Hill’s Lunar Problem

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## Abstract
This Master’s thesis shows that the secondary’s or moon’s flow of Hill’s Lunar Problem with certain initial points leaves the planetary neighborhood around the second primary (earth) for large time and does not return. Additionally, it proves the existence of a minimal energy $E_0$, so that there are initial points for every energy $E$ greater than $E_0$ whose flow crosses the planetary neighborhood in configuration space from one Lagrange point to the other.
In Section 2, coordinate transformations given by Meyer [1981], Meyer and Schmidt [2000] and Kummer [1983] are used in order to transform the Three-Body Problem into Hill’s Lunar Problem. The theorems about the Birkhoff normal form in Section 3 are from Arnold [1997] and Uzer et al. [2002] with newly elaborated proofs by the author of this thesis. The statements in Section 4 and Section 5 can be found in Waalkens et al. [2005], the proofs of the application to the quadratic part of Hill’s Lunar Problem were added by the author. The theorems and their proofs in Section 6 and Section 7 are from the author of this thesis.
1 Introduction

“In this [Hill’s] work, one is allowed to perceive the germ of most of the progress that Science has made ever since.”

Poincaré (1905)\(^1\)

George William Hill was born 1838 in New York and started his academic life in Rutgers College, New Jersey. There he studied the classical mathematical work of the 18th and early 19th century and eventually joined the staff of scientists working in Cambridge, Massachusetts, on the “American Ephemeris and Nautical Almanac”.

He started working on the theory of Jupiter and Saturn in 1877 and his eventual results “New Theory of Jupiter and Saturn” were a cornerstone in the great project of revising all the data for the orbits in the solar system. This publication is Volume III with more than 500 pages of Hill’s Collected Works (see Hill [1905-1907]), which was published in 1905 and consists of more than 80 papers.

The data resulting from “New Theory of Jupiter and Saturn” were the best ones available to verify Einstein’s theory of general relativity. What is more, his greatest contribution to (celestial) mechanics is the 1877 paper “On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motion of the Sun and Moon” and the 1878 paper “Researches in Lunar Theory”.

His inspirations might be connected to Euler’s “Second Theory of the Moon” (1772), which introduced two new ideas: Euler described the Moon in Cartesian coordinates with respect to the ecliptic and this reference system rotates around the Earth with the Moon’s mean motion in longitude. Hill changed that theory by letting the coordinates turn with respect to the Sun rather then the Moon.

Eventually Hill’s theory gave a complete description of the flow in phase space near the real lunar trajectory. He retired in 1892 after a ten year stay at Washington D.C. (see Gutzwiller [1998])

Hill’s Lunar Problem is a limiting case of the circular restricted Three-Body Problem. The latter describes the behavior of a very light particle (like the moon) or secondary (its mass can be neglected) within the potential of two heavier masses (the so-called primaries). These two masses may also have different weight; the first primary can be the sun, the second primary might be the earth.

Furthermore, the circular restricted Three-Body Problem implies that both primaries rotate around their center of mass in one common plane. Thus, in the circular case one can move the center of mass to the origin and introduce coordinates which rotate around it with a constant velocity. In this new coordinate system the primaries remain at the same points; the first weight (sun) \(1 - \mu\) at \((-\mu, 0, 0)^T\), the second weight (earth) \(\mu\) at \((1 - \mu, 0, 0)^T\), with constant distance for all time.

Hill’s Lunar Problem shifts the lighter primary mass to the origin and sets its weight infinitesimal small. One can introduce new variables so that the distance between the remaining heavy mass and the one at the origin tends to infinity. The

\(^1\)from Poincaré’s introduction to Hill’s Collected Works according to Gutzwiller [1998].
1 INTRODUCTION

A small particle is near the origin and only influenced by the resulting simplified potential.

This Master’s thesis is concerned with the dynamic of Hill’s Lunar Problem. In Section 2, the Hamiltonian of the restricted Three-Body Problem as well as of Hill’s Lunar Problem are derived from the general Three-Body Problem. The Birkhoff normal form for Hamiltonians with a stable or unstable equilibrium is discussed in Section 3, so that Hill’s Lunar Problem can be reduced to a Birkhoff-like normal form of degree 2 in Section 4.

Additionally, in order to describe the secondary’s or moon’s escape from the lighter primary’s or earth’s planetary neighborhood, several submanifolds are defined and discussed in Section 4 for that Birkhoff-like normal form of degree 2, based on Waalkens et al. [2005]. The proofs were added to the statements given in that article. At the end of this section, the flow of Hill’s Lunar Problem is shown in several figures with two degrees of freedom.

A formula for the volume of the flow, which crosses the planetary neighborhood from one side to the other is derived in Section 5 as well as a definition of the planetary neighborhood for the whole Hamiltonian.

Section 6 compares the flow of the undisturbed part of Hill’s Lunar Problem with the disturbed one. The aim is to prove a similar dynamic of the disturbed flow for large time for certain initial points.

Lastly, the goal of Section 7 is to show existence of initial points whose flow crosses the planetary neighborhood from one Lagrange point to the other. The thesis concludes with a short summary in Section 8.

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2 Derivation of the Hamiltonians

To derive the Hamiltonian of Hill’s Lunar Problem one has to start with the Hamiltonian of the full Three-Body Problem on the symplectic manifold \((\mathcal{P}, \omega_0)\) as follows (see Definition 10.7 and Section 11.4 in Knauf [2012]):

\[
H: \mathcal{P} \to \mathbb{R}
\]

\[
H(p, q) = \frac{1}{2} \sum_{j=1}^{3} \frac{\|p_j\|^2}{m_j} - \sum_{1 \leq i < j \leq 3} G \frac{m_i m_j}{\|q_j - q_i\|} \quad (G, m_j > 0),
\]

\[
\mathcal{P} := T^* \mathcal{M} \cong \mathbb{R}^9 \times \tilde{\mathcal{M}}, \quad \mathcal{M} := \{(q_1, q_2, q_3) \in \mathbb{R}^3 \mid q_k \neq q_l \text{ for } k \neq l\}.
\]

2.1 The restricted Three-Body Problem

If one assumes that the primaries rotate with constant velocity \(\omega \in \mathbb{R} \setminus \{0\}\) around their center of mass, the coordinates can be changed into a rotating system. As a consequence, both primaries rest at the same place. Using the transformation given in Meyer [1981]

\[
J := \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \exp(-\omega J t) = \begin{pmatrix}
\cos(\omega t) & -\sin(\omega t) & 0 \\
\sin(\omega t) & \cos(\omega t) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

the following continuously differentiable coordinate transformation can be defined

\[
\Xi(p, q, t) := (\exp(-\omega J t)p_1, \ldots, \exp(-\omega J t)q_3) =: (\tilde{p}, \tilde{q}).
\]

Lemma 2.1 (statement by Meyer [1981]). The transformation \(\Xi|_t(p, q) := \Xi(p, q, t)\) is symplectic for every \(t \in \mathbb{R}\).

Proof by author. The transformation fulfills \(D \Xi|_t(p, q) \cdot J (D \Xi|_t(p, q))^T = J\) for every \(t \in \mathbb{R}\) with \(J := \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \in \text{Mat}(18),
\]

\[
D \Xi|_t(p, q) = \exp(-\omega J t) \oplus \ldots \oplus \exp(-\omega J t),
\]

\[
(D \Xi|_t(p, q))^T = \exp(\omega J t) \oplus \ldots \oplus \exp(\omega J t).
\]

Hence, it is symplectic.

The matrix \(D \Xi|_t(p, q)\) is symplectic and therefore has a symplectic inverse. So the inverse function theorem yields that \(\Xi|_t\) is locally invertible and the inverse \(\Xi|^{-1}_t(\tilde{p}, \tilde{q})\) symplectic where defined. With this transformation, the Hamiltonian changes into \(\tilde{H}(\tilde{p}, \tilde{q}) := H(\Xi|^{-1}_t(\tilde{p}, \tilde{q}))\), independent of \(t\), and the differential
2 DERIVATION OF THE HAMILTONIANS

Lemma 2.2 (statement by Meyer [1981])

plectic as well.

With the function $\omega = 1 = G, m_1 = 1 - \mu$ (first primary, the sun), $m_2 = \mu$ (second primary, the earth), $M_1 := 1 + m_3, M_2 := \frac{m_3}{1 + m_3}$. Symplectic Jacobi coordinates are defined in Meyer [1981] as

$$\Gamma(\tilde{p}, \tilde{q}) := \begin{pmatrix} \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3, m_1 \tilde{p}_2 - m_2 \tilde{p}_1, \frac{1}{M_1} \tilde{p}_3 - M_2 (\tilde{p}_1 + \tilde{p}_2), \\ m_1 \tilde{q}_1 + m_2 \tilde{q}_2 + m_3 \tilde{q}_3, \tilde{q}_2 - \tilde{q}_1, \tilde{q}_3 - m_1 \tilde{q}_1 - m_2 \tilde{q}_2 \end{pmatrix} =: (v, u).$$

Lemma 2.2 (statement by Meyer [1981]). The coordinate transformation $\Gamma$ is symplectic as well.
Proof by author. The Jacobian of $\Gamma$ is

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -m_2 & 0 & 0 & m_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -m_2 & 0 & 0 & m_3 & 0 & 0 & 0 & 0 \\
-m_2 & 0 & 0 & -m_2 & 0 & 0 & m_3 & 0 & 0 & 0 \\
0 & -m_2 & 0 & 0 & -m_2 & 0 & 0 & m_3 & 0 & 0 \\
0 & 0 & -m_2 & 0 & 0 & -m_2 & 0 & 0 & m_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

hence, the equation $D\Gamma(\tilde{p}, \tilde{q}) \cdot J(D\Gamma(\tilde{p}, \tilde{q}))^T = J$ holds.

Then $\tilde{H}(v, u) := \tilde{H}(\Gamma^{-1}(v, u))$ and according to Meyer and Schmidt [2000] a

$$
\tilde{H}(v, u) = \frac{\|v_1\|^2}{2M_1} - \langle u_1, Jv_1 \rangle + \frac{\|v_2\|^2}{2\mu(1 - \mu)} - \frac{\langle u_2, Jv_2 \rangle - \mu(1 - \mu)}{\|u_2\|} \\
+ \frac{\|v_3\|^2}{2M_2} - \langle u_3, JV_3 \rangle - \frac{\mu m_3}{\|u_3 - (1 - \mu)u_2\|} - \frac{(1 - \mu)m_3}{\|u_3 + \mu u_2\|}.
$$

Set without loss of generality $u_1 = 0 = v_1$ (invariant due to $\dot{u}_1 = 0 = \dot{v}_1$, center of mass and total linear momentum) and the secondary’s (moon) mass $m_3$ small. The restricted Hamiltonian can then be written as

$$
\hat{H}(v, u) = K_0(v, u) + H_0(v, u)
$$

where

$$
K_0(v, u) := \frac{\|v_2\|^2}{2\mu(1 - \mu)} - \langle u_2, Jv_2 \rangle - \frac{\mu(1 - \mu)}{\|u_2\|} \\
H_0(v, u) := \frac{1 + m_3}{2m_3} \|v_3\|^2 - \langle u_3, Jv_3 \rangle - \frac{\mu m_3}{\|u_3 - (1 - \mu)u_2\|} - \frac{(1 - \mu)m_3}{\|u_3 + \mu u_2\|}
$$

With new coordinates by the transformation

$$
\Phi(v, u) := (u, \mu^{-1}(1 - \mu)^{-1}v) =: (\tilde{v}, \tilde{u}),
$$

energy $\tilde{H}_0 := \mu^{-1}(1 - \mu)^{-1}H_0$ and mass $\tilde{m}_3 := m_3 \mu^{-1}(1 - \mu)^{-1}$ from Meyer and Schmidt [2000] they transform to

$$
\tilde{K}_0(\tilde{v}, \tilde{u}) = \frac{1}{2}\|\tilde{v}_2\|^2 - \langle \tilde{u}_2, J\tilde{v}_2 \rangle - \frac{1}{\|u_2\|},
$$

$$
\tilde{H}_0(\tilde{v}, \tilde{u}) = \frac{1 + \mu(1 - \mu)\tilde{m}_3}{2m_3} \|\tilde{v}_3\|^2 - \langle \tilde{u}_3, J\tilde{v}_3 \rangle - \frac{\tilde{m}_3 \mu}{\|u_3 - (1 - \mu)u_2\|} - \frac{\tilde{m}_3 (1 - \mu)}{\|u_3 + \mu u_2\|}
$$

With different definition of $m_1$ and $m_2$.

