them to be expressed in terms of one fixed complex conjugation $C: \mathcal{F} \to \mathcal{F}$ given by Definition 1.1. We recapitulate here Definition 1.45 because of its importance in this chapter:
Definition 4.4. Let $\mathcal{H}$ be Hilbert space with real structure $\mathcal{C}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called even real or odd real if $S^*AS = A$ for some even or odd symmetry operator $S$, respectively. An operator $A \in \mathcal{B}(\mathcal{H})$ is called even symmetric or odd symmetric if $S^*A^tS = A$ with an even or odd symmetry operator $S$, respectively. An orthogonal projection $P$ is called even Lagrangian or odd Lagrangian if $S^*PS = P^\perp$ with an even or odd symmetry $S$, respectively. Synonymously to Lagrangian we use the term symplectic.

Let us add a few comments here. An odd real operator can rightfully be called quaternionic since $CS$ gives a quaternionic structure on $\mathcal{H}$ in the sense of Definition 1.2. For self-adjoint operators, the notions of symmetry and reality obviously coincide. Moreover, an operator $A$ is odd symmetric with respect to $S$ if and only if $SA$ is an anti-symmetric operator. The range of the Lagrangian projections is half dimensional and a Lagrangian subspace in the Krein space $(\mathcal{H}, J)$ where $J = S$ or $J = iS$ is the fundamental symmetry, cf. Definition 1.4 and Definition 1.7.

In the literature, the term Lagrangian is sometimes only used for the real odd Lagrangian case. In connection with topological insulators, the Lagrangian nature of $P$ is also called an even or odd particle hole symmetry. Some further comments on the Lagrangian structure follow in Section 4.2.2 below.

As already indicated above, there will $\mathbb{Z}$-index, $2\mathbb{Z}$-index and $\mathbb{Z}_2$-index theorems. Moreover, it will be explained in the subsections below how to further distinguish these index theorems into real, quaternionic, odd Lagrangian and even Lagrangian types, where we will often used the letters R, Q, S and O, respectively. Here, S comes from symplectic and describes the odd Lagrangian type and O alludes to the complex orthogonal groups that is connected to even Lagrangian projections. This nomenclature is chosen according to the symmetry relation of the projection involved in the pairing. The following result specifies which type of index is well-defined for the given symmetries. The proof is spread over the next sections.

Theorem 4.5. Let $P \in \mathcal{P}(\mathcal{H})$ and $F \in \mathcal{U}(\mathcal{H})$ such that $T = PFP + P^\perp$ is a Fredholm operator, and let $S, \Sigma$ and $\tilde{S}$ be three commuting symmetry operators. In the respective cases, the index pairing is of the type indicated in Table 4.2.
There, for each entry 0 one has $\text{ind}(T) = 0$, and, moreover, there exists a path of index pairings with the given symmetries in an augmented Hilbert space that connects the pairing to a pairing with trivial $\mathbb{Z}_2$-index.

\begin{table}[h]
\centering
\begin{tabular}{ |c|c|c|c|c|c| }
\hline
 & $d = 2$ & $d = 4$ & $d = 6$ & $d = 8$ \\
\hline
$[S,F] = 0$ & $\Sigma^2 = 1$ & $\Sigma^2 = -1$ & $\Sigma^2 = -1$ & $\Sigma^2 = 1$ \\
$j$ & $\Sigma^* F^t \Sigma$ & $\Sigma^* \widetilde{F} \Sigma$ & $\Sigma^* F^t \Sigma$ & $\Sigma^* \widetilde{F} \Sigma$ \\
$[\hat{S},F] = 0$ & & & & \\
\hline
0 & $S^* \overline{P} S = P$ & $S^2 = 1$ & 0 & R-2 Z & R-Z$_2$ & Z \\
1 & $S^* \overline{P} S = P$ & $S^2 = 1$ & 0 & 0 & 0 & O-Z$_2$ \\
 & $\hat{S}^* \overline{P} \hat{S} = P^\perp$ & $\hat{S}^2 = 1$ & & & & \\
2 & $S^* \overline{P} S = P^\perp$ & $S^2 = 1$ & Z & 0 & O-2 Z & O-Z$_2$ \\
3 & $S^* \overline{P} S = P^\perp$ & $S^2 = 1$ & Q-Z$_2$ & 0 & 0 & 0 \\
 & $\hat{S}^* \overline{P} \hat{S} = P$ & $\hat{S}^2 = -1$ & & & & \\
4 & $S^* \overline{P} S = P$ & $S^2 = -1$ & Q-Z$_2$ & Z & 0 & Q-2 Z \\
5 & $S^* \overline{P} S = P$ & $S^2 = -1$ & 0 & S-Z$_2$ & 0 & 0 \\
 & $\hat{S}^* \overline{P} \hat{S} = P^\perp$ & $\hat{S}^2 = -1$ & & & & \\
6 & $S^* \overline{P} S = P^\perp$ & $S^2 = -1$ & S-Z$_2$ & S-Z$_2$ & Z & 0 \\
7 & $S^* \overline{P} S = P^\perp$ & $S^2 = -1$ & 0 & 0 & R-Z$_2$ & 0 \\
 & $\hat{S}^* \overline{P} \hat{S} = P$ & $\hat{S}^2 = 1$ & & & & \\
\hline
\end{tabular}
\caption{Index pairing for Theorem 4.5.}
\end{table}

In Table 4.2, the labels $j$ and $d$ are introduced for pure convenience for the moment being, but it will become apparent later on that they have the following interpretations: The number $j$ labels the real $K \mathcal{R}_j$-groups of projections having the symmetries listed in the table, and the number $d$ labels the symmetry class of the $K \mathcal{R}$-cycle. Here, we only listed the four even ones, whereas the odd ones are dealt with later on, see Section 4.4 for details. Furthermore, if we look into the theory of topological insulators, then $j$ labels the Cartan-Altland-Zirnbauer
classes and \(d\) the dimension of physical space, see Section 4.3. Topological insulators give us now, by definition, operators with non-vanishing invariants. However, let us briefly sketch how such examples can be constructed from basic mathematical objects.

**Example 4.6.** (See also [Schu15b].) Let \(\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})\) be a Hilbert space with real structure that is given by the standard complex conjugation. Furthermore, let \(\pi\) be the Hardy projection on \(\ell^2(\mathbb{N})\), and let \(V\) be the left shift on \(\ell^2(\mathbb{Z})\). Then we consider on \(\mathcal{H} \oplus \mathcal{H}\) the operators \(P = (\pi 0 0 \pi)\) and \(F = (V 0 V^* 0)\). These operators satisfy \(P = \overline{P}\) and \(F = \overline{F}\). If one introduces \(S_\pm = \Sigma_\pm = (0 \mp 1 1 0)\), then also \(S_\pm^* \overline{P} S_\pm = P\) and \(\Sigma_\pm^* F^t \Sigma_\pm = F\). Then \(T = PFP + P^\perp\) is a Fredholm operator with \(\text{ind}(T) = 0\) and \(\text{ind}_2(T) = 1\). The pairing is of the types \((j,d) = (0,2)\) and \((j,d) = (4,6)\), where only commuting symmetries are considered. The index inquires a different stability for the two cases. For \((j,d) = (0,2)\), all perturbations have a non-trivial \(\mathbb{Z}_2\)-index. On the other hand, for \((j,d) = (4,6)\) it is possible to find a perturbation \(T' = P'F'P' + (P')^\perp\) respecting the symmetries such that \(\text{ind}(T') = 0\) and \(\text{ind}_2(T') = 0\).

Now let us construct examples with \(j = 2, 6\). Set \(P = (\pi 0 \pi 0)\). Then, with \(S_\pm\) as above, \(S_\pm^* \overline{P} S_\pm = P^\perp\). Now, with \(F\) and \(\Sigma_\pm\) also as above, the Fredholm operator \(T = PFP + P^\perp\) has \(\text{ind}(T) = 2\) and \(\text{ind}_2(T) = 0\). This is stable in the cases \(d = 2, 6\), however, strictly speaking, for \((j,d) = (2,6)\) and \((j,d) = (6,2)\) the symmetries are anti-commuting, but this can be fixed as in the proof of Proposition 4.19. It is possible to construct non-trivial examples for all remaining cases in a similar manner. If \(F\) is interpreted as fixing an element in the \(KR\)-groups of the \(C^*\)-algebra \(C(S^1)\) equipped with the involutive isomorphism \((\tau f)(z) = f(\overline{z})\), then these examples provide index theorems detecting all the non-trivial \(KR\)-elements.

The next comments on the theorem concern the connection to the classifying spaces of Atiyah and Singer [AS69], that were already presentend in Section 3.3 and are recapitulated here. These eight classifying spaces were firstly introduced in [AS69] as sets of skew-adjoint Fredholm operators on a real Hilbert space having essential spectrum to both sides of the origin and anti-commuting with a
representation of a real Clifford algebra. Here it is more convenient to work with a representation of these spaces on an infinite dimensional complex Hilbert space $\mathcal{H}$ equipped with complex conjugation $C$ and a fixed odd symmetry operator $\Theta$, see Theorem 3.17 for more details where we, however, used the letter $I$ instead of $\Theta$. Let $\mathcal{F}(\mathfrak{g})$ denote the Fredholm operators on $\mathfrak{g}$ and $\mathcal{F}_{sa,\text{ess}}(\mathfrak{g})$ the self-adjoint Fredholm operators having essential spectrum both in $\mathbb{R}^+$ and $\mathbb{R}^-$. Then we know from Theorem 3.17 how to identify these topological spaces of [AS69] with the following spaces:

\[
\begin{align*}
\mathcal{F}^1 &= \{ T = -T^t \in \mathcal{F}_{sa,\text{ess}}(\mathfrak{g}) \}, \\
\mathcal{F}^2 &= \{ T = -T^t \in \mathcal{F}(\mathfrak{g}) \}, \\
\mathcal{F}^3 &= \{ T = \Theta^*T\Theta \in \mathcal{F}_{sa,\text{ess}}(\mathfrak{g}) \}, \\
\mathcal{F}^4 &= \{ T = \Theta^*T\Theta \in \mathcal{F}(\mathfrak{g}) \}, \\
\mathcal{F}^5 &= \{ T = -\Theta^*T\Theta \in \mathcal{F}_{sa,\text{ess}}(\mathfrak{g}) \}, \\
\mathcal{F}^6 &= \{ T = T^t \in \mathcal{F}(\mathfrak{g}) \}, \\
\mathcal{F}^7 &= \{ T = T^t \in \mathcal{F}_{sa,\text{ess}}(\mathfrak{g}) \}, \\
\mathcal{F}^8 &= \{ T = T^t \in \mathcal{F}(\mathfrak{g}) \}.
\end{align*}
\]

Here, we separated the odd labels, which represent the self-adjoint Fredholm operators, from the even labels. Moreover, we have the following table, as already noted in Section 3.3:

\[
\begin{array}{cccccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\pi_0(\mathcal{F}^i) & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} \\
\end{array}
\]

Table 4.3: Connected components for Fredholm operators.

Now it can readily be checked that the Fredholm operators $T$ in row $j = 0$ of Table 4.2 lie in the classifying space $\mathcal{F}^6$, $\mathcal{F}^4$, $\mathcal{F}^2$ and $\mathcal{F}^8$ respectively, while those in the row $j = 4$ lie in $\mathcal{F}^2$, $\mathcal{F}^8$, $\mathcal{F}^6$ and $\mathcal{F}^4$, where one uses $\Theta = \Sigma S$ and verifies that one can use $\Theta = 1$ whenever $\Theta^2 = 1$. In this chapter, we give short and independent proofs that are not invoking [AS69], that the index pairings are indeed well-defined in rows $j = 0$ and $j = 4$. However, what is not shown and investigated here is that the operator pairs $(P, F)$ and $(P', F')$ with same index can actually be homotopically deformed into each other without violating the symmetries and the Fredholm property. Such connectedness statements are known for the classifying spaces. The entries of the table carrying an S or
an O as prefix require different arguments, which are presented in the sections below. They provide new index pairings to our best knowledge, and we are not able to reduce them to the classifying spaces.

The remainder of this section is mainly devoted to the proof of Theorem 4.5. The basic well-known facts used here are that $\ker(T) = \ker(T^*T)$ and that $T$ is a Fredholm operator if and only if $0$ does not lie in the essential spectrum of $T^*T$, cf. Lemma 3.7 and Lemma 3.8. A basic property useful for assuring the existence of $2\mathbb{Z}$ and $\mathbb{Z}_2$-indices is a Kramers degeneracy, meaning multiplicity for eigenvalues, of the low lying eigenvalues of $T^*T$. This is an elementary result from linear algebra arguments, which can become somewhat intricate though, see the paragraphs below. This Kramers degeneracy is sufficient for the proof of all $\mathbb{Z}_2$-indices when combined with the following result.

**Proposition 4.7.** Let $P \in \mathcal{P}(\mathfrak{g})$ and $F \in \mathcal{U}(\mathfrak{g})$ such that $T = PFP + P^\perp$ is a Fredholm operator. Then one has $\text{ind}(T) = 0$ in the following two situations:

(a) $P$ is even or odd real and $F$ is even or odd symmetric.

(b) $P$ is even or odd Lagrangian and $F$ is even or odd real.

**Proof.** To (a): The hypothesis implies that $(\Sigma S)^*T^t\Sigma S = T$ holds so that $\ker(T^*) = C\Sigma S \ker(T)$.

To (b): Here one checks $(\Sigma S)^*T^t\Sigma S = P^\perp F^*P^\perp + P$. Combined with the fact that

$$\widetilde{F} : \ker(PFP + P^\perp) \to \ker(P^\perp F^*P^\perp + P), \quad x \mapsto Fx$$

is a bijection, see Lemma 1.46, this implies the claim. □

**Quaternionic $2\mathbb{Z}$-indices**

In this paragraph, the two $2\mathbb{Z}$-index theorems in the rows $j = 0$ and $j = 4$ of Table 4.2 in Theorem 4.5 are proved. In both cases, one uses the odd symmetry $\Theta = \Sigma S$ which then induces the relation $T = \Theta^*\overline{T}\Theta$, namely the matrix entries of $T$ are quaternionic in the grading of $\Theta$. Hence, $T$ lies in the classifying space $F_4$ for which it is now shown, cf. also [Schu15b], that the index is even. This is
4.2 Index pairings with symmetries

again based on the standard Kramers degeneracy argument presented below. It applies, in particular, to the kernel of $T$ and its cokernel, where we can use Lemma 3.7 for showing the Fredholm property and consider the self-adjoint operators $T^*T$ and $TT^*$, showing the R-2Z and Q-2Z entries in the table of Theorem 4.5.

**Proposition 4.8.** Let $\Theta$ be an odd symmetry operator, and let $T \in B(\mathcal{H})$ with $T = \Theta^*\mathcal{T}\Theta$. Then each eigenvalue of $T^*T$ has even multiplicity. To specify this: For each $\lambda \in \text{spec}_p(T^*T) \subset [0, \infty)$ and associated eigenvector $v \in \mathcal{H} \setminus \{0\}$, which means

$$T^*Tv = \lambda v,$$

there is a linearly independent eigenvector $w \in \mathcal{H} \setminus \{0\}$ given by

$$w = \Theta Cv.$$

**Proof.** With $\lambda \geq 0$ and $v, w \in \mathcal{H}$ from above and by using $\Theta C = C\Theta$ and $\Theta^* = -\Theta$, we can calculate:

$$T^*Tw = -(\Theta^*CT^*C\Theta)T\Theta^*Cv = -\Theta^*CT^*Tv = -\Theta^*C\lambda v = \lambda w.$$

Suppose now that there is $\mu \in \mathbb{C}$ such that $v = \mu w$. Then one would have

$$v = \mu \Theta Cv = \mu \Theta C \mu \Theta Cv = |\mu|^2 \Theta^2 v = -|\mu|^2 v$$

since $\Theta$ is an odd symmetry operator. This is in contradiction with $v \neq 0$. Clearly, $L = \text{span}\{v, w\}$ is an invariant subspace of $T^*T$ and so is its orthogonal complement $L^\perp$. One now goes on and restricts $T^*T$ to this orthogonal complement $L^\perp$ and repeats the argument above. In this argument, we have to use that $\Theta CL \subset L$ and $\Theta CL^\perp \subset L^\perp$, which implies that the vector $w$ lies in $L^\perp$ if $v \in L^\perp$. \qed

**Corollary 4.9.** Let $T \in F(\mathcal{H})$ be a Fredholm operator with $T = \Theta^*\mathcal{T}\Theta$ for an odd symmetry operator $\Theta$. Then the Noether index $\text{ind}(T)$ is always even.

**Proof.** By Lemma 3.7 we have $\text{Ker}(T) = \text{Ker}(T^*T)$ and in the same manner
Ker($T^*$) = Ker($TT^*$). Since $T^* \in \mathcal{F}(\mathcal{H})$ with $T^* = \Theta^* T^* \Theta$ as well, we can apply Proposition 4.8 on both operators.

\[ \text{Odd symmetric } \mathbb{Z}_2\text{-indices} \]

In the cases of the R-$\mathbb{Z}_2$ and Q-$\mathbb{Z}_2$ entries in Table 4.2 of Theorem 4.5, the Fredholm operator $T$ is odd symmetric $\Theta^* T^t \Theta = T$ with an odd symmetry operator given by $\Theta = \Sigma S$. In other words, we have $T \in \mathcal{F}^2$. By Proposition 4.7 this implies $\text{ind}(T) = 0$. That the parity of the dimension of the kernel of an odd symmetric Fredholm operator is a homotopy invariant was already proven in [Schu15b], and also follows from the results of [AS69]. In [GS16] a simpler new proof was presented and we will repeat it in the same way here. The core of the proof is showing that all positive eigenvalues of $T^* T$ have an even multiplicity which implies that under homotopies the parity of the nullity is conserved. This proves the entries R-$\mathbb{Z}_2$ and Q-$\mathbb{Z}_2$ of Table 4.2. Let us point out the following result does not apply for $\lambda = 0$. Otherwise if it applied, all $\mathbb{Z}_2$-indices would be trivial.

**Proposition 4.10.** Let $\Theta$ be an odd symmetry and let $\Theta^* T^t \Theta = T \in \mathcal{B}(\mathcal{H})$. Then each non-zero eigenvalue of $T^* T$ has even multiplicity. More precisely: For each $\lambda \neq 0$ and $v \in \mathcal{H} \setminus \{0\}$

\[ T^* Tv = \lambda v, \]

implies that

\[ w = \Theta C T v \neq 0, \]

is linearly independent of $v$ and also satisfies $T^* Tw = \lambda w$.

**Proof.** For $\lambda > 0$ we set $v, w \in \mathcal{H}$ as above, and by using $\Theta C = C \Theta$ and $\Theta^* = -\Theta$ one has:

\[ T^* Tw = \Theta(\Theta^* T^* \Theta)(\Theta^* T \Theta) \overline{Tv} = \Theta T T^* \overline{Tv} = \lambda \Theta \overline{T v} = \lambda w. \]

Suppose now that there is a $\mu \in \mathbb{C}$ such that $v = \mu w$. Then one would have

\[ v = \mu \Theta T C v = \mu \Theta T \overline{\mu} \Theta T v = -|\mu|^2 T^* Tv = -|\mu|^2 \lambda v, \]
which is a contradiction to \( v \neq 0 \). Now, \( L = \text{span}\{v, w\} \) is an invariant subspace of \( T^*T \) and so is its orthogonal complement \( L^\perp \). One goes on and restricts \( T^*T \) to this orthogonal complement \( L^\perp \) and repeats the above argument again. In this argument we have to use that \( \Theta C T L \subset L \) and \( \Theta C T L^\perp \subset L^\perp \), which implies that the vector \( w \) lies in \( L^\perp \) if \( v \in L^\perp \).

**Corollary 4.11.** Let \( T \in \mathcal{F}(\mathfrak{g}) \) be a Fredholm operator with \( \Theta^*T^t\Theta = T \) for an odd symmetry operator \( \Theta \). Then the Noether index \( \text{ind}(T) \) is always zero and it has a well-defined \( \mathbb{Z}_2 \)-index \( \text{ind}_2(T) \).

*Proof.* Obviously, \( \text{Ker}(T^*) = \Theta^*C\text{Ker}(T) \) and therefore the Noether index is zero. By the fact \( \text{Ker}(T) = \text{Ker}(T^*T) \), cf.

Lemma 3.7, we can consider the self-adjoint Fredholm operator \( T^*T \). By an homotopy within the class of Fredholm operators with this symmetry relation only eigenvalues can move into zero. Since they are always of even multiplicity, the kernel \( \text{Ker}(T^*T) \) can change only in steps of 2. Therefore \( \text{ind}_2(T) \) is a homotopy-invariant. \( \square \)

**Symplectic \( 2\mathbb{Z} \)-indices**

In this section, we consider the two commuting symmetry operators \( S \) and \( \Sigma \) that satisfy \( S^2 = \eta 1 \) and \( \Sigma^2 = -\eta 1 \) for \( \eta \in \{-1, +1\} \). Then \( P \) is supposed to be Lagrangian with respect to \( S \), which means \( S^*PS = P^\perp \). Moreover, we assume \( F \) to be symmetric with respect to \( \Sigma \), that is \( \Sigma^*F^t\Sigma = F \), and both commuting with the other symmetry, that is \( [F, S] = 0 \) and \( [P, \Sigma] = 0 \). As always \( T = PFP + P^\perp \) is supposed to be a Fredholm operator. This corresponds to the O-\( 2\mathbb{Z} \) and S-\( 2\mathbb{Z} \) entries in rows \( j = 2 \) and \( j = 6 \) of Table 4.2. For the proofs below it will be convenient to work with frames instead of projections. These frames here are analogously defined to \( J \)-Lagrangian frames in Section 1.2 about Krein spaces, but here we have a Hilbert space with real structure and so the definition is slightly different.

**Definition 4.12.** Let \( S \) be a symmetry operator on the Hilbert space \( \mathfrak{g} \) with real structure and \( \mathfrak{g}_0 \subset \mathfrak{g} \) a closed subspace. A linear map \( \Phi : \mathfrak{g}_0 \rightarrow \mathfrak{g} \) is called a \( S \)-Lagrangian frame if we have

\[
\Phi^*\Phi = 1_{\mathfrak{g}_0}, \quad \Phi\Phi^* + (S\Phi)(S\Phi)^* = 1_{\mathfrak{g}}.
\]
Note that each $S$-Lagrangian frame $\Phi$ is a partial isometry. One immediately deduces $\Phi^* S \Phi = 0$ or equivalently $\Phi^t S \Phi = 0$, namely column vectors in $\Phi$ and $S \Phi$ are orthogonal in $\mathcal{H}$. The usefulness of $S$-Lagrangian frames is that they give a Lagrangian projection $P = \Phi \Phi^*$ in $\mathcal{H}$.

**Remark 4.13.** For a given Lagrangian projection $P$ with respect to a given $S$ there is always a $S$-Lagrangian frame, just by defining $\mathcal{H}_0 := \text{Ran}(P)$ and $\Phi : \mathcal{H}_0 \to \mathcal{H}$ with $\Phi(x) = x$ for all $x \in \text{Ran}(P)$.

Next let us introduce for $T = PFP + P \perp$ the auxiliary operator:

$$T_0 = \Phi^* T \Phi = \Phi^* F \Phi \in \mathcal{B}(\mathcal{H}_0).$$

Clearly $T$ is a Fredholm operator if and only if $T_0$ is a Fredholm operator and the Noether indices coincide. Therefore the following result, applied to the kernel and cokernel of $T_0$, implies that entries $S$-$2\mathbb{Z}$ and $O$-$2\mathbb{Z}$ in rows $j = 2$ and $j = 6$ of Table 4.2 are well-defined.

**Proposition 4.14.** Let $S, \Sigma$ commuting symmetries, where one is odd and the other even, $P \in \mathcal{P}(\mathcal{H})$, $F \in \mathcal{U}(\mathcal{H})$ such that $S^* \Phi^* S = P \perp$ and $\Sigma^* F^t \Sigma = F$ and the commuting property $[F, S] = 0$ and $[P, \Sigma] = 0$. Furthermore, let $\Phi$ be a $S$-Lagrangian frame with $P = \Phi \Phi^*$, let $T := PFP + P \perp$ and $T_0 := \Phi^* T \Phi$. Then each eigenvalue in $[0, 1)$ of $T_0^* T_0$ has even multiplicity. In particular for $v \in \mathcal{H} \setminus \{0\}$

$$T_0^* T_0 v = \lambda v, \quad \lambda \in [0, 1),$$

implies that

$$w = \mathcal{C}(S \Phi)^* \Sigma^* F \Phi v,$$

is linearly independent of $v$ and also satisfies $T_0^* T_0 w = \lambda w$.

**Proof.** First of all note the following:

$$T_0^* T_0 = \Phi^* F^* \Phi \Phi^* F \Phi = 1 - \Phi^* F^* (S \Phi^*) (S \Phi^*)^* F \Phi$$

$$= 1 - \mathcal{C}(S \Phi^*)^* \Sigma^* F \Phi \Phi^* F \Phi \Sigma \Phi^* C$$

$$= 1 - (S \Phi^*)^* \Sigma^* F \Phi \Phi^* F \Phi \Sigma (S \Phi^*).$$
Now we choose an eigenvalue $\lambda \in [0,1)$ with an eigenvector $v$ and set $w$ as above. Then

\[
T_0^* T_0 w = (S\Phi)^* \Sigma^* F\Phi \left(1 - \Phi^* F^* \Sigma (S\Phi)^* \Sigma^* F\Phi\right) Cv
= (S\Phi)^* \Sigma^* F\Phi \left(C\Phi^* F^* \Sigma \Sigma^* F\Phi\right) v
= (S\Phi)^* \Sigma^* F\Phi \left(C\Phi^* F^* \Phi \Phi^* F\Phi\right) v
= C (S\Phi)^* \Sigma^* F\Phi T_0^* T_0 v = \lambda w ,
\]

where we used $\Phi \Phi^*$ and the properties of the $S$-Lagrangian frame. Now let $\mu \in \mathbb{C} \setminus \{0\}$ be such that $v = \mu w$. Then

\[
v = \mu C (S\Phi)^* \Sigma^* F\Phi v = |\mu|^2 (S\Phi)^* \Sigma^* F\Phi (S\Phi)^* \Sigma^* F\Phi v
= \eta |\mu|^2 (S\Phi)^* F^* \Phi (S\Phi)^* F\Phi v = -|\mu|^2 \eta^2 \Phi^* (S\Phi) (S\Phi)^* F\Phi v
= -|\mu|^2 (1 - T_0^* T_0) v = -|\mu|^2 (1 - \lambda) v ,
\]

which is impossible because $\lambda \in [0,1)$. This last calculation has also shown that $L := \text{span}\{v, w\}$ is invariant under $A := C (S\Phi)^* \Sigma^* F\Phi$. It is an easy calculation to show that $A^* = -A$ holds and therefore also $L^\perp$ is invariant under $A$. Therefore we can repeat the argument above for the operator $T_0^* T_0 |_{L^\perp}$ and iteratively get the wanted result.

Remark 4.15. When $S^2 = 1$, the above proof also goes through with minor modifications if the condition $\Sigma^* F^t \Sigma = F$ with $\Sigma^2 = -1$ is replaced by $F^t = -F$. Hence, the Fredholm operator $T = PFP + P^\perp$ for even real $P$ and antisymmetric $F$ has even dimensional kernel and cokernel.