The coordinate transformation satisfies $D\Phi \cdot J(D\Phi)^T = \mu^{-1}(1 - \mu)^{-1}J$. Hence, the necessary scaling of the Hamiltonian.
2 DERIVATION OF THE HAMILTONIANS

The function $\tilde{K}_0$, which only depends on $\bar{u}_2$ and $\bar{v}_2$, has a critical point at $(1, 0, 0, 0, 1, 0)^T$ and $\tilde{H}_0$ only depends on $\bar{u}_2, \bar{u}_3$ as well as $\tilde{v}_3$. With new coordinates\(^4\) and energy\(^5\) $H_R = \tilde{m}_3^{-1} \tilde{H}_0$ from Meyer and Schmidt [2000]

$$
\begin{align*}
\Omega(\tilde{v}_2, \tilde{v}_3, \bar{u}_2, \bar{u}_3) &= \left( \tilde{m}_3^{-1/2} \tilde{v}_2, \tilde{m}_3^{-1} \tilde{v}_3, \tilde{m}_3^{-1/2} (\bar{u}_2 - (1, 0, 0)^T), \bar{u}_3 \right) \\
&= (P_R, P_Q, R, Q),
\end{align*}
$$

the Hamiltonian $\tilde{H}_0$ becomes

$$
H_R(P_Q, R, Q) = \frac{1 + \mu(1 - \mu)\tilde{m}_3}{2}\|P_Q\|^2 - \langle Q, J P_Q \rangle - \frac{\|Q - (1 - \mu)((1, 0, 0)^T + \tilde{m}_3^{1/2}R)\|}{1 - \mu - \|Q + \mu((1, 0, 0)^T + \tilde{m}_3^{1/2}R)\|}.
$$

The dynamic of $R$ is invariant in the limit $\tilde{m}_3 \propto m_3 \rightarrow 0$, according to the Hamiltonian ODE ($H_R$ is independent of the canonical conjugated coordinate $P_R$ of $R$)

$$
\begin{align*}
\dot{P}_R &= -\frac{\partial H_R}{\partial R}, \\
\frac{\partial H_R}{\partial R_1} &= -\mu(Q_1 - (1 - \mu) - (1 - \mu)m_3^{1/2}R_1)(1 - \mu)m_3^{1/2} \\
&+ \frac{(1 - \mu)(Q_1 + \mu + \mu m_3^{1/2}R_1)m_3^{1/2}}{\|Q + \mu((1, 0, 0)^T + \tilde{m}_3^{1/2}R)\|^3}, \\
\frac{\partial H_R}{\partial R_2} &= -\mu(Q_2 - (1 - \mu)m_3^{1/2}R_2)(1 - \mu)m_3^{1/2} \\
&+ \frac{(1 - \mu)(Q_2 + \mu m_3^{1/2}R_2)m_3^{1/2}}{\|Q + \mu((1, 0, 0)^T + \tilde{m}_3^{1/2}R)\|^3}, \\
\frac{\partial H_R}{\partial R_3} &= -\mu(Q_3 - (1 - \mu)m_3^{1/2}R_3)(1 - \mu)m_3^{1/2} \\
&+ \frac{(1 - \mu)(Q_3 + \mu m_3^{1/2}R_3)m_3^{1/2}}{\|Q + \mu((1, 0, 0)^T + \tilde{m}_3^{1/2}R)\|^3}, \\
\dot{R} &= \frac{\partial H_R}{\partial P_R} = 0.
\end{align*}
$$

\(^4\)Without the definition of $P_R$.  
\(^5\)The coordinate transformation satisfies $D \Omega \cdot J (D \Omega)^T = \tilde{m}_3^{-1} J$. Hence, the necessary scaling of the Hamiltonian.
2 DERIVATION OF THE HAMILTONIANS

One can take the limit of the secondary’s mass $\tilde{m}_3 \propto m_3 \to 0$, so that $H_R$ becomes independent of $R$, set $P := PQ$. The dynamic of the primaries is no longer influenced by the secondary, if it does not collide with them.

$$H_R(P, Q) = \frac{1}{2}||P||^2 - (Q, JP) - \frac{\mu}{||Q - (1 - \mu, 0, 0)^T||} - \frac{1 - \mu}{||Q + (\mu, 0, 0)^T||}.$$

This defines the Hamiltonian of the restricted Three-Body Problem where $\mu$ is the mass of the primaries with position $(1 - \mu, 0, 0)^T$ respectively $(-\mu, 0, 0)^T$.

2.2 Hill’s Lunar Problem

The secondary (moon) is assumed to stay close to second primary (earth) with mass $\mu$, so one can shift the earth’s position into the origin via $Q \to Q + (1 - \mu, 0, 0)^T$.

$$\tilde{H}_R(P, Q) = \frac{1}{2}||P||^2 + P_1Q_2 - P_2Q_1 - (1 - \mu)P_2 - \frac{\mu}{||Q||} - \frac{1 - \mu}{||Q + (1, 0, 0)^T||}.$$

This Hamiltonian yields the following differential equations

$$\dot{P}_1 = P_2 - \frac{\mu Q_1}{||Q||^3} - \frac{(1 - \mu)(1 + Q_1)}{||Q + (1, 0, 0)^T||^3},$$

$$\dot{P}_2 = -P_1 - \frac{\mu Q_2}{||Q||^3} - \frac{(1 - \mu)Q_2}{||Q + (1, 0, 0)^T||^3},$$

$$\dot{Q}_1 = P_1 + Q_2 = -\dot{P}_2 - \frac{\mu Q_2}{||Q||^3} - \frac{(1 - \mu)Q_2}{||Q + (1, 0, 0)^T||^3} + Q_2,$n

$$\dot{Q}_2 = P_2 - Q_1 - (1 - \mu)$$

$$= \dot{P}_1 + \frac{\mu Q_1}{||Q||^3} + \frac{(1 - \mu)(1 + Q_1)}{||Q + (1, 0, 0)^T||^3} - Q_1 - (1 - \mu).$$

If one eliminates $P$ from the equations of $Q$, it becomes clear that they coincide with the ones induced by the Hamiltonian from Section 5 in Kummer [1983]

$$\tilde{H}_R(P, Q) := \frac{1}{2}||P||^2 + P_1Q_2 - P_2Q_1 + (1 - \mu)(1 - Q_1) - \frac{\mu}{||Q||} - \frac{1 - \mu}{||Q + (1, 0, 0)^T||},$$

which are

$$\dot{P}_1 = P_2 + (1 - \mu) - \frac{\mu Q_1}{||Q||^3} - \frac{(1 - \mu)(1 + Q_1)}{||Q + (1, 0, 0)^T||^3},$$

$$\dot{P}_2 = -P_1 - \frac{\mu Q_2}{||Q||^3} - \frac{(1 - \mu)Q_2}{||Q + (1, 0, 0)^T||^3},$$

$$\dot{Q}_1 = P_1 + Q_2 = -\dot{P}_2 - \frac{\mu Q_2}{||Q||^3} - \frac{(1 - \mu)Q_2}{||Q + (1, 0, 0)^T||^3} + Q_2,$n

$$\dot{Q}_2 = P_2 - Q_1$$

$$= \dot{P}_1 + \frac{\mu Q_1}{||Q||^3} + \frac{(1 - \mu)(1 + Q_1)}{||Q + (1, 0, 0)^T||^3} - Q_1 - (1 - \mu).$$
The projection of these two different flows onto the configuration space are equal. So one can analyze the Hamiltonian $\tilde{H}_R$ instead of the original $\hat{H}_R$. Let the energy $\tilde{E}_R$ of $\tilde{H}_R$ be small enough that $\|Q\| < 1$ in order to avoid collision with the mass $1 - \mu$. That is the secondary moves inside a neighborhood of the second primary. Then one can expand $\|Q + (1, 0, 0)^T\|$ in a Taylor series around $(0, 0, 0)^T$

$$\frac{1}{\|Q + (1, 0, 0)^T\|} = 1 - Q_1 + Q_1^2 - \frac{1}{2} Q_2^2 - \frac{1}{2} Q_3^2 + \mathcal{O} (\|Q\|^3).$$

The Hamiltonian $\tilde{H}_R$ becomes with this result

$$\tilde{E}_R \equiv \tilde{H}_R(P, Q) = \frac{1}{2} \|P\|^2 + P_1 Q_2 - P_2 Q_1 - \frac{\mu}{\|Q\|} - (1 - \mu) \left( Q_1^2 - \frac{1}{2} Q_2^2 - \frac{1}{2} Q_3^2 \right) + \mathcal{O} (\|Q\|^3).$$

Using new coordinates

$$\Psi(P, Q) := (\mu^{-1/3} P, \mu^{-1/3} Q) =: (Y, X)$$

and energy\footnote{The coordinate transformation satisfies $D\Psi \cdot J (D\Psi)^T = \mu^{-2/3} J$. Hence, the necessary scaling of the Hamiltonian.} $H_H := \mu^{-2/3} \tilde{H}_R$ from Section 5 in Kummer [1983] the Hamiltonian becomes

$$E \equiv H_H(Y, X) = \frac{1}{2} \|Y\|^2 + Y_1 X_2 - Y_2 X_1 - \frac{1}{\|X\|} X_1^2 - \frac{1}{2} X_2^2 - \frac{1}{2} X_3^2 + \mathcal{O} \left( \mu^{1/3} \|X\|^3 \right).$$

Hill’s Lunar Problem asks for the limit $\mu \to 0$, that is the first primary (sun) is infinitely heavy and infinitely far away in comparison to the second primary

$$H_H(Y, X) = \frac{1}{2} \|Y\|^2 + Y_1 X_2 - Y_2 X_1 - \frac{1}{\|X\|} X_1^2 + \frac{1}{2} X_2^2 + \frac{1}{2} X_3^2. \quad (2.2)$$

The former position $(-1, 0, 0)^T$ of the mass $1 - \mu$ in $Q$-coordinates, became $(-\mu^{-1/3}, 0, 0)^T$ in $X$-coordinates, which diverges to minus infinity on the $X_1$-axis as $\mu \to 0$. What is more, the error term $\mathcal{O} \left( \mu^{1/3} \|X\|^3 \right)$ is neglected in $H_H$. 

### 3 Birkhoff normal form

The article Waalkens et al. [2005] uses different manifolds in order to define a planetary neighborhood and analyze the secondary’s (moon) flow around the second primary (earth). For this, the authors use the Birkhoff normal form and this section introduces its general ideas.
3.1 Birkhoff normal form with a Liapunov-stable equilibrium

At the beginning, one starts with a general Hamiltonian $H \in C^{s+1}(\mathbb{R}^{2n}, \mathbb{R})$ with $n \in \mathbb{N}$ degrees of freedom and one Liapunov-stable non-degenerated equilibrium $L_1 \in \mathbb{R}^{2n}$. The linearization of the Hamiltonian vector field $X = J\nabla H$ at this point is infinitesimally symplectic, hence, the eigenvalues must be imaginary $\pm i\omega_j, \omega_j > 0 (j = 1, \ldots, n)$, in order to ensure that the equilibrium is Liapunov-stable (Remark 6.25 in Knauf [2012]). Furthermore, assume that the characteristic frequencies $\pm i\omega_j, \omega_j > 0 (j = 1, \ldots, n)$ differ from one another.

$L_1$ can be shifted into the origin without loss of generality and $H$ will be analyzed in the corresponding coordinate system with $L_1 = 0$ and $H(L_1) = 0$. If the algebraic and geometric multiplicities of each eigenvalue are equal, the quadratic part of the Hamiltonian can be written in suitable coordinates $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ as (according to Equation 6.3.2 in Knauf [2012])

$$H_2(p, q) = \frac{1}{2} \sum_{j=1}^{n} \omega_j (p_j^2 + q_j^2).$$

The whole Hamiltonian with usage of Taylor expansion around the origin is

$$H(p, q) = \frac{1}{2} \sum_{j=1}^{n} \omega_j (p_j^2 + q_j^2) + H_3(p, q) + H_4(p, q) + \ldots + R_{s+1}(p, q),$$

where $H_N \in C^{s+1}(\mathbb{R}^{2n}, \mathbb{R})$ is a homogeneous polynomial of degree $N$ (every term has the same degree) and $R_{s+1}$ denotes the remaining terms of order larger than $s$. These requirements are fulfilled if $H$ is a positive definite quadratic form, see Lemma 6.29 in Knauf [2012].

**Definition 3.1** (by Arnold [1997]). The characteristic frequencies $\omega_j$ satisfy a resonance relation of order $K \in \mathbb{N}$, if there exist integers $k_j \in \mathbb{Z} (j = 1, \ldots, n)$ not all equal to zero such that

$$k_1 \omega_1 + \ldots + k_n \omega_n = 0 \text{ where } |k_1| + \ldots + |k_n| = K.$$

Any system in Birkhoff normal form is completely integrable. If the characteristic frequencies do not satisfy a resonance relation of a specific order $s$ or smaller, the Hamiltonian can be reduced to a Birkhoff normal form up to terms of degree $s + 1$.

**Definition 3.2.** A Hamiltonian is in Birkhoff normal form of degree $s$, if it is a polynomial of degree $s$ in canonical variables $(P, Q) = (P_1, \ldots, P_n, Q_1, \ldots, Q_n)$, which is actually a polynomial of degree $\lfloor \frac{s}{2} \rfloor$ in the variables $J_j := \frac{1}{2} (P_j^2 + Q_j^2)$ $(j = 1, \ldots, n)$. 
Example 3.3. For $n = 1$ and $s = 2m \in 2\mathbb{N}$ is

$$\mathcal{H}_{2m}(P, Q) = \sum_{j=1}^{m} c_j \left( \frac{1}{2} (P^2 + Q^2) \right)^j,$$

with coefficients $c_j \in \mathbb{R}$.

In particular, a Birkhoff normal form of degree $s$ is independent of $\varphi_j$ ($j = 1, \ldots, n$), the canonical conjugated angle coordinates of $J_j$. After reducing $H$ to a Birkhoff normal form of degree $s$, the dynamic near the equilibrium can be analyzed more easily. The $J_j$ are constants of motion for the truncated normal form (only considering terms up to order $s + 1$ without rest terms).

Example 3.4. The Hamiltonian ODE for the truncated normal form of degree two can be written in the action-angle coordinates $(J, \varphi) = (J_1, \ldots, J_n, \varphi_1, \ldots, \varphi_n)$

$$\dot{J}_j = -\frac{\partial \mathcal{H}_2}{\partial \varphi_j} = 0 \text{ and } \dot{\varphi}_j = \frac{\partial \mathcal{H}_2}{\partial J_j} = \omega_j.$$

The motion on invariant tori is

$$J_j(t) = J_j(0) \text{ and } \varphi_j(t) = \omega_j t + \varphi_j(0) \quad (j = 1, \ldots, n).$$

So the dynamic near the equilibrium can be approximated by the motion on tori given by the truncated Birkhoff normal form.

Generally, a Hamiltonian cannot be reduced completely to a Birkhoff normal form. Nevertheless, canonical coordinates can be defined which reduce the Hamiltonian to a Birkhoff normal form of degree $s$ up to terms of degree $s + 1$ under certain circumstances. The following proof is a new elaboration of first proposition in Appendix 7.A in Arnold [1997].

**Theorem 3.5** (by Arnold [1997]). Assume that the frequencies $\omega_j$ of a Hamiltonian with a Liapunov-stable equilibrium at 0 do not satisfy any resonance relation of order $s$ or smaller. Then there is a canonical coordinate system in a neighborhood of the equilibrium position such that the Hamiltonian $H$ is reduced to a Birkhoff normal form of degree $s$ up to terms of order $s + 1$. In particular,

$$H(p, q) = \mathcal{H}_s(P, Q) + R_{s+1}(P, Q),$$

where $R_{s+1}(P, Q) = O\left( (\|P\| + \|Q\|)^{s+1} \right)$.

**Proof by author.** Taylor expansion around the equilibrium at 0 yields $H = H_2 + H_3 + \ldots + R_{s+1}$ where $H_N \in C^{s+1}(\mathbb{R}^{2n}, \mathbb{R})$ is a homogeneous polynomial in $(p, q)$ of degree $N$. This theorem is proven by inductive increase of the degree $2 < N \leq s$ up to which $H$ has already been reduced to a Birkhoff normal form.
Hence, assume \( \mathcal{H}_{N-1} \) is a Birkhoff normal form of degree \( N - 1 \) greater than two in appropriate coordinates and

\[
H = \mathcal{H}_2 + \mathcal{H}_{N-1} + H_N + \ldots + R_{s+1}
\]

in these coordinates.

It is sufficient to construct a generating function of a canonical transformation which reduces the homogeneous term \( H_N \) to Birkhoff normal form. In order to do this, one can at first apply a general canonical transformation to \( \mathcal{H}_2 \). Then choose that transformation’s coefficients appropriate, so that terms of \( H_N \), which are not symmetrical in exchanging of \( p_j \) and \( q_j \), are eliminated.

The quadratic part (see Equation 6.3.2 in Knauf [2012])

\[
H_2(p, q) = \frac{1}{2} \sum_{j=1}^{n} \omega_j \left( p_j^2 + q_j^2 \right),
\]

is already a Birkhoff normal form, so \( H_2 \) equals \( H_2 \) and the first transformation does not change the coordinates \((p, q)\).

The proof becomes easier in complex coordinates \( z_j := q_j + i p_j \) and \( w_j := q_j - i p_j \) \((j = 1, \ldots, n)\), so the quadratic part of the Hamiltonian in \((z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_n)\) is

\[
\mathcal{H}_2(z, w) = \sum_{j=1}^{n} i \omega_j z_j w_j.
\]

Upon passing to these coordinates, one must multiply the Hamiltonian with \( 2i \) because the Wirtinger derivatives yield

\[
\dot{z}_j = q_j + i p_j = \frac{\partial H}{\partial p_j} - i \frac{\partial H}{\partial q_j} = -2i \left( \frac{i}{2} \frac{\partial H}{\partial p_j} + \frac{1}{2} \frac{\partial H}{\partial q_j} \right) = -2i \frac{\partial H}{\partial w_j},
\]

\[
\dot{w}_j = q_j - i p_j = 2i \left( -\frac{i}{2} \frac{\partial H}{\partial p_j} + \frac{1}{2} \frac{\partial H}{\partial q_j} \right) = 2i \frac{\partial H}{\partial z_j}.
\]

In order to transform the canonical coordinates \((z, w) \mapsto (Z, W)\), one can use the following generating function \((Z, W) + S_N(Z, W)\), where \( S_N \) describes a homogeneous polynomial of degree \( N \) in \((Z, W) = (Z_1, \ldots, Z_n, w_1, \ldots, w_n)\). The transformation with generating function \((Z, W) + S_N(Z, W)\) changes only terms of order \( N \) or higher and one obtains the new coordinates according to Section 10.5 in Knauf [2012] by

\[
W_j = w_j + \frac{\partial S_N}{\partial Z_j} \quad \text{and} \quad z_j = Z_j + \frac{\partial S_N}{\partial w_j} \quad (j = 1, \ldots, n).
\]

In general, the polynomials can be written as

\[
H_N(z, w) = \sum_{(k,l) \in \mathcal{K}_N} i c_{k,l} M_{k,l}(z, w),
\]

\[
S_N(Z, W) = \sum_{(k,l) \in \mathcal{K}_N} d_{k,l} M_{k,l}(Z, W).
\]
The Hamiltonian $H$ can be written in terms of $J$ and $d$:

$$H_{N} = \{ (\mathbf{a}, \mathbf{b}) \in \mathbb{N}_{0}^{2n} \ | \ |\mathbf{a}| + |\mathbf{b}| = N \},$$

$$M_{k,l}(\mathbf{z}, \mathbf{w}) := z_{1}^{k_{1}} \cdots z_{n}^{k_{n}} \cdot w_{1}^{l_{1}} \cdots w_{n}^{l_{n}}.$$  

The Hamiltonian $H_{2}$ is converted with $(\mathbf{Z}, \mathbf{W}) + S_{N}(\mathbf{Z}, \mathbf{W})$ (3.2) into

$$H_{2}(\mathbf{z}, \mathbf{w}) = H_{2}(\mathbf{Z}, \mathbf{W}) + \sum_{j=1}^{n} i\omega_{j} \left( W_{j} \frac{\partial S_{N}}{\partial w_{j}} - Z_{j} \frac{\partial S_{N}}{\partial Z_{j}} \right) - \sum_{j=1}^{n} i\omega_{j} \frac{\partial S_{N}}{\partial w_{j}} \frac{\partial S_{N}}{\partial Z_{j}}$$

$$= H_{2}(\mathbf{Z}, \mathbf{W}) + \sum_{j=1}^{n} i\omega_{j} \left( W_{j} \frac{\partial S_{N}}{\partial W_{j}} - Z_{j} \frac{\partial S_{N}}{\partial Z_{j}} \right) - R_{2N-2}(\mathbf{Z}, \mathbf{W})$$

$$\overset{(3.4)}{=} H_{2}(\mathbf{Z}, \mathbf{W}) + \sum_{(k,l) \in K_{N}} d_{k,l} \sum_{j=1}^{n} i\omega_{j} (l_{j} - k_{j}) M_{k,l}(\mathbf{Z}, \mathbf{W})$$

$$- R_{2N-2}(\mathbf{Z}, \mathbf{W}).$$

The resulting error of passing from $\frac{\partial S_{N}}{\partial w_{j}}(\mathbf{Z}, \mathbf{W})$ to $\frac{\partial S_{N}}{\partial W_{j}}(\mathbf{Z}, \mathbf{W})$ has degree larger than $N$ and is included in the remaining term $R_{2N-2}(\mathbf{Z}, \mathbf{W})$. Since the frequencies do not satisfy a resonance relation of order $s$ and smaller, $\sum_{j=1}^{n} i\omega_{j} (l_{j} - k_{j})$ is equal to zero if and only if $k_{j} = l_{j}$ for all $j$. Every homogeneous term, which fulfills $\sum_{j=1}^{n} i\omega_{j} (l_{j} - k_{j}) = 0$ depends only on the action coordinates $J_{j} (j = 1, \ldots, n)$.

By defining

$$d_{k,l} := \frac{-c_{k,l}}{\sum_{j=1}^{n} \omega_{j} (l_{j} - k_{j})} \text{ if } \sum_{j=1}^{n} i\omega_{j} (l_{j} - k_{j}) \neq 0$$

with $c_{k,l}$ from (3.3) and equal to zero otherwise, the terms of these homogeneous polynomials $M_{k,l}$, which could not yet be written in $J_{j}$ ($j = 1, \ldots, n$), are eliminated by this transformation.

Altogether, the whole Hamiltonian (3.1) becomes under the transformation and above defined coefficients $d_{k,l}$

$$H(\mathbf{z}, \mathbf{w}) = H_{2}(\mathbf{Z}, \mathbf{W}) + H_{N-1}(\mathbf{Z}, \mathbf{W})$$

$$+ \sum_{(k,l) \in K_{N}} d_{k,l} \sum_{j=1}^{n} i\omega_{j} (l_{j} - k_{j}) M_{k,l}(\mathbf{Z}, \mathbf{W}) + H_{N}(\mathbf{Z}, \mathbf{W})$$

$$+ \tilde{H}_{N+1}(\mathbf{Z}, \mathbf{W}) + \ldots + \tilde{H}_{s+1}(\mathbf{Z}, \mathbf{W})$$

$$= H_{2}(\mathbf{Z}, \mathbf{W}) + H_{N-1}(\mathbf{Z}, \mathbf{W}) + \sum_{(k,k) \in K_{N}} i c_{k,k} M_{k,k}(\mathbf{Z}, \mathbf{W})$$

$$+ \tilde{H}_{N+1}(\mathbf{Z}, \mathbf{W}) + \ldots + \tilde{H}_{s+1}(\mathbf{Z}, \mathbf{W})$$

$$= H_{2}(\mathbf{Z}, \mathbf{W}) + H_{N}(\mathbf{Z}, \mathbf{W}) + H_{N+1}(\mathbf{Z}, \mathbf{W}) + \ldots + \tilde{H}_{s+1}(\mathbf{Z}, \mathbf{W}).$$
So \( H = H_2 + H_N + \tilde{H}_{N+1} + \ldots + \tilde{R}_{s+1} \) is reduced to a Birkhoff normal form of degree \( N \) in these coordinates.

Here \( \tilde{H}_{N+1} \) and \( \tilde{R}_{s+1} \) denotes the transformed part of degree \( N + 1 \) respectively the transformed rest term of degree \( s + 1 \) by \( S_N \). Setting \( N = 3, \ldots, s \) inductively yields transformation of higher order and concludes this proof. \( \square \)

The dynamic around the equilibrium is mainly determined by the Birkhoff normal form and the error term can at first be neglected near that point. If the precision is too low, the degree may be raised as long as the frequencies satisfy no resonance relation.