Corollary 4.16. Let $T = PFP + P^\perp$ be a Fredholm operator with $\Sigma^* F^t \Sigma = F$ and $S^* P S = P^\perp$ for two commuting symmetries $S, \Sigma$, where one is odd and the other even. In addition, assume $[F, S] = 0$ and $[P, \Sigma] = 0$. Then the Noether index of $T$ is always even.

Proof. We use Proposition 4.14 for $T$ and $T^*$ which also fulfills $\Sigma^* (F^*)^t \Sigma = F^*$ and conclude with Lemma 3.7, that the index of $T$ has to be even.
Symplectic \( \mathbb{Z}_2 \)-indices

In this paragraph the two commuting symmetry operators \( S \) and \( \Sigma \) satisfying \( S^2 = \eta 1 \) and \( \Sigma^2 = \eta 1 \) for \( \eta \in \{-1,+1\} \). Then \( P \) is supposed to be Lagrangian projection with respect to \( S \) and \( F \) real with respect to \( \Sigma \), and both commuting with the other symmetry. As always \( T = PFP + P^\perp \) is supposed to be a Fredholm operator. Hence the aim is to analyse the \( \mathbb{Z}_2 \) entries in the rows \( j = 2 \) and \( j = 6 \) of Theorem 4.5. In the following result, it is allowed to work with a second Lagrangian projection \( Q \) also commuting with \( \Sigma \). Choosing \( P = Q \) and reasoning as in the first paragraph of Section 4.2.2 then shows that the \( O-\mathbb{Z}_2 \) and \( S-\mathbb{Z}_2 \) entries in rows \( j = 2 \) and \( j = 6 \) of Table 4.2 are well-defined. Again it is important to note that the argument does not apply to the eigenvalue \( \lambda = 0 \).

**Proposition 4.17.** Let \( S, \Sigma \) commuting symmetry operators, either both odd or both even, \( P, Q \in \mathbb{P}(\mathfrak{H}), F \in \mathbb{U}(\mathfrak{H}) \) such that \( S^*PS = P^\perp \), \( S^*QS = Q^\perp \) and \( \Sigma^*F\Sigma = F \). Moreover assume \( [F, S] = 0 \) and \( [P, \Sigma] = [Q, \Sigma] = 0 \). Let \( \Phi \) and \( \Psi \) be Lagrangian frames for \( P \) and \( Q \), respectively, and set

\[
T_0 = \Psi^*F\Phi .
\]

Then each non-vanishing eigenvalue of \( T_0^*T_0 \) has even multiplicity. To specify this: For \( v \in \mathfrak{H} \setminus \{0\} \)

\[
T_0^*T_0 v = \lambda v , \quad \lambda > 0 ,
\]

implies that

\[
w = \mathcal{C}(S\Phi)^*\Sigma F^*\Psi\Psi^*F\Phi v \neq 0 ,
\]

is linearly independent of \( v \) and also satisfies \( T_0^*T_0 w = \lambda w \).

**Proof.** Choose \( \lambda > 0 \) and \( v \) as an eigenvector and \( w \) defined as above. Let us introduce the auxiliary operator

\[
R_0 := \Psi^*F\Sigma\Phi .
\]

Then we also write the vector \( w \) in the form \( w = \mathcal{C}R_0^*T_0 v = R_0^t T_0^t Cv \). Further-
more, one has the identities

\[ T_0^t T_0 = 1 - R_0^* R_0 , \quad R_0^t T_0 = - \eta T_0^* R_0 , \quad T_0 T_0^* = 1 - R_0 R_0^* . \]

Let us start with the first one:

\[
T_0^t T_0 = \Phi^t \Sigma^* F^* \Psi^t F \Sigma \Phi = (S\Phi)^* \Sigma^* F^* (S\Psi) (S\Psi)^* F \Sigma (S\Phi) \\
= (S\Phi)^* \Sigma^* F^* (1 - \Psi \Psi^*) F \Sigma (S\Phi) \\
= 1 - (S\Phi)^* \Sigma^* F^* \Phi \Psi^* F \Sigma (S\Phi) \\
= 1 - R_0^* R_0 ,
\]

Moreover, we get:

\[
R_0^t T_0 = (S\Phi)^* F^t \Psi^\dagger F \Phi = (S\Phi)^* \Sigma^* F^* \Phi^\dagger F \Sigma (S\Phi) \\
= \eta \Phi^* \Sigma^* F^* (S\Psi) (S\Psi)^* F \Sigma (S\Phi) = \eta \Phi^* \Sigma^* F^* (1 - \Psi \Psi^*) F \Sigma (S\Phi) \\
= - \eta T_0^* R_0 .
\]

The third one follows in a similar manner. Using these identities, it can now be verified that \( w \) is an eigenvector:

\[
T_0^* T_0 w = T_0^t T_0 R_0^* T_0 v = (1 - R_0^* R_0) R_0^* T_0 v = R_0^* (1 - R_0^* R_0) T_0 v \\
= R_0^* T_0 T_0^* T_0 v = \lambda w .
\]

Suppose now that there is \( \mu \in \mathbb{C} \) such that \( v = \mu w \). Then one would have

\[
T_0^* T_0 v = \mu T_0^* T_0 w = \mu T_0^* T_0 R_0^t T_0 v = \eta |\mu|^2 T_0^* T_0 (R_0^t T_0) R_0^* T_0 v \\
= - \eta^2 |\mu|^2 T_0^* T_0 R_0^* R_0^* T_0 v = - |\mu|^2 T_0^* T_0 T_0^* (1 - T_0 T_0^*) T_0 v \\
= - |\mu|^2 (\lambda^2 - \lambda^3) v ,
\]

in contradiction to the positivity of \( T_0^* T_0 \) and \( \|T_0\| \leq 1 \) (so that \( \lambda \leq 1 \)). Hence, \( \text{span}\{v, w\} \) is a two-dimensional invariant subspace of \( T_0^* T_0 \). Its orthogonal complement is also invariant. Restricting \( T_0^* T_0 \) to this orthogonal complement allows to repeat the argument. \( \square \)
Vanishing index pairings with one symmetry each

The aim of this paragraph is to show the four entries 0 in Theorem 4.5 with even \( j \), namely for \((j, d) \in \{(0, 2), (2, 4), (4, 6), (6, 8)\}\). They explain that the Noether index vanishes and the \( \mathbb{Z}_2 \)-index is no homotopy invariant. By Proposition 4.7, the Noether index vanishes in each of these cases so that only remains to show that the \( \mathbb{Z}_2 \)-index changes to the trivial value along a path of index pairings respecting the symmetries imposed. In all examples that we considered before such a path could be constructed within the given Hilbert space and we suspect this to be true in general, but here only the last claim of Theorem 4.5 will be proven, which means that we show that the path can be constructed in an augmented Hilbert space. Let us mainly focus on the case \((j, d) = (0, 2)\) since the others are dealt with in a similar manner. Set

\[
\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}^2, \quad \tilde{\Sigma} = \Sigma \oplus \sigma_1, \quad \tilde{S} = S \oplus 1_2, \quad \tilde{C} = C \oplus \overline{\cdot},
\]

where \(\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) and the last overline denotes the complex conjugation in \(\mathbb{C}^2\). Therefore \(\tilde{\mathcal{H}}\) becomes a Hilbert space with real structure \(\tilde{C}\). Furthermore with \(r_\lambda = \begin{pmatrix} \cos(\lambda) & -\sin(\lambda) \\ \sin(\lambda) & \cos(\lambda) \end{pmatrix}\) and \(p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) let us set

\[
\tilde{F}_\lambda = F \oplus r_\lambda, \quad \tilde{P} = P \oplus p, \quad \tilde{T}_\lambda = \tilde{P} \tilde{F}_\lambda \tilde{P} + 1 - \tilde{P}.
\]

By construction \(\tilde{P}\) and \(\tilde{F}_\lambda\) satisfy all the symmetries of the index pairing \((j, d) = (0, 2)\), and \(\tilde{T}_\lambda\) is a Fredholm operator for all \(\lambda\). Now \(\tilde{T}_0 = T \oplus 1_2\) so that, in particular, \(\text{ind}_2(\tilde{T}_0) = \text{ind}_2(T)\). On the other hand, at \(\lambda = \frac{\pi}{2}\) the dimension of the kernel changes by 1 so that \(\text{ind}_2(\tilde{T}_{\pi/2}) = \text{ind}_2(T) + 1\).

Index pairings with three symmetries

Finally, let us consider the cases in Theorem 4.5 where three symmetries \(S\), \(\hat{S}\) and \(\Sigma\) are involved. These obviously correspond to the rows with odd \(j\). Therefore all these rows are placed between two other rows which are specified by one of the symmetries for the projection \(P\) each. Of course we use here the periodicity and therefore the symmetries of \(j = 7\) are given by those of \(j = 6\) and \(j = 0\). As the symmetries \(S\) and \(\hat{S}\) are supposed to commute, the
Fredholm operators $T$ in rows with odd $j$ inherit all the properties from the neighbouring rows with an even label $j$. This implies that for each odd $j$ there is only one possibly non-vanishing entry in $\mathbb{Z}_2$ which appears for $d$ such that for the neighbouring label $j$ there are $\mathbb{Z}$- and $\mathbb{Z}_2$-indices. Indeed, if one of the neighbouring $j$ has a $2\mathbb{Z}$ entry, automatically the $\mathbb{Z}_2$-index vanishes. This is relevant for eight of the 16 cases, e.g. $(j,d) = (4,1), (6,1)$. The 0 entries in the four cases $(j,d) = (2,1), (4,3), (6,5), (8,7)$ follow from the arguments in Section 4.2.2 because the homotopies constructed there merely modify $F$ and not $P$, so that they extend directly to the neighbouring cases with an extra symmetry for $P$.

4.2.3 Pairing projections with projections

In the last section, we analysed the pairing of a projection with a unitary and got all possible situations in Table 4.2. In this section, we collect all the possible pairings between two projection $P,E \in \mathbb{P}(\mathfrak{g})$ and prove the associated index theorems for the Fredholm operator in the upper right of Table 4.1.

**Proposition 4.18.** Let $P$ and $E$ be two orthogonal projections such that $T = P(1 - 2E)P + P^\perp$ is a Fredholm operator. Then $\text{ind}_2(T)$ takes the following values of Table 4.4.

**Proof.** First recall from Proposition 4.2 that $\text{ind}(T) = 0$. Let us first look at the entries in the column $d = 1$. Set $F = 1 - 2E$ which is then unitary and satisfies both $\Sigma^*F\Sigma = F$ and $\Sigma^*F^t\Sigma = F$ with $\Sigma^2 = 1$. Hence, the pairing $PFP - 1 - P$ lies both in the column $d = 2$ and $d = 8$ of the table of Theorem 4.5. Consequently, for $j = 0$ the entry is both 0 and $\mathbb{Z}$, leading to the entry 0. Similarly, one argues for $j = 6$. For $j = 2$, the entry is both $\mathbb{Z}$ and O-$\mathbb{Z}_2$, leading to O-$\mathbb{Z}_2$ for $(j,d) = (2,1)$. For $j = 4$ the entries are Q-$\mathbb{Z}_2$ and Q-$2\mathbb{Z}$, implying an entry 0 because Q-$2\mathbb{Z}$ actually means that the kernel is even dimensional. This concludes all cases of the column $d = 1$. For column $d = 5$ one can proceed in a similar manner using columns $d = 4,6$ of Theorem 4.5. Furthermore, these arguments applies also to the lines $j = 0$ and $j = 4$. Hence, it only remains to consider the four entries $(j,d) \in \{(2,3), (2,7), (6,3), (6,7)\}$. For $(j,d) = (2,3)$, let us note that $\text{Ker}(T) = \text{Ker}(P_i(1 - 2E)P + P^\perp)$. Setting now $F = i(1 - 2E)$
Table 4.4: Pairings for Proposition 4.18

one has $\Sigma^*F\Sigma = F$ and hence the entry $(j, d) = (2, 4)$ allows to conclude. For $(j, d) = (6, 7)$ one can proceed in the same manner. Next let us consider $(j, d) = (2, 7)$. Using $[P, \Sigma] = 0$ one finds $\text{Ker}(T) = \text{Ker}(P\Sigma(1 - 2E)P + P^\perp)$. Now $F = \Sigma(1 - 2E) = -F^t$ and by Remark 4.15 one concludes that $\text{Ker}(T)$ is even dimensional so that the $\mathbb{Z}_2$-index vanishes. Again the case $(j, d) = (6, 3)$ works similarly.

4.2.4 Pairing unitaries with unitaries

The only remaining pairing is now the left bottom case in Table 4.1, which means that we pair two unitaries $U, F \in \mathcal{U}(\mathfrak{g})$.

Proposition 4.19. Let $U$ and $F$ be two unitaries such that $T$ given by (4.4), which means

$$T = P \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} P + P^\perp,$$
is a Fredholm operator. Then \( \text{ind}_2(T) \) takes the following values:

<table>
<thead>
<tr>
<th></th>
<th>( d = 2 )</th>
<th>( d = 4 )</th>
<th>( d = 6 )</th>
<th>( d = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [S, F] = 0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Sigma^* F^t \Sigma = F )</td>
<td>( \Sigma^* F^t \Sigma = F )</td>
<td>( \Sigma^* F^t \Sigma = F )</td>
<td>( \Sigma^* F^t \Sigma = F )</td>
<td></td>
</tr>
<tr>
<td>( j )</td>
<td>( \Sigma^2 = 1 )</td>
<td>( \Sigma^2 = -1 )</td>
<td>( \Sigma^2 = -1 )</td>
<td>( \Sigma^2 = 1 )</td>
</tr>
<tr>
<td>( [\Sigma, P] = 0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{S}, F = 0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

1. \( S^* \bar{U} S = U \)
   - \( S^2 = 1 \)
   - \( O - \mathbb{Z}_2 \)

2. \( S^* U^t S = U \)
   - \( S^2 = -1 \)
   - \( Q - \mathbb{Z}_2 \)

3. \( S^* \bar{U} S = U \)
   - \( S^2 = -1 \)
   - \( S - \mathbb{Z}_2 \)

4. \( S^* U^t S = U \)
   - \( S^2 = 1 \)
   - \( R - \mathbb{Z}_2 \)

**Proof.** By Proposition 4.3 one has \( \text{ind}(T) = 0 \). For \( j = 1, 5 \) the projection \( P = \frac{1}{2} (\begin{smallmatrix} 1 & U \end{smallmatrix} \begin{smallmatrix} U^* & 1 \end{smallmatrix}) \) satisfies

\[
\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}^* P \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} = P, \quad \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}^* P \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix} = 1 - P.
\]

Consequently \( P \) indeed satisfies the relations given in rows \( j = 1, 5 \) of Theorem 4.5 respectively and consequently one can copy these rows. For \( j = 3, 7 \) the relations are

\[
\begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix}^* P \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} = P, \quad \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}^* P \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} = P^\perp.
\]

These are again the relations given in rows \( j = 3, 7 \) of Table 4.2, with the
sole difference that the two symmetry operators anti-commute. This can be fixed using the Cayley transformation \( C = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i \frac{1}{2} \\ 1 & -1 \end{array} \right) \) in the added fiber. The projections \( P' = CPC^* = \frac{1}{2} \left( \begin{array}{cc} 1-b & -ia \\ ia & 1+b \end{array} \right) \) with \( a = \frac{1}{2}(U + U^*) \) and \( b = \frac{1}{2i}(U - U^*) \) then satisfy

\[
\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}^* P' \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} = P', \quad \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}^* P' \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} = (P')^\perp.
\]

Now the symmetries commute and are of the same type. Thus the index of \( CT'C^* = P'(F_0^0) P' + (P')^\perp \) is given by the index in the rows \( j = 3, 7 \) of Theorem 4.5.

Due to Proposition 4.3 it is also possible to study instead of \( T \) also the operator \( T'' = UF + FU \). This lead to different index types. For example, for \( (j, d) = (3, 2) \) or \( (j, d) = (7, 6) \), one has \( (S\Sigma)^*(T'')^t(S\Sigma) = T'' \) with \( (S\Sigma)^2 = -\mathbf{1} \). Consequently, \( T'' \) is in the classifying space \( F_2 \) and has a non-vanishing index of type considered in Section 4.2.2.

### 4.3 Index pairings for topological insulators

In this section, we will use our index pairings from above for the classification of topological insulators. We will review the three symmetry relations, give a short overview for zero-dimensional systems and explain the implications of Theorem 4.5 for the periodic table of topological insulators. We will also construct some non-trivial examples.

#### 4.3.1 Topological insulators and their classification

Here we consider a physical system consisting of independent Fermions and is described by a bounded one-particle Hamiltonian that is a self-adjoint operator \( H \in \mathcal{B} \) acting on a Hilbert space with real structure \( \mathcal{C} \). This Hamiltonian can have one or several of the following symmetries, cf. Section 1.4, implemented by \emph{commuting} symmetry operators \( S_{ch}, S_{tr} \) and \( S_{ph} \) in the sense of Definition 1.43 as already introduced in Definition 1.44. We summarise these possible symmetry
relations:

\[ S^*_{\text{ch}} H S_{\text{ch}} = -H, \quad S^2_{\text{ch}} = \pm 1, \quad \text{(chiral symmetry, CHS)} \]
\[ S^*_{\text{tr}} H S_{\text{tr}} = H, \quad S^2_{\text{tr}} = \pm 1, \quad \text{(time-reversal symmetry, ±TRS)} \quad (4.5) \]
\[ S^*_{\text{ph}} H S_{\text{ph}} = -H, \quad S^2_{\text{ph}} = \pm 1, \quad \text{(particle-hole symmetry, ±PHS)} \]

Both the time-reversal and the particle-hole symmetry can be even or odd but for the chiral symmetry the sign is irrelevant. The CHS is sometimes also called a sublattice symmetry because that is the way it often appears in particular models, and as an alternatively fermionic parity. The cases in which \( H \) has no symmetry or only a CHS are called the complex classes since there is no real structure involved. These are the first two rows in Table 4.5 below and they will not be further considered here.

The main focus is on the other eight so-called real classes because they invoke the real structure on \( \mathcal{H} \). There are four cases where there is only one symmetry and there are four with a combination of a TRS with a PHS. In these latter cases, their product \( S_{\text{ch}} = S_{\text{tr}} S_{\text{ph}} \) always also induces a chiral symmetry, so there cannot be a case with exactly two symmetries. The symmetry classification into these 10 classes has been known and widely used since the work of Altland and Zirnbauer \([AZ97]\). With reference of this work each class is associated to a so-called CAZ label, as in Cartan-Altland-Zirnbauer label. With respect to the physical background, this classification applies to metals, insulators and mesoscopic systems in the same manner.

In the theory of topological insulators one now considers only Fermion systems for which the Fermi level lies either in a gap of \( H \) or at least in a region of strong Anderson localisation. Mathematically speaking the first thing can be realised by having a spectral gap around zero for \( H \), meaning \( H \) is invertible. The main new feature is that within several of the CAZ classes there are topologically different ground states which can be distinguished by so-called strong invariants. The possible values of these strong invariants are given in Table 4.5. From the physical point of view integer quantum Hall systems, that means \( d = 2 \) and \( j = 0 \), are the best known examples with non-vanishing strong invariants, see \([BES94]\) for a mathematical treatment. The first new topological insulators in
### Table 4.5: List of symmetry classes ordered by TRS, PHS and CHS as well as the CAZ label. Then follow the strong invariants in dimension $d = 0, \ldots, 7$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>TRS PHS CHS</th>
<th>CAZ</th>
<th>$d=0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0 0</td>
<td>A</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0 0 0 1</td>
<td>AIII</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>+1 0 0 0</td>
<td>AI</td>
<td>$\mathbb{Z}$</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>+1 +1 +1</td>
<td>BDI</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0 +1 0</td>
<td>D</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>$2\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>−1 +1 +1</td>
<td>DIII</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>$2\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>−1 0 0</td>
<td>AII</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>−1 −1 +1</td>
<td>CII</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0 −1 0</td>
<td>C</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>+1 −1 +1</td>
<td>CI</td>
<td>$2\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Dimension $d = 2$ with odd TRS were theoretically discovered by Kane and Mele [KM05b]. Moreover a complete list was found by Schnyder et al. [Sch+08] and the periodic ordering of Table 4.5 was put forward by Kitaev [Kit09]. In the same paper, Kitaev also showed that each entry in the lower part of the table has an interpretation in realm of Real $K$-theory, cf. [Kar78] and [Schr93]. Let us briefly indicate how this goes. One considers $\mathbb{R}^d = \mathbb{R}^d_\tau$ as the momentum space on which complex conjugation is implemented by the involution $\tau(k) = -k$ for $k \in \mathbb{R}^d$. This makes $C_0(\mathbb{R}^d_\tau)$ into a $C^*$-algebra with symmetry of which the $KR$-groups $KR_j(C_0(\mathbb{R}^d_\tau))$, $j = 0, \ldots, 7$, can be defined. Each element of these $KR$-groups is interpreted as the class specified by a Fermi projection (of a translation invariant system) with a particular combination of the symmetries in (4.5). More precisely, the $(j,d)$th entry of Table 4.5 is an invariant given by

$$\text{Inv}(j, d) \in KR_j(C_0(\mathbb{R}^d_\tau)) = \pi_{j-1-d}(O), \quad (4.6)$$
where the fundamental groups of the stabilised orthogonal group $O$ are $8$-periodic and given by Table 4.3. The second equality in (4.6) can be found in textbooks [Kar78], [Schr93] and has been explained in various more recent works in the context of topological insulators [SCR10], [FM13], [Thi16], [KZ16].

The formula (4.6) indeed explains the 8 periodicity of the lower part of Table 4.5 both in $d$ and $j$. For periodic rather than translation invariant systems, the invariants take values in the larger group $KR_j(C_0(\mathbb{T}^d_T))$. The supplementary elements are then the so-called weak invariants [Kit09] which allow to further distinguish ground states with equal strong invariants.

From now on, it will be supposed that the Hamiltonian $H$ acts on the tight-binding Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ with $N$-dimensional fibers and that the symmetry operators $S_{ch}$, $S_{tr}$ and $S_{ph}$ act on these fibers only. Furthermore a Fermi level $\mu \in \mathbb{R}$ is given. In all cases with either a CHS or a PHS it is supposed to be $\mu = 0$. It specifies the Fermi projection $P = \chi(H \leq \mu)$ as the spectral projection of $H$ on energies below the Fermi level.

### 4.3.2 Invariants for zero-dimensional systems

The main aim of the present work is to calculate the invariant $\text{Inv}(j,d)$ not as a $K$-group element as in (4.6), but rather as numerical invariant from an index pairing for systems submitted to the basic symmetries. As a warm-up let us consider the case of dimension $d = 0$, following [Kit09], [HL11], see also [Lor15]. Here the Hamiltonian $H$ is merely a finite dimensional self-adjoint matrix, where the size of this matrix is the dimension of the fiber over a point constituting the 0-dimensional system, which satisfies some of the fundamental symmetries (4.5) implemented by matrices $S_{tr}$ and $S_{ph}$. Moreover, the Fermi level is supposed to lie in a gap of the spectrum of $H$. For CAZ classes A, AI and AII the invariant distinguishing different systems is simply the signature of $H - \mu 1$ given by $\text{Inv}(j,0) = \text{Tr}(P) - \text{Tr}(P^\perp)$ for $j = 0, 4$. This is clearly a homotopy invariant which can only change when $\mu$ is an eigenvalue of $H$ (so that the system is not an insulator any more). Moreover, in the case of odd TRS (Class AII) a Kramers degeneracy argument indeed shows that the signature is always even, leading to the $2\mathbb{Z}$ entry for $(j,d) = (5,0)$. If one considers a Hamiltonian with PHS, then the Fermi projection $P = \chi(H \leq 0)$
satisfies
\[ S_{\text{ph}}^* P S_{\text{ph}} = P^\perp, \]
namely \( P \) is Lagrangian with respect to the symmetry \( S_{\text{ph}} \). See Example 1.48 for more details. Taking the trace of the identity immediately implies that the signature vanishes. A similar argument also applies for chiral systems, in particular Class AIII. Hence, it only remains to explain the two, secondary, \( \mathbb{Z}_2 \)-invariants appearing in Classes D and BDI. In some typical models, which are stemming from second quantised operators that are quadratic in the creation and annihilation operators, the even PHS and Hamiltonian are of the form
\[
H = \begin{pmatrix} h & \Delta \\ \Delta^* & -\bar{h} \end{pmatrix}, \quad S_{\text{ph}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Hence, we have \( S_{\text{ph}} = \sigma_1 \otimes 1 \). Now the PHS, meaning \( S_{\text{ph}} \overline{H} S_{\text{ph}} = -H \), is equivalent to the so-called *BdG equation* \( \Delta^t = -\Delta \). The invariant is best defined in the Majorana representation, which is obtained after a Cayley transformation in the grading of \( S_{\text{ph}} \):
\[
H_{\text{Maj}} = C^t H \overline{C}, \quad \text{with} \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i1 \\ 1 & i1 \end{pmatrix}.
\]

Then the PHS becomes \((C^* S_{\text{ph}} \overline{C})^* \overline{H_{\text{Maj}}} (C^* S_{\text{ph}} \overline{C}) = -H_{\text{Maj}}\) so that the new symmetry operator is \( C^* S_{\text{ph}} \overline{C} = C^* \sigma_1 \overline{C} = 1 \). Hence,
\[
H_{\text{Maj}} = H_{\text{Maj}}^* = -\overline{H_{\text{Maj}}} = -H_{\text{Maj}}^t.
\]

Therefore, \( H_{\text{Maj}} \) is a purely imaginary and antisymmetric matrix. The matrix entries of \( H_{\text{Maj}} \) can be readily expressed in terms of the real and imaginary parts of the matrix entries of \( H \). Now one has, as for every real skew-symmetric matrix, the relation \( \det(iH_{\text{Maj}}) = \text{Pf}(iH_{\text{Maj}})^2 \), where \( \text{Pf} \) denotes the Pfaffian of the skew-symmetric matrix. The \( \mathbb{Z}_2 \)-invariant is then defined as the sign of the Pfaffian \( \text{Inv}(2, 0) = \text{sgn}(\text{Pf}(iH_{\text{Maj}})) \in \mathbb{Z}_2 \). For systems of Class BDI \((j = 1)\) that, moreover, have an even TRS, this Pfaffian invariant given by \( \text{Inv}(1, 0) = \text{sgn}(\text{Pf}(iH_{\text{Maj}})) \in \mathbb{Z}_2 \) is still well-defined and may be non-trivial.
This completes the discussion of 0-dimensional systems.

### 4.3.3 Reordering of the symmetries of the Hamiltonian

As discussed in Section 4.3 above, the Hamiltonian is supposed to have one or two of the commuting physical symmetries $S_{tr}$ and $S_{ph}$ given by (4.5). Two of these symmetries then induce a chiral symmetry $S_{ch} = S_{tr}S_{ph}$. The first task is to order these symmetries in the way they appear in the periodic table, that is Table 4.5 above. This is written out in the following Table 4.6.