### 3.2 Birkhoff normal form with an unstable equilibrium

Now consider a Hamiltonian \( H \in C^{s+1}(\mathbb{R}^{2n}, \mathbb{R}) \) with one equilibrium of a saddle-center-\ldots-center type, like the Lagrange-points \( L_1, L_2 \) of Hill’s Lunar Problem (4.1) on the \( q_{11} \)-axis, which define the places where the centrifugal force neglects the gravitational one. Then the linearization of the vector field \( X = J\nabla H \) at this point has eigenvalues \( \pm \lambda, \lambda > 0 \) and \( \pm i \omega_j, \omega_j > 0 (j = 2, \ldots, n) \). Once again, the frequencies \( \omega_j (j = 2, \ldots, n) \) must not satisfy a resonance relation of order \( s \) or smaller and should differ from one another. Without loss of generality \( L_1 \) is shifted into the origin.

A similar transformation like in Theorem 3.5 can be found which reduces \( H \) into a Birkhoff-like normal form \( H_s \in C^{s+1}(\mathbb{R}^{2n}, \mathbb{R}) \) up to terms of degree \( s + 1 \).

The quadratic part can, according to Appendix 6 in Arnold [1997], be written in appropriate symplectic coordinates as

\[
H_2(p, q) = \frac{1}{2} \lambda \left( p_1^2 - q_1^2 \right) + \frac{1}{2} \sum_{j=2}^{n} \omega_j \left( p_j^2 + q_j^2 \right) .
\]  

(3.5)

Rotating of the \( p_1 - q_1 \)-plane clockwise by \( \frac{\pi}{4} \) results in the following function, which is again the truncated Birkhoff normal form of order 2 (hence, the first transformation does not change the coordinates \( (p, q) \))

\[
H_2(p, q) = \lambda \tilde{p}_1 \tilde{q}_1 + \frac{1}{2} \sum_{j=2}^{n} \omega_j \left( p_j^2 + q_j^2 \right) .
\]  

(3.6)

The Taylor expansion at the equilibrium \( L_1 = 0 \) yields

\[
H(p, q) = \lambda \tilde{p}_1 \tilde{q}_1 + \frac{1}{2} \sum_{j=2}^{n} \omega_j \left( p_j^2 + q_j^2 \right) + H_3(p, q) + H_4(p, q) + \ldots + R_{s+1}(p, q) ,
\]

The resulting normal form does not coincide with the Definition 3.2, due to the different eigenvalue structure. Such a Hamiltonian is considered to be in a Birkhoff-like normal form of degree \( s \), if it is a polynomial of degree \( \left\lfloor \frac{s}{2} \right\rfloor \) in the newly defined action coordinates \( I := P_1Q_1 \) and \( J_j := \frac{1}{2} (P_j^2 + Q_j^2) \) \( (j = 2, \ldots, n) \).
where \( H_N \in O^{s+1}(\mathbb{R}^{2n}, \mathbb{R}) \) is once more a homogeneous polynomial in \((p, q) = (\tilde{p}_1, p_2, \ldots, p_n, \tilde{q}_1, q_2, \ldots, q_n)\) of degree \( N \).

One can prove a theorem similar to 3.5 for Hamiltonians with an unstable equilibrium of saddle-center-. . . -center type (see Theorem 2.3 with a different proof in Uzer et al. [2002]).

**Theorem 3.6** (statement by Uzer et al. [2002]). *If the frequencies \( \omega_j (j = 2, \ldots, n) \) of a Hamiltonian with a saddle-center-. . . -center equilibrium at 0 do not satisfy any resonance relation of order \( s \) or smaller, then there exist canonical coordinates in a neighborhood of the equilibrium such that the Hamiltonian \( H \) can be reduced to a Birkhoff-like normal form of degree \( s \) up to terms of degree \( s + 1 \). In particular,

\[
H(p, q) = \mathcal{H}_s(P, Q) + R_{s+1}(P, Q),
\]

where \( R_{s+1}(P, Q) = O \left( (\|P\| + \|Q\|)^{s+1} \right) \).

During the proof of this statement, one has to pay attention to the imaginary part and the coefficients \( c_{k,l} \) and \( d_{k,l} \) might be complex.

**Proof by author.** This proof is similar to the one of Theorem 3.5 and uses the same approach. Again this theorem is proven by inductive increase of the degree \( 2 < N \leq s \) up to which \( H \) has already been reduced to a Birkhoff-like normal form. Hence, assume \( H_{N-1} \) is a Birkhoff normal form of degree \( N - 1 \) greater than two in appropriate coordinates and

\[
H = H_2 + H_{N-1} + H_N + \ldots + R_{s+1} \quad (3.7)
\]

in these coordinates.

It is sufficient to construct a generating function of a canonical transformation which reduces the homogeneous term \( H_N \) to Birkhoff normal form. In order to do this, one can at first apply a general canonical transformation to \( H_2 \). Then choose that transformation’s coefficients appropriate, so that terms of \( H_N \), which are not symmetrical in exchanging of \( p_j \) and \( q_j \), are eliminated.

The quadratic part (3.6) is already a Birkhoff normal form, so \( H_2 \) equals \( H_2 \) and the first transformation does not change the coordinates \((p, q)\).

Using again complex coordinates \( z_j := q_j + i p_j \) and \( w_j := q_j - i p_j (j = 2, \ldots, n) \) and \( z_1 := \tilde{p}_1, w_1 := \tilde{q}_1 \), the quadratic part changes into

\[
\mathcal{H}_2(z, w) = \lambda z_1 w_1 + \sum_{j=2}^{n} i \omega_j z_j w_j.
\]

Upon passing to these coordinates, one must again multiply the Hamiltonian with \( 2i \) due to the Wirtinger derivatives.

In order to transform the canonical coordinates \((z, w) \mapsto (Z, W)\), one can use the following generating function \((Z, W) + S_N(Z, W)\), where \( S_N \) describes a
homogeneous polynomial of degree $N$ in $(\mathbf{Z}, \mathbf{w}) = (Z_1, \ldots, Z_n, w_1, \ldots, w_n)$. The transformation with generating function $(\mathbf{Z}, \mathbf{w}) + S_N(\mathbf{Z}, \mathbf{w})$ changes only terms of order $N$ or higher and one obtains the new coordinates according to Section 10.5 in Knauf [2012] by (3.2) using the same notation for the general polynomials as (3.3) and (3.4).

The coefficients $c_{k,l} \in \mathbb{R}$ of $H_N$ result from the Taylor expansion of $H$ at 0 and $d_{k,l} \in \mathbb{R}$ are going to be defined below such that terms of $H_N$ which could not yet be written in $J_j$, are eliminated. The notation used above is similar to Theorem 3.5.

The Hamiltonian $\mathcal{H}_2$ is converted with $(\mathbf{Z}, \mathbf{w}) + S_N(\mathbf{Z}, \mathbf{w})$ into

$$\mathcal{H}_2(\mathbf{z}, \mathbf{w}) = \mathcal{H}_2(\mathbf{Z}, \mathbf{W})$$

$$+ \left( \lambda (l_1 - k_1) + \sum_{j=2}^{n} i \omega_j (l_j - k_j) \right) \left( W_j \frac{\partial S_N}{\partial w_j} - Z_j \frac{\partial S_N}{\partial Z_j} \right)$$

$$- R_{2N-2}(\mathbf{Z}, \mathbf{W})$$

$$= \mathcal{H}_2(\mathbf{Z}, \mathbf{W})$$

$$+ \sum_{(k,l) \in K_N} d_{k,l} \left( \lambda (l_1 - k_1) + \sum_{j=2}^{n} i \omega_j (l_j - k_j) \right) M_{k,l}(\mathbf{Z}, \mathbf{W})$$

$$- R_{2N-2}(\mathbf{Z}, \mathbf{W}).$$

The resulting error of passing from $\frac{\partial S_N}{\partial w_j}$ to $\frac{\partial S_N}{\partial Z_j}$ has degree larger than $N$ and is included in the term $R_{2N-2}(\mathbf{Z}, \mathbf{W})$. Since the frequencies do not satisfy a resonance relation of order $s$ and smaller, the coefficients $\lambda (l_1 - k_1) + \sum_{j=2}^{n} i \omega_j (l_j - k_j)$ is equal to zero if and only if $k_j = l_j$ for all $j$. Every homogeneous term, which fulfills that depends only on the action coordinates $I = P_1 Q_1$ and $J_j (j = 2, \ldots, n)$.

By defining

$$d_{k,l} := \frac{-c_{k,l}}{\lambda (l_1 - k_1) + \sum_{j=2}^{n} i \omega_j (l_j - k_j)} \text{ if } \lambda (l_1 - k_1) + \sum_{j=2}^{n} i \omega_j (l_j - k_j) \neq 0$$

with $c_{k,l}$ from similar to (3.3) and equal to zero otherwise, the terms of these homogeneous polynomials $M_{k,l}$, which could not yet be written in $I$ and $J_j$, are eliminated by this transformation.

Altogether, the whole Hamiltonian (3.7) becomes a Birkhoff normal form of degree $N$ with the transformation and above defined coefficients $d_{k,l}$

$$H(\mathbf{z}, \mathbf{w}) = \mathcal{H}_2(\mathbf{Z}, \mathbf{W}) + \mathcal{H}_N(\mathbf{Z}, \mathbf{W}) + \tilde{H}_{N+1}(\mathbf{Z}, \mathbf{W}) + \ldots + \tilde{R}_{s+1}(\mathbf{Z}, \mathbf{W}).$$

Here $\tilde{H}_{N+1}$ and $\tilde{R}_{s+1}$ denotes the transformed part of degree $N + 1$ respectively the transformed rest term of degree $s + 1$ by $S_N$. Setting $N = 3, \ldots, s$ inductively yields transformation of higher order and concludes this proof. \qed
The truncated Birkhoff normal form is integrable up to every degree, it only depends on the action coordinates $I, J_j$ (constants of motion). However, the remaining term $R_{s+1}$ might, additionally, depend on the conjugated angle coordinates $\varphi_j$ ($j = 1, \ldots, n$).

4 The quadratic part of Hill’s Lunar Problem

With the Birkhoff-like normal form for a Hamiltonian with a saddle-center-. . .-center equilibrium (see Section 3.2), the dynamic of Hill’s equations around the Lagrange points can be analyzed. In Uzer et al. [2002] the necessary submanifolds are defined, however, the proofs in this section for the quadratic part $H_2$ of Hill’s Lunar Problem around $L_1$ (3.6) were added to the statements of that article.

In contrast to (2.2), Hill’s Lunar Problem is described in Waalkens et al. [2005] by the following Hamiltonian

$$H : \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}$$

$$H(p, q) = \frac{1}{2}||p||^2 + p_1q_2 - p_2q_1 - q_1^2 + \frac{1}{2}(q_2^2 + q_3^2) - \frac{3}{||q||}$$

which leads to the vector field

$$\left( \begin{array}{c} \dot{p} \\ \dot{q} \end{array} \right) = X(p, q) = \mathbb{J} \nabla H(p, q) = \left( \begin{array}{c} p_2 + 2q_1 - \frac{3q_1}{||q||^3} \\ -p_1 - q_2 - \frac{3q_2}{||q||^3} \\ -q_3 - \frac{3q_3}{||q||^3} \\ p_1 + q_2 \\ p_2 - q_1 \\ p_3 \end{array} \right).$$

In this definition, the term $-\frac{3}{||q||}$ differs from $-\frac{1}{||q||}$ in (2.2). Both notations are common in literature and one can transfer (4.1) into (2.2) by using the coordinates

$$Y_j = \left( \frac{1}{3} \right)^{1/3} p_j, \quad X_j = \left( \frac{1}{3} \right)^{1/3} q_j, \quad H_H = \left( \frac{1}{3} \right)^{2/3} H.$$

**Remark 4.1.** In contrast to the circular restricted Three-Body Problem (see Figure 11.3.3 in Knauf [2012]), this system has only two remaining equilibria at

$L_1 = (0, -1, 0, -1, 0, 0), \quad L_2 = (0, 1, 0, 1, 0, 0), \quad H(L_j) = -4.5 \quad (j = 1, 2)$

and the linearization of the vector field at these points has eigenvalues $\pm \lambda, \pm i \omega_1, \pm i \omega_2$ with

$$\lambda = \sqrt{2\sqrt{7} + 1}, \quad \omega_2 = 2, \quad \omega_3 = \sqrt{2\sqrt{7} - 1}.$$

The equilibria define the places where the centrifugal force neglects the gravitational one. $L_2$ lies in configuration space between the primaries on the $q_1$-axis,
whereas $L_1$ is on the left side of the lighter primary (earth) on the $q_1$-axis. Both of them are a saddle-center-center type according to the eigenvalues above.