<table>
<thead>
<tr>
<th>$j$</th>
<th>CAZ</th>
<th>$S^*\bar{P}S$</th>
<th>$S$</th>
<th>$\hat{S}^*\bar{P}\hat{S}$</th>
<th>$\hat{S}$</th>
<th>$R^2$</th>
<th>$R^*PR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>AI</td>
<td>$P$</td>
<td>+TRS</td>
<td>$P$</td>
<td>+TRS</td>
<td>$-1$</td>
<td>$P^\perp$</td>
</tr>
<tr>
<td>7</td>
<td>CI</td>
<td>$P^\perp$</td>
<td>−PHS</td>
<td>$P$</td>
<td>+TRS</td>
<td>$1$</td>
<td>$P^\perp$</td>
</tr>
<tr>
<td>6</td>
<td>C</td>
<td>$P^\perp$</td>
<td>−PHS</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>CII</td>
<td>$P$</td>
<td>−TRS</td>
<td>$P^\perp$</td>
<td>−PHS</td>
<td>$1$</td>
<td>$P^\perp$</td>
</tr>
<tr>
<td>4</td>
<td>AII</td>
<td>$P$</td>
<td>−TRS</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>DIII</td>
<td>$P^\perp$</td>
<td>+PHS</td>
<td>$P$</td>
<td>−TRS</td>
<td>$-1$</td>
<td>$P^\perp$</td>
</tr>
<tr>
<td>2</td>
<td>D</td>
<td>$P^\perp$</td>
<td>+PHS</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>BDI</td>
<td>$P$</td>
<td>+TRS</td>
<td>$P^\perp$</td>
<td>+PHS</td>
<td>$1$</td>
<td>$P^\perp$</td>
</tr>
</tbody>
</table>

**Table 4.6: Reordering of the Symmetries**

In this table $S$ and $\hat{S}$ are either $S_{tr}$ or $S_{ph}$ and $R = S\hat{S} = S_{ch}$. Let us note that it is not possible to choose $S = S_{tr}$ or $S = S_{ph}$ throughout for all $j$ because both a single TRS and a single PHS appear as single symmetries for $j$ even. On the other hand for odd $j$ we made a particular, arbitrary, choice as to what is $S$ and what is $\hat{S}$. It turns out that this ordering is obtained precisely by reversing the ordering of the Dirac operator in Section 4.3.4 below, which is inherited from the Clifford group structure, namely $j = 9 - d$ for $d = 1, \ldots, 8$.

For odd $j$, one needs to reduce out the Fermi projection to a unitary, see e.g. [Ryu+10] and [PS16b] by going to a basis in which the chiral symmetry is proportional to the third Pauli matrix $\sigma_3$. In Proposition 1.63 we explained
how symmetry operators can be transformed into the normal forms. With this result and the hypothesis that the eigenspaces of all symmetry operators are of equal dimension, there exists a real unitary basis transformation $O$ such that the following table holds:

<table>
<thead>
<tr>
<th>$j$</th>
<th>7</th>
<th>5</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O^*RO =$</td>
<td>$i\sigma_2 \otimes 1$</td>
<td>$\sigma_3 \otimes 1$</td>
<td>$i\sigma_2 \otimes 1$</td>
<td>$\sigma_3 \otimes 1$</td>
</tr>
<tr>
<td>$O^*SO =$</td>
<td>$i\sigma_2 \otimes 1$</td>
<td>$1 \otimes i\sigma_2$</td>
<td>$i\sigma_2 \otimes i\sigma_2$</td>
<td>$1 \otimes 1$</td>
</tr>
<tr>
<td>$iCO^<em>ROC^</em> =$</td>
<td>$\sigma_3 \otimes 1$</td>
<td></td>
<td>$\sigma_3 \otimes 1$</td>
<td></td>
</tr>
<tr>
<td>$iCO^*SOC^t =$</td>
<td>$i\sigma_2 \otimes 1$</td>
<td></td>
<td>$i\sigma_2 \otimes i\sigma_2$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.7: Normal forms for symmetry operators for odd $j$

In the last two columns a supplementary Cayley transformation was carried out so that the chiral symmetry is proportional to the Pauli matrix $\sigma_3$. Let us point out that the Cayley transform is taken in the first factor, and defined exactly as always, see for example (4.7), so we can use the identities:

$$C^t\sigma_1 C = 1, \quad C^t\sigma_2 C = i\sigma_2, \quad C^t\sigma_3 C = -i\sigma_1, \quad C^tC = \sigma_3, \quad CC^t = \sigma_1.$$  

The effect of the Cayley transform is to change the symmetry operators from commuting to anti-commuting, just as in the proof of Proposition 4.19 and Section 1.6. This results from the fact that the TRS and PHS have to be transformed as $S \mapsto CSC^t$, while the chiral symmetry as $R \mapsto CRC^*$, see again Section 1.6. In the new basis, one can now reduce out $P$ to a unitary $U$. The

<table>
<thead>
<tr>
<th>$j$</th>
<th>7</th>
<th>5</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CO^<em>(2P - 1)OC^</em> =$</td>
<td>\begin{pmatrix} 0 &amp; U \ U^* &amp; 0 \end{pmatrix}</td>
<td>\begin{pmatrix} 0 &amp; U^* \ U &amp; 0 \end{pmatrix}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U =$</td>
<td>$U^t$</td>
<td>$\sigma_2^*U\sigma_2$</td>
<td>$\sigma_2^*U^t\sigma_2$</td>
<td>$U$</td>
</tr>
<tr>
<td>$O^*(2P - 1)O =$</td>
<td>\begin{pmatrix} 0 &amp; U \ U^* &amp; 0 \end{pmatrix}</td>
<td>\begin{pmatrix} 0 &amp; U^* \ U &amp; 0 \end{pmatrix}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8: Relations for odd $j$
symmetry $S$, which is a TRS for $j = 1, 5$ and a PHS for $j = 3, 7$, now implies that the unitary $U$ has a symmetry as listed in Table 4.8 above.

In conclusion, for every chiral system, which namely means $j$ is odd, a unitary operator $U$ is needed to specify the Fermi projection $P$. This unitary operator inherits symmetries from $P$. Before going further, let us point out that arbitrary unitary transformations of the Hamiltonian may change the unitary $U$, e.g. for $j = 1, 5$ by using

$$
\begin{pmatrix}
1 & 0 \\
0 & U
\end{pmatrix} O^* (2P - 1) O \begin{pmatrix}
1 & 0 \\
0 & U
\end{pmatrix}^* = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

which appears to radically change the $K$-theoretic data. Indeed, this unitary transformation shifts the two chiral components with respect to each other by $U$, and this does change the model in an essential manner. It is therefore not allowed. The basis transformations $O$ and $C$ in Table 4.8 are going to be local for tight-binding models on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ considered below, namely they commute with all position operators.

### 4.3.4 The Dirac operator and its symmetries

In the previous section, the Hamiltonian together with its Fermi level lead to a projection $P$ on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ and for odd $j$ to a unitary $U$ on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{\frac{N}{2}}$. This is the data that will constitute the $K$-theoretic input to the index pairings in Section 4.3.5 below. The $K$-homological part of the pairing is deduced from the unbounded self-adjoint Dirac operator, already known from Section 1.6,

$$D = \sum_{j=1}^{d} X_j \otimes 1 \otimes \Gamma_j,$$  

(4.8)

which acts on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^{d'}$ where $d' = 2\lfloor \frac{d}{2} \rfloor$. Here $X_1, \ldots, X_d$ are the components of the position operator on $\ell^2(\mathbb{Z}^d)$, and $\Gamma_1, \ldots, \Gamma_d \in \mathbb{C}^{d' \times d'}$ are anti-commuting matrices that square to 1 and are such that $\Gamma_{2n}$ is imaginary and $\Gamma_{2n+1}$ is real. The latter fact allows to implement complex conjugation in a convenient manner. In Section 1.6, it was already
shown how these matrices are constructed as a particular irreducible representation of the Clifford algebra $\mathcal{C}_d$. The Dirac operator $D$ defined in (4.8) may not look like the usual Dirac operator, but after a discrete Fourier transform $\mathcal{F}$ it takes the more familiar form

$$\mathcal{F}D\mathcal{F}^* = \sum_{j=1}^{d} i\partial_{k_j} \otimes 1 \otimes \Gamma_j,$$

acting on the Hilbert space $L^2(\mathbb{T}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^{d'}$ over the Brillouin torus. Due to Proposition 1.62, the Dirac operator inherits from the $\Gamma$-matrices the following symmetries:

$$\Sigma^* \bar{D} \Sigma = \kappa D, \quad \Omega^* D \Omega = -D, \quad \hat{\Sigma}^* \bar{D} \hat{\Sigma} = -\kappa D,$$  \hspace{1cm} (4.9)

where the latter two are only given for even $d$. Here $\Sigma, \Omega$ and $\hat{\Sigma}$ are symmetry operators which are defined in Section 1.6.4 and $\kappa = (-1)^{\lfloor \frac{d}{2} \rfloor}$. In particular, $\Omega \Sigma = \Sigma \Omega$ for $d = 4, 8$ and $\Omega \Sigma = -\Sigma \Omega$ for $d = 2, 6$. Using the terminology introduced for the Hamiltonian in Section 4.3, $\kappa = 1$ leads to a PHS for $\Sigma$ and a TRS for $\hat{\Sigma}$, while $\kappa = -1$ implies that $\Sigma$ is a TRS and $\hat{\Sigma}$ a PHS. Pending on the signs of $\Sigma^2$ and $\hat{\Sigma}^2$ these symmetries are even or odd. All these properties together allow to interpret $D$ as a $KR$-cycle, see Section 4.4. Associated to $D$ is again a spectral projection

$$E = \chi_{(0,\infty)}(D) + e_0 \otimes 1 \otimes \chi_{(0,\infty)}(\Gamma_1)$$

where $e_0$ is the projection on the state $\ell^2(\mathbb{Z}^d)$ over the origin in $\mathbb{Z}^d$. The projection $E$ will be called Hardy projection as in dimension $d = 1$ it projects on the Hardy space of positive frequencies. Now the properties (4.9) and Proposition 1.62 can be summarised in the following Table 4.9.

In the last two rows the orthogonal basis change $O$ of Proposition 1.64 is used, so that the chiral symmetry allows to deduce a unitary $F$ from $E$ that then has the symmetries indicated in an analogous manner as the Fermi projection $P$ leads to a unitary $U$ in Section 4.3.3. In the index theorems for the even dimensions, it will be assumed below that this representation is chosen. These
4.3 Index pairings for topological insulators

<table>
<thead>
<tr>
<th>$d \mod 8$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma^* \bar{E} \Sigma = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; \cdots &amp; 0 \end{pmatrix}$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
</tr>
<tr>
<td>$\Sigma^2 = \begin{pmatrix} 1 &amp; -1 &amp; -1 &amp; -1 &amp; 1 &amp; 1 &amp; 1 \end{pmatrix}$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
</tr>
<tr>
<td>$\hat{\Sigma}^* \bar{E} \hat{\Sigma} = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; \cdots &amp; 0 \end{pmatrix}$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
</tr>
<tr>
<td>$\hat{\Sigma}^2 = \begin{pmatrix} 1 &amp; -1 &amp; -1 &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
</tr>
<tr>
<td>$\Omega^* E \Omega = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; \cdots &amp; 0 \end{pmatrix}$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
</tr>
<tr>
<td>$2 O^* E O - 1 = \begin{pmatrix} 0 &amp; F \end{pmatrix}$</td>
<td>$(0 \ 0 \ F \ 0)$</td>
<td>$(0 \ F^* \ 0)$</td>
<td>$(0 \ 0 \ F \ 0)$</td>
<td>$(0 \ 0 \ F \ 0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F = \begin{pmatrix} F^t \ \sigma_2^* \bar{F} \sigma_2 \end{pmatrix}$</td>
<td>$F^t$</td>
<td>$\sigma_2^* \bar{F} \sigma_2$</td>
<td>$-F^t$</td>
<td>$\sigma_1^* \bar{F} \sigma_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.9: Symmetry relations for the Dirac operator.

considerations already conclude the analysis of the symmetries of the Dirac operator, the Hardy projection and the Dirac phase, namely the $K$-homological part of the pairing.

4.3.5 Fredholm operators for topological insulators

In this section the numerical invariants $\text{Inv}(j, d)$ for higher dimensions $d \geq 1$ are going to be calculated as index pairings of $P$ and $U$ paired with $E$ and $F$, all given by the Tables 4.6, 4.8 and 4.9. These index pairings are listed in Table 4.1, namely one sets

$$T = \begin{cases} 
PFP + P^\perp, & \text{d even}, \\
EUE + E^\perp, & \text{d odd and } j \text{ odd}, \\
E(2P - 1)E + E^\perp, & \text{d odd and } j \text{ even}. \end{cases}$$ (4.10)

The first two operators act on the Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{N_{d'}}$ and the last one on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{Nd'}$. Let us note the symmetry of $T$ upon exchange $(P, F) \leftrightarrow (E, U)$, under which the $K$-theoretic and $K$-homological parts exchange roles. Because of Proposition 4.3 this symmetry also holds for the case of odd $d$ and even $j$. Of course, as we already emphasised, the following will be crucial:
Standing Hypothesis \( T \) is a Fredholm operator on \( \mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{N_d'} \).

It is shown in [BES94], [PLB13] and [PS16b] that this Fredholm property actually holds when the Fermi level \( \mu \) lies in a gap or, more generally, in a region of dynamical Anderson localisation for covariant random models. Now (4.10) are exactly the pairings considered in Section 4.2. The conclusions are resumed in the following theorem.

**Theorem 4.20.** Each of the entries of the periodic Table 4.5 of topological insulators can be understood as \( \mathbb{Z}, 2\mathbb{Z} \) or \( \mathbb{Z}_2 \)-index theorem in the sense of Theorem 4.5 associated to the Fredholm operators (4.10). In the terminology of Section 4.2, the type R, Q, S or O of the index theorem is given in the following table:

<table>
<thead>
<tr>
<th>( d )</th>
<th>( j = 0 )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
<th>( j = 5 )</th>
<th>( j = 6 )</th>
<th>( j = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>R-( \mathbb{Z} )</td>
<td>( \mathbb{Z} )</td>
<td>R-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>O-( \mathbb{Z}_2 )</td>
<td>O-( \mathbb{Z}_2 )</td>
<td>R-( \mathbb{Z} )</td>
<td>S-( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>2</td>
<td>R-( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z} )</td>
<td>O-( \mathbb{Z}_2 )</td>
<td>O-( \mathbb{Z}_2 )</td>
<td>R-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>3</td>
<td>R-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z} )</td>
<td>O-( \mathbb{Z}_2 )</td>
<td>O-( \mathbb{Z}_2 )</td>
<td>R-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>4</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z} )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>5</td>
<td>R-( \mathbb{Z} )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z} )</td>
<td>R-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>6</td>
<td>S-( \mathbb{Z} )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>S-( \mathbb{Z}_2 )</td>
<td>Q-( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>7</td>
<td>S-( \mathbb{Z} )</td>
<td>Q-( \mathbb{Z}_2 )</td>
<td>R-( \mathbb{Z}_2 )</td>
<td>Z</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** For a given \((j, d)\) one reads of the symmetries of \( P \) or \( U \) and \( E \) or \( F \) from Table 4.6, Table 4.8 and Table 4.9. A careful check then allows to reduce all entries of the table above to those of Theorem 4.5 or Proposition 4.18. In particular, the sub-table obtained by considering only even \( d \) is precisely the table of Theorem 4.5. \( \square \)

Let us note that there is little difference between \( \mathbb{Z}_2 \) and \( 2\mathbb{Z}_2 \)-indices if one is merely interested in distinguishing ground states since group theoretical
they are the same. There is, however, a meaning associated to the evenness of indices. For example [DS16], in dimension $d = 2$ a BdG Hamiltonian, that means Class D or C, $j = 2$ or $j = 6$, has a Majorana zero mode attached to a vortex defect. While indeed there is such a zero mode in Class D, there is none for a Class C system because the index is always even. This is also reflected by the fact that going two dimensions to the left in the periodic table, where the vortex point defect is an effectively zero-dimensional system, one finds a $\mathbb{Z}_2$-index for $(j, d) = (2, 0 \simeq 8)$, but no entry for $(j, d) = (6, 0 \simeq 8)$.

### 4.3.6 Examples

Here, we will present some examples and give general recipes for building non-trivial models that have a time-reversal or a particle hole symmetry. These examples are taken, like most of this chapter, from [GS16] and may be helpful for readers that are familiar with such models. Especially, the focus here is on topological insulators with $\mathbb{Z}_2$-indices in low dimensions since these are of experimental relevance, cf. [HK10]. For example we find Class D in dimension $d = 1$, the Kitaev chain from [Kit01], Class AII in dimension $d = 2$, the Kane-Mele model from [KM05b], and Class AII in $d = 3$, the Fu-Kane-Mele strong topological insulator from [FKM07].

**One-dimensional examples**

The infinite and clean Kitaev chain is described by a Hamiltonian on the Hilbert space $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ given by

$$H = \frac{1}{2} \begin{pmatrix} V + V^* + 2\mu & i(V - V^*) \\ i(V - V^*) & -(V + V^* + 2\mu) \end{pmatrix}, \quad S_{\text{ph}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here, $V$ stand for the left shift and $\mu \in \mathbb{R}$ is to be interpreted as a chemical potential. This model has an even PHS given by the first Pauli matrix $S_{\text{ph}} = \sigma_1 \otimes 1$ and an even TRS given by the third Pauli matrix $S_{\text{tr}} = \sigma_3 \otimes 1$. Consequently, this model lies in the Class BDI. According to Theorem 4.20, it has a well-defined $\mathbb{Z}$-index. Note that it there is no problem with the fact that the two symmetries actually anti-commute since upon Cayley transform they become
commuting without leaving the Class BDI.

For the calculation of the $Z$-index, we have to diagonalise the chiral symmetry $S_{ch} := S_{ph}S_{tr} = i\sigma_2 \otimes 1$, and this is realised by the well-known Cayley transform $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i1 \\ i1 & 1 \end{pmatrix}$, meaning $CS_{ch}C^* = -i\sigma_3 \otimes 1$. Then

$$CHC^* = \begin{pmatrix} 0 & V + \mu \\ V^* + \mu & 0 \end{pmatrix}.$$  

$$C(2P - 1)C^* = \begin{pmatrix} 0 & (V + \mu)|V + \mu|^{-1} \\ (V^* + \mu)|V^* + \mu|^{-1} & 0 \end{pmatrix}.$$  

Now using the unitary $U = (V + \mu)|V + \mu|^{-1}$ and the so-called Hardy projection $E = \chi_{(0,\infty)}(X)$, we define a Fredholm operator $T = EU E + E^\perp$ for $\mu \notin \{-1, 1\}$. Note that for $\mu \in \{-1, 1\}$ the gap is closed and $T$ is not Fredholm. However, for $|\mu| < 1$ one has $\text{ind}(T) = \text{ind}(EU E|_{\text{Ran}(E)}) = \text{ind}(E(V + \mu)E|_{\text{Ran}(E)}) = 1$ since $EVE|_{\text{Ran}(E)}$ is the unilateral shift on $\ell^2(\mathbb{N})$. For $|\mu| > 1$, we find $\text{ind}(T) = 0$.

Now, we add a supplementary term to the Hamiltonian that breaks the time-reversal symmetry but not the even particle-hole symmetry. In this case, $H$ has no chiral symmetry any more, and therefore it lies in Class D. By Theorem 4.20, the $Z_2$-index $\text{ind}_2(E(2P - 1)E + E^\perp)$ is still well-defined, and it can be read off the formula above that it is actually equal to the non-trivial value 1 for $|\mu| < 1$. Moreover, let us emphasise that this $Z_2$-index is stable under (random) perturbations of the Hamiltonian that conserve the even PHS. This $Z_2$-index is equal to $\text{ind}_2(EH E + E^\perp)$, namely the multiplicity of the zero modes of the half-space Hamiltonian $E H E$. This gives us an interesting physical interpretation of the $Z_2$-index.

**Examples in even dimension**

In even dimensions $d$, there are models of Class A, meaning there is no symmetry involved, which have non-vanishing, higher even, Chern numbers, see [Ryu+10]. By an index theorem in [PLB13], they are equal to the index of a Fredholm operator described next. Let the Hamiltonian $h$ act on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ and let $p = \chi_{(-\infty,\mu]}(h)$ be the associated Fermi projection where the Fermi energy $\mu$
lies in a region of dynamical localisation. If $F$ is the Dirac phase as constructed in Section 4.3.4, then the Chern number is equal to $\text{ind}(pFp + p^\perp)$.

From this model, let us now construct further non-trivial models with symmetries, similar as in [DS16] and inspired by the construction of the Kane-Mele model from two copies of the Haldane model. The new Hamiltonian on the Hilbert space $\mathcal{H} = \mathcal{H}(\mathbb{Z}^2) \otimes \mathbb{C}^N \otimes \mathbb{C}^2$ is of the form

$$H = \begin{pmatrix} h & g \\ g^* & \bar{h} \end{pmatrix}, \tag{4.11}$$

in the grading of the spin degree of freedom $\mathbb{C}^2$. Then, it has an odd TRS given by $S_{tr} = i\sigma_2 \otimes 1$ whenever $g^t = -g$. The next aim is to calculate the invariant for $H$ from then one of $h$. For vanishing $g$, the Fermi projection $P = \chi_{(-\infty, \mu]}(H)$ is given by $P = \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix}$. Therefore

$$PFP = \begin{pmatrix} pFp & 0 \\ 0 & \bar{p}F\bar{p} \end{pmatrix}.$$ 

Now, we assume that $\text{ind}(pFp + P^\perp)$ is odd. Hence, if $\text{dim}(\text{Ker}(pFp))$ is odd (resp. even), then $\text{dim}(\text{Ker}(pF^*p)) = \text{dim}(\text{Ker}(pFp)) = \text{dim}(\text{Ker}(\bar{p}F\bar{p}))$ is even (resp. odd). It follows that $\text{dim}(\text{Ker}(PFP))$ is indeed odd so that $\text{ind}_2(PFP + P^\perp) = 1$. Now, in the case $d = 2$ and if $g = -g^t$ is added homotopically, then this non-trivial $\mathbb{Z}_2$-index is conserved by homotopy invariance. If $g$ is chosen to be the Rashba coupling and $h$ the Haldane Hamiltonian, then the model obtained in this manner is precisely the Kane-Mele model. In a similar way for $d = 6$, one can build a model with an even TRS and non-trivial $\mathbb{Z}_2$-invariant.

Now let us indicate the changes needed to construct a topologically non-trivial model with a PHS. One starts from

$$H' = \begin{pmatrix} h & f \\ f^* & -\bar{h} \end{pmatrix},$$

and then imposes $f = -f^t$ for an even PHS $S_{ph} = \sigma_1 \otimes 1$, or $f = f^t$ for an odd
4 Index pairings and topological insulators

The given system with an even PHS is of particular interest in dimension $d = 8$, where a $\mathbb{Z}_2$-topological insulator can be constructed. On the other hand, the odd PHS is of relevance in $d = 4$.

Examples in odd dimension

The procedure in the paragraph about examples in even dimensions above can be modified to produce topological models in odd dimension $d$ that have two commuting symmetries, namely the cases $j = 1, 3, 5, 7$. For this special purpose, we start from a Hamiltonian $h$ on the Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^2$ with a chiral symmetry $\sigma_3 h \sigma_3 = -h$, where we shortly write $\sigma_3$ instead of $\sigma_3 \otimes 1$. In the grading of $\sigma_3$, the Hamiltonian is of the form $h = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$, and under a localisation hypothesis it has an index $\text{ind}(EaE + E^\perp)$ that is connected to odd higher Chern numbers by an index theorem, see [Ryu+10] and [PS16b]. Now one can construct systems with TRS similar as in (4.11) by setting

$$H = \begin{pmatrix} h & g \\ g^* & \sigma_1 h \sigma_1 \end{pmatrix}.$$ 

This Hamiltonian has a CHS $S_{\text{ch}} = \sigma_3 \otimes 1$ as long as $\sigma_3 g \sigma_3 = -g$ holds. Furthermore, it has also an odd TRS $S_{\text{tr}} = \sigma_1 \otimes i \sigma_2$ whenever $g = -\sigma_1 g^t \sigma_1$, that is $g = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with anti-symmetric $b$ and $c$. Hence, this Hamiltonian lies in Class CII, $j = 5$. Now one checks

$$W^* HW = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c^* & a^t \\ a^* & c & 0 & 0 \\ b^* & \overline{a} & 0 & 0 \end{pmatrix}.$$ 

The unitary $U$ of Table 4.8 is the phase of the invertible $(a^* \ c^* \ b \ \overline{a}^t)$, which is the upper right entry of $W^* HW$. For $b$ and $c$ small enough, one has $\text{ind}(T) = 0$ for $T = EUE + E^\perp$. However, it is possible that there is a secondary $\mathbb{Z}_2$-invariant. For $d = 3$, this $\mathbb{Z}_2$-invariant is non-trivial provided that $\text{ind}(EaE + E^\perp)$ is odd by an argument similar to the one given above in this section.

Finally let us construct a three-dimensional model in Class AII that has
a non-trivial $\mathbb{Z}_2$-invariant. This will be achieved by the Hamiltonian by the tight-binding version in [Ryu+10, Eq. (82)]. Let $(\Gamma_j)_{j=1,\ldots,5}$ be an irreducible complex representation of the real Clifford algebra $\mathcal{C} \ell_5$ on $\mathbb{C}^4$, exactly as given in Section 1.6.3. Then, consider the following Hamiltonian on the Hilbert space $\ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^4$:

$$H = \sum_{j=1}^{3} \frac{1}{2i}(V_j - V_j^*) \otimes \Gamma_j + \left( m + \sum_{j=1}^{3} \frac{1}{2}(V_j + V_j^*) \right) \otimes \Gamma_4,$$

where $V_j$ are the shift operators on $\ell^2(\mathbb{Z}^3)$ in the three spatial directions and $m \in \mathbb{R}$ is interpreted as a mass term. Due to the anti-commutation relations the Hamiltonian has a CHS $S_{\text{ch}} = \Gamma_5$ and an odd TRS $S_{\text{tr}} = i\Gamma_2$. Thus $H$ also has an even PHS and is in Class DIII, $j = 3$. As such, it has a $\mathbb{Z}$-invariant. Now diagonalizing $W^*S_{\text{ch}}W = 1 \otimes \sigma_3$, the Hamiltonian is of the form $W^*HW = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ and the integer invariant is $\text{ind}(EAE + E^\perp)$. It can be checked, see [Ryu+10], that the values are $1$ for $|m| \in (1, 3)$ and $-2$ for $m \in (-1, 1)$, and $0$ otherwise. However, at $m = -3, -1, 1, 3$ the gap is closed.

Now, we could add a term to the Hamiltonian that breaks the CHS but keeps the odd TRS intact, e.g. the perturbation $\frac{\lambda}{2i}(V_1 - V_1^*) \otimes \Gamma_5$ for some $\lambda \in \mathbb{R}$. In this case, the model falls into Class AII and therefore has a $\mathbb{Z}_2$-invariant in dimension $d = 3$. If $\text{ind}(EAE + E^\perp)$ is odd, as for $1 < |m| < 3$, this $\mathbb{Z}_2$-invariant given by $\dim(\ker(EHE)) \mod 2$ is non-trivial.