The eigenvalues do not satisfy any resonance relation, therefore, the Hamiltonian can be reduced to a Birkhoff normal form up to arbitrary finite degree. Above Hamiltonian (4.1) is invariant under following smooth transformation of the phase space

$$S: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$$

$$S(p, q) := (p_1, -p_2, -p_3, -q_1, q_2, q_3),$$

with $S(L_1) = L_2$. However, it is not invariant under time reversal due to the rotating coordinate system and the mixed terms $p_1q_2 - p_2q_1$.

**Lemma 4.2** (statement by Waalkens et al. [2005]). The sets with $p_3 = 0 = q_3$ or $P_3 = 0 = Q_3$ are invariant for $H$ respectively the truncated Birkhoff normal form $\mathcal{H}_2$ of degree 2 and defines the planar Hill Problem with two degrees of freedom.

**Proof by author.** This statement is a direct result from the corresponding ODE

$$\dot{p}_3 = -q_3 - \frac{3q_3}{\|q\|} \quad \text{and} \quad \dot{q}_3 = p_3,$$

$$\dot{P}_3 = -\omega_3 Q_3 \quad \text{and} \quad \dot{Q}_3 = \omega_3 P_3.$$

Therefore, $p_3 = 0 = q_3$ and $P_3 = 0 = Q_3$ for all time.

**Remark 4.3.** The linearization of the vector field at the equilibria in the planar case ($p_3 = 0 = q_0$) has the following eigenvalues

$$\pm \sqrt{2\sqrt{7} + 1}, \quad \pm i \sqrt{2\sqrt{7} + 1},$$

in contrast to Remark 4.1.

### 4.1 Submanifolds of the phase space

The truncated Birkhoff-like normal form of Hill’s Lunar Problem $H$ of degree two at the equilibrium $0$ has been derived in Section 3.2. The Hamiltonian in appropriate coordinates is (3.6)

$$\mathcal{H}_2(P, Q) = \lambda P_1 Q_1 + \frac{1}{2} \sum_{j=2}^{3} \omega_j (P_j^2 + Q_j^2) = \lambda I + \sum_{j=2}^{3} \omega_j J_j.$$

The corresponding ODE is as follows

$$\dot{P}_1 = -\lambda P_1, \quad \dot{Q}_1 = \lambda Q_1, \quad \dot{P}_j = -\omega_j Q_j, \quad \dot{Q}_j = \omega_j P_j \text{ for } j = 2, 3.$$  

(4.4)
It results in the following flow for all $t \in \mathbb{R}$

$$
\begin{align*}
P_1(t) &= e^{-\lambda t} P_1(0), \quad Q_1(t) = e^{\lambda t} Q_1(0), \\
P_j(t) &= \cos(\omega_j t) P_j(0) - \sin(\omega_j t) Q_j(0), \\
Q_j(t) &= \sin(\omega_j t) P_j(0) + \cos(\omega_j t) Q_j(0) \quad \text{for } j = 2, 3.
\end{align*}
$$

(4.5)

Altogether, the flow is for $x \in \mathbb{R}^3 \times \mathbb{R}^3$, $t \in \mathbb{R}$

$$
\Phi_t(x) = \left( e^{-\lambda t} x_1, \cos(\omega_2 t)x_2 - \sin(\omega_2 t)x_5, \cos(\omega_3 t)x_3 - \sin(\omega_3 t)x_6, \\
e^{\lambda t} x_4, \sin(\omega_2 t)x_2 + \cos(\omega_2 t)x_5, \sin(\omega_3 t)x_3 + \cos(\omega_3 t)x_6 \right).
$$

It is evident that $I$ and $J_j$ are constants of motion

$$
\begin{align*}
\dot{I} &= \lambda \left( \dot{P}_1 Q_1 + \dot{P}_1 Q_1 \right) = \lambda (\dot{P}_1 Q_1 + \lambda P_1 Q_1) = 0, \\
\dot{J}_j &= \omega_j \left( \dot{P}_j P_j + \dot{Q}_j Q_j \right) = \omega_j (\dot{Q}_j P_j + \omega_j P_j Q_j) = 0 \quad \text{for } j = 2, 3.
\end{align*}
$$

As a result of the eigenvalues calculated in Remark 4.1, the equilibrium $L_1$ is of saddle-center-center type with energy $E_0$. What is more, the $P_1 - Q_1$-plane is the saddle plane and the area with $P_1 > Q_1$ is defined as the “unbounded” part, the area with $P_1 < Q_1$ as the “bounded” part (see Figure 4.4). In these terms, a secondary’s (moon) transition from $P_1 > Q_1$ to $P_1 < Q_1$ describes an orbit, which is captured inside the planetary neighborhood of the second primary (earth). In contrast, the secondary escapes the neighborhood during the contrary transition.

The secondary mass can only traverse from one side to the other through a passage in the phase space which depends on the energy. If the energy is slightly higher than $E_0$, the passage is small. Due to the fact that $E_0$ is the energy of the equilibrium, the possible impulse’s range is small near that point (the energy is almost equal to the energy at the equilibrium $E_0$).

### 4.1.1 Energy surface

For regular values $E \gtrless E_0$ of $\mathcal{H}_2$, the energy surface is a $(2n - 1) = 5$-dimensional submanifold $\Sigma_E$ of the phase space (see A.46 in Knauf [2012])

$$
\Sigma_E := \left\{ (P, Q) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \mathcal{H}_2(P, Q) \equiv E \right\} = \mathcal{H}_2^{-1}(E).
$$

Using the constant of motion $I$ and $J_j$, it can be written as

$$
\left\{ (P, Q) \mid \lambda P_1 Q_1 = E_1, \quad \frac{1}{2} \omega_j (P_j^2 + Q_j^2) = E_j \quad (j = 2, 3) \quad \text{with } \sum_{j=1}^{3} E_j = E \right\}.
$$

**Lemma 4.4** (statement by Waalkens et al. [2005]). The secondary mass (moon) can escape from the planetary neighborhood or can be captured inside the neighborhood only for energies $E_1 > 0$ (see Figure 4.4).
**Proof by author.** Let the energy $E_1 > 0$, then this statement is a direct result of sign $(P_1(0)) = \text{sign} (Q_1(0))$ together with the flow (4.5). It is obvious that the sign of $P_1(t)$ and $Q_1(t)$ remains constant and for $P_1(0) \neq 0 \neq Q_1(0)$ there exists a time $t_0 := \frac{1}{2\lambda} \ln \left(\frac{P_1(0)}{Q_1(0)}\right)$, so that $P_1(t_0) = Q_1(t_0)$. For time $t_1$ greater or less than $t_0$, the flow is either inside the neighborhood or outside.

In contrast, negative energies $E_1 < 0$ yield $\text{sign} (P_1(0)) = -\text{sign} (Q_1(0))$ and the secondary either remains inside or outside the planetary neighborhood for all time $t$, since the flow cannot traverse from the area with $P_1 > Q_1$ to $P_1 < Q_1$ or backwards (the sign is constant for all time according to the flow (4.5)). □

**Lemma 4.5** (statement by Waalkens et al. [2005]). *Locally the energy surface $\Sigma_E$ is isomorph to $I \times S^{2n-2} = I \times S^4$ where $I \subset \mathbb{R}$ is an interval.*

**Proof by author.**

- This interval is given by
  \[
  I \subset \mathbb{R} \text{ if } E \geq 0,
  \]
  \[
  I \subset \left[\sqrt{\frac{2}{\lambda}}|E|, \infty\right] \text{ or } I \subset \left(-\infty, -\sqrt{\frac{2}{\lambda}}|E|\right] \text{ if } E < 0.
  \]

The Lemma can be proven more easily, if the normal form $\mathcal{H}_2$ is analyzed within a coordinate system, which has not been rotated clockwise. That is $\mathcal{H}_2(\mathbf{P}, \mathbf{Q}) = \frac{1}{2} \lambda (P_1^2 - Q_1^2) + \frac{1}{2} \sum_{j=2}^{3} \omega_j \left(P_j^2 + Q_j^2\right)$ (3.5).

Consider the following smooth map for a $c \in I$, where $S^4_{\sqrt{E + 0.5\lambda c^2}}$ denotes the 4-dimensional sphere with radius $\sqrt{E + 0.5\lambda c^2}$

\[
F: \Sigma_E \cap \{Q_1 = c\} \rightarrow S^4_{\sqrt{E + 0.5\lambda c^2}}
\]

\[
F(\mathbf{P}, \mathbf{Q}) := \left(\sqrt{\frac{\lambda}{2}} P_1, \sqrt{\frac{\omega_2}{2}} P_2, \sqrt{\frac{\omega_3}{2}} P_3, \sqrt{\frac{\omega_2}{2}} Q_2, \sqrt{\frac{\omega_3}{2}} Q_3\right).
\]

The map is well-defined, because $\frac{1}{2} \lambda P_1^2 + \frac{1}{2} \sum_{j=2}^{3} \omega_j \left(P_j^2 + Q_j^2\right) = E$ and $\frac{1}{2} \lambda Q_1^2 = E + \frac{1}{2} \lambda c^2$. With the smooth inverse map of $F$

\[
F^{-1}: S^4_{\sqrt{E + 0.5\lambda c^2}} \rightarrow \Sigma_E \cap \{Q_1 = c\}
\]

\[
F^{-1}(P_1, P_2, P_3, Q_2, Q_3) := \left(\sqrt{\frac{\lambda}{2}} P_1, \sqrt{\frac{2}{\omega_2}} P_2, \sqrt{\frac{2}{\omega_3}} P_3, c, \sqrt{\frac{2}{\omega_2}} Q_2, \sqrt{\frac{2}{\omega_3}} Q_3\right)
\]

the isomorphism $\Sigma_E \cap \{Q_1 = c\} \cong S^4_{\sqrt{E + 0.5\lambda c^2}}$ is true for every $c \in I$.

- As a matter of fact, this isomorphism is not only true for the quadratic part of the Birkhoff normal form, but also for $H$ and the submanifold $H^{-1}(E)$ within an appropriate neighborhood around $L_1$. Denote the remaining terms of degree greater than two with $R_3$, then Taylor expansion at $L_1$ and $H_2 = \mathcal{H}_2$ (see proof of Theorem 3.6) yield

\[
H(\mathbf{x}) = E_0 + \mathcal{H}_2(\mathbf{x}) + R_3(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})
\]
The quadratic part’s signature is $(5, 1)$ and can be written as
\[ \mathcal{H}_2(x) = \frac{1}{2} \langle x, Ax \rangle \] with 
\[ A = \text{diag} (\lambda, \omega_2, \omega_3, -\lambda, \omega_2, \omega_3) \in \text{GL}(6, \mathbb{R}). \]

Matrix $D^2 \mathcal{H}_2 = A$ has maximal rank and the equilibrium is non-degenerated. The norm of the gradient $\nabla \mathcal{H}_2(x) = Ax$ for $x \in \mathbb{R}^3 \times \mathbb{R}^3$ can be estimated with
\[ \| \nabla \mathcal{H}_2(x) \| \geq s \| x \| \] where 
\[ s := \min |\text{spec}(A)| > 0. \]

The inequality above leads to the following result for the Hamiltonian $H$ and $c > 0$ with $x \in \mathbb{R}^3 \times \{0\}$
\[ \| \nabla H(x) \| = \| \nabla \mathcal{H}_2(x) + \nabla R_3(x) \| \geq \| \nabla \mathcal{H}_2(x) \| - \| \nabla R_3(x) \| \] as long as 
\[ s > c > \| x \|. \]

Therefore, $L_1$ is the only critical point within an appropriate neighborhood and it is also non-degenerated for $H$. With Morse’s lemma G.6 in Knauf [2012] a map $\varphi \in C^1(U, V)$ can be defined, in which $U \subset \mathbb{R}^3 \times \{0\}$ is an open neighborhood of $L_1$ (equilibrium with index 1). This map fulfills
\[ H \circ \varphi^{-1}(x) = E_0 - x_1^2 + \sum_{j=2}^{6} x_j^2 \text{ for } x \in V := \varphi(U). \]

Using the arguments from above, the level surfaces of
\[ \tilde{\mathcal{H}}_2(x) := H \circ \varphi^{-1}(x) = E_0 + \left( -x_1^2 + x_2^2 \right) + \left( x_3^2 + x_5^2 \right) + \left( x_4^2 + x_6^2 \right) \]
are isomorphic to $I \times S^4$ as well.