### 4.4 $KR$-cycles

This short section serves mainly to show how the results from above can be integrated in the realm of non-commutative geometry of Alain Connes, see [Con95], [Con94] and [GVF01]. For this purpose, one first of all needs a $C^*$-algebra $\mathcal{A}$ with symmetry $\tau : \mathcal{A} \to \mathcal{A}$, see Definition 2.45. Here, we will use the short notation $\tau(A) = \overline{A}$. For sake of concreteness, one may suppose that $\mathcal{A}$ is given by a concrete subset of operators on a Hilbert space $\mathcal{H}$ with real structure $\mathcal{C}$, and that $\tau(A) = \mathcal{C}AC$ for $A \in \mathcal{A}$. Moreover, choose $\mathcal{A}_0$ as a dense subalgebra of $\mathcal{A}$. Associated to these algebras, $KR$-cycles and $KR$-groups can
be defined, and their pairing can be calculated in an abstract manner as a Kasparov product, cf. [Kas81].

This leads to the concrete index pairings obtained from the Fredholm operators in Section 4.2.1. These operators have the invariants \( \text{ind} \) and \( \text{ind}_2 \) taking values in \( \mathbb{Z}, 2\mathbb{Z} \) or \( \mathbb{Z}_2 \), which are, by the results of Section 4.2, indeed independent of the choice of representative of the \( KR \)-group.

Then the results of Section 4.3 appear as a concrete realisation of the abstract theory to the \( C^* \)-algebraic theory of aperiodic media, see e.g. [Bel86] and [BES94]. There, the \( C^* \)-algebra \( \mathcal{A} \) is the algebra of covariant families of local operators in \( \mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \), cf. also Section 2.2. Alternatively, it can be constructed as a reduced crossed product, tensorised with finite matrix algebras. Moreover, \( \mathcal{A}_0 \) is the set of covariant families of finite range, also introduced in Section 1.3 and Section 2.2. Then, the Fermi projection specifies a \( KR \)-group element and the Dirac operator provides a \( KR \)-cycle.

In this section, it is assumed that \( \mathcal{A} \) is a unital \( C^* \)-algebra of operators on a Hilbert space with complex conjugation \( C \), which means we have a \( C^* \)-algebra with symmetry. According to [GVF01, Definition 9.18] a (reduced) \( KR^i \)-cycle for \( (\mathcal{A}_0, \tau) \) and \( i = 0, \ldots, 7 \) is given by a self-adjoint operator \( D \) on \( \mathcal{H} \), a symmetry operator \( \bar{\Sigma} \) and, in addition for even \( i \), an even symmetry operator \( \Omega \). The operator \( D \), which is a Dirac operator, has compact resolvent and bounded commutators \([A, D]\) for all \( A \in \mathcal{A}_0 \). The even symmetry operator \( \Omega \) is called a grading and fulfils \( \Omega^* D \Omega = -D \). Moreover, the following table has to hold:

<table>
<thead>
<tr>
<th>( i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\Sigma}^2 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \bar{\Sigma}^* D \bar{\Sigma} )</td>
<td>( D )</td>
<td>( -D )</td>
<td>( D )</td>
<td>( D )</td>
<td>( D )</td>
<td>( -D )</td>
<td>( D )</td>
<td>( D )</td>
</tr>
<tr>
<td>( \Omega \bar{\Sigma} \Omega )</td>
<td>( \bar{\Sigma} )</td>
<td>( -\bar{\Sigma} )</td>
<td>( \bar{\Sigma} )</td>
<td>( -\bar{\Sigma} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

These so-called reduced \( KR \)-cycles are equivalent to Kasparov’s \( KR \)-cycles defined in [Kas81], see also [GVF01, Theorem 9.19]. The above definition and table is verbatim the definition found in [GVF01] except for the slight difference that in [GVF01] the anti-linear operator \( \bar{\Sigma} C \) appears instead of \( \bar{\Sigma} \). However, this
is just a rewriting to match the prior notations in this work. Let us also note that 
KR-cycles are also called *spectral triples with real structure*, see e.g. [Con94],
or *KR-homology elements*. The Dirac operator from equation (4.8) together 
with the symmetries (4.9) constitute an example of a KR$^i$-cycle provided the 
following identifications are made: $i = 8 - d$ and the symmetry operator $\Sigma$ is 
chosen as $\hat{\Sigma} = \Sigma$ for $i = 2, 6$ and $\hat{\Sigma} = \Sigma$ for all other $i$. 
5 Sturm-Liouville oscillation theory

The oscillation theory is known for almost two centuries and mainly introduced by the Swiss and French mathematician Jacques Charles François Sturm (1803–1855) and the French mathematician Joseph Liouville (1809–1882). However, this oscillation theory mostly refers to continuous Sturm-Liouville problems and not to the discrete analogon in form of Jacobi operators considered here. We first want to give a short introduction in the classical theory, see also [AM10] and [AHP05] for more details.

Many new results of this chapter, in particular the results about high-dimensional random Schrödinger operators, are based on the recent joint work [GSV17] with H. Schulz-Baldes and C. Villegas-Blas. These results concern Jacobi operators with block entries from a unital C*-algebra. Considering the covariant families of local operators on an infinite dimensional Hilbert space, one can show that the integrated density of states is approximated by a generalised winding number of a special associated unitary element, called the Prüfer phase. These new insights are presented in Section 5.3 and Section 5.5 below.

5.1 Introduction and Sturm-Liouville problems

On the unit interval $[0, 1] \subset \mathbb{R}$, the classical continuous Sturm-Liouville-problem in given by

$$H = \frac{d}{dt} a \frac{d}{dt} + V,$$

(5.1)
where \( t \mapsto a(t) \) is a continuously differentiable function with \( a < 0 \) on \([0, 1]\) and \( V \) a multiplication operator given by a continuous real function \( t \mapsto V(t) \). Therefore, \( H : L^2([0, 1]) \supset D \to L^2([0, 1]) \) is an unbounded operator that is self-adjoint by a suitable choice of the domain \( D \). For example, if we want Dirichlet-boundary conditions, we will choose the well-known Sobolev spaces and set \( D = H^2([0, 1]) \cap H_0^1([0, 1]) \). In general, we could also consider more general boundary condition for a solution \( x : [0, 1] \to \mathbb{R} \) of the form:

\[
\beta_0 x(0) + \gamma_0 a(0) x'(0) = 0, \quad \beta_1 x(1) + \gamma_1 a(1) x'(1) = 0
\]  

(5.2)

with \( \beta_i, \gamma_i \in \mathbb{R} \) and \(|\beta_i| + |\gamma_i| \neq 0\) for \( i = 0, 1 \).

Naturally, there are many generalisations of the presented Sturm-Liouville-problem that is considered here. For example, one can drop the condition of the continuity of the functions \( a \) and \( V \) by requiring that the functions are integrable and \( ax' \) is absolutely continuous. Indeed, this does not change the main results and one speaks even in this setting of a regular Sturm-Liouville-problem. However, if one considers unbounded intervals or cases where the function \( a \) has zeros, then one speaks of a singular Sturm-Liouville-problem, but these are not considered in this work. As a side note, one should mention that choosing \( a < 0 \) is just a convention for including the negative Laplace operator, given as \( -\Delta = -\frac{d^2}{dt^2} \), and there may be other authors that set \( a > 0 \), but this is obviously equivalent.

In the presented context, one usually sees the next theorem as the classical Sturm-Liouville oscillation theory:

**Theorem 5.1.** For \( a \in C^1([0, 1]) \) with \( a < 0 \) and \( V \in C([0, 1]) \) the eigenvalue equation \( Hx = \lambda x \) with boundary conditions (5.2) gives countable many simple eigenvalues, which accumulate only at infinity:

\[
\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty.
\]

An eigenfunction \( x_n : [0, 1] \to \mathbb{R} \) associated to \( \lambda_n \) has exactly \( n \) zeros in the interval \((0, 1)\). Moreover, the zeros of \( x_n \) in \((0, 1)\) lie interlacing between the zeros of \( x_{n+1} \) in \((0, 1)\).
Proof. See [Schm14, Section 15.1].

Consequently, the differentiable eigenfunctions for an eigenvalue $\lambda_n$ oscillate between positive and negative values, and the number of oscillations gives the number of eigenvalues below this value. Besides [Schm14], also [AlG08], [GH08] and [Wal13] are good references for explanations and elementary proofs of Theorem 5.1 above. An elaborate presentation of continuous Sturm-Liouville problems that go beyond the setting above can be found in [AM10].

5.2 Jacobi operators and Jacobi matrices

The discrete analogue of the Sturm-Liouville problem (5.1) is the so-called Jacobi operator. It is a tridiagonal matrix that is here infinite in one direction:

$$H = \begin{pmatrix}
  v_1 & a_2 & & & \\
  a_2 & v_2 & a_3 & & \\
  & a_3 & v_3 & \ddots & \\
  & & \ddots & \ddots & \ddots
\end{pmatrix}.$$

In other words, $H$ is a linear operator $\ell^2(\mathbb{N}) \supset D \to \ell^2(\mathbb{N})$ with a suitable chosen domain $D$, where one mostly assumes $v_n \in \mathbb{R}$ and $a_n > 0$ in analogy to the continuous Sturm-Liouville problem above. If one rewrites the action of $H$ componentwise,

$$(Hx)_n = a_{n+1}x_{n+1} + a_n x_{n-1} + v_n x_n, \quad n \geq 2,$$

one recognises the first two summands as one possibility of a discrete Laplacian or rather a second derivative as a central difference with weights $a_n$. The last summand is again a multiplication operator. In order to get an analogous case as before with two boundary conditions, we will truncate this one-sided infinite matrix and get an ordinary matrix $H_N$ for each $N \in \mathbb{N}$, which is known as a Jacobi matrix. By fixing the boundary conditions, one finds an oscillation theory similar to the results in Theorem 5.1, and we will derive this in the
following as a special case from a general result. For more details about the classical case, see, for example, [Sim05].

However, we will generalise this setting to reach new results. Therefore, we consider an arbitrary unital C*-algebra \( \mathcal{A} \) and choose sequences of coefficients \((a_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\) from \( \mathcal{A} \), where \( v_n \) has to be self-adjoint and \( a_n \) has to be invertible for each \( n \in \mathbb{N} \). The main object in this chapter will be the following Jacobi matrix

\[
H_N = \begin{pmatrix}
  v_1 & a_2 & & & \\
  a_2^* & v_2 & a_3 & & \\
    & a_3^* & v_3 & & \\
    & & \ddots & \ddots & \\
    & & & a_{N-1}^* & a_N \\
    & & & v_N & a_N^*
\end{pmatrix}
\]

(5.3)

where \( N \in \mathbb{N} \). We will always set \( a_1 := 1 \), where \( 1 \) denotes the unit in \( \mathcal{A} \). Note that this framework includes the case \( \mathcal{A} = \mathbb{B}(\mathcal{H}) \), where \( \mathcal{H} \) is a complex Hilbert space. Then the element \( H_N \in M_N(\mathcal{A}) \) is self-adjoint, where \( M_N(\mathcal{A}) = \mathbb{C}^{N \times N} \otimes \mathcal{A} \), cf. also Definition 2.12. This framework naturally includes the one-dimensional case from above with real entries.

In the following analysis, we can distinguish three different interesting cases, namely \( \mathcal{A} = \mathbb{C} \), \( \mathcal{A} = \mathbb{C}^{L \times L} \) for \( L \in \mathbb{N} \) and \( \mathcal{A} \) as an infinite dimensional C*-algebra. Often we will just consider \( \mathcal{A} \subset \mathbb{B}(\mathcal{H}) \) to explicitly use the operator nature of the entries. Moreover, we will also point out which differences occur between these three cases. The symbol \( T \) always stands for a chosen normalised trace on \( \mathcal{A} \), cf. Definition 2.26.

### 5.3 Analysis for Jacobi matrices

In this section, we will show that the spectrum of the Jacobi matrix \( H_N \) has a connection to a rotation number given by a unitary element and in the same way to an intersection number for Lagrangian subspaces that is known as the Bott-Maslov index. The finite-dimensional matrix case, meaning \( \mathcal{A} = \mathbb{C}^{L \times L} \), was recently considered in [Schu07] and [Schu12] and is included as a special
Now, we consider the eigenvalue equation for the Jacobi matrix $H_N$ defined in (5.3) when the entries are operators:

$$H_N y = \lambda y \quad \text{for } y = (y_1, \ldots, y_N) \in \mathcal{H}^N \text{ if } A \subset B(\mathcal{H}) \, . \quad (5.4)$$

In a general formulation and an arbitrary abstract C*-algebra $A$, we can use this operator theoretic view and write the eigenvalue equation as

$$H_N x = \lambda x \quad \text{for } x = (x_1, \ldots, x_N) \in A \oplus \cdots \oplus A \, . \quad (5.5)$$

This is of course a generalisation since in the case $A \subset B(\mathcal{H})$ a solution of (5.4), and, on the other hand, if there is a solution $y$ of (5.4) one can just define an operator $x \in B(\mathcal{H}^N)$ that has exactly the range span$(y)$ and gives a solution of (5.5).

For each $n \in \{2, \ldots, N-1\}$ one finds that the general eigenvalue equation (5.5) above can be written as a block matrix recursion formula

$$\begin{pmatrix} a_{n+1} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} (\lambda 1 - v_n) a_n^{-1} & -a_n^* \\ a_n^{-1} & 0 \end{pmatrix} \begin{pmatrix} a_n x_n \\ x_{n-1} \end{pmatrix} \, , \quad (5.6)$$

where $1$ denotes the unit in $A$. However, we will often omit the symbol and just write $\lambda$ instead of $\lambda 1$. The $2 \times 2$ block matrix in (5.6) is called a transfer operator and here denoted for each $n \in \mathbb{N}$ by:

$$T_n(\lambda) = \begin{pmatrix} (\lambda - v_n) a_n^{-1} & -a_n^* \\ a_n^{-1} & 0 \end{pmatrix} \quad \text{with } a_1 := 1 \, . \quad (5.7)$$

By using induction on equation (5.6), one gets a product of these transfer operators and we can use this result to choose $A$-valued functions $\lambda \mapsto x_n(\lambda)$ defined by:

$$\begin{pmatrix} a_{n+1} x_{n+1}(\lambda) \\ x_n(\lambda) \end{pmatrix} := T_n(\lambda) \cdots T_1(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, , \quad \text{for } n = 1, \ldots, N \, , \quad (5.8)$$

where $a_{N+1} \in A$ is arbitrary but also invertible. Moreover, this definition
means that we set \( x_0 = 0 \), which corresponds to Dirichlet boundary conditions on the left side, and we choose \( x_1 = 1 \). Indeed, the parameter \( x_1 \) is a degree of freedom in this equation as we will see later. However, in the classical one-dimensional case, one can set \( x_1 = 1 \) without loss of generality, and the maps \( \lambda \mapsto x_n(\lambda) \) are complex-valued polynomials of order \( n - 1 \). For this case, meaning the entries of the Jacobi matrix are complex numbers, we obtain the classical oscillation theory:

**Proposition 5.2.** For \( A = \mathbb{C} \) the eigenvalues of \( H_N \) coincide with the zeros of the polynomial \( x_{N+1} \). All eigenvalues are simple.

**Proof.** Simply by definition of the polynomials in (5.8), we find for all \( \lambda \in \mathbb{R} \):

\[
(H_N - \lambda) \begin{pmatrix} x_1(\lambda) \\ x_2(\lambda) \\ \vdots \\ x_N(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

Therefore, the vector on the left side is an eigenvector for \( H_N \) associated to the eigenvalue \( \lambda \) if and only if \( x_{N+1}(\lambda) = 0 \). Obviously, by the recursion formula (5.6), every other eigenvector of \( H_N \) has to be a multiple of one of these eigenvectors, so each eigenvalue is simple.

**Proposition 5.3.** For \( \mathfrak{H} = \mathbb{C} \) the eigenvalues of \( H_N \) and \( H_{N+1} \) are completely different and interlacing, that is

\[
\lambda_1(N + 1) < \lambda_1(N) < \lambda_2(N + 1) < \lambda_2(N) < \cdots < \lambda_N(N) < \lambda_{N+1}(N + 1),
\]

where \( \lambda_1(N + 1) \) denotes the smallest eigenvalue of \( H_{N+1} \) and analogously the other ones.

**Proof.** We use ideas from [Sim05] and the min-max principle, see, e.g. [Schm14, Theorem 12.1] and [HJ85, Theorem 4.2.11]. Accordingly, the eigenvalues of \( H_N \) are given by:

\[
\lambda_k(N) = \max_{D \in \mathcal{F}_{k-1}^N} \min \left\{ \langle y, H_N y \rangle \mid y \in \mathbb{C}^N, y \perp D, \|y\| = 1 \right\}, \quad (5.9)
\]
\[
\lambda_k(N) = \min_{D \in \mathcal{F}_N} \max \left\{ \langle y, H_N y \rangle \mid y \in \mathbb{C}^N, y \perp D, \|y\| = 1 \right\}. \quad (5.10)
\]

Here, \( \mathcal{F}_N \) stands for the set of \( m \)-dimensional subspaces of \( \mathbb{C}^N \) for \( m \in \mathbb{N} \). We start by using (5.9): Since \( H_N \) can be identified with \( H_{N+1} \) restricted to the subspace \( \mathbb{C}^N \oplus 0 \), we get the following inequality for a subspace \( D \in \mathcal{F}_k \) and \( k = 1, \ldots, N \) by

\[
\min \left\{ \langle y, H_{N+1} y \rangle \mid y \in \mathbb{C}^{N+1}, y \perp D, \|y\| = 1 \right\} \\
\leq \min \left\{ \langle y, H_{N+1} y \rangle \mid y = z \oplus 0 \in \mathbb{C}^N \oplus \mathbb{C}, y \perp D, \|y\| = 1 \right\} \\
= \min \left\{ \langle z, H_N z \rangle \mid z \in \mathbb{C}^N, z \perp P_N(D), \|z\| = 1 \right\},
\]

where \( P_N(D) \) is the orthogonal projection of \( D \) to \( \mathbb{C}^N \). Taking the maximum on both sides, we get:

\[
\lambda_k(N + 1) \leq \lambda_k(N) \quad \text{for } k = 1, \ldots, N.
\]

Now with an analogue argument, we show the interlacing property by using (5.10). For a subspace \( D \in \mathcal{F}_{(N+1)-k} \), we get:

\[
\max \left\{ \langle y, H_{N+1} y \rangle \mid y \in \mathbb{C}^{N+1}, y \perp D, \|y\| = 1 \right\} \\
\geq \max \left\{ \langle z, H_N z \rangle \mid z \in \mathbb{C}^N, z \perp P_N(D), \|z\| = 1 \right\}.
\]

Then taking the minimum on both sides, we obtain:

\[
\lambda_{k-1}(N) \leq \lambda_k(N + 1) \quad \text{for } k = 2, \ldots, N + 1.
\]

Now we show that, in fact, the equality never holds. This follows by the definition of the polynomials \( x_n \) in (5.8) and Proposition 5.2 above. If there would be a \( \lambda \in \mathbb{R} \) with \( x_{N+1}(\lambda) = x_N(\lambda) = 0 \), then, by the recursion formula (5.6), we would have \( x_n(\lambda) = 0 \) for all \( n \), in contradiction to \( x_1 = 1 \). \( \Box \)

Naturally, in the general case of an arbitrary C*-algebra the chosen map \( \lambda \mapsto x_{N+1}(\lambda) \) is also an \( \mathcal{A} \)-valued-polynomial, but the notion of zeros would
be too restrictive, and, seeing $H_N$ as a self-adjoint operator, there may be
even continuous spectrum that is not given by isolated zeros of a polynomial.
However, by the same argument as before, one can describe at least the set of
eigenvalues of $H_N$ if $\mathcal{A} \subset \mathcal{B}(\mathfrak{H})$.

**Proposition 5.4.** Let $\mathcal{A} \subset \mathcal{B}(\mathfrak{H})$. A real number $\lambda$ is an eigenvalue of $H_N$ if
and only if the intersection

$$T_N(\lambda) \cdots T_1(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathfrak{H} \cap \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathfrak{H}$$

is non-trivial. In this case, the dimension of this intersection coincides with
the multiplicity of the eigenvalue $\lambda$.

**Proof.** This is similar to the proof of Proposition 5.2 in the case $\mathcal{A} = \mathbb{C}$ above.
Since $\mathcal{A} \subset \mathcal{B}(\mathfrak{H})$, the elements $x_{N+1}$ and $x_N$ in equation (5.8) are operators on
a Hilbert space, and they act on a vector $y_1 \in \mathfrak{H}$. Therefore, we can define the
vector-valued polynomials:

$$\begin{pmatrix} a_{N+1} y_{N+1}(\lambda) \\ y_N(\lambda) \end{pmatrix} := T_N(\lambda) \cdots T_1(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} y_1 .$$

They obviously depend on $y_1$. If one can choose $y_1 \in \mathfrak{H}$ in such a way that
$y_{N+1}$ vanishes, then we have an eigenvector by the same argument as in the
proof of Proposition 5.2. Again, on the other hand, each eigenvector has to be
of this form and this immediately gives the multiplicity.

The proof above is indeed more or less the same as in the one-dimensional
case, but before going further, let us explain the claim of this proposition
more concretely. The space $(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \mathfrak{H}$ is a so-called *Lagrangian subspace* in the
Krein space $(\mathfrak{H} \oplus \mathfrak{H}, I)$, see Definition 1.7, where $I = (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$. The transfer
operators $T_n(\lambda)$ are $I$-unitaries, cf. Definition 1.18, and this is proved in
Proposition 5.7 in the next section. As a reminder, they are also sometimes
called symplectic operators. Therefore, the theory of Krein spaces implies
that the transfer operators transform Lagrangian subspaces into Lagrangian
subspaces, see Lemma 1.19 or the whole Section 1.2 for more details. The left
Analysis for Jacobi matrices

side in the equation from Proposition 5.4 above describes a discrete dynamical evolution of a Lagrangian subspace, and then we ask about the intersection of two Lagrangian subspaces. The dimension of this intersection is known as the Bott-Maslov index in the finite dimensional case, cf. [Bot56] and [MBA72]. We will explore this further in Section 5.3.7.

Now we define the main object in this investigation, namely a unitary element in \( \mathcal{A} \) that we call the Prüfer phase, motivated by the name from the classical oscillation theory:

\[
  u_N(\lambda) = 
  \begin{bmatrix}
    1 \\
    i \\
  \end{bmatrix}^* \begin{bmatrix}
    T_N(\lambda) & \cdots & T_1(\lambda) \\
    0 & \cdots & 1
  \end{bmatrix}
  \begin{bmatrix}
    1 \\
    0
  \end{bmatrix}
  \begin{bmatrix}
    T_N(\lambda) & \cdots & T_1(\lambda) \\
    0 & \cdots & 1
  \end{bmatrix}^{-1}
\]

(5.11)

In the one-dimensional case, \( \mathcal{A} = \mathbb{C} \), this is just a complex number on the unit circle, which can be written via the polynomials \( x_N \) and \( x_{N+1} \) as

\[
  u_N(\lambda) = \frac{a_{N+1} x_{N+1}(\lambda) - ix_N(\lambda)}{a_{N+1} x_{N+1}(\lambda) + ix_N(\lambda)},
\]

where this is well-defined by the combination of Proposition 5.2 and Proposition 5.3. One easily sees, using Proposition 5.2, that the eigenvalues of \( H_N \) correspond to the real points \( \lambda \) where \( u_N(\lambda) = -1 \). This fact will be generalised later.

The main result of this section will be the next theorem. It summarises all facts about the Prüfer phase that can be proven in this general setting of a C*-algebra \( \mathcal{A} \). In the next sections below, we will expand these result when there are special cases, for example finite-dimensional matrices and high dimensional random operators.

**Theorem 5.5.** Let \( \mathcal{A} \) be a unital C*-algebra and \( N \in \mathbb{N} \). Assume \( H_N \) is given by (5.3) with self-adjoint \( v_n \) and invertible \( a_n \) for all \( n \). For \( \lambda \in \mathbb{R} \), let \( u_N(\lambda) \) be defined by (5.11). Then:

(a) For all \( \lambda \in \mathbb{R} \), the element \( u_N(\lambda) \) is well-defined and unitary. The map \( \lambda \mapsto u_N(\lambda) \) is real analytic.
(b) We have the following norm convergence:

\[
\lim_{\lambda \to -\infty} u_N(\lambda) = 1 \quad \text{from above} \quad \text{and} \quad \lim_{\lambda \to \infty} u_N(\lambda) = 1 \quad \text{from below}.
\]

More precisely, \( u_N(\lambda) = 1 - 2i\lambda^{-1} + \mathcal{O}(\lambda^{-2}) \) for \( \lambda \to \pm\infty \).

(c) \( s_N(\lambda) := \frac{1}{iN} (u_N(\lambda))^* \partial_\lambda u_N(\lambda) \) is a strictly positive element.

(d) The map \( \lambda \mapsto s_N(\lambda) \) is norm-integrable and integrable with respect to each normalised trace \( \mathcal{T} : \mathcal{A} \to \mathbb{C} \), that is

\[
\int_{\mathbb{R}} \| s_N(\lambda) \| \, d\lambda < \infty, \quad \int_{\mathbb{R}} \mathcal{T}(s_N(\lambda)) \, d\lambda < \infty .
\]

**Proof.** For (a), (b) and (c) combine just Proposition 5.10, Proposition 5.16 and Proposition 5.14 below. The claim in (d) is just Corollary 5.15 and Corollary 5.19.

From part (c), we will conclude that the spectrum of \( u_N(\lambda) \) rotates in the positive sense around 0 when \( \lambda \) increases. Again, this is easily explainable in the one-dimensional case \( \mathcal{A} = \mathbb{C} \) since \( u_N(\lambda) \) is just a complex number that moves on the unit circle by varying \( \lambda \). When considering the path \( \gamma : [a, b] \ni \lambda \mapsto u_N(\lambda) \) in the complex plane, then the winding number with respect to the origin is given by

\[
\nu(\gamma) := \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} = \frac{1}{2\pi i} \int_a^b (u_N(\lambda))^* \partial_\lambda u_N(\lambda) \, d\lambda = \frac{N}{2\pi} \int_a^b s_N(\lambda) \, d\lambda. \quad (5.12)
\]

Since the integrand is positive by part (c) above, this explains the rotation in the positive sense of the complex number \( u_N(\lambda) \). As already noted above, each time it goes through \(-1\), there is an eigenvalue of \( H_N \). See Figure 5.1. This means that this winding number \( \nu(\gamma) \) is related to the number of eigenvalues of \( H_N \) in the interval \([a, b]\). By taking the limit \( a \to -\infty \), we get the estimate:

\[
\left| \#\{ \text{eigenvalue of } H_N \text{ below } \lambda \} - \frac{N}{2\pi} \int_{-\infty}^\lambda s_N(t) \, dt \right| \leq \frac{1}{2}.
\]
Figure 5.1: For $\mathcal{A} = \mathbb{C}$: Prüfer phase on the unit circle.

All this will be proven below. For more about the classical one-dimensional case, see [Sim05].

In Section 5.4, we will generalise this winding number for the finite dimensional case $\mathcal{A} = \mathbb{B}(\mathbb{C}^L)$ when having $L$ eigenphases that rotate around the unit circle. They also count the eigenvalues of $H_N$ in a similar sense as above explained for the case $L = 1$. In Section 5.5, we will even expand this notion of a rotation number to an infinite dimensional case.

For convenience and better referring, we will now fix the general assumptions of the framework introduced in this section.