### 4.1.2 Normally hyperbolic invariant manifold

Setting $P_1 = 0 = Q_1$ in $\Sigma_E$ defines a normally hyperbolic invariant manifold
\[ S_{\text{NHIM}} := \{ (P, Q) \in \Sigma_E \mid P_1 = 0 = Q_1 \}, \]
which is a $(2n - 3)=3$-dimensional sphere.

The corresponding stable and unstable manifolds (four dimensional cylinders) are given by
\[ W_s \left( S_{\text{NHIM}} \right) = \{ (P, Q) \in \Sigma_E \mid Q_1 = 0 \}, \]
\[ W_u \left( S_{\text{NHIM}} \right) = \{ (P, Q) \in \Sigma_E \mid P_1 = 0 \}. \]

**Definition 4.6.** Let $M$ be a smooth manifold and $f$ a diffeomorphism on $M$. A $f$-invariant submanifold $U \subset M$ is called normally hyperbolic if
\[ T_U M = TU \oplus E_s \oplus E_u. \]
In the notation above, $E_s$ respectively $E_u$ are the $T_f$-invariant stable and unstable vector bundles. Furthermore, there must exist constants $0 < \frac{1}{\mu} < \nu < 1$, $c > 0$, so that

\begin{align*}
\|Tf^t(x)\| \leq c\|x\| & \text{ for all } x \in TU \text{ and } t \in \mathbb{R}, \\
\|Tf^t(x)\| \leq c\nu^t \|x\| & \text{ for all } x \in E_s \text{ and } t > 0, \\
\|Tf^{-t}(x)\| \leq c\nu^t \|x\| & \text{ for all } x \in E_u \text{ and } t > 0.
\end{align*}

Lemma 4.7 (statement by Waalkens et al. [2005]). The set $S^3_{\text{NHIM}} := \{(P, Q) \in \Sigma_E \mid P_1 = 0 = Q_1\}$ is normally hyperbolic and $\mathcal{H}_2$-invariant. In order to prove that $S^3_{\text{NHIM}}$ is normally hyperbolic, one has to check the definition with $M := \Sigma_E$, $U := S^3_{\text{NHIM}}$ and $f^t := \Phi_t$. The corresponding vector bundles are

\begin{align*}
TS^3_{\text{NHIM}} \Sigma_E &= \{(u, v) \in S^3_{\text{NHIM}} \times \mathbb{R}^6 \mid d\mathcal{H}_2(u)(v) = 0\}, \\
TS^3_{\text{NHIM}} &= \{(u, v) \in S^3_{\text{NHIM}} \times \mathbb{R}^6 \mid v_1 = 0 = v_4, \ d\mathcal{H}_2(u)(v) = 0\}, \\
E_s &= \{(u, v) \in S^3_{\text{NHIM}} \times \mathbb{R}^6 \mid v_4 = 0, \ d\mathcal{H}_2(u)(v) = 0 \\
&\text{and let } \langle v, w \rangle_H = 0 \text{ for all } w \in TuS^3_{\text{NHIM}}\}, \\
E_u &= \{(u, v) \in S^3_{\text{NHIM}} \times \mathbb{R}^6 \mid v_1 = 0, \ d\mathcal{H}_2(u)(v) = 0 \\
&\text{and let } \langle v, w \rangle_H = 0 \text{ for all } w \in TuS^3_{\text{NHIM}}\}.
\end{align*}

Here $\langle u, v \rangle_H := \lambda(u_1v_1 + u_4v_4) + \omega_2(u_2v_2 + u_5v_5) + \omega_3(u_3v_3 + u_6v_6)$ denotes a scalar product on $\mathbb{R}^6$.

\textbf{Proof by author.} \bullet Since $P_1 = 0 = Q_1$, the manifold $S^3_{\text{NHIM}}$ is $\mathcal{H}_2$-invariant.

As an example, the following arguments show the derivation of the stable manifold’s definition. One has to claim $v_4 = 0$, so that there is no direction left which leads away from $W_s(S^3_{\text{NHIM}})$. Additionally, $E_s$ has to be a subset of $TS^3_{\text{NHIM}} \Sigma_E$, so $d\mathcal{H}_2(u)(v) = 0.$ By using following sum

\begin{align*}
TS^3_{\text{NHIM}} \Sigma_E = TS^3_{\text{NHIM}} \oplus (TS^3_{\text{NHIM}})^\perp,
\end{align*}

where

\begin{align*}
(TS^3_{\text{NHIM}})^\perp &= \{(u, v) \in S^3_{\text{NHIM}} \times \mathbb{R}^6 \mid d\mathcal{H}_2(u)(v) = 0 \\
&\text{and let } \langle v, w \rangle_H = 0 \text{ for all } w \in TuS^3_{\text{NHIM}}\},
\end{align*}

one has to claim that for all $w \in TuS^3_{\text{NHIM}}$ the equation $\langle v, w \rangle_H = 0$ holds. That ensures that $E_s$ is a subset of $(TS^3_{\text{NHIM}})^\perp$.

\textbullet The total derivative $d\mathcal{H}_2(u)(v) = \sum_{j=1}^6 \frac{\partial \mathcal{H}_2}{\partial x_j}(u)v_j$ of $\mathcal{H}_2$ at $u \in \mathbb{R}^6$ in direction $v \in \mathbb{R}^6$ is

\begin{align*}
d\mathcal{H}_2(u)(v) &= \lambda(u_4v_1 + u_1v_4) + \omega_2(u_2v_2 + u_5v_5) + \omega_3(u_3v_3 + u_6v_6).
\end{align*}
Let \((u, v) \in S^3_{\text{NHIM}} \times \mathbb{R}^6\) then it becomes
\[
d\mathcal{H}_2(u)(v) = \omega_2(u_2v_2 + u_5v_5) + \omega_3(u_3v_3 + u_6v_6).
\]

For \((u, v) \in E_s\) according to its definition, it is claimed that for every \(w \in T_uS^3_{\text{NHIM}}\) (that is, every \(w \in \mathbb{R}^6\) with \(w_1 = 0 = w_4\) and \(d\mathcal{H}_2(u)(w) = 0 = (u, w)_H\)) the equation \((v, w)_H = 0\) holds. One can prove that this is true if and only if \(v_2 = v_3 = v_5 = v_6 = 0\); Choose \((u, v) \in E_s\) generally
\[
u = (0, u_2, u_3, 0, u_5, u_6), \quad v = (v_1, v_2, v_3, 0, v_5, v_6).
\]

Write \(v = \tilde{v}_1 + \tilde{v}_2\) where
\[
\tilde{v}_1 = (v_1, 0, 0, 0, 0, 0), \quad \tilde{v}_2 = (0, v_2, v_3, 0, v_5, v_6).
\]

Obviously \(\tilde{v}_2 \in T_uS^3_{\text{NHIM}}\), since \(d\mathcal{H}_2(u)(\tilde{v}_2) = d\mathcal{H}_2(u)(v) = 0\) and the first and fourth entry is zero. Since \((u, v) \in E_s\) it must be true that for all \(w \in T_uS^3_{\text{NHIM}}\), and, in particular for \(\tilde{v}_2\), the scalar product \((v, w)_H\) is zero. That is
\[
(v, \tilde{v}_2)_H = (\tilde{v}_1, \tilde{v}_2)_H + (\tilde{v}_2, \tilde{v}_2)_H = 0 + \|\tilde{v}_2\|_H^2 = 0
\]
and, hence, \(\tilde{v}_2 = 0\). On the whole,
\[
E_s = \left\{ (u, v) \in T_{S^3_{\text{NHIM}}} \Sigma_E \mid v_2 = v_3 = v_4 = v_5 = v_6 = 0 \right\}
\]
and only \(v_1\) is not pre-set yet. The same arguments for
\[
E_u = \left\{ (u, v) \in T_{S^3_{\text{NHIM}}} \Sigma_E \mid v_1 = v_2 = v_3 = v_5 = v_6 = 0 \right\}
\]
leads to the fact that \(v_1\) is not pre-defined. Both \(E_s\) and \(E_u\) are 1-dimensional, whereas \(T_{S^3_{\text{NHIM}}} \Sigma_E\) has dimension eight and \(T S^3_{\text{NHIM}}\) has dimension six.

• The corresponding tangential function of \(\Phi_t\) at \((u, v) \in T \Sigma_E\) maps \(T_u \Sigma_E\) onto \(T_{\Phi_t(u)} \Sigma_E\) and is given by
\[
T_u \Phi_t(v) = (\Phi_t(u), D\Phi_t(u)(v))
\]
where
\[
D\Phi_t(u)(v) = \begin{pmatrix}
e^{-\lambda t}v_1, \cos(\omega_2 t)v_2 - \sin(\omega_2 t)v_5, \cos(\omega_3 t)v_3 - \sin(\omega_3 t)v_6, \\
e^{\lambda t}v_4, \sin(\omega_2 t)v_2 + \cos(\omega_2 t)v_5, \sin(\omega_3 t)v_3 + \cos(\omega_3 t)v_6
\end{pmatrix}.
\]

Both the stable \(E_s\) and the unstable \(E_u\) manifolds are \(T \Phi\)-invariant and the map’s norm is given by
\[
\|T_u \Phi_t(v)\|^2 = e^{-2\lambda t}v_1^2 + v_2^2 + v_3^2 + e^{2\lambda t}v_4^2 + v_5^2 + v_6^2.
\]
For \((u, v) \in TS^3_{NHIM}\) that formula changes into
\[
\|T_u \Phi_t(v)\|^2 = v_2^2 + v_3^2 + v_5^2 + v_6^2 = \|v\|^2 \leq \mu^2 |t| \|v\|^2.
\]
For \((u, v) \in E_s\) that formula becomes
\[
\|T_u \Phi_t(v)\|^2 = e^{-2\lambda t} v_1^2.
\]
For \((u, v) \in E_u\) that formula is
\[
\|T_u \Phi_t(v)\|^2 = e^{-2\lambda t} v_4^2.
\]
On the whole, the Definition 4.6 is fulfilled with \(c = 1, \nu = e^{-\lambda} < 1\) and a \(\mu > e^\lambda > 1\). Therefore, \(S^3_{NHIM}\) is normally hyperbolic.

Lastly, one has to prove that \(W_s(S^3_{NHIM})\) and \(W_u(S^3_{NHIM})\) define the stable respectively unstable manifolds of \(S^3_{NHIM}\). For every \((P(0), Q(0)) \in W_s(S^3_{NHIM})\) the following is true
\[
P_1(t) = e^{-\lambda t} P_1(0) \to 0 \text{ if } t \to \infty \text{ and } Q_1(t) = e^{\lambda t} Q_1(0) \equiv 0.
\]
That is, every trajectory with such initial points converge towards \(S^3_{NHIM}\) in positive time.

In contrast, for all \((P(0), Q(0)) \in W_u(S^3_{NHIM})\) one can compute
\[
Q_1(t) \to 0 \text{ if } t \to -\infty \text{ and } P_1(t) = e^{-\lambda t} P_1(0) \equiv 0.
\]
Hence, trajectories with such initial point converge towards \(S^3_{NHIM}\) in negative time.

Using the same arguments like the ones for the energy surface (see proof of Lemma 4.5)
\[
\{ (p, q) \in H^{-1}(E) \mid p_1 = 0 = q_1 \}
\]
is a normally hyperbolic invariant manifold of the whole Hamiltonian \(H\) within certain neighborhood around \(L_1\). With Morse’s lemma, the Hamiltonian can be written in appropriate coordinates defined on a neighborhood \(U \subset \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\) of \(L_1 = 0\) as
\[
\tilde{H}_2(x) := H \circ \varphi^{-1}(x) = E_0 + (-x_1^2 + x_2^2) + (x_3^2 + x_5^2) + (x_4^2 + x_6^2),
\]
where \(x \in V := \varphi(U)\).

Altogether, similar arguments as above result locally in a normally hyperbolic invariant manifold for \(H\) in appropriate coordinates. \qed
Considering the direction of the flow, the unstable and stable manifold can be further subdivided into cylinders

\[
W_s^\pm (S^3_{\text{NHIM}}) := \{ (P, Q) \in W_s (S^3_{\text{NHIM}}) | \pm P_1 \geq 0 \},
\]

\[
W_u^\pm (S^3_{\text{NHIM}}) := \{ (P, Q) \in W_u (S^3_{\text{NHIM}}) | \pm Q_1 \geq 0 \}.
\]

The area of \( W_s (S^3_{\text{NHIM}}) \) with \( P_1(0) > 0 \) consists of initial points which will converge towards \( S^3_{\text{NHIM}} \). Due to \( P_1(0) > Q_1(0) = 0 \) and the stability \( (P_1(t) \to 0) \) for positive time, they start within the unbounded part and converge towards \( S^3_{\text{NHIM}} \) and remain there (invariant) in positive time.

With similar argumentation, initial points within the area \( P_1(0) < Q_1(0) = 0 \) converge inside the neighborhood towards \( S^3_{\text{NHIM}} \) and stay there as well.

The same is true for \( W_u (S^3_{\text{NHIM}}) \) and \( Q_1(0) > P_1(0) = 0 \) (initial points that converge inside the neighborhood towards \( S^3_{\text{NHIM}} \)) respectively \( Q_1(0) < P_1(0) = 0 \) (points which converge outside towards \( S^3_{\text{NHIM}} \)) in negative time due to the stability.

**Lemma 4.8** (statement by Waalkens et al. [2005]). The area of \( \Sigma_E \) consisting of all initial points, which will escape from the planetary neighborhood eventually or escaped it already, is bounded by

\[
W_s^- (S^3_{\text{NHIM}}) \cup W_u^- (S^3_{\text{NHIM}}).
\]

Whereas the area consisting of all initial points, that will be captured or have been captured, is bounded by

\[
W_s^+ (S^3_{\text{NHIM}}) \cup W_u^+ (S^3_{\text{NHIM}}).
\]

These are not manifolds anymore, because \( W_u \) is transversal to \( W_s \) (see Figure 4.2).

**Proof by author.**

- The energy surface has dimension five, both \( W_s^\pm (S^3_{\text{NHIM}}) \) and \( W_u^\pm (S^3_{\text{NHIM}}) \) have dimension four, hence, the unions’ codimension are one and both surround a subset of \( \Sigma_E \).

Let \( M \) be a connected manifold of dimension \( n \) and \( U \subset M \) a submanifold with codimension one. Two points \( x, y \in M \setminus U \) inside the complement can be connected within \( M \) by a curve which does not intersect with \( U \). In contrast, the possible boundary \( \partial V \) of a submanifold \( V \subset M \) of codimension zero has codimension one. So in some cases, submanifolds of codimension one might separate areas.

- Defining the following subspaces, which contain all initial points that will escape or escaped \( A_e \) or will be captured or have been captured \( A_c \) (that is, in positive or negative time)

\[
A_c := \{ (P, Q) \in \Sigma_E | P_1, Q_1 > 0 \}, \quad A_e := \{ (P, Q) \in \Sigma_E | P_1, Q_1 < 0 \}.
\]
Initial points with \( \text{sign} (P_1(0)) = -\text{sign} (Q_1(0)) \) cannot traverse from one side to the other, according to the the section about the energy surface 4.1.1, hence, the signs have to be equal. The secondary mass (moon) starts inside the unbounded area, if \( (P(0), Q(0)) \in A_c \) and \( P_1(0) \geq Q_1(0) > 0 \). The dynamic \( P_1(t) = e^{-\lambda t} P_1(0) \) as well as \( Q_1(t) = e^{\lambda t} Q_1(0) \) yields the convergence for \( t \to \infty \) towards zero in \( P_1 \)-direction and towards infinity in \( Q_1 \)-direction. In particular, there exists a time \( t_0 := \frac{1}{2\lambda} \ln \frac{P_1(0)}{Q_1(0)} > 0 \) so that \( P_1(t_0) = Q_1(t_0) \). What is more \( 0 < P_1(t_1) < Q_1(t_1) \) if \( t_1 > t_0 \), that is the secondary mass is then located inside the bounded area.

One can see that trajectories with initial points \( (P(0), Q(0)) \in A_c \) and \( 0 < P_1(0) < Q_1(0) \) inside the bounded area, have been captured in negative time, with similar arguments. For \( t_1 < t_0 < 0 \) is \( P_1(t_1) > Q_1(t_1) > 0 \) and the flow diverges towards infinity in \( P_1 \)-direction and converges to zero in \( Q_1 \)-direction in the limit \( t \to -\infty \). Altogether, the secondary mass has traversed from the unbounded to the bounded area in the past.

The same applies to initial points inside \( A_e \) with \( P_1(0) \leq Q_1(0) < 0 \) which will escape the planetary neighborhood in the future and initial points inside \( A_e \) with \( 0 > P_1(0) > Q_1(0) \) which have been traversed from the neighborhood into the unbounded space in the past.

The boundaries of \( A_c \) and \( A_e \) are given by

\[
\partial A_c = \{ (P, Q) \in \Sigma_E \mid P_1 \geq 0, \, Q_1 = 0 \} \cup \{ (P, Q) \in \Sigma_E \mid P_1 = 0, \, Q_1 \geq 0 \}, \\
\partial A_e = \{ (P, Q) \in \Sigma_E \mid P_1 \leq 0, \, Q_1 = 0 \} \cup \{ (P, Q) \in \Sigma_E \mid P_1 = 0, \, Q_1 \leq 0 \}
\]

\[
= W_s^+ (S_{\text{NHIM}}) \cup W_u^+ (S_{\text{NHIM}}) \\
= W_s^- (S_{\text{NHIM}}) \cup W_u^- (S_{\text{NHIM}})
\]

Hence, the unions \( W_s^+ (S_{\text{NHIM}}) \cup W_u^+ (S_{\text{NHIM}}) \) respectively \( W_s^- (S_{\text{NHIM}}) \cup W_u^- (S_{\text{NHIM}}) \) surround every initial points, which will escape or will be captured in positive or negative time.

\[\bullet\] Lastly, one can prove that \( W_s (S_{\text{NHIM}}) \subset \Sigma_E \) and \( W_u (S_{\text{NHIM}}) \subset \Sigma_E \) are transversal, in particular,

\[ T_u \Sigma_E = T_u W_s (S_{\text{NHIM}}) + T_u W_u (S_{\text{NHIM}}) \]

for all \( \mathbf{u} \in W_s (S_{\text{NHIM}}) \cap W_u (S_{\text{NHIM}}) = S_{\text{NHIM}} \). This is true since

\[ T_{S_{\text{NHIM}}} \Sigma_E = \{ (\mathbf{u}, \mathbf{v}) \in S_{\text{NHIM}} \times \mathbb{R}^6 \mid d\mathcal{H}_2(\mathbf{u})(\mathbf{v}) = 0 \}, \]

\[ T_{S_{\text{NHIM}}} W_s = \{ (\mathbf{u}, \mathbf{v}) \in S_{\text{NHIM}} \times \mathbb{R}^6 \mid v_4 = 0, \, d\mathcal{H}_2(\mathbf{u})(\mathbf{v}) = 0 \}, \]

\[ T_{S_{\text{NHIM}}} W_u = \{ (\mathbf{u}, \mathbf{v}) \in S_{\text{NHIM}} \times \mathbb{R}^6 \mid v_1 = 0, \, d\mathcal{H}_2(\mathbf{u})(\mathbf{v}) = 0 \}, \]

hence, the subspaces are transversal. \(\square\)
4.1.3 Dividing surface

The dividing surface \( S_{DS}^{2n-2} = S_{DS}^{4} \) (4-dimensional sphere in \( \Sigma_E \)) is given by

\[
S_{DS}^{4} := \{ (P, Q) \in \Sigma_E \ | \ P_1 = Q_1 \}.
\]

It is obvious that every trajectory which passes through this subspace either escapes the planetary neighborhood (they pass through the part with \( P_1 = Q_1 > 0 \)) or is captured inside it (by passing through the part with \( P_1 = Q_1 < 0 \)) in positive or negative time. These two parts are divided by the normally hyperbolic invariant manifold \( S_{NHIM}^{3} \subset S_{DS}^{4} \). In particular, the dividing surface parts the energy surface into two disjoint areas.

**Lemma 4.9** (statement by Waalkens et al. [2005]). Additionally, it is a so-called sphere of no return, that is every trajectory which passes \( S_{DS}^{4} \) cannot return to it (save those with initial point within \( S_{NHIM}^{3} \)).

**Proof by author.** This is a direct result of the ODE (4.4). The flow \( Q_1(t) = e^{\lambda t} Q_1(0) \) diverges towards plus or minus infinity in the limit \( t \to \infty \), depending on whether \( Q_1(0) > 0 \) or \( Q_1(0) < 0 \). Considering the same limit, \( P_1(t) \) converges towards zero, because of \( P_1(t) = e^{-\lambda t} P_1(0) \). Hence, trajectories with initial points within \( S_{DS}^{4} \setminus S_{NHIM}^{3} \) and, therefore, \( P_1(0) = Q_1(0) \neq 0 \) leave this set and can never return. □

**Lemma 4.10** (statement by Waalkens et al. [2005]). The flow \( \Phi_1 \) is transversal to \( S_{DS,0}^{4} := S_{DS}^{4} \setminus S_{NHIM}^{3} \).

**Proof by author.** For \( u \in \Sigma_E \) the Hamiltonian vector field \( X_2(u) = J \nabla H_2(u) \in T_u \Sigma_E \) (4.4) is

\[
X_2(u) = (\lambda u_1, -\omega_2 u_5, -\omega_3 u_6, \lambda u_4, \omega_2 u_2, \omega_3 u_3).
\]

If \( u \in S_{DS,0}^{4} = \{ (P, Q) \in \Sigma_E \ | \ P_1 = Q_1 \neq 0 \} \) then

\[
X_2(u) = (-\lambda u_1, -\omega_2 u_5, -\omega_3 u_6, \lambda u_4, \omega_2 u_2, \omega_3 u_3).
\]

The last vector is not part of \( T_u S_{DS,0}^{4} = \{ v \in T_u \Sigma_E \ | \ v_1 = v_4 \neq 0 \} \), which means that the vector field is transversal to \( T_u S_{DS,0}^{4} \) at every \( u \in S_{DS,0}^{4} \).

It is important to spare out the normally hyperbolic invariant manifold, because for \( u \in S_{NHIM}^{3} \subset S_{DS}^{4} \) the vector field is

\[
X_2(u) = (0, -\omega_2 u_5, -\omega_3 u_6, 0, \omega_2 u_2, \omega_3 u_3).
\]

That vector is part of \( T_u S_{DS}^{4} = \{ v \in T_u \Sigma_E \ | \ v_1 = v_4 \} \), hence, the vector field is not transversal to \( T_u S_{DS}^{4} \). □

The discussed division of the energy surface in this section is shown in Figure 4.1.
5-dimensional energy surface $\Sigma_E$

- $S^4_{DS}$ dividing surface
- $S^3_{\text{NHIM}}$

- $S^4_{DS}$: $P_1 = Q_1 > 0$
- $S^4_{DS}$: $P_1 = Q_1 < 0$

Figure 4.1: Division of the energy surface

4.2 Visualization of the flow

All of the above defined surfaces are shown on the following Figures 4.2-4.5 using the special case $P_3 = 0 = Q_3$ (see Lemma 4.2) (thus, only two remaining degrees of freedom $n = 2$) with one constant $E \gtrless E_0$. Figure 4.2 and 4.4 show the flow’s projection onto the $P_1 - Q_1$-saddle plane, whereas Figure 4.3 and 4.5 show the projection onto the $P_2 - Q_2$-center plane. For every constant chosen $E$, both coordinates $I$ and $J_2$ are defined implicitly by $H(I, 0) \equiv E$ and $H(0, J_2) \equiv E$, which means $I \equiv \frac{E}{\lambda}$ and $J_2 \equiv \frac{E}{\omega_2}$.

In Figure 4.2, the energy surface (light blue) is bounded by the hyperbola $I \equiv \frac{E}{\lambda}$ within the first and third quadrant. The sphere $S^2_{DS}$ divides the inner and the outer area and is itself halved by $S^3_{\text{NHIM}}$. The arrows at the stable and unstable manifolds indicate the flow’s dynamic. The trajectories with initial points within the second quadrant stay outside, the ones with initial points inside the fourth quadrant remain inside the planetary neighborhood, since $\lambda P_1(0)Q_1(0) = E_1 < 0$ (see Lemma 4.4).

In Figure 4.3, the energy surfaces span the whole background and the projection of $S^2_{DS}$ is the red circular plane with radius $\sqrt{2}J_2$. Since $P_1 = Q_1$ are not preset, the possible range of $J_2$ is exactly this area. In contrast, $S^3_{\text{NHIM}}$ only consists of the bordering circle line, because $P_1 = 0 = Q_1$ are already defined.