**Hypothesis 5.6.** Let $\mathcal{A}$ be a unital $C^*$-algebra, and for all $n \in \mathbb{N}$ let $a_n, v_n \in \mathcal{A}$ with $a_1 = 1$, $v_n$ self-adjoint and $a_n$ invertible. For $N \in \mathbb{N}$ let $H_N$ denote the self-adjoint element in $M_N(\mathcal{A}) = \mathbb{C}^{N \times N} \otimes \mathcal{A}$ given by (5.3). In addition, we suppose $\xi := \sup_{n \in \mathbb{N}} \{ \|a_n\|, \|a_n^{-1}\|, \|v_n\| \} < \infty$.

### 5.3.1 On transfer matrices

For the proof of Theorem 5.5, we will need some knowledge about the transfer matrices defined in (5.7). We start with their symplectic nature. Recall therefore the notions of $J$-unitaries and $I$-unitaries in $C^*$-algebras as given in Definition 2.62, as well as the Cayley transform $C$ from equation (2.5).
Proposition 5.7. Let \( v \in \mathcal{A} \) be self-adjoint and \( a \in \mathcal{A} \) invertible. For any \( \lambda \in \mathbb{R} \), one has

\[
T(\lambda) := \begin{pmatrix} (\lambda - v)a^{-1} & -a^* \\ a^{-1} & 0 \end{pmatrix} \in \mathcal{U}(\mathcal{A}, I).
\]

Furthermore,

\[
CT(\lambda)C^* = \frac{1}{2} \begin{pmatrix} (\lambda - v)a^{-1} - i(a^* + a^{-1}) & (\lambda - v)a^{-1} + i(a^* - a^{-1}) \\ (\lambda - v)a^{-1} - i(a^* - a^{-1}) & (\lambda - v)a^{-1} + i(a^* + a^{-1}) \end{pmatrix}
\]

and lies in \( \mathcal{U}(\mathcal{A}, J) \). In the representation \( CT(\lambda)C^* = \begin{pmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{pmatrix} \), the block matrix entries \( c_\lambda \) and \( d_\lambda \) satisfy

\[
\|d_\lambda^{-1}c_\lambda\|^2 \leq 1 - 4\xi^{-2}(|\lambda| + \xi + 2)^{-2},
\]

where \( \xi := \max\{|v|, |a|, |a^{-1}|\} \).

Proof. First note that the matrix \( T(\lambda) \) with entries in \( \mathcal{A} \) is invertible with inverse given by

\[
T(\lambda)^{-1} = \begin{pmatrix} 0 & a \\ -(a^*)^{-1} & (a^*)^{-1}(\lambda - v) \end{pmatrix}.
\]

Moreover, after a short calculation, one sees \( T(\lambda)^*IT(\lambda) = I \), which is the first claim. Then the second claim directly follows from \( iI = C^*JC \). It remains only to show that the estimate holds. Therefore note that we have for the norm of \( d_\lambda \), where we omit the index \( \lambda \) from now on, the following:

\[
\|d\| = \frac{1}{2}\|(\lambda - v)a^{-1} + i(a^* + a^{-1})\| \leq \frac{1}{2}(|\lambda| + \xi)\xi + \xi =: \kappa^{-1}.
\]

We conclude \( dd^* \leq \kappa^{-2} \mathbf{1} \) and therefore also \( d^{-1}(d^{-1})^* \geq \kappa^2 \mathbf{1} \). However, this implies \( 1 - d^{-1}(d^{-1})^* \leq (1 - \kappa^2) \mathbf{1} \). Since we have \( d^{-1}c(d^{-1}c)^* = \mathbf{1} - d^{-1}(d^{-1})^* \) for all \( J \)-unitaries, see Remark 2.67, the norm inequality \( \|d^{-1}c\|^2 \leq 1 - \kappa^2 \) holds as wanted. \( \square \)
Lemma 5.8. Assume Hypothesis 5.6. The product of the transfer matrices given by (5.7) has the following representation for $N \geq 2$:

$$T_N(\lambda) \cdots T_1(\lambda) = \begin{pmatrix} p_N(\lambda) & r_{N-1}(\lambda) \\ q_{N-1}(\lambda) & s_{N-2}(\lambda) \end{pmatrix}.$$  

Here, $p_N, q_{N-1}, r_{N-1}$ and $s_{N-2}$ are polynomials of order $N$, $N-1$ and $N-2$, respectively, and with coefficients generated by $v_n, a_n^*, a_n^{-1}$ for $n = 1, \ldots, N$. The leading coefficient of $p_N$ is given by $a_1^{-1} \cdots a_1^{-1}$. Each polynomial can be estimated by

$$\|p_N(\lambda)\| \leq 5^{N-1}\xi^{2N}(1 + |\lambda|^N),$$

and the same estimate holds for the polynomials $q_{N-1}, r_{N-1}$ and $s_{N-2}$. Furthermore, we have for the derivatives

$$\|\partial_\lambda p_N(\lambda)\| \leq 9^{N-1}\xi^{2N}(1 + |\lambda|^{N-1}),$$

and again the same estimate holds for the polynomials $q_{N-1}, r_{N-1}$ and $s_{N-2}$.

Proof. This is a straight-forward induction argument, whereas the base case is obvious. Therefore, we just give the induction step from $N$ to $N + 1$. In order to do that, we take the polynomials that build $T_N(\lambda) \cdots T_1(\lambda)$ and calculate:

$$T_{N+1}(\lambda)T_N(\lambda) \cdots T_1(\lambda) = \begin{pmatrix} \lambda - v_{N+1} + a_{N+1}^{-1} & -a_{N+1}^* \\ a_{N+1}^{-1} & 0 \end{pmatrix} \begin{pmatrix} p_N(\lambda) & r_{N-1}(\lambda) \\ q_{N-1}(\lambda) & s_{N-2}(\lambda) \end{pmatrix} = \begin{pmatrix} (\lambda - v_{N+1})a_{N+1}^{-1}p_N - a_{N+1}^*q_{N-1} & (\lambda - v_{N+1})a_{N+1}^{-1}r_{N-1} - a_{N+1}^*s_{N-2} \\ a_{N+1}^{-1}p_N & a_{N+1}^{-1}r_{N-1} \end{pmatrix}.$$  

Here, we omitted the argument $\lambda$ of the polynomials. So obviously, the order of the polynomials has increased by one and the norm estimate is also true, which we will explicitly check for $p_{N+1}$, where we set $\alpha = 5^{N-1}\xi^{2N}$:

$$\|p_{N+1}(\lambda)\| \leq \left(\|(\lambda - v_{N+1})a_{N+1}^{-1}\| + \|a_{N+1}^*\|\right)\alpha(1 + |\lambda|^N) \leq (|\lambda|\xi + \xi^2 + \xi)\alpha(1 + |\lambda|^N) \leq (2 + |\lambda|)\alpha\xi^2(1 + |\lambda|^N).$$
= (2 + |λ|^{N+1})αξ^2 + 2 |λ|^N αξ^2 + |λ| αξ^2.

Now we distinguish two different cases: First consider the case |λ| ≥ 1, which implies |λ|^{N+1} ≥ |λ|. Hence we get:

\[\|p_{N+1}(λ)\| \leq (2 + 4 |λ|^{N+1})αξ^2 \leq 4αξ^2(1 + |λ|^{N+1}).\]

In the second case |λ| < 1, we have:

\[\|p_{N+1}(λ)\| \leq (5 + |λ|^{N+1})αξ^2 \leq 5αξ^2(1 + |λ|^{N+1}).\]

In summary, we get the wanted estimate for \(p_{N+1}\). Of course for \(r_N\), the proof is exactly the same, and, for the other two polynomials, the estimate is then obvious.

The proof for the estimate of the derivatives is more or less the same. Just note that

\[\|∂_λ p_{N+1}\| \leq ξ |λ - v_{N+1}| |∂_λ p_N| + ξ |p_N| + ξ |∂_λ q_{N-1}|.\]

By using the induction hypothesis and the estimate for \(\|p_N\|\), one gets

\[\|∂_λ p_{N+1}\| \leq βξ(1 + |λ|^{N-1})\left(|λ - v_{N+1}| + (1 + |λ|) + 1\right),\]

where we set \(β = 9^{N-1}ξ^{2N}\). Distinguishing again \(|λ| < 1\) and \(|λ| ≥ 1\), one gets the wanted estimate. And as before, for \(r_N\) the proof is exactly the same, and, for the other two polynomials, the estimate is obvious after showing the first ones.

Proposition 5.9. Assume Hypothesis 5.6 and \(λ ∈ ℝ\). For each \(N ∈ ℕ\) define \(a_λ, b_λ, c_λ, d_λ ∈ \mathcal{A}\) by

\[CT_N(λ) \cdots T_1(λ) C^* = \begin{pmatrix} a_λ & b_λ \\ c_λ & d_λ \end{pmatrix}. \quad (5.14)\]

Then there exists a positive constant \(κ_N\), not depending on \(\{v_n, a_n\}_{n ∈ ℕ}\) but on
\[ \xi \text{ and } \lambda, \text{ such that} \]
\[ \|d^{-1}_\lambda c_\lambda\|^2 \leq 1 - \kappa_N^2. \]

In particular, the constant can be chosen as \( \kappa_N = 5^{-N}\xi^{-2N}(1 + |\lambda|^N)^{-1}. \)

**Proof.** For \( N = 1 \) this is precisely Proposition 5.7 with \( \kappa_1 = \xi^{-1}|\lambda| + \xi + 2 \). For larger \( N \) we use Lemma 5.8 and see that

\[ d_\lambda = \frac{1}{2}\left(p_N(\lambda) + s_{N-2}(\lambda) + i(q_{N-1}(\lambda) - r_{N-1}(\lambda))\right). \]

With the norm estimate for the polynomials of Lemma 5.8, we get

\[ \|d_\lambda\| \leq 5^N\xi^{2N}(1 + |\lambda|^N) = : \kappa_N^{-1}, \]

and we see that we can fully repeat the argument of the proof of Proposition 5.7 and this leads to the stated bound. \( \square \)

### 5.3.2 On Lagrangian frames and the Prüfer phase

Before, we have considered the eigenvalue equation for \( H_N \), which is also known as the stationary Schrödinger equation, given by

\[ H_N x = \lambda x \quad (5.15) \]

for a real energy \( \lambda \in \mathbb{R} \) with \( x \in \mathcal{A} \oplus \cdots \oplus \mathcal{A} \). As seen above, we can rewrite this in terms of the transfer operators \( T_n(\lambda) \). The Schrödinger equation (5.15) is then satisfied if and only if

\[ \begin{pmatrix} a_{n+1}x_{n+1} \\ x_n \end{pmatrix} = T_n(\lambda) \begin{pmatrix} a_nx_n \\ x_{n-1} \end{pmatrix}, \quad \text{for all } n = 1, \ldots, N, \quad (5.16) \]

and the boundary conditions hold: \( x_{N+1} = x_0 = 0 \). Here, (5.16) will be used to generate formal solutions \( \Phi_n(\lambda) \) as \( 2 \times 1 \) matrices with entries in \( \mathcal{A} \), which means

\[ \Phi_n(\lambda) = T_n(\lambda)\Phi_{n-1}(\lambda), \quad \Phi_0(\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (5.17) \]
In the case $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, the range of $\Phi_0(\lambda)$ is an $I$-Lagrangian subspace and since $T_n(\lambda)$ is $I$-unitary, one knows that for all $\Phi_n(\lambda)$ the range is an $I$-Lagrangian subspace as well, see Lemma 1.19 for more details. Of course, we can formulate this in a more abstract sense by interpreting $\Phi_0(\lambda)$ as an $I$-Lagragian frame in an abstract $C^*$-algebra, cf. Section 2.5, and using Proposition 2.66. Then we get:

**Proposition 5.10.** Assume Hypothesis 5.6. Then for $N \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, equation (5.17) defines an $I$-Lagrangian frame $\Psi_N(\lambda) := \Phi_N(\lambda) |\Phi_N(\lambda)|^{-1}$. The stereographic projection gives a unitary

$$u_N(\lambda) = \Pi_I([\Psi_N(\lambda)])$$

that coincides with the Prüfer phase in (5.11) and depends real-analytic on $\lambda$.

**Proof.** Since $\Phi_0(\lambda)$ is an $I$-Lagragian frame and all $T_n(\lambda)$ are $I$-unitaries by Proposition 5.7, $\Psi_N(\lambda)$ is also an $I$-Lagrangian frame. From Proposition 2.65 we know that $u_N(\lambda)$ is well-defined and gives rise to equation (5.11), which also shows in combination with Lemma 5.8 that $u_N$ is the product

$$u_N(\lambda) = [p_N(\lambda) - iq_{N+1}(\lambda)] [p_N(\lambda) + iq_{N+1}(\lambda)]^{-1},$$

where $p_N$ and $q_{N-1}$ are polynomials of order $N$ and $N - 1$, respectively. Using Proposition 2.68, we conclude that $\lambda \mapsto u_N(\lambda)$ is real-analytic. \hfill $\square$

We should emphasise that we have hereby proved part (a) in Theorem 5.5. Moreover, we can give an alternative formulation here, when using the left action $\mathbb{U}(\mathcal{A}, I) \times \mathbb{L}(\mathcal{A}, I)$ given by $T \cdot [\Phi] = [T\Phi |T\Phi|^{-1}]$ and the Cayley transform $C$. This also comes from Proposition 1.22, which also holds in the $C^*$-algebra stetting. Then the recursion definition is given by

$$u_N(\lambda) = \Pi_I \left( T_N(\lambda) \cdot [\Psi_{N-1}(\lambda)] \right) = CT_N(\lambda)C^* \cdot \Pi_I([\Psi_{N-1}(\lambda)])$$

$$= CT_N(\lambda)C^* \cdot u_{N-1}(\lambda),$$

(5.19)
where \( \bullet \) denotes the Möbius transform given by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet u = (au + b)(cu + d)^{-1}.
\]

In the same way, when considering this for the product of the transfer matrices
\( T(N, \lambda) := T_N(\lambda) \cdots T_1(\lambda) \), we also see that the Prüfer phase satisfies
\[
u_N(\lambda) = \Pi_I \left( T(N, \lambda) \cdot [\Phi_0(\lambda)] \right) = CT(N, \lambda)C^* \bullet 1,
\]
where we have used \( \Pi_I \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 1 \).

**Lemma 5.11.** Assume Hypothesis 5.6 and let \( T \) be a normalised trace on \( \mathcal{A} \). Then the complex-valued function
\[
\lambda \mapsto T \left[ (\Phi_N^* \Phi_N)^{-1} \Phi_N^* I \partial_\lambda \Phi_N \right]
\]
is integrable over \( \mathbb{R} \).

**Proof.** By Lemma 5.8 we have \( \Phi_N(\lambda) = \left( \begin{pmatrix} p_N(\lambda) \\ q_{N-1}(\lambda) \end{pmatrix} \right) \) for polynomials with coefficients in \( \mathcal{A} \) of order \( N \) and \( N - 1 \), respectively. Hence, we have:
\[
\Phi_N^* \Phi_N = p_N(\lambda)^* p_N(\lambda) + q_{N-1}(\lambda)^* q_{N-1}(\lambda).
\]
This is a polynomial of order \( 2N \), where the positive leading coefficient is \( (a_N^{-1} \cdots a_1^{-1})^*(a_N^{-1} \cdots a_1^{-1}) \). Obviously, \( \Phi_N^* \Phi_N \) is an invertible element for all \( \lambda \) by definition of Lagrangian frames. On the other hand, we have
\[
\Phi_N^* I \partial_\lambda \Phi_N = p_N(\lambda)^* \partial_\lambda q_{N-1}(\lambda) + q_{N-1}(\lambda)^* \partial_\lambda p_N(\lambda),
\]
which is a polynomial of order \( 2N - 2 \). In summary, since \( T \) is a continuous functional, then \( f : \lambda \mapsto T \left[ (\Phi_N^* \Phi_N)^{-1} \Phi_N^* I \partial_\lambda \Phi_N \right] \) also is a continuous function on \( \mathbb{R} \) with asymptotics
\[
\lambda^2 |f(\lambda)| \xrightarrow{\lambda \to \pm \infty} \text{const.} \in \mathbb{C}
\]
and therefore integrable.
Note that the frame $\Phi_N(\lambda)$ is technically speaking not an $I$-Lagrangian frame in our terms since in general $\Phi_N(\lambda)^* \Phi_N(\lambda) \neq 1$. For this reason, we have defined the normalised frame $\Psi_N(\lambda)$, but often it will be easier to bypass this normalisation, and therefore we will need the following general formulas.

**Lemma 5.12.** Let $\mathcal{A}$ be a unital $C^*$-algebra and $a, b \in \mathcal{A}$ with $\Phi = \left( \begin{smallmatrix} a & b \\ \ast & \ast \end{smallmatrix} \right)$ such that $\Phi | \Phi |^{-1}$ exists as an $I$-Lagrangian frame and define $\phi_{\pm} = (1, \pm i) \Phi$. Then:

$$\phi_{-}^* \phi_{-} + \phi_{+}^* \phi_{+} = \Phi^* \Phi.$$

**Proof.** Setting $c := |\Phi|^{-1} = (\Phi^* \Phi)^{-1/2}$, then $\Psi := \left( \begin{smallmatrix} a & c \\ b & c \end{smallmatrix} \right)$ is an $I$-Lagrangian frame, and by Lemma 1.12 we know that $(a \pm ib)c$ is unitary. This implies $(a \pm ib)^* (a \pm ib) = c^{-2} = \Phi^* \Phi$ as stated. \qed

### 5.3.3 On estimates for the Prüfer phase

**Proposition 5.13.** Assume Hypothesis 5.6. For any $\lambda_0 > 0$ and $j \in \mathbb{N}$, there exists a constant $K_\xi$, not depending on $\{a_n, v_n\}_{n \in \mathbb{N}}$ but on $\xi$, such that the Prüfer phase satisfies

$$\sup_{\lambda \in [-\lambda_0, \lambda_0]} \left\| \partial_\lambda^j u_N(\lambda) \right\| \leq K_\xi.$$

**Proof.** Let $a_\lambda, b_\lambda, c_\lambda, d_\lambda$ be the matrix entries of $CT(N, \lambda) C^* \in U(\mathcal{K}, J)$ as given in (5.14). Then we find by Lemma 5.8:

$$a_\lambda = \frac{1}{2} \left( p_N(\lambda) + s_{N-2}(\lambda) - iq_{N-1}(\lambda) + ir_{N-1}(\lambda) \right),$$

$$b_\lambda = \frac{1}{2} \left( p_N(\lambda) - s_{N-2}(\lambda) - iq_{N-1}(\lambda) - ir_{N-1}(\lambda) \right),$$

$$c_\lambda = \frac{1}{2} \left( p_N(\lambda) - s_{N-2}(\lambda) + iq_{N-1}(\lambda) - ir_{N-1}(\lambda) \right),$$

$$d_\lambda = \frac{1}{2} \left( p_N(\lambda) + s_{N-2}(\lambda) + iq_{N-1}(\lambda) - ir_{N-1}(\lambda) \right),$$

where $p_N, q_{N-1}, r_{N-1}$ and $s_{N-2}$ are polynomials of order $N$, $N-1$ and $N-2$, respectively. Again by Lemma 5.8, we know that the coefficients are generated
5.3 Analysis for Jacobi matrices

by \( \{v_n, a_n^*, a_n^{-1}\}_{n=1,...,N} \). Moreover, due to equation (5.20) we have:

\[
u_N(\lambda) = (a_\lambda + b_\lambda)(c_\lambda + d_\lambda)^{-1}.
\]

Hence, we can use the following derivative formula for a differentiable path \( t \mapsto x(t) \in \mathcal{A} \), where \( x(t) \) is for all \( t \) invertible:

\[
\partial_t (x(t)^{-1}) = -x(t)^{-1}(\partial_t x(t))x(t)^{-1}.
\]

(5.21)

Then, we can write:

\[
\partial_\lambda u_N(\lambda) = \left( (\partial_\lambda a_\lambda + \partial_\lambda b_\lambda) - u_N(\lambda)(\partial_\lambda c_\lambda + \partial_\lambda d_\lambda) \right)(d_\lambda^{-1}c_\lambda + 1)^{-1}d_\lambda^{-1}.
\]

Since the inverse exists and we only have polynomials in \( \lambda \), we just have to estimate the norm here. We use Remark 2.67, which says \( \|d_\lambda\| \leq 1 \), and we need Proposition 5.9 for the asymptotic behaviour of \( \|(d_\lambda^{-1}c_\lambda + 1)^{-1}\| \). Then we get:

\[
\|\partial_\lambda u_N(\lambda)\| \leq (\|\partial_\lambda a_\lambda\| + \|\partial_\lambda b_\lambda\| + \|\partial_\lambda c_\lambda\| + \|\partial_\lambda d_\lambda\|) \frac{1}{1 - \|d_\lambda^{-1}c_\lambda\|}.
\]

By Lemma 5.8, we can estimate the derivatives of all polynomials by the same formula and therefore obtain:

\[
\|\partial_\lambda u_N(\lambda)\| \leq \left( 8 \cdot 9^{N-1}\xi^{2N}(1 + |\lambda|^{N-1}) \right) \frac{1}{1 - \sqrt{1 - 5^{-2N}\xi^{-4N} (1 + |\lambda|^{2N})^{-1}}}.
\]

This formula gives us for each \( \lambda_0 > 0 \) the constant \( K_\xi \). Deriving several times and using the same bounds provides the general estimate.

We have not explicitly written down the proof for the higher derivatives because lengthy formulas arise and the ideas are the same. More importantly, in the upcoming proofs, we will only need the bound for the first derivative.
5.3.4 Monotonicity of the Prüfer phase

In this section, we verify part (c) of the main theorem and, hence, the key monotonicity property that the Prüfer phase has in the energy parameter $\lambda$.

**Proposition 5.14.** Assume Hypothesis 5.6. For any $\lambda \in \mathbb{R}$ the element

$$s_N(\lambda) := \frac{1}{iN} (u_N(\lambda))^* \partial_\lambda u_N(\lambda) \in \mathcal{A}$$

is strictly positive.

**Proof.** We already know that the Prüfer phase is given via the stereographic projection by $u_N(\lambda) = \Pi_I([\Phi_N(\lambda) \mid \Phi_N(\lambda)|^{-1}])$. If we set $\phi_\pm(\lambda) = (1, \pm i) \Phi_N(\lambda)$, then we have the short formula $u_N(\lambda) = \phi_-(\lambda)(\phi_+(\lambda))^{-1}$. From Lemma 5.12 it follows

$$(\phi_-(\lambda))^* \phi_-(\lambda) = (\phi_+(\lambda))^* \phi_+(\lambda) \quad \text{and} \quad (u_N(\lambda))^* = \phi_+(\lambda)(\phi_-(\lambda))^{-1}.$$  

Clearly, this also follows by the representation of $u_N$ by the polynomials as in equation (5.18). Therefore, we can write the derivative in the following form, where we omit the dependence on $\lambda$ here and in the rest of the proof,

$$u_N^* \partial_\lambda u_N = (\phi_+^{-1})^* \left[ \phi_-^* \partial_\lambda \phi_- - \phi_+^* \partial_\lambda \phi_+ \right] \phi_+^{-1}. \quad (5.22)$$

Hence, it is sufficient to verify that the following element is strictly positive:

$$\frac{1}{i} \left[ \phi_-^* \partial_\lambda \phi_- - \phi_+^* \partial_\lambda \phi_+ \right] = 2 \Phi_N^* I \partial_\lambda \Phi_N.$$

Now using the product rule for the derivative, we get:

$$\partial_\lambda \Phi_N = \sum_{n=1}^{N} T_N \cdots T_{n+1} (\partial_\lambda T_n) T_{n-1} \cdots T_1 \Phi_0.$$

This implies that

$$(\Phi_N)^* I \partial_\lambda \Phi_N = \sum_{n=1}^{N} \Phi_0^* (T_{n-1} \cdots T_1)^* T_n^* I (\partial_\lambda T_n) (T_{n-1} \cdots T_1) \Phi_0.$$
As one can easily check that
\[ T_n^* I (\partial_{\lambda} T_n) = \begin{pmatrix} (a_n a_n^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } n = 1, \ldots, N, \]
and thus
\[ (\Phi_N)^* I \partial_{\lambda} \Phi_N = \sum_{n=1}^{N} (\Phi_0)^* (T_{n-1} \cdots T_1)^* \begin{pmatrix} (a_n a_n^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} (T_{n-1} \cdots T_1) \Phi_0. \]
Clearly each of the summands is a positive element. In order to prove a strict lower bound, it is sufficient that the first two terms \( n = 1, 2 \) give a strictly positive contribution, meaning they are invertible. Hence let us verify that
\[ \begin{pmatrix} (a_1 a_1^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + T_1^* \begin{pmatrix} (a_2 a_2^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} T_1 \]
is invertible. As \((a_n a_n^*)^{-1} \geq \xi^{-2} 1\) for all \( n \), this invertibility is equivalent to the invertibility of the following operator:
\[ X := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + T_1^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_1. \]
Using the element \( b = (\lambda - v_1) a_1^{-1} \), which in norm is bounded above by a constant depending on \( \lambda \), one thus just has to note the invertibility of
\[ X = \begin{pmatrix} 1 + b^* b & -(a_1 b)^* \\ -a_1 b & a_1 a_1^* \end{pmatrix} = \begin{pmatrix} 1 & -b^* \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & a_1^* \end{pmatrix}, \]
but that immediately follows by the block decomposition above.

**Corollary 5.15.** Assume Hypothesis 5.6 and let \( T : \mathcal{A} \to \mathbb{C} \) be a normalised trace. Then \( \lambda \mapsto T(s_N(\lambda)) \) is integrable:
\[ \int_{\mathbb{R}} T(s_N(t)) \, dt < \infty. \]

**Proof.** As often we suppress the argument \( \lambda \). Note that we have by equa-
tion (5.22) and Lemma 5.12:

\[ \mathcal{T}[(u_N)^*\partial_\lambda u_N] = \mathcal{T}[(\phi_+^{-1}(\phi_-^{-1})^*2\Phi_N^*I\partial_\lambda \Phi_N] = \mathcal{T}[(\Phi_N^*\Phi_N)^{-1}2\Phi_N^*I\partial_\lambda \Phi_N]. \]

Then using Lemma 5.11, we conclude that \( \mathcal{T}(s_N(\cdot)) \) is integrable. \( \square \)

5.3.5 Asymptotics of the Prüfer phase

**Proposition 5.16.** Assume Hypothesis 5.6. Then there is a constant \( \lambda_0 \), not depending on \( \{v_n, a_n\}_{n \in \mathbb{N}} \) but on \( \xi \), such that for all \( N \in \mathbb{N} \) and \( |\lambda| > \lambda_0 \)

\[ \|u_N(\lambda) - (1 - 2i\lambda^{-1})\| \leq \frac{K\xi^2}{\lambda^2}, \]

where \( K > 0 \) is an independent constant. In particular,

\[ \lim_{\lambda \to \pm \infty} u_N(\lambda) = 1. \]

**Proof.** Without loss of generality, we will always assume \( \lambda > 0 \). We set the constant \( K = 28 \) and \( \lambda_0 > K\xi^3 \) and prove that this is a suitable choice but of course this is far away from an optimal bound. For a better understanding, we will first explicitly consider the case \( N = 1 \). There we have:

\[ u_1(\lambda) = \left((\lambda - v_1) - i\right)\left((\lambda - v_1) + i\right)^{-1} = \left(1 - \lambda^{-1}(v_1 + i)\right)\left(1 - \lambda^{-1}(v_1 - i)\right)^{-1}. \]

We can use the Neumann series for the inverse operator on the right hand as long as \( \lambda > 2(\xi + 1) \), which is fulfilled for \( \lambda > \lambda_0 \). Therefore we find:

\[ (1 - \lambda^{-1}(v_1 - i))^{-1} = \sum_{k=0}^{\infty} \frac{(v_1 - i)^k}{\lambda^k}. \]

Accordingly, we have for the Prüfer phase

\[ u_1(\lambda) = 1 - 2i\lambda^{-1} + \lambda^{-2}\left(-v_1(v_1 - i) - i(v_1 - i) + (1 - (v_1 + i)\lambda^{-1}) \sum_{k=0}^{\infty} \frac{(v_1 - i)^{k+2}}{\lambda^k}\right) \]
as wanted. Hence, the norm estimate with \(\|v_1\| \leq \xi\) gives us immediately

\[
\|u_1(\lambda) - (1 - 2i\lambda^{-1})\| \leq \lambda^{-2}(\xi(\xi + 1) + (\xi + 1) + 6(\xi + 1)^2) \leq \frac{K\xi^2}{\lambda^2}
\]

with \(K = 28\).

Now we will do an induction over \(N\), and therefore let us set

\[
r_N(\lambda) := u_N(\lambda) - (1 - 2i\lambda^{-1}). \tag{5.23}
\]

Assume now that \(r_{N-1}\) fulfils the estimate from the proposition for \(K = 28\) and some constant \(\lambda_0 > K\xi^3\). First we want to derive an iterative equation for this operator, that is, express it in terms of \(r_{N-1}(\lambda)\). This will be obtained from the equation (5.19), namely \(u_N(\lambda) = CT_N(\lambda)C^* \bullet u_{N-1}(\lambda)\), which becomes explicitly when using (5.13):

\[
u_N(\lambda) = (\lambda - v_N)a_N^{-1} - ia_N^* - ia_N^{-1} (\lambda - v_N)a_N^{-1} + ia_N^* - ia_N^{-1}) \bullet u_{N-1}(\lambda)
\]

\[
= \left[1 - w_1(N, \lambda)]]a_N^{-1} \left[1 - w_2(N, \lambda)]a_N^{-1} \bullet u_{N-1}(\lambda),
\]

where we set

\[
w_1(N, \lambda) = \lambda^{-1}(v_N + ia_N^*a_N + i),
\]

\[
w_2(N, \lambda) = \lambda^{-1}(v_N - ia_N^*a_N + i),
\]

\[
w_3(N, \lambda) = \lambda^{-1}(v_N + ia_N^*a_N - i),
\]

\[
w_4(N, \lambda) = \lambda^{-1}(v_N - ia_N^*a_N - i).
\]

To shorten notations, we will from now on drop the arguments of \(w_j\), \(u_N\) and \(r_N\) and also the index \(N\) on \(a_N\) and \(v_N\). Since \(\xi \geq 1\), \(\|v\| \leq \xi\) and \(\|a\| \leq \xi\), the operators \(w_j\) satisfy

\[
\|w_j\| \leq \lambda^{-1}(\xi + \xi^2 + 1) \leq 3\lambda^{-1}\xi^2 \leq \xi, \tag{5.24}
\]

for \(\lambda > 3\xi\), which it satisfied for \(\lambda_0 > K\xi^3\). Actually, later on we will suppose
that $\lambda_0$ is indeed larger than $K\xi^3$. Now by writing out the Möbius action, we obtain the short formula:

$$u_N = [(1 - w_1)a^{-1}u_{N-1}a + 1 - w_2] [(1 - w_3)a^{-1}u_{N-1}a + 1 - w_4]^{-1}.$$ 

For getting the connection to $r_{N-1}(\lambda)$, we have to expand the inverse operator on the right hand side and express $u_{N-1}(\lambda)$ in these formulas by $r_{N-1}(\lambda)$ according to (5.23). Therefore we define

$$d_N = 2 [(1 - w_3)a^{-1}u_{N-1}a + 1 - w_4]^{-1}$$

$$= [1 - \frac{1}{2}w_3 - \frac{1}{2}w_4 - i(1 - w_3)\lambda^{-1} + \frac{1}{2}(1 - w_3)a^{-1}r_{N-1}a]^{-1}$$

$$= [1 - b_N]^{-1},$$

where $b_N$ is defined as

$$b_N = \frac{1}{2}w_3 + \frac{1}{2}w_4 + i(1 - w_3)\lambda^{-1} - \frac{1}{2}(1 - w_3)a^{-1}r_{N-1}a$$

$$= \lambda^{-1}(v - iw_3) - \frac{1}{2}(1 - w_3)a^{-1}r_{N-1}a. \quad (5.25)$$

The last equation holds since $w_3 + w_4 = 2\lambda^{-1}v - 2\lambda^{-1}i$. Now, we can expand this inverse operator and simply use the resolvent identity, see Lemma 2.69, and obtain:

$$d_N = (1 - b_N)^{-1} = 1 + b_N d_N. \quad (5.26)$$

Now we can write $u_N$ in the following form

$$u_N = [(1 - w_1)a^{-1}u_{N-1}a + 1 - w_2] \frac{1}{2} d_N$$

$$= [1 - \frac{1}{2}w_1 - \frac{1}{2}w_2 - i(1 - w_1)\lambda^{-1} + \frac{1}{2}(1 - w_1)a^{-1}r_{N-1}a] d_N \quad (5.27)$$

$$= [1 - (v + 2i - iw_1)\lambda^{-1} + \frac{1}{2}(1 - w_1)a^{-1}r_{N-1}a] d_N.$$

Setting $k_N = -(v - iw_1)\lambda^{-1} + \frac{1}{2}(1 - w_1)a^{-1}r_{N-1}a$, one gets the short formula:

$$u_N = [1 - 2i\lambda^{-1} + k_N] d_N.$$
Using this and the equation (5.26), we can calculate the rest term:

\[ r_N = [1 - 2i\lambda^{-1} + k_N] [1 + b_N d_N] - (1 - 2i\lambda^{-1}) \]

\[ = k_N + [1 - 2i\lambda^{-1} + k_N] b_N d_N. \]

Here, we have successfully eliminated the second part of \( r_N \) and, in this regard, also the term of order \( \lambda^0 \). Now we can use equation (5.26) again to eliminate all terms of order \( \lambda^{-1} \) and \( \lambda^{-2} \):

\[ r_N = k_N + [1 - 2i\lambda^{-1} + k_N] b_N [1 + b_N d_N] \]

\[ = k_N + b_N b_N d_N + [-2i\lambda^{-1} + k_N] b_N [1 + b_N d_N] \]

\[ = k_N + b_N b_N d_N + [-2i\lambda^{-1} + k_N] b_N d_N. \] (5.28)

In the last equation, we used (5.26) to simplify the right-hand side again. Now we will take care of the first two summands in (5.28), where the terms with \( v \) cancel out by definition of \( k_N \) and \( b_N \). Consequently, we obtain:

\[ k_N + b_N = iw_1 \lambda^{-1} + \frac{1}{2} (1 - w_1) a^{-1} r_{N-1} a - iw_3 \lambda^{-1} - \frac{1}{2} (1 - w_3) a^{-1} r_{N-1} a \]

\[ = iw_1 \lambda^{-1} - \frac{1}{2} w_1 a^{-1} r_{N-1} a - iw_3 \lambda^{-1} + \frac{1}{2} w_3 a^{-1} r_{N-1} a. \]

Using the fact that \( w_1 - w_3 = 2i\lambda^{-1} \), we find the short expression:

\[ k_N + b_N = i\lambda^{-1} (w_1 - w_3) - \frac{1}{2} (w_1 - w_3) a^{-1} r_{N-1} a \]

\[ = -2\lambda^{-2} - i\lambda^{-1} a^{-1} r_{N-1} a. \]

We combine this with our expression of \( r_N \) in (5.28) and estimate the norm in the following:

\[ \|r_N\| \leq \|k_N + b_N\| + \|b_N\|^2 \|d_N\| + \| - 2i\lambda^{-1} + k_N \| \|b_N\| \|d_N\| \]

\[ \leq \|k_N + b_N\| + \|b_N\|^2 \|d_N\| + (\|k_N\| + 2\lambda^{-1}) \|b_N\| \|d_N\|. \] (5.29)

Now we have to estimate the norm of \( b_N \) from (5.25) and also of \( k_N \) using
\( (5.24) \):

\[
\|b_N\| \leq \lambda^{-1}2\xi + \xi^3\|r_{N-1}\|, \quad \|k_N\| \leq \lambda^{-1}2\xi + \xi^3\|r_{N-1}\|. \tag{5.30}
\]

Note that by Lemma 2.69 \( \|d_N\| \leq 2 \) if \( \|b_N\| \leq \frac{1}{2} \), and this last part we will show by an induction argument. However, first we estimate:

\[
\|r_N\| \leq \|k_N + b_N\| + \|b_N\|^2\|d_N\| + (\|k_N\| + 2\lambda^{-1})\|b_N\|\|d_N\|
\leq 2\lambda^{-2} + \lambda^{-1}\xi^2\|r_{N-1}\| + 2\|d_N\|(\lambda^{-1}2\xi + \xi^3\|r_{N-1}\|)^2
\tag{5.31}
\]

\[ + (2\lambda^{-1}\|d_N\|)(\lambda^{-1}2\xi + \xi^3\|r_{N-1}\|). \]

Clearly, in this equation we want to write the estimate in a polynomial of \( r_{N-1} \), and therefore we first prove that \( \|d_N\| \leq 2 \) always holds. By using equation (5.30) and the induction hypothesis, we get

\[
\|b_N\| \leq \lambda^{-1}2\xi + \xi^3\|r_{N-1}\| \leq \lambda^{-1}2\xi + K\lambda^{-2}\xi^5 \leq \frac{1}{2} \quad \text{for} \quad \lambda > \lambda_0,
\]

where \( \lambda_0 > K\xi^3 \). Hence by Lemma 2.69, we also have \( \|d_N\| \leq 2 \). We can use this to simplify equation (5.31):

\[
\|r_N\| \leq 2\lambda^{-2} + \lambda^{-1}\xi^2\|r_{N-1}\| + 4(\lambda^{-1}2\xi + \xi^3\|r_{N-1}\|)^2
\]

\[
+ (4\lambda^{-1})(\lambda^{-1}2\xi + \xi^3\|r_{N-1}\|)
\]

\[
= 2\lambda^{-2} + \lambda^{-1}\xi^2\|r_{N-1}\|
\]

\[
+ 4\left(\lambda^{-2}4\xi^2 + \lambda^{-2}2\xi + (4\lambda^{-1}\xi^4 + \lambda^{-1}\xi^3)\|r_{N-1}\| + \xi^6\|r_{N-1}\|^2\right)
\]

\[
\leq 26\xi^2\lambda^{-2} + 21\lambda^{-1}\xi^4\|r_{N-1}\| + 4\xi^6\|r_{N-1}\|^2. \tag{5.32}
\]

For the final contraction argument, let us set

\[
z_N := \lambda^2\|r_N\|
\]

and introduce the function \( f_\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by

\[
f_\lambda(x) := 26\xi^2 + (21\lambda^{-1}\xi^4)x + (4\lambda^{-2}\xi^6)x^2.
\]
Then, since $\|d_N\| \leq 2$ holds, (5.32) now reads

$$z_N \leq f_\lambda(z_{N-1}).$$

If $\lambda$ is sufficiently large, then $f_\lambda$ has an attractive fixed point $x_\lambda$ with a basin of attraction $[0, t_\lambda)$ where $t_\lambda$ is a second fixed point, a repeller, cf. Figure 5.2. Obviously, one has $t_\lambda \to \infty$ monotonously for $\lambda \to \infty$. So choose $\lambda_0 > K \xi^3$ such that $t_{\lambda_0} > K \xi^2$. Since $z_{N-1} \leq K \xi^2$ for all $\lambda > \lambda_0$ by the induction hypothesis, we know that $z_{N-1}$ is in the basin of attraction, and we obtain $z_N \leq f(z_{N-1}) \leq z_{N-1}$. This implies $\|r_N\| \leq \lambda^{-2} K \xi^2$ for all $\lambda > \lambda_0$ and finishes the induction proof.

\[ \square \]

### 5.3.6 Asymptotics for derivatives of Prüfer phases

The proof of Proposition 5.13 implies that $\partial_\lambda u_N(\lambda)$ grows at most polynomially in $\lambda$. In the following, it will be shown that it actually decreases for large $\lambda$. In principle, one could be as explicit about the constants and dependence on $\xi$ in the following, but we refrain from doing so since we will not need the explicit constants.
Lemma 5.17. Assume Hypothesis 5.6. There are constants $\lambda_0$ and $\widehat{K}$, not depending on $\{a_n, v_n\}_{n \in \mathbb{N}}$ but on $\xi$, such that for all $N \geq 1$ and $|\lambda| > \lambda_0$:

$$\|\partial_\lambda u_N(\lambda) - 2i\lambda^{-2}\| \leq \frac{\widehat{K}}{|\lambda|^3}.$$

Proof. We reuse several notations from the proof of Proposition 5.16, in particular $w_j$, $d_N$ and $b_N$. Furthermore, $a = a_N$ and $v = v_N$ do not carry an index and we often suppress the dependence on $\lambda$. Let us also introduce a notation for the term that needs to be estimated:

$$p_N = \partial_\lambda u_N - 2i\lambda^{-2}.$$

For the calculation of the derivative $\partial_\lambda u_N$, let us start from the representation of $u_N$ in (5.27):

$$\partial_\lambda u_N = \frac{1}{2} \left[-(\partial_\lambda w_1) a^{-1} u_{N-1} a + (1 - w_1) a^{-1} \partial_\lambda u_{N-1} a - \partial_\lambda w_2 \right] d_N$$

$$+ \frac{1}{2} \left[(1 - w_1) a^{-1} u_{N-1} a + 1 - w_2 \right] \partial_\lambda d_N.$$

Using the derivative formula for the inverse, cf. equation (5.21), and calculate:

$$p_N = \frac{1}{2} \left[-(\partial_\lambda w_1) a^{-1} u_{N-1} a + (1 - w_1) a^{-1} \partial_\lambda u_{N-1} a - \partial_\lambda w_2 \right] d_N$$

$$- \frac{1}{2} \left[(1 - w_1) a^{-1} u_{N-1} a + 1 - w_2 \right] d_N$$

$$\cdot \frac{1}{2} \left[-(\partial_\lambda w_3) a^{-1} u_{N-1} a + (1 - w_3) a^{-1} \partial_\lambda u_{N-1} a - \partial_\lambda w_4 \right] d_N - 2i\lambda^{-2}.$$

Now let us replace $\partial_\lambda u_{N-1} = p_{N-1} + 2i\lambda^{-2}$ and split $u_{N-1} = (u_{N-1} - 1) + 1$:

$$p_N = \frac{1}{2} \left[-(\partial_\lambda w_1) a^{-1} (u_{N-1} - 1) a$$

$$+ (1 - w_1) (a^{-1} p_{N-1} a + 2i\lambda^{-2}) - \partial_\lambda (w_1 + w_2) \right] d_N$$

$$- \frac{1}{2} \left[(1 - w_1) a^{-1} (u_{N-1} - 1) a + 2 1 - (w_1 + w_2) \right] d_N.$$
\[ 5.3 \text{ Analysis for Jacobi matrices} \]

\[
\cdot \frac{1}{2} \left[ - (\partial_{\lambda} w_3) a^{-1} (u_{N-1} - 1) a \\
+ (1 - w_3) (a^{-1} p_{N-1} a + 2i\lambda^{-2} 1) - \partial_{\lambda} (w_3 + w_4) \right] d_N \\
- 2i\lambda^{-2}.
\]

Now from the definitions

\[ w_1 + w_2 = 2\lambda^{-1}(v + i), \quad w_3 + w_4 = 2\lambda^{-1}(v - i), \]

we obtain

\[ \partial_{\lambda} (w_1 + w_2) = -2\lambda^{-2}(v + i), \quad \partial_{\lambda} (w_3 + w_4) = -2\lambda^{-2}(v - i). \]

Let us replace these latter two equations in the expression for \( p_N \):

\[
p_N = \frac{1}{2} \left[ - (\partial_{\lambda} w_1) a^{-1} (u_{N-1} - 1) a \\
+ (1 - w_1) a^{-1} p_{N-1} a - 2i\lambda^{-2} w_1 + 2\lambda^{-2}(v + 2i) \right] d_N \\
- \left[ 1 + \frac{1}{2} (1 - w_1) a^{-1} (u_{N-1} - 1) a - \frac{1}{2} (w_1 + w_2) \right] d_N \\
\cdot \frac{1}{2} \left[ - (\partial_{\lambda} w_3) a^{-1} (u_{N-1} - 1) a + (1 - w_3) a^{-1} p_{N-1} a \\
- 2i\lambda^{-2} w_3 + 2\lambda^{-2} v \right] d_N - 2i\lambda^{-2}.
\]

To see that the terms of order \( \lambda^{-2} \) indeed cancel out, we can now replace \( d_N = 1 + b_N d_N \) as given in (5.26) several times:

\[
p_N = \frac{1}{2} \left[ -(\partial_{\lambda} w_1) a^{-1} (u_{N-1} - 1) a + (1 - w_1) a^{-1} p_{N-1} a - 2i\lambda^{-2} w_1 \right] \\
+ \frac{1}{2} \left[ - (\partial_{\lambda} w_1) a^{-1} (u_{N-1} - 1) a \\
+ (1 - w_1) a^{-1} p_{N-1} a - 2i\lambda^{-2} w_1 + 2\lambda^{-2}(v + 2i) \right] b_N d_N \\
- \left[ 1 + \frac{1}{2} (1 - w_1) a^{-1} (u_{N-1} - 1) a - \frac{1}{2} (w_1 + w_2) \right] b_N d_N \\
\cdot \frac{1}{2} \left[ - (\partial_{\lambda} w_3) a^{-1} (u_{N-1} - 1) a \\
+ (1 - w_3) a^{-1} p_{N-1} a - 2i\lambda^{-2} w_3 + 2\lambda^{-2} v \right] d_N \\
- 2i\lambda^{-2}.
\]
The only terms that still have to be cancelled out are \( \frac{1}{2} a^{-1} p_{N-1} a \) appearing in the first line and the second to last summand:

\[
\begin{align*}
\lambda^{-2} v 
\end{align*}
\]

For the following estimates, we now use \( \|u_N - 1\| \leq c \lambda^{-1} \) as follows from Proposition 5.16, for some constant \( c \), which will take increasing values in the following. Furthermore, \( \|w_j\| \leq c \lambda^{-1} \) and \( \|\partial_\lambda w_j\| \leq c \lambda^{-2} \) as well as \( \|b_N\| \leq c \lambda^{-1} \) and \( \|d_N\| \leq 2 \), see the proof of Proposition 5.16. Carefully
checking all terms, this leads to
\[ \|p_N\| \leq c\lambda^{-3} + c\lambda^{-1}\|p_{N-1}\|. \]
Setting now \( z_N = \lambda^3\|p_N\| \), it follows that \( z_N \leq c + c\lambda^{-1}p_{N-1} \leq c(1 - c\lambda^{-1})^{-1} \). This concludes the proof.

**Proposition 5.18.** Assume Hypothesis 5.6. There are constants \( K' \) and \( \lambda_0 \), not depending on \( \{a_n, v_n\}_{n \in \mathbb{N}} \) but on \( \xi \), such that for all \( N \geq 1 \) and \( |\lambda| > \lambda_0 \) the speed operator \( s_N = \frac{1}{iN} u_N^* \partial_\lambda u_N \) fulfills
\[ \|s_N(\lambda) - 2N^{-1}\lambda^{-2}\| \leq \frac{K'}{N|\lambda|^3}. \]

*Proof.* Let us begin by using the triangle inequality and splitting the contributions as follows
\[
N\|s_N(\lambda) - 2N^{-1}\lambda^{-2}\| \leq \|u_N^* \partial_\lambda u_N - 2iu_N^* \lambda^{-2}\| + \|2iu_N^* \lambda^{-2} - 2i\lambda^{-2}\|
\leq \|u_N^*\|\|\partial_\lambda u_N - 2i\lambda^{-2}\| + 2\lambda^{-2}\|u_N^* - 1\|
= \|\partial_\lambda u_N - 2i\lambda^{-2}\| + 2\lambda^{-2}\|u_N - 1\|.
\]
Now for the first part, we can use Lemma 5.17, and for the second part, we conclude with Proposition 5.16 that \( \|u_N^* - 1\| \leq 2|\lambda|^{-1} + K\xi\lambda^{-2} \) for constants \( \lambda_0 \) and \( K \). Combining the two estimates gives the wanted result. \qed

**Corollary 5.19.** Assume Hypothesis 5.6. Then the norm of \( s_N \) is integrable. Moreover, there is a constant \( K' \) and \( \lambda_0 > 0 \), not depending on \( \{a_n, v_n\}_{n \in \mathbb{N}} \) but on \( \xi \), such that for all \( N \geq 1 \) and \( \lambda \geq \lambda_0 \):
\[
\int_{-\infty}^{-\lambda} \|s_N(t)\| \, dt \leq \frac{K'}{N\lambda}, \quad \int_{\lambda}^{\infty} \|s_N(t)\| \, dt \leq \frac{K'}{N\lambda}.
\]

*Proof.* This follows directly from Proposition 5.18 above. \qed
5.3.7 Intersections of Lagrangian subspaces

Now we focus on the case $A \subset B(\mathcal{H})$, for which we have the Lagrangian subspaces in the Krein space $(\mathcal{H} \oplus \mathcal{H}, I)$. Recall Proposition 5.4, which characterises the point spectrum of $H_N$ with $I$-Lagrangian frames and transfer operators that are $I$-unitaries. Here we explain especially the point spectrum with finite multiplicities as the intersection of two Lagrangian frames. This will especially important in the next section when we consider the finite dimensional case $A = B(\mathbb{C}^L)$. We already used the Lagrangian frame that describes Dirichlet boundary conditions on the right side. It is given by $\hat{\Psi} := (\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})$ and we define the associated singular cycle:

$$\hat{L}_\mathcal{H} := \{ [\Phi] \in L(\mathcal{H} \oplus \mathcal{H}, I) \mid \text{Ran}(\Phi) \cap \text{Ran}(\hat{\Psi}) \neq \{0\} \}.$$ 

In a generalised version, this singular cycle was defined by Bott in [Bot56] and Maslov in [MBA72]. Therefore the intersection theory below is known as the Bott-Maslov-index, cf. [Schu07]. Now we write this set of Lagrangian subspaces $\hat{L}_\mathcal{H}$ as the union of

$$\hat{L}_\mathcal{H}^m := \{ [\Phi] \in L(\mathcal{H} \oplus \mathcal{H}, I) \mid \dim(\text{Ran}(\Phi) \cap \text{Ran}(\hat{\Psi})) = m \},$$

for $m \in \mathbb{N} \cup \{\infty\}$.

**Lemma 5.20.** Let $\Phi, \Psi$ two $I$-Lagrangian frames. Then

$$\dim (\text{Ran}(\Phi) \cap \text{Ran}(\Psi)) = \dim (\ker(\Phi^* I \Psi)).$$

**Proof.** This is a straightforward calculation where we use Corollary 1.14. Hence, the $I$-Langragian frames can be written as $\Phi = \frac{1}{2}(u^+ + 1)_{(iu_1 - i)u}$ and $\Psi = \frac{1}{2}(v^+ + 1)_{(iv_1 - i)v}$ for unitaries $u, v \in B(\mathcal{H})$. Now, it is easy to see that $y \in \text{Ran}(\Phi) \cap \text{Ran}(\Psi)$ if and only if there are $x, \bar{x} \in \mathcal{H}$ with $(u \pm 1)(x) = (v \pm 1)(\bar{x})$. However, these two equations are only simultaneously satisfied in the case $x = \bar{x}$. Therefore we
obtain:
\[
x \in \text{Ker}(\Phi^*I\Psi) \iff u^*v(x) = x \iff (u \pm \mathbf{1})(x) = (v \pm \mathbf{1})(x)
\]
\[
\iff \frac{1}{2} \left( u^* + v^* \right) (x) = \frac{1}{2} \left( u - v \right) (x) \in \text{Ran}(\Phi) \cap \text{Ran}(\Psi).
\]

This is the wanted isomorphism, and hence the dimensions coincide. \hfill \Box

**Lemma 5.21.** Under the stereographic projection \( \Pi_I : \mathbb{L}(\mathfrak{H} \oplus \mathfrak{H}, I) \to \mathbb{U}(\mathfrak{H}) \), one has
\[
\Pi_I(\hat{\mathbb{L}}_\mathfrak{H}^m) = \left\{ u \in \mathbb{U}(\mathfrak{H}) \mid -1 \text{ is eigenvalue of } u \text{ with multiplicity } m \right\}.
\]

**Proof.** We use ideas from [Schu07]. Again by Corollary 1.14, each \( I \)-Langragian frame can be written as \( \Phi = \frac{1}{2}\left( u^* + v^* \right) \) for a unitary \( u \in \mathbb{B}(\mathfrak{H}) \), and then we have \( \Pi_I([\Phi]) = u \). By Lemma 5.20 for each \( I \)-Lagrangian frame \( \Phi \), one has \( [\Phi] \in \hat{\mathbb{L}}_\mathfrak{H}^m \) if and only if \( \dim \text{Ker}(u + \mathbf{1}) = m \) where \( \Pi_I([\Phi]) = u \). This shows the equality since the stereographic projection is bijective. \hfill \Box

Recall that for the Jacobi matrix \( H_N \), we define \( \Phi_N(\lambda) = T_N(\lambda) \cdots T_1(\lambda) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) and the \( I \)-Lagrangian frame \( \Psi_N(\lambda) = \Phi_N(\lambda) |\Phi_N(\lambda)|^{-1} \), cf. Proposition 5.10.

**Proposition 5.22.** Assume Hypothesis 5.6 for \( A \subset \mathbb{B}(\mathfrak{H}) \). We define
\[
\gamma(\lambda) := \dim \text{Ker}(\Psi_N(\lambda)^* I\widehat{\Psi})
\]
with \( \widehat{\Psi} := \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \) and \( \Psi_N(\lambda) \) from above. Then \( \lambda \in \mathbb{R} \) is an eigenvalue of \( H_N \) if and only if \( \gamma(\lambda) \neq 0 \). In this case, \( \gamma(\lambda) \) is the multiplicity of the eigenvalue \( \lambda \).

**Proof.** We have already shown in Proposition 5.4 that \( \lambda \) is an eigenvalue of \( H_N \) of multiplicity \( m \) if and only if \( \dim \text{Ran}(\Psi_N(\lambda)) \cap \text{Ran}(\widehat{\Psi}) \) has dimension \( m \). This last condition is equivalent to \( [\Psi_N(\lambda)] \in \hat{\mathbb{L}}_\mathfrak{H}^m \) and by Lemma 5.20 this implies \( \dim \text{Ker}(\Psi_N(\lambda)^* I\widehat{\Psi}) = m \). \hfill \Box

**Theorem 5.23.** Assume Hypothesis 5.6 for \( A \subset \mathbb{B}(\mathfrak{H}) \). Then \( \lambda \in \mathbb{R} \) is an eigenvalue of \( H_N \) of multiplicity \( m \) if and only if \( -1 \) is an eigenvalue of \( u_N(\lambda) \) of multiplicity \( m \).
Proof. By Proposition 5.22, $\lambda \in \mathbb{R}$ is an eigenvalue of $H_N$ with multiplicity $m$ if and only if $\gamma(\lambda) = m$, which means $[\Psi_N(\lambda)] \in \mathbb{D}_{\beta}^m$. By Lemma 5.21, this is equivalent to the unitary $u_N(\lambda) = \Pi_j([\Psi_N(\lambda)])$ having $-1$ as an eigenvalue of multiplicity $m$. \hfill $\Box$

5.4 Oscillation theory in finite dimensions

In this section, we will show and generalise the matrix oscillation theory presented in [Schu07]. If the C*-algebra is given by the matrices $A = B(\mathbb{C}^L)$ for $L \in \mathbb{N}$, then the spectrum of the Jacobi matrix and the spectrum of the Prüfer phase $u_N$ consist only of eigenvalues. These eigenvalues rotate on the unit circle, and the flow through $-1$ can be counted via spectral flow theory.

We define

$$\Theta_N(\lambda) := \frac{1}{i N} \int_{-\infty}^{\lambda} \frac{1}{L} \text{Tr}_L [(u_N(t))^* \partial_t u_N(t)] \, dt = \int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt \quad (5.33)$$

and call $\Theta_N$ the total rotation number. Of course, the normalised trace $\mathcal{T}$ in the C*-algebra is here given by the usual trace $\text{Tr}_L$ for matrices, meaning $\mathcal{T} = \frac{1}{L} \text{Tr}_L$. Here, we will deduce interesting facts about the rotation number and later see that

$$\Theta_N(\lambda) \xrightarrow{\lambda \to -\infty} 0, \quad \Theta_N(\lambda) \xrightarrow{\lambda \to \infty} 2\pi$$

hold, cf. Proposition 5.31 below. For the proof of these asymptotics, we will need some results coming from analytic perturbation theory of matrices.

Proposition 5.24. Let $\mathcal{H}$ be a finite dimensional Hilbert space with $L = \dim \mathcal{H}$ and $\mathbb{R} \ni \lambda \mapsto u(\lambda)$ be a real analytic path in the unitaries $U(\mathcal{H})$. Then there are real analytic maps

$$\theta_k : \mathbb{R} \to \mathbb{R}, \quad x_k : \mathbb{R} \to \mathcal{H}, \quad \text{for } k = 1, \ldots, L$$

with $u(\lambda)x_k(\lambda) = e^{i\theta_k(\lambda)}x_k(\lambda)$ and $\langle x_k(\lambda), x_j(\lambda) \rangle = \delta_{kj}$ for all $\lambda$.

Proof. See, for example, [Kato, Theorem II.1.10] or [Rel69, Chapter II.2]. \hfill $\Box$
This means here with $\mathcal{H} = C^L$ and by Theorem 5.5 that we have the $L$ eigenvalues $e^{i\theta_k(\lambda)}$ of $u_N(\lambda)$ and that the corresponding eigenphases $\theta_k(\lambda)$ can be chosen to be real analytic and even to fulfil:

$$\theta_k(\lambda) \xrightarrow{\lambda \to -\infty} 0.$$  

The last part follows from the asymptotic $u_N(\lambda) \to 1$ for $\lambda \to -\infty$. In the one-dimensional case, $u_N$ is exactly given by the only eigenvalue $e^{i\theta_1(\lambda)}$, and we already know by Theorem 5.5 and equation (5.12) that this eigenvalue rotates in the positive sense around zero. We will now prove the same thing separately for the eigenvalues of $u_N$ in the finite dimensional case.

**Theorem 5.25.** Assume Hypothesis 5.6 for $A = B(\mathbb{C}^L)$ with $L \in \mathbb{N}$, and let $u_N$ be the Prüfer phase.

(a) There are real analytic functions $\theta_k : \mathbb{R} \to \mathbb{R}$ for $k = 1, \ldots, L$ such that

$$\left\{ e^{i\theta_k(\lambda)} \mid k = 1, \ldots, L \right\} = \text{spec}(u_N(\lambda))$$

for all $\lambda$ with $\theta_k(\lambda) \to 0$ for $\lambda \to -\infty$.

(b) For these eigenphases, one has $\theta_k(\lambda) \to 2\pi N$ for $\lambda \to \infty$.

(c) For all $k = 1, \ldots, L$, the map $\theta_k : \mathbb{R} \to \mathbb{R}$ is an increasing function.

(d) A real number $\lambda$ is an eigenvalue of $H_N$ of multiplicity $m$ if and only if there are exactly $m$ indices $k$ with $\theta_k(\lambda) = \pi \mod 2\pi$.

(e) The total rotation number $\Theta_N(\lambda) := \int_{-\infty}^{\lambda} \frac{1}{L} \text{Tr}(s_N(t)) dt$ has the limit $\Theta_N(\infty) = 2\pi$ and fulfills

$$\left| N_{N,L}(\lambda) - \frac{1}{2\pi} \Theta_N(\lambda) \right| \leq \frac{1}{2N},$$

where $N_{N,L}(\lambda) := \frac{1}{NL} \text{Tr}(\chi(-\infty,\lambda](H_N))$ is the integrated density of states for the matrix $H_N$.

**Proof.** See the next section below. Part (a) is proven in Lemma 5.26. Part (b) is given in Proposition 5.30. Part (c) is to be found in Proposition 5.27. Part (d)
follows from part (a) in combination with Theorem 5.23. Finally, part (e) is
given in Proposition 5.31 and Remark 5.34. □

5.4.1 On the eigenphases of the Prüfer phase

Lemma 5.26. Assume Hypothesis 5.6 for $A = B(\mathbb{C}^L)$ with $L \in \mathbb{N}$. There are
real analytic functions $\theta_k : \mathbb{R} \to \mathbb{R}$ for $k = 1, \ldots, L$ such that
\[
\left\{ e^{i\theta_k(\lambda)} \bigg| k = 1, \ldots, L \right\} = \text{spec}(u_N(\lambda))
\]
for all $\lambda$ with $\theta_k(\lambda) \to 0$ for $\lambda \to -\infty$.

Proof. By Proposition 5.24, there are real analytic functions $\theta_k : \mathbb{R} \to \mathbb{R}$ with
$u(\lambda)x_k(\lambda) = e^{i\theta_k(\lambda)}x_k(\lambda)$ for some eigenvectors $x_k(\lambda)$. By Theorem 5.5 part (b),
we can look at the limit $\lambda \to -\infty$ and conclude that the limit $\theta_k(-\infty)$ has to
be an integer multiple of $2\pi$. Without loss of generality, we can choose this
multiple to be zero. □

Proposition 5.27. Assume Hypothesis 5.6 for $A = B(\mathbb{C}^L)$ with $L \in \mathbb{N}$. Then
\[
\text{Tr}_L\left((u_N(\lambda))^* \partial \lambda u_N(\lambda)\right) = i \sum_{k=1}^{L} \theta'_k(\lambda),
\]
and $\theta'_k(\lambda) > 0$ for all $\lambda \in \mathbb{R}$ and $k = 1, \ldots, L$.

Proof. By the perturbation result of Proposition 5.24, we know that $u_N(\lambda)$
can be diagonalised with a unitary $v(\lambda)$ such that $\lambda \mapsto v(\lambda)^*u_N(\lambda)v(\lambda)$ is also
real-analytic. Therefore we calculate while dropping the argument and the
index $N$:
\[
\text{Tr}_L\left((v^*uv)^* \partial \lambda (v^*uv)\right) = \text{Tr}_L\left(v^*u^*v(\partial \lambda v^*)uv + v^*u^*(\partial \lambda u)v + v^*(\partial \lambda v)\right)
\]
\[
= \text{Tr}_L\left(v(\partial \lambda v^*)\right) + \text{Tr}_L\left(u^*(\partial \lambda u)\right) + \text{Tr}_L\left(v^*(\partial \lambda v)\right)
\]
\[
= \text{Tr}_L\left(\partial \lambda (v^*v)\right) + \text{Tr}_L\left(u^*(\partial \lambda u)\right)
\]
Here we used the properties of the trace and can now conclude with $v^*v = 1$
that
\[ \text{Tr}_L \left( u^* (\partial_\lambda u) \right) = \text{Tr}_L \left( (v^* uv)^* \partial_\lambda (v^* uv) \right) = i \sum_{k=1}^L \theta_k' (\lambda), \]

where \( \theta_k \) are the eigenphases. Knowing that \( s_N \) is a positive operator, we conclude that the sum on the right-hand side is positive.

For showing the second part, we use the same diagonalisation as before and choose \( e_k \) as the \( k \)th vector of standard basis in \( \mathbb{C}^L \). Then we get:

\[
\begin{align*}
    i \theta_k (\lambda) &= \langle e_k, (v^* uv)^* \partial_\lambda (v^* uv) e_k \rangle \\
    &= \langle e_k, (v^* u v (\partial_\lambda v^*) uv + v^* u (\partial_\lambda u) v v^* (\partial_\lambda v^* e_k) e_k \rangle \\
    &= \langle u v e_k, v (\partial_\lambda v^*) u v e_k \rangle + \langle v e_k, u^* (\partial_\lambda u) v v^* v e_k \rangle + \langle v e_k, (\partial_\lambda v^* v e_k) \rangle.
\end{align*}
\]

Using that \( v e_k \) is an eigenvector of \( u \), simplifies the first summand, and using the strict positivity of \( \frac{1}{i} u^* (\partial_\lambda u) \), we get:

\[
\theta_k (\lambda) > \frac{1}{i} \langle v e_k, [v (\partial_\lambda v^* + (\partial_\lambda v^* v^* v e_k) \rangle = 0.
\]

The last equation holds since the operator inside the brackets is self-adjoint, and so the right-hand side is purely imaginary. That is only possible if the whole term vanishes.

\[ \theta_k (\lambda) \xrightarrow{\lambda \to -\infty} 0, \quad \theta_k (\lambda) \xrightarrow{\lambda \to \infty} 2\pi N_k \]

with \( N_k \in \mathbb{N} \). In the one-dimensional case, \( L = 1 \), one has \( N_1 = N_L = N \).

**Proof.** The first limit is just given by definition and the second one follows from Proposition 5.13 since \( u_N \) converges to 1 and the limit of \( \theta_k \) has to be a multiple of \( 2\pi \). It remains only to show the one-dimensional case. In this case, \( \mathcal{A} = \mathbb{C} \), we have as already mentioned

\[
u_N (\lambda) = \frac{a_{N+1} x_{N+1}(\lambda) - i x_N (\lambda)}{a_{N+1} x_{N+1}(\lambda) + i x_N (\lambda)} = e^{i \theta_1 (\lambda)} \]
with the polynomials $x_{N+1}$ and $x_N$. The zeros of $x_{N+1}$ coincide with the simple eigenvalues of $H_N$ by Proposition 5.2 and Proposition 5.3. Therefore each eigenvalue $\tilde{\lambda}$ of $H_N$ is given by $u_N(\tilde{\lambda}) = -1$ or, in other words, we have $\theta_1(\tilde{\lambda}) = \pi \mod 2\pi$. Since $\theta_1$ is increasing, we reach this value exactly $N$ times, and this shows the limit. \hfill \Box

**Lemma 5.29.** For $k = 1, \ldots, L$, the function $\gamma_k : [-1, 1] \to \mathbb{C}$, given by

\[
t \mapsto \exp[i \theta_k(\tan(\frac{\pi}{2} t))], \quad \gamma_k(\pm 1) = 1,
\]

is a continuous closed path in the complex plane. It has a well-defined winding number around zero, denoted by $\nu(\theta_k)$. For all $k$, one has $\nu(\theta_k) = N_k$ with $N_k$ given by Lemma 5.28 above.

**Proof.** Obviously, the path is continuous and closed since $\theta_k(\lambda)$ is continuous and converges for $|\lambda| \to \infty$ to an integer multiple of $2\pi$. The winding number can be calculated by

\[
\nu(\theta_k) := \frac{1}{2\pi i} \int_{-1}^{1} \frac{\gamma_k'(t)}{\gamma_k(t)} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta_k'(x) dx = N_k,
\]

where $N_k$ is chosen like in Lemma 5.28. \hfill \Box

**Proposition 5.30.** For all $k = 1, \ldots, N$, one has $\nu(\theta_k) = N$ and, in particular,

\[
\theta_k(\lambda) \xrightarrow{\lambda \to \infty} 2\pi N.
\]

**Proof.** We know that $u_N$ can be locally represented by a power series where the coefficients are generated by $\{a_n^*, v_n, a_n^{-1}\}$. Now we can use the homotopy

\[
v_n(\lambda, t) = (1-t)v_n(\lambda), \quad t \in [0, 1].
\]

and a real analytic homotopy map $t \mapsto a_n(\lambda, t)$ with $a_n(\lambda, 0) = a_n(\lambda)$ and $a_n(\lambda, 1) = 1$. This gives rise to a homotopy $u_N(\lambda, t)$ where $t \mapsto u_N(\lambda, t)$ is real analytic as well. Accordingly, we get a real analytic homotopy for the eigenphases $t \mapsto \theta_k(\lambda, t)$. It is well known that the winding number of paths in the complex plane is homotopy-invariant. Therefore, for all $k$, we have
\[ \nu(\theta_k(\cdot, 0)) = \nu(\theta_k(\cdot, 1)). \]
Hence, we just have to calculate the winding numbers for the Schrödinger operator
\[
H_N(1) = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
& 1 & 0 \\
& & & \ddots \\
& & & & 0 & 1 \\
& & & & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}.
\]
However, this operator is unitarily equivalent to the sum of \( L \) one-dimensional operators \( \tilde{H}_N \oplus \cdots \oplus \tilde{H}_N \). By Lemma 5.28, we can conclude \( \nu(\theta_k) = N \) for all \( k = 1, \ldots, N \).

**Proposition 5.31.** For all \( \lambda \in \mathbb{R} \) the total rotation number \( \Theta_N(\lambda) \), given by
\[
\Theta_N(\lambda) := \int_{-\infty}^{\lambda} \frac{1}{L} \text{Tr}_L(s_N(t)) \, dt = \frac{1}{NL} \sum_{k=1}^{L} \theta_k(\lambda),
\]
is well-defined, and we have
\[
\Theta_N(\lambda) \xrightarrow{\lambda \to -\infty} 0, \quad \Theta_N(\lambda) \xrightarrow{\lambda \to \infty} 2\pi.
\]

**Proof.** That \( \Theta_N(\lambda) \) exists for all \( \lambda \) and has finite limits follows from Theorem 5.5 (d). Moreover by Proposition 5.27, the trace is given by the sum of the derivatives of \( \theta_k \), and accordingly, we get the following. Here, as often, we suppress the argument of the function \( u_N \).
\[
\Theta_N(\lambda) = \frac{1}{iNL} \int_{-\infty}^{\lambda} \text{Tr}_L[(u_N)^* \partial_t u_N] \, dt
= \frac{1}{NL} \sum_{k=1}^{L} \int_{-\infty}^{\lambda} \theta_k'(t) \, dt = \frac{1}{NL} \sum_{k=1}^{L} \theta_k(\lambda).
\]
Using the limits from the Lemmas above, we find \( \lim_{\lambda \to \infty} \Theta_N(\lambda) = 2\pi \).
5.4.2 Counting function and rotation number

Here, we will explicitly use the finite dimensionality for describing the whole spectrum of $H_N$ with the intersection theory described in Section 5.3.7. There, we have found the correspondence of eigenvalues of $H_N$ with the points $\lambda$ where $u_N(\lambda)$ has eigenvalues in $-1$, cf. Theorem 5.23.

**Proposition 5.32.** Assume Hypothesis 5.6 for $A = \mathbb{C}$, meaning $L = 1$. Then the number of eigenvalues of $H_N$ below a value $\lambda$ is given by the number of intersection of $(-\infty, \lambda] \ni t \mapsto u_N(t)$ with $-1$. Moreover, we have the estimate:

$$\left| \#\{\mu \in \mathbb{R} \mid \mu \leq \lambda \text{ eigenvalue of } H_N\} - \frac{N}{2\pi} \int_{-\infty}^{\lambda} s_N(t) \, dt \right| \leq \frac{1}{2}. \quad (5.34)$$

![Visualisation of (5.34) for $N = 6$.](image)

*Figure 5.3: Visualisation of (5.34) for $N = 6$.*

**Proof.** By Proposition 5.27, we know $N \int_{-\infty}^{\lambda} s_N(t) \, dt = \theta_1(\lambda)$, where $\theta_1(\lambda)$ is the only eigenphase of the complex number $u_N(\lambda)$. Now we denote the eigenvalues of $H_N$ by $\lambda_1 < \lambda_2 < \ldots < \lambda_{NL}$. From Theorem 5.25, we conclude that the smallest eigenvalue $\lambda_1$ of $H_N$ satisfies $\theta_1(\lambda_1) = \pi$. Moreover, since $\theta_1$
is an increasing function, we have

\[ \theta_1(\lambda) \in [\pi + 2\pi(j - 1), \pi + 2\pi j] \quad \text{for} \quad \lambda \in [\lambda_j, \lambda_{j+1}] \]

by part (d) of Theorem 5.25. This means that we have \( |j - \frac{1}{2\pi} \theta_1(\lambda)| \leq \frac{1}{2} \) for all \( \lambda \in [\lambda_j, \lambda_{j+1}] \) and all \( j \). Using in addition the limit values of \( \theta_1 \) concludes the proof.

**Proposition 5.33.** Assume Hypothesis 5.6 for \( A = B(\mathbb{C}^L) \). Then the number of eigenvalues of \( H_N \) below a value \( \lambda \) is exactly the spectral flow of the path \( (-\infty, \lambda] \ni t \mapsto u_N(t) \) through \(-1\). Moreover, we have the estimate:

\[
\left| \# \{ \mu \in \mathbb{R} \mid \mu \leq \lambda \text{ eigenvalue of } H_N \} - \frac{NL}{2\pi} \int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt \right| \leq \frac{L}{2}. \quad (5.35)
\]

Here, \( \mathcal{T} = \frac{1}{L} \text{Tr}_L \) is the usual normalised trace and the eigenvalues are counted with multiplicities.

![Visualisation of (5.35) for \( N = 3 \) and \( L = 2 \).]
Proof. The estimate can be proven analogously to the one-dimensional case. Therefore, we split the counting function for $H_N$ into

$$Z_k(\lambda) := \#\{\mu \leq \lambda \mid \theta_k(\mu) = -1\}, \text{ for } k = 1, \ldots, L.$$ 

By part (d) of Theorem 5.25, we know that $\sum_k Z_k(\lambda)$ is the whole counting function for $H_N$. Using the same argument as in the proof of Proposition 5.32 above, we conclude

$$\left| Z_k(\lambda) - \frac{1}{2\pi} \theta_k(\lambda) \right| \leq \frac{1}{2},$$

for all $k = 1, \ldots, L$. Now, we just use the triangle inequality and get:

$$\left| \sum_{k=1}^{L} Z_k(\lambda) - \frac{1}{2\pi} \sum_{k=1}^{L} \theta_k(\lambda) \right| \leq \sum_{k=1}^{L} \left| Z_k(\lambda) - \frac{1}{2\pi} \theta_k(\lambda) \right| \leq \frac{L}{2}.$$

On the left-hand side, the first sum is the full counting function, and the second sum is given by the total rotation number, see Proposition 5.31. Therefore, we immediately obtain $\left| \#\{\mu \leq \lambda \text{ eigenvalue of } H_N\} - \frac{NL}{2}\Theta_N(\lambda) \right| \leq \frac{L}{2}$. \hfill \qed

Remark 5.34. By introducing the integrated density of states for $H_N$, see Definition 1.41, where we write $N_{N,L} := N_{H_N} = \frac{1}{NL} \text{Tr}(\chi(-\infty, \lambda](H_N))$ for given $H_N \in \mathbb{C}^{N \times N} \otimes \mathbb{C}^{L \times L}$, the formula (5.35) reads:

$$\left| N_{N,L}(\lambda) - \frac{1}{2\pi} \Theta_N(\lambda) \right| \leq \frac{1}{2N}.$$

Remark 5.35. The bound in (5.35) is optimal. Consider for $N = 2$ and $A = \mathbb{C}^{L \times L}$ the Jacobi matrix

$$H_N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It has spectrum $\text{spec}(H_N) = \{-1, 1\}$ with multiplicity $L$ for each eigenvalue. A straightforward calculation shows for $\lambda < -1$:

$$\int_{-\infty}^{\lambda} T(s_N(t)) \, dt = \arctan \left( \lambda \frac{\lambda}{1 - \lambda^2} \right) \xrightarrow{\lambda \to -1^-} \frac{\pi}{2}.$$
Therefore, we can look at the counting function for $\lambda < -1$ and get

\[
\left| \#\{\mu \leq \lambda \text{ eigenvalue of } H_N\} - \frac{NL}{2\pi} \int_{-\infty}^{\lambda} T(s_N(t)) \, dt \right| \xrightarrow{\lambda \to -1^-} \frac{L}{2}.
\]

## 5.5 Oscillation theory for high dimensional random Jacobi operators

While in last section, we have considered the C*-algebra $\mathcal{A} = B(\mathbb{C}^L)$ in combination with the usual trace for matrices to count the eigenvalues, we now expand these results to the C*-algebra of covariant families of local operators $\mathcal{A}_{\text{cov},c}$ defined in Section 2.2, together with the finite trace $T$ given by an invariant and ergodic probability measure $\mathbf{P}$. For more details, see also Section 1.3 and Proposition 2.39. The finite trace $T$ is needed to define a total rotation number $\Theta_N$ as in (5.33).

In this case, it seems that there are not any known results connecting the Prüfer phase to spectral properties of the initial operator, except for [ASV13] where a Fredholm condition allowed to focus on a single discrete eigenvalue of the Prüfer phase. In general, the spectrum of the unitary Prüfer phase can be essential and spread out over the whole unit circle, so that, on first sight, it seems of little use for oscillation theory. However, we have shown in Theorem 5.5 that the whole spectrum of the Prüfer phase moves around the unit circle in the same direction, and therefore, under certain conditions on the entries of the Jacobi matrix, one can define an associated spectral flow in a generalised sense like in [Ben+06]. Our framework of covariant families of operators allows to study higher-dimensional discrete random Schrödinger operators and calculate their integrated density of states (IDOS) via the generalised spectral flow of the Prüfer phases. This is an essential novel extension of oscillation theory and based on the recent publication [GSV17].

### 5.5.1 Overview

Let us now describe the framework and results in a more detailed manner and provide some technical insights to its proof. We will consider the Jacobi matrix
$H_N$ of equation (5.3) and will be interested in the situation where the entries $v_n$ and $a_n$ are elements of the C*-algebra $A_{\text{cov,c}} \cong C(\Omega) \times \mathbb{Z}^d$, which means they are covariant and local operator families on $\ell^2(\mathbb{Z}^d)$ to a dynamical system $\mathbb{Z}^d \times \Omega \to \Omega$. Here, $\Omega$ is a compact space (of disorder configurations) equipped with commuting homeomorphisms $T_1, \ldots, T_d$, see Section 1.3 and Section 2.2.

Other sources are, e.g. [Bel86] and [PF92]. However, we recapitulate here the important facts: Such a covariant family $a = (A_\omega)_{\omega \in \Omega}$ consists of local operators $A_\omega$ on $\ell^2(\mathbb{Z}^d)$ that are strongly continuous in $\omega \in \Omega$. The locality property means that the matrix elements $\langle e_n, A_\omega e_m \rangle$ decay sufficiently fast with the distance $\|n - m\|_\infty$ between points $n, m \in \mathbb{Z}^d$. Moreover, the covariance property with respect to the right shifts $S_1, \ldots, S_d$ on $\ell^2(\mathbb{Z}^d)$ holds:

$$S_j A_\omega S_j^* = A_{T_j(\omega)}, \quad j = 1, \ldots, d.$$  \hspace{1cm} (5.36)

If $P$ is an invariant and ergodic probability measure on $\Omega$, one obtains a normalised and finite trace on $\mathcal{A}$ by setting

$$\mathcal{T}(a) = E \langle e_0, A_\omega e_0 \rangle = \int \langle e_0, A_\omega e_0 \rangle dP(\omega), \quad a = (A_\omega)_{\omega \in \Omega},$$

where $E$ denotes just the average with respect to $P$. For sake of concreteness, we will stick to the framework just described, but note that one can consider the covariance relation with respect to magnetic translations, but then one has to work with a twisted crossed product algebra $\mathcal{A}$, and can choose the space $\Omega$ as well as the probability $P$ to be dependent on $n$ (at least in the first part of Theorem 5.37 below). We reformulate the general Hypothesis 5.6 for this case:

**Hypothesis 5.36.** Let $\mathcal{A} = A_{\text{cov,c}}$ be the unital C*-algebra of covariant families of local operators on $\ell^2(\mathbb{Z}^d)$ with respect to a compact space $\Omega$ and commuting homeomorphisms $T_1, \ldots, T_d$. Moreover, let $P$ be an invariant and ergodic Borel probability measure on $\Omega$ and $\mathcal{T}$ the corresponding trace on $\mathcal{A}$. For all $n \in \mathbb{N}$ let $a_n, v_n \in \mathcal{A}$ with $a_n$ invertible and $v_n$ self-adjoint. We also set $a_1 = 1$. For all $N \in \mathbb{N}$ let $H_N$ denote the self-adjoint element in $M_N(\mathcal{A}) = \mathbb{C}^{N \times N} \otimes \mathcal{A}$ given by (5.3). In addition, we suppose $\xi = \sup_{n \geq 1} \{\|a_n\|, \|a_n^{-1}\|, \|v_n\|\} < \infty$ and that there are finite volume restrictions $a_{n,L}$ and $v_{n,L}$ for all $n$ and $L \in \mathbb{N}$. 
such that there is an $L_0$ with

$$
\xi_L := \sup_{n \geq 1} \left\{ \|a_{n,L}\|, \|a_{n,L}^{-1}\|, \|v_{n,L}\| \right\} \leq 2 \xi < \infty
$$

for all $L \geq L_0$.

How to choose the finite volume restrictions such that they satisfy this norm estimate will be explained in more detail in Section 5.5.2 below.

Resuming, $H_N$ is a discrete Schrödinger operator on the strip Hilbert space $\ell^2(\{1, \ldots, N\} \times \mathbb{Z}^d)$ with matrix elements of range 1 in the finite direction $\{1, \ldots, N\}$ and a covariance relation in the fiber $\mathbb{Z}^d$. A typical example where all the above holds is the standard Anderson model on the strip $\{1, \ldots, N\} \times \mathbb{Z}^d$. There, $H_N$ is of the form (5.3) with $a_n = 1$ the identity on $\ell^2(\mathbb{Z}^d)$ and

$$
v_n = \sum_{j=1}^{d} (S_j + S_j^*) + \sum_{m \in \mathbb{Z}^d} \bar{v}_{n,m} e_m \langle e_m, \cdot \rangle : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d),
$$

where $(\bar{v}_{n,m})_{n \in \mathbb{N}, m \in \mathbb{Z}^d}$ are i.i.d. real random variables distributed by a compactly supported probability measure, cf. Example 1.27.

As a matrix-valued covariant operator family, $H_N$ has a well-defined IDOS given by

$$
\mathcal{N}_N(\lambda) = \frac{1}{N} \text{Tr}_N \otimes \mathcal{T}(\chi_{(-\infty,\lambda]}(H_N)),
$$

where $\chi$ is the characteristic function and $\text{Tr}_N$ the standard trace on $\mathbb{C}^{N \times N}$. Note that $\mathcal{N}_N$ is right-continuous and increasing with limits $\mathcal{N}_N(-\infty) = 0$ and $\mathcal{N}_N(\infty) = 1$, hence it specifies a probability measure called the density of states (DOS) of the covariant operator family $H_N$ on the strip Hilbert space $\ell^2(\{1, \ldots, N\} \times \mathbb{Z}^d)$. In the limit $N \to \infty$, one obtains the IDOS of the $(d+1)$-dimensional covariant Hamiltonian on $\ell^2(\mathbb{Z}^{d+1})$:

$$
\mathcal{N}(\lambda) = \lim_{N \to \infty} \mathcal{N}_N(\lambda).
$$

The main aim of this chapter is to calculate the integrated density of states $\mathcal{N}_N$ by means of Sturm-Liouville type oscillation theory, namely to approximate it by an average rotation number. We can use all general results from Section 5.3
and can moreover show the following:

\[ \int_{-\infty}^{\infty} \mathcal{T}(s_N(\lambda)) \, d\lambda = 2\pi. \]

This generalises the result for the total rotation number from the finite dimensional case. There the Prüfer phase is a finite dimensional unitary matrix and with eigenvalues all rotating in the positive sense on the unit circle, and as we have seen each time that one passes by \(-1\) there is an eigenvalue of \(H_N\). In the recent situation of infinite dimensional fibers, however, the spectrum of \(u_N\) is mostly essential and in interesting examples actually may fill at some points the whole unit circle. We do not attempt to define intersection theory in this situation. On the other hand, it does make sense to consider the spectral flow around the unit circle with respect to the trace \(\mathcal{T}\). That this spectral flow indeed approximates the IDOS of \(H_N\) is the one of the main result of this dissertation. It is proven in Section 5.5.3 below.

**Theorem 5.37.** Assume Hypothesis 5.36. Then there exists a constant \(K\), only depending on \(\xi\) not on \(\{a_n, v_n\}_{n \in \mathbb{N}}\), such that for all \(\lambda \in \mathbb{R}\) and all \(N \geq 1\):

\[ \left| N_N(\lambda) - \frac{1}{2\pi} \int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt \right| \leq \frac{K}{N}. \]  

(5.37)

In particular, the \((d+1)\)-dimensional IDOS satisfies

\[ N(\lambda) = \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt. \]

The one-dimensional case coincides in this framework with \(d = 0\) since \(\mathbb{Z}^0\) consists only of one point and the covariance relation (5.36) is then empty. In this case, we have \(\int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt\) as the rotation number and Theorem 5.37 is a well-known result, cf. [PF92] and [AHP05]. For general \(d\), it is the integrated phase velocity density with respect to \(\mathcal{T}\), and we also refer to it as a rotation number. It is a type II_1 spectral flow in the finite von Neumann algebra \(L^\infty(\mathcal{A}, \mathcal{T})\) as defined in [Ben+06, Section 5.1], see also Definition 2.35 above. The core of the proof of Theorem 5.37 is the result about the finite
dimensional case, see Proposition 5.33 and Remark 5.34, which states that the estimate (5.37) holds for Jacobi matrices with finite dimensional matrix entries, \textit{independently} of the size of these matrix entries. This allows to reduce the proof of Theorem 5.37 to a control of finite size approximations of the IDOS and the rotation number. While this is standard for the IDOS, cf. [PF92], the approximation of the rotation number requires the precise high energy estimates for the Prüfer phases and its derivatives that we stated in Section 5.3.

\subsection{5.5.2 Finite volume approximations}

The strategy for the proof of Theorem 5.37 is to control the finite volume approximations of both $N_N$ and $s_N$ in the vertical, infinite direction. Such approximations follow from standard ergodic properties of the covariant operators in $\mathcal{A}$. On the other hand we can use Sturm-Liouville-type oscillation theory for the finite dimensional from Section 5.4.

We already defined the finite volume approximants with Dirichlet conditions in Section 1.3 by using the surjective partial isometry

$$\widetilde{\pi}_L : \ell^2(\mathbb{Z}^d) \to \ell^2(\{-L, \ldots, L\}^d) \cong \mathbb{C}^{(2L+1)^d}$$

be the surjective partial isometry given by the restriction of the sequences. Then the restrictions of $a_n$ and $v_n$ with Dirichlet boundary condition are $\widetilde{\pi}_L a_n \widetilde{\pi}_L^*$ and $\widetilde{\pi}_L v_n \widetilde{\pi}_L^*$ and this of course to be understood pointwise in the family $a_n$ and $v_n$. Up to boundary terms localised close to the boundary of the cube $\Lambda_L := \{-L, \ldots, L\}^d \subset \mathbb{Z}^d$, these restrictions are the approximants $a_{n,L}$ and $v_{n,L}$ to be used in the following. We will suppose that the boundary terms can be chosen such that for some $L_0$

$$\xi_L := \sup_{n \geq 1} \{ \|a_{n,L}\|, \|a_{n,L}^{-1}\|, \|v_{n,L}\| \} \leq 2\xi < \infty, \quad \text{for all } L \geq L_0. \quad (5.38)$$

This excludes topological systems with protected boundary states which can possibly destroy the invertibility of $a_{n,L}$, but is satisfied for the restrictions in the standard Anderson model discussed in the introduction. Once the approximants $a_{n,L}$ and $v_{n,L}$ are chosen such that (5.38) holds, one can define
the finite volume approximation of the Hamiltonian:

\[ H_{N,L} = \begin{pmatrix} v_{1,L} & a_{2,L} \\ a_{2,L}^* & v_{2,L} & a_{3,L} \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \\ & & & v_{N-1,L} & a_{N,L} \\ a_{N,L}^* & v_{N,L} \end{pmatrix}. \]

It is family of self-adjoint matrices of size \( N(2L + 1)^d \), where we will write \( H_{N,L} = (H_{N,L}^\omega)_{\omega \in \Omega} \). As such, each matrix \( H_{N,L}^\omega \) in this family has a normalised eigenvalue counting function, cf. Section 1.3 and Definition 1.41:

\[ N_{N,L}^\omega(\lambda) := N_{H_{N,L}^\omega}(\lambda) = \frac{1}{N(2L + 1)^d} \text{Tr}(\chi(H_{N,L}^\omega \leq \lambda)). \]

Here, we have emphasised the dependence on \( \omega \). The function \( \lambda \in \mathbb{R} \mapsto N_{N,L}^\omega(\lambda) \) is the integrated density of states (IDOS) for the matrix \( H_{N,L}^\omega \) and therefore increasing with asymptotics \( N_{N,L}^\omega(-\infty) = 0 \) as well as \( N_{N,L}^\omega(\infty) = 1 \). For fixed \( N \) we can use Theorem 1.40 or rather Remark 1.42. Note that these results hold for the covariant local algebra \( A_{\text{cov},c} \) as the completion of the finite range operators, see also [BES94] and [PF92].

**Proposition 5.38.** Assume Hypothesis 5.36. For any fixed \( N \in \mathbb{N} \) there is a Borel set \( \hat{\Omega} \) with \( P(\hat{\Omega}) = 1 \) such that for all \( \lambda \in \mathbb{R} \) and all \( \omega \in \hat{\Omega} \) the following holds:

\[ \lim_{L \to \infty} N_{N,L}^\omega(\lambda) = N_N(\lambda). \]

The convergence also holds in expectation.

**Proof.** With analogues proofs as in Section 1.3, cf. Theorem 1.35 and Theorem 1.40, one gets an analogues result as in Remark 1.42. \( \square \)

Uniform estimates in the energy parameter on the convergence of the integrated density of states have recently been obtained in [SSV17], but this will not be needed here. As \( N_N \) is one of the quantities in the main result, namely
Theorem 5.37, we will now calculate $\mathcal{N}_{N,L}^{\omega}$ by means of matrix-valued oscillation theory of Section 5.4. For this purpose, we use the following notations:

**Definition 5.39.** Let $L \in \mathbb{N}$. Assume Hypothesis 5.6 with $\mathcal{A}$ as the algebra consisting of families of matrices in $\mathcal{B}(\mathbb{C}^{2L+1}^{d})$ and choose the entries $\{a_{n,L}, v_{n,L}\}_{n \in \mathbb{N}}$. Then denote the associated transfer matrices by $T_{n,L}$, the associated Prüfer phase by $u_{N,L}$ and the associated speed operator by $s_{N,L}$.

Again note that this definition has to be understood for families of operators, for example $a_{n,L} = (a_{n,L}^{\omega})_{\omega \in \Omega}$, but we will often suppress the dependence of $\omega$. By Section 5.4, we get all bounds of Propositions 5.13, 5.16 and 5.18 uniformly in $L$ since the estimate (5.38) holds. All these propositions provide pointwise estimates not requiring any ergodicity properties. These latter are used again for the next result, which is similar to Proposition 5.38. As the techniques of proof are standard, see Section 1.3, we do not provide lengthy details.

**Proposition 5.40.** Assume Hypothesis 5.36 and denote by $s_{N,L} = (s_{N,L}^{\omega})_{\omega \in \Omega}$ the finite volume restriction of $s_{N}$. Then there is a Borel set $\hat{\Omega}$ with $P(\hat{\Omega}) = 1$ such that for all $\lambda \in \mathbb{R}$ and for all $\omega \in \hat{\Omega}$ the following holds:

$$
\lim_{L \to \infty} \frac{1}{(2L+1)^{d}} \text{Tr}(s_{N,L}^{\omega}(\lambda)) = T(s_{N}(\lambda)).
$$

The convergence also holds in expectation.

**Sketch of proof.** Since this can be shown analogously to Section 1.3, we just emphasise some steps. By construction, the matrix entries $a_{n,L}$ and $v_{n,L}$ of $T_{n,L}(\lambda)$ are finite volume restrictions of covariant operators $a_{n}$ and $v_{n}$, up to boundary terms. Since these operators are also local, the geometric resolvent identity shows that also $(a_{n,L})^{-1}$ is the finite volume restriction of $(a_{n})^{-1}$ up to boundary terms. It is compact, but not necessarily of finite range any more. As this property is inherited to products of such operators, also the matrix entries of $T_{N,L}(\lambda) \cdots T_{1,L}(\lambda)$ are finite volume restrictions of the matrix entries of $T_{N}(\lambda) \cdots T_{1}(\lambda)$, up to boundary terms. In conclusion, also $u_{N,L}(\lambda)$ is a finite volume restriction of $u_{N}(\lambda)$, up to boundary terms. As the Prüfer phases are rational functions in $\lambda$, also $s_{N,L}(\lambda)$ is a finite volume restriction of $s_{N}(\lambda)$ up
to boundary terms. Therefore Birkhoff’s theorem implies, as in the proof of Theorem 1.34, the stated almost sure convergence.

The next remark in the section is the already known main results of oscillation theory for matrix valued Jacobi matrices as developed in Section 5.4 and in [Schu07] and [Schu12]. The statement below is precisely the estimate given by Remark 5.34 but now formulated for the finite volume restrictions.

**Remark 5.41.** For the Jacobi matrix $H_{N,L}^\omega$ with matrix entries, the eigenvalue counting function $N_{N,L}^\omega$ is approximated by the speed matrix $s_{N,L}^\omega$ uniformly in the size $L$:

$$\left| N_{N,L}^\omega(\lambda) - \frac{1}{2\pi} \int_{-\infty}^{\lambda} \frac{1}{(2L+1)^d} \text{Tr}(s_{N,L}^\omega(\lambda)) \, d\lambda \right| \leq \frac{1}{2N}.$$  

**Proposition 5.42.** Assume Hypothesis 5.36 and denote by $s_N$ the velocity operator given by $s_N(\lambda) = \frac{1}{iN}(u_N(\lambda))^* \partial_\lambda u_N(\lambda)$. Then

$$\int_{-\infty}^{\infty} \mathcal{T}(s_N(t)) \, dt = 2\pi.$$  

**Proof.** For the finite volume restrictions $s_{N,L} = (s_{N,L}^\omega)_{\omega \in \Omega}$, which is a family of matrices of size $(2L+1)^d \times (2L+1)^d$, we have

$$\int_{-\infty}^{\infty} \frac{1}{(2L+1)^d} \text{Tr}(s_{N,L}^\omega(t)) \, dt = 2\pi$$

for all $\omega \in \Omega$ by Proposition 5.31. Then we can just use Proposition 5.40 when choosing $\omega_0 \in \hat{\Omega}$ and get

$$\lim_{L \to \infty} \frac{1}{(2L+1)^d} \text{Tr}(s_{N,L}^{\omega_0}(\lambda)) = \mathcal{T}(s_N(\lambda))$$

for all $\lambda \in \mathbb{R}$. Since $\lambda \mapsto \mathcal{T}(s_N(\lambda))$ is integrable by Theorem 5.5 (d), we conclude by Fatou’s Lemma $\int_{\mathbb{R}} \mathcal{T}(s_N(t)) \, dt \leq 2\pi$. For the other inequality, we use condition (5.38) in combination with Proposition 5.13 and Corollary 5.19.
Let \( \epsilon > 0 \) and choose \( \lambda_0 \) such that
\[
\int_{-\infty}^{-\lambda} \| s_{N,L}(t) \| \, dt + \int_{\lambda}^{\infty} \| s_{N,L}(t) \| \, dt < \epsilon
\]
for all \( L \in \mathbb{N} \) and \( \lambda \geq \lambda_0 \) by Corollary 5.19. By condition (5.38), we can use Lebesgue’s Dominated Convergence Theorem for the bounded interval \([-\lambda, \lambda]\), and we get
\[
2\pi = \lim_{L \to \infty} \int_{-\infty}^{\infty} \frac{1}{(2L + 1)^d} \text{Tr}(s_{N,L}^\omega(t)) \, dt \leq \epsilon + \int_{-\lambda}^{\lambda} \mathcal{T}(s_N) \, dt
\]
for all \( \lambda \geq \lambda_0 \). Therefore \( \int_{\mathbb{R}} \mathcal{T}(s_N) \, dt \geq 2\pi \).

5.5.3 Proof of the main result

In this section, we finally resemble the estimates and facts stated in Section 5.3 to provide a proof of the main result that is Theorem 5.37. We will use the finite dimensional oscillation theory of Section 5.4 and the general result of Corollary 5.19. Most importantly, the last result holds uniformly in \( L \) when applied to the finite volume restrictions.

Proof of Theorem 5.37. Here, we will suppress all dependencies on \( \omega \). Let us begin by estimating the expression in Theorem 5.37 by a sum of three terms, namely with \( \mathbf{E} \) denoting the average with respect to the probability measure \( \mathbf{P} \), and we choose \( L \) sufficiently large, what will be clear later. Hence for \( \lambda \in \mathbb{R} \), we get:
\[
\left| \mathcal{N}_N(\lambda) - \frac{1}{2\pi} \int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt \right| \leq |\mathcal{N}_N(\lambda) - \mathbf{E} \mathcal{N}_{N,L}(\lambda)|
+ \left| \mathbf{E} \mathcal{N}_{N,L}(\lambda) - \mathbf{E} \frac{1}{2\pi} \int_{-\infty}^{\lambda} \frac{1}{(2L + 1)^d} \text{Tr}(s_{N,L}(t)) \, dt \right| \quad (5.39)
+ \frac{1}{2\pi} \left| \int_{-\infty}^{\lambda} \mathbf{E} \frac{1}{(2L + 1)^d} \text{Tr}(s_{N,L}(t)) \, dt - \int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt \right|.
\]

The first contribution on the right hand side of (5.39) can be made arbitrarily
small, say less than $N^{-1}$, by choosing $L$ sufficiently large, see Proposition 5.38. The second summand in (5.39) is less than or equal to $(2N)^{-1}$ for any $L$ by the finite dimensional oscillation theory, see Remark 5.41. The last summand has again to be split in several contributions, namely for large $\lambda_0 > 0$ with $\lambda > -\lambda_0$, we estimate it in the following way:

$$
\int_{-\infty}^{\lambda} \left| \mathbf{E} \frac{1}{(2L + 1)^d} \text{Tr}(s_{N,L}(t)) - T(s_N(t)) \right| dt \leq \int_{-\infty}^{-\lambda_0} T(s_N(t)) dt \\
+ \int_{-\infty}^{-\lambda_0} \mathbf{E} \frac{1}{(2L + 1)^d} \text{Tr}(s_{N,L}(t)) dt \\
+ \int_{-\lambda}^{-\lambda_0} \left| \mathbf{E} \frac{1}{(2L + 1)^d} \text{Tr}(s_{N,L}(t)) - T(s_N(t)) \right| dt .
$$

(5.40)

The first summand can be estimated by using $|T(a)| \leq \|a\|$ for $a \in \mathcal{A}$ combined with Corollary 5.19. Then we obtain

$$
\int_{-\infty}^{-\lambda_0} T(s_N(t)) dt \leq \frac{K'}{N\lambda_0},
$$

for some constant $K'$ only depending on $\xi$. Since we also have the inequality $(2L + 1)^{-d} \text{Tr}(s_{N,L}(t)) \leq \|s_{N,L}(t)\|$, we use Corollary 5.19 again for the second summand and get the same estimate.

As to the last summand in (5.40), Proposition 5.40 assures the pointwise convergence of the integrand for each $t$ when $L \to \infty$. Moreover, we know by Proposition 5.13 that there is a constant $K_\xi$, not depending on the coefficients \(\{a_n, v_n\}_{n \in \mathbb{N}}\), such that $\sup_{t \in [-\lambda_0, \lambda]} \|s_N(t)\| \leq K_\xi$. Since we assumed (5.38), this estimate holds uniformly in $L$ for $s_{N,L}$ and a constant $K_2\xi$. So we get

$$
\sup_{L > L_0} \left( \sup_{t \in [-\lambda_0, \lambda]} \|s_{N,L}(t)\| \right) \leq K_2\xi ,
$$

which shows that integrand is bounded by the integrable function $t \mapsto K_\xi + K_2\xi$. Using Lebesgue’s Dominated Convergence Theorem, we conclude that the integral vanishes for $L \to \infty$. In particular, it can be made smaller than $N^{-1}$ for $L$ sufficiently large. In conclusion, all terms on the right hand side of (5.39) can be made smaller than a constant times $N^{-1}$ by choosing $L$ sufficiently
large, and this constant only depends on $\xi$. This concludes the proof. \qed

5.6 Some numerical analysis of the Anderson model

It is interesting to study the relation of the $(d+1)$-dimensional IDOS $\mathcal{N}$ given by Theorem 5.37 in a numerical setting. We have already seen in the proof of Theorem 5.37 above, meaning equation (5.39), that the estimate of

$$\left| \mathcal{N}_N(\lambda) - \frac{1}{2\pi} \int_{-\infty}^{\lambda} \mathcal{T}(s_N(t)) \, dt \right|$$

consists of three parts, which can be hold arbitrary small for large $N$ and $L$. We will now consider the middle part

$$\left| \mathbf{E} \mathcal{N}_{N,L}(\lambda) - \mathbf{E} \frac{1}{2\pi} \int_{-\infty}^{\lambda} \frac{1}{(2L+1)^d} \text{Tr}(s_{N,L}(t)) \, dt \right| \quad (5.41)$$

since here we have only matrices and can use the finite dimensional oscillation theory of Section 5.4. This means by Remark 5.41 that the term in (5.41) can be made smaller than $N^{-1}$ for all $L$. For numerical reasons it would be better to look at each integrand instead of the whole integral on the right hand side. However, $\mathcal{N}_{N,L}(\lambda)$ is a step function and cannot be written as an integral over the Lebesgue measures. This means that the associated DOS is not absolutely continuous with respect to the Lebesgue measure. Indeed, we find just Dirac measures on each eigenvalue. Therefore, we have to smooth and average the DOS to an absolutely continuous function. This will not be a problem since, nevertheless, we are interested in finite differences for the numerical analysis.

First of all, we present the main object of research here, the Anderson model, the one-dimensional version of which was already considered in Section 1.3. There, we had a covariant family $(H_\omega)_{\omega \in \Omega} \subset \ell^2(\mathbb{Z})$ of finite range consisting of self-adjoint operators. Now we choose this family for every entry $v_n$ on the diagonal of the Jacobi matrix.
Definition 5.43. For given dimension \( d \in \mathbb{N} \), we consider the Jacobi matrix

\[
H_N = \begin{pmatrix}
  v_1 & 1 & & \\
  1 & v_2 & & \\
  & 1 & v_3 & \\
  & & \ddots & \\
  & & & v_{N-1} & 1 \\
  & & & 1 & v_N
\end{pmatrix}
\]

where each \( v_n = (V_n^\omega)_{\omega \in \Omega} \) is the \( d \)-dimensional Anderson model for \( \kappa \geq 0 \), that is

\[
V_n^\omega = \Delta + \kappa \sum_{m \in \mathbb{Z}^d} \tilde{v}_{n,m}(\omega) e_m(\cdot) e_m(\cdot)
\]

with i.i.d. real random variables \( \tilde{v}_{n,m} \) on \([- \frac{1}{2}, \frac{1}{2}]\). For programming reasons, we have to use the finite volume restrictions and set \( v_{n,L} = (V_{n,L}^\omega)_{\omega \in \Omega} \). This means that, for a given \( \omega \in \Omega \), we have the \((2L + 1) \times (2L + 1)\)-sized matrix:

\[
V_{n,L}^\omega = \begin{pmatrix}
  \tilde{v}_{n,-L} & 1 & & \\
  1 & \tilde{v}_{n,-L+1} & & \\
  & 1 & \ddots & \\
  & & \ddots & \tilde{v}_{n,L-1} \\
  & & & 1 & \tilde{v}_{n,L}
\end{pmatrix}
\]

Note that we now emphasise the dependence on the parameter \( \omega \) since we will calculate the matrices for different values of \( \omega \). In the same manner and as in Section 5.5, we also denote the associated Jacobi matrix by \( H_{N,L}^\omega \) and the eigenvalue counting function by \( N_{N,L}^\omega \).

5.6.1 Numerical setup

We give a short overview about the main objects. First of all, we choose a partition \( \{ \lambda_i \} \) of the interval \([-5, 5]\), which always overlaps the whole spectrum of \( H_N \). We do this for a given \( \varepsilon > 0 \) and use an equidistant grid given by
\( \lambda_{i+1} - \lambda_i = \varepsilon \) for all \( i \). To approximate the density of states, we take the central difference quotient:

\[
\frac{\Delta N_{N,L}^\omega(\lambda_i)}{\Delta \lambda} := \frac{N_{N,L}^\omega(\lambda_{i+1}) - N_{N,L}^\omega(\lambda_{i-1})}{2\varepsilon}.
\]

For the approximation of the derivative of \( u_{N,L} \), we do the same and set:

\[
\frac{\Delta u_{N,L}^\omega(\lambda_i)}{\Delta \lambda} := \frac{u_{N,L}^\omega(\lambda_{i+1}) - u_{N,L}^\omega(\lambda_{i-1})}{2\varepsilon}.
\]

Note that we have two different approaches here. On the one hand, we have the analytic function \( u_{N,L} \) that can be qualitatively good approximated by finite differences, and on the other hand, we want to smooth the non-continuous step function \( N_{N,L} \). Therefore, we choose \( \varepsilon \) for the DOS much larger than for \( u_{N,L} \) such that we obtain good results. The approximation for the trace of the speed operator \((2\pi(2L + 1)^d)^{-1}\text{Tr}(s_{N,L}(\lambda))\) is then defined by:

\[
\sigma_{N,L}^\omega(\lambda_i) := \frac{1}{2\pi i(2L + 1)^d N} \text{Tr}\left[ u_{N,L}^\omega(\lambda_i)^* \frac{\Delta u_{N,L}^\omega}{\Delta \lambda}(\lambda_i) \right].
\]

Finally, we do an average to approximate the expectation value and set:

\[
\text{DOS}(\lambda) := \frac{1}{K} \sum_{j=1}^{K} \frac{\Delta N_{N,L}^\omega}{\Delta \lambda}(\lambda), \quad \Sigma(\lambda) := \frac{1}{K} \sum_{j=1}^{K} \sigma_{N,L}^\omega(\lambda).
\]

The difference between these two quantities is our approximation of equation (5.41). By construction, we have the following parameters for the analysis: \( N, L, d, \kappa, K, \varepsilon \).

We close this chapter by some visualisations of the numerical analysis of the density of states for the Anderson model. We will set \( d = 1 \) for simplicity and choose \( N \) and \( L \) such that we get a square lattice in both directions. Then we can see what happens at different values for the disorder \( \kappa \).
5.6.2 Two-dimensional Anderson model for $\kappa = 0$

The disorder parameter $\kappa$ is set to zero, which means that there is no random potential. For this reason, a mean value does not change anything and we will naturally choose $K = 1$. In this setting, the Anderson model should converge to the $(d + 1)$-dimensional Laplacian for $N \to \infty$, where as stated $d = 1$.

![Figure 5.5: Approximation for $N = 61$, $L = 30$, $\kappa = 0$ and $K = 1$.](image)

We see in Figure 5.5 that there is a peak in the centre. On the boundary of the interval, the DOS immediately increases and decreases, respectively. Also, as expected, the density of states is fully symmetric. We also expect this when we turn the symmetric random potential on. If one increases $N$ and $L$, one will see that the peak in the centre will get sharper and that there are higher amplitudes on the boundary of the interval. Note also that the density of states lies exactly inside the interval $[-4, 4]$. 
However, one already recognises in this picture that there are a lot of numerical errors. For this reason, we will concentrate on the cases where there is a random potential, meaning $\kappa > 0$, since then the individual numerical errors can be reduced by an average process.

### 5.6.3 Two-dimensional Anderson model for $\kappa = 0.5$

In the case $\kappa = 0.5$, see Figure 5.6, the randomness is slightly on and the sharp edges on the boundary of the interval, one could see in the example above, are now more unclear. This means that the spectrum is drifting out of the interval $[-4, 4]$ and the peak in the centre is getting lower. Also the fluctuations are less significant here since we are able to take the mean of $K = 50$ different simulations.

![Figure 5.6: Approximation for $N = 61$, $L = 30$, $\kappa = 0.5$ and $K = 50$.](image-url)
5.6.4 Two-dimensional Anderson model for $\kappa = 1$

Now by setting $\kappa = 1$, the perturbation of the random potential is stronger such that the maximum at 0 is getting lower. See Figure 5.7. One can also notice that the fluctuations are almost gone in this case.

![Figure 5.7: Approximation for $N = 61, L = 30, \kappa = 1$ and $K = 50$.](image)

5.6.5 Two-dimensional Anderson model for $\kappa = 2$

In Figure 5.8, you can see the simulation with disorder parameter $\kappa = 2$. As expected the spectrum is also distributed outside of $[-4, 4]$ and the peak at zero is weaker than before.
5.6 Some numerical analysis of the Anderson model

Figure 5.8: Approximation for $N = 61, L = 30, \kappa = 2$ and $K = 50$. 
Bibliography