As discussed before, trajectories which will be captured or have been captured cross the part of $S^2_{DS}$ with $P_1 = Q_1 > 0$. The area consisting of these initial points is bounded by $W^+_s(S^1_{\text{NHIM}}) \cup W^+_u(S^1_{\text{NHIM}})$ and the hyperbola $I \equiv \frac{E}{\lambda}$ (green pattern). Whereas trajectories that can escape or escaped cross the part of $S^2_{DS}$ with $P_1 = Q_1 < 0$. This part is bounded by $W^-_s(S^1_{\text{NHIM}}) \cup W^-_u(S^1_{\text{NHIM}})$ and $I \equiv \frac{E}{\lambda}$ (orange pattern) in Figure 4.4.

The projection of some orbits onto the center plane is shown in Figure 4.5. Their colors coincide with the ones in the other figures.

One can envision the dynamic as a flow on a saddle, like in Figure 4.6. Every
Figure 4.2: Projection onto the $P_1 - Q_1$-saddle plane with manifolds

Figure 4.3: Projection onto the $P_2 - Q_2$-center plane with manifolds
Figure 4.4: Projection onto the $P_1 - Q_1$-saddle plane with flow

Figure 4.5: Projection onto the $P_2 - Q_2$-center plane with flow
trajectory which traverse from inside the planetary neighborhood to outside or visa versa, crosses the dividing surface.

In Figure 4.7, one can see the projections onto the $Q_1 - Q_2$-plane. Then $S_{\text{NHIM}}^1$ is a line on the $Q_2$-axis and $S_{\text{DS}}^2$ is depicted as an ellipsoid. Furthermore, the unstable manifold $W_u (S_{\text{NHIM}}^1)$ of $S_{\text{NHIM}}^1$ looks like a cylinder in $Q_1$-direction. The next proof evidences these statements.

**Proof by author.** For $(P, Q) \in \Sigma_E = \mathcal{H}^{-1}_2(E)$ and constant $P_1$, $Q_1$, $P_2$ the following equation is evident $Q_2 = \pm \sqrt{\frac{2}{\omega_2} (E - \lambda P_1 Q_1) - P_2^2}$. Since that equation is only well-defined for values of $P_2$ between $\pm \sqrt{\frac{2}{\omega_2} (E - \lambda P_1 Q_1)}$, the possible range of $Q_2$ is given by

$$Q_2 \in \left[ -\sqrt{\frac{2}{\omega_2} (E - \lambda P_1 Q_1)}, \sqrt{\frac{2}{\omega_2} (E - \lambda P_1 Q_1)} \right].$$

Using the definition of $S_{\text{NHIM}}^1$, that is $P_1 = 0 = Q_1$ yields

$$Q_2 \in \left[ -\sqrt{\frac{2}{\omega_2} E}, \sqrt{\frac{2}{\omega_2} E} \right] = \left[ -\sqrt{2J_2}, \sqrt{2J_2} \right].$$

For $S_{\text{DS}}^2$ with $P_1 = Q_1$ the following is true

$$Q_2 \in \left[ -\sqrt{\frac{2}{\omega_2} (E - \lambda Q_1^2)}, \sqrt{\frac{2}{\omega_2} (E - \lambda Q_1^2)} \right].$$

The projection of $W_u^\pm (S_{\text{NHIM}}^3)$ is calculated by setting $P_1 = 0$, $\pm Q_1 > 0$, then

$$Q_2 \in \left[ -\sqrt{2J_2}, \sqrt{2J_2} \right].$$
Lastly, the energy surface’s projection, which is bordered by the two curves in this figure, becomes

\[ Q_2 \in \left[ -\sqrt{\frac{2}{\omega_2}(E + \lambda|P_1Q_1|)}, \sqrt{\frac{2}{\omega_2}(E + \lambda|P_1Q_1|)} \right]. \]

This figure only depicts the projections of the discussed surfaces, in particular, the dividing surface is not a part of the $Q_1 - Q_2$-plane.

### 4.3 Dynamic around the Lagrange points

Until now, every proof has only considered the Birkhoff-like normal form $\mathcal{H}_2$ at $L_1$, which could be shifted into the origin. In order to analyze the whole lunar problem (see Remark 4.1) with two equilibria and, therefore, the possible escape from the planetary neighborhood, one has to make use of the $S$-invariance of $H$ (4.3). The Lagrange points are located on different sides of the lighter primary (earth) on the $Q_1$-axis.

In the following statements, the unbounded area is considered to be on the left side of $L_1$ and on the right side of $L_2$. More precisely, it is the subspace with $P_1 > Q_1+1$ and $P_1 > -Q_1+1$. The subset between the dividing surfaces $S_{DS}^4(L_1)$ at $L_1$ and $S_{DS}^4(L_2)$ at $L_2$ is the bounded part and within lies the lighter primary.
The corresponding ODE is

\[ H_{(L_1)}(P, Q) = -4.5 + \lambda P_1 (Q_1 + 1) + \frac{1}{2} \omega_2 \left( (P_2 + 1)^2 + Q_2^2 \right) + \frac{1}{2} \omega_2 \left( P_3^2 + Q_3^2 \right). \]

The flow \( \Phi_{L_1}(P, Q) \) for \( (P, Q) \in \mathbb{R}^3 \times \mathbb{R}^3 \) and all \( t \in \mathbb{R} \) is

\[
\begin{align*}
P_1^{(L_1)}(t) &= e^{-\lambda t} P_1(0), & Q_1^{(L_1)}(t) &= e^{\lambda t} Q_1(0) - 1, \\
P_2^{(L_1)}(t) &= \cos(\omega_2 t) P_2(0) - \sin(\omega_2 t) Q_2(0) - 1, \\
P_3^{(L_1)}(t) &= \cos(\omega_3 t) P_3(0) - \sin(\omega_3 t) Q_3(0), \\
Q_j^{(L_1)}(t) &= \sin(\omega_j t) P_j(0) + \cos(\omega_j t) Q_j(0) \text{ for } j = 2, 3.
\end{align*}
\]

Accordingly the manifolds from Section 4.1 have to be defined newly as

\[
\begin{align*}
\Sigma_E(L_1) &= \{ (P, Q) \in \mathbb{R}^6 \times \mathbb{R}^6 \mid H_{(L_1)}(P, Q) = E \}, \\
S_{\text{NHIM}}^E(L_1) &= \{ (P, Q) \in \Sigma_E(L_1) \mid P_1 = 0, Q_1 = -1 \}, \\
W_s(L_1) &= \{ (P, Q) \in \Sigma_E(L_1) \mid Q_1 = -1 \}, \\
W_u(L_1) &= \{ (P, Q) \in \Sigma_E(L_1) \mid P_1 = 0 \}, \\
W_s^±(L_1) &= \{ (P, Q) \in W_u(L_1) \mid \pm P_1 \geq 0 \}, \\
W_u^+(L_1) &= \{ (P, Q) \in W_u(L_1) \mid Q_1 \geq -1 \}, \\
W_u^-(L_1) &= \{ (P, Q) \in W_u(L_1) \mid Q_1 \leq -1 \}, \\
S_{\text{DS}}^E(L_1) &= \{ (P, Q) \in \Sigma_E \mid P_1 = Q_1 + 1 \}.
\end{align*}
\]

Using the \( S \)-invariance and symmetry of \( H \), the phase space around \( L_1 \) is mapped onto the one around \( S(L_1) = L_2 \). With \( H_{(L_2)}(P, Q) = H_{(L_1)} \circ S(P, Q) \), one gets the quadratic part of \( H \) at \( L_2 \) as

\[
H_{(L_2)}(P, Q) = -4.5 + \lambda P_1 (-Q_1 + 1) + \frac{1}{2} \omega_2 \left( (-P_2 + 1)^2 + Q_2^2 \right) + \frac{1}{2} \omega_2 \left( -P_3^2 + Q_3^2 \right).
\]

Figure 4.8 shows the benefit of using the symmetry instead of plain translation. The transformation \( S \) not only maps \( L_1 \) onto \( L_2 \), but also mirrors the phase space.
Using the extended smooth transformation given in Waalkens et al. [2005]

\[ \tilde{S}: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \]

\[ \tilde{S}(\mathbf{p}, \mathbf{q}, t) := (p_1, -p_2, -p_3, -q_1, q_2, q_3, -t), \]

the flow satisfies \( \Phi(L_2)(\mathbf{P}, \mathbf{Q}, t) = \Phi(L_1) \circ \tilde{S}(\mathbf{P}, \mathbf{Q}, t) \). Furthermore, the different subspaces of the energy surface transform accordingly to

\[ S_{NHIM}^3(L_2) = S(S_{NHIM}^3(L_1)) = \{(\mathbf{P}, \mathbf{Q}) \in \Sigma_E(L_2) \mid P_1 = 0, \ Q_1 = 1\}, \]

\[ W_u(L_2) = S(W_u(L_1)) = \{(\mathbf{P}, \mathbf{Q}) \in \Sigma_E(L_2) \mid P_1 = 0\}, \]

\[ W_s(L_2) = S(W_s(L_1)) = \{(\mathbf{P}, \mathbf{Q}) \in \Sigma_E(L_2) \mid Q_1 = 1\}, \]

\[ W^+(L_2) = S(W^+_u(L_1)) = \{(\mathbf{P}, \mathbf{Q}) \in W_u(L_2) \mid Q_1 \geq 1\}, \]

\[ W^-(L_2) = S(W^-_u(L_1)) = \{(\mathbf{P}, \mathbf{Q}) \in W_u(L_2) \mid Q_1 \leq 1\}, \]

\[ W^+(L_2) = S(W^+_u(L_1)) = \{(\mathbf{P}, \mathbf{Q}) \in W_u(L_2) \mid \pm P_1 \geq 0\}, \]

\[ S_{DS}^4(L_2) = S(S_{DS}^4(L_1)) = \{(\mathbf{P}, \mathbf{Q}) \in \Sigma_E \mid P_1 = -Q_1 + 1\}. \]

From now on, the whole Hamiltonian of Hill’s Lunar Problem (4.1) will be analyzed, instead of the quadratic parts or Birkhoff-like normal forms at \( L_j \) (\( j = 1, 2 \)). The subsets defined in the last section for \( H_2 \) also exist for the Birkhoff normal form \( H_s \) of degree \( s \) in appropriate coordinates \((\mathbf{P}, \mathbf{Q})\) and are given accordingly.
Due to the eigenvalues (see Remark 4.1) which do not fulfill any resonance relation, $H$ can be reduced to a Birkhoff normal form of any arbitrary finite degree. Then there exists an explicit formula for these submanifolds in $(P, Q)$-coordinates, whose phase space structures can be mapped into the original coordinates $(p, q)$ by the inverse of normal transformation (see Waalkens et al. [2005]). The energy surface and dividing surfaces are $(E$ regular value of $H_s)$

$$\Sigma_E = \{(P, Q) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid H_s(P, Q) \equiv E\} = H_s^{-1}(E),$$

$$S_{DS}^1(L_1) = \{(P, Q) \in \Sigma_E \mid P_1 = Q_1 + 1\},$$

$$S_{DS}^4(L_2) = \{(P, Q) \in \Sigma_E \mid P_1 = -Q_1 + 1\}.$$

The paper Waalkens et al. [2005] uses a formula for the volume of the flow which escapes the planetary neighborhood of the second primary (earth) (5.7) and the authors cite Pollak [1981] for the proof. However, the proof could not be found in this paper, so the formula will be verified with Lemma 5.1, Lemma 5.2 and Theorem 5.4.

### 5.1 Definition of the planetary neighborhood

In order to define the planetary neighborhood of Hill’s Lunar Problem in configuration space precisely, the authors of Waalkens et al. [2005] utilize the so called zero-velocity lines. These are the lines on which the flow has vanishing velocity. The Hamiltonian (4.1) of Hill’s Lunar Problem in the invariant planar case has to satisfy

$$\dot{q} = \frac{\partial H}{\partial p} = \left(\begin{array}{c} p_1 + q_2 \\ p_2 - q_1 \end{array}\right) \equiv 0. \quad (5.1)$$

This is true, if and only if $p_1 = -q_2$ and $p_2 = q_1$. Using these equalities yields

$$H(p, q) = -\frac{3}{2} q_1^2 - \frac{3}{\|q\|} \equiv E.$$

The equation $\frac{3}{2} q_1^2 + \frac{3}{\|q\|} + E_0 \equiv 0$ is fulfilled, if $q_1 = \pm 1, q_2 = 0$. For $q_1 = \pm 1$ and energies less than 1.5 one gets the possible range of $q_2$ as

$$q_2 = \pm \sqrt{\frac{36}{(3 + 2E)^2} - 1}.$$

The last equation is not defined, if the energy is less than $E_0 = -4.5$. In particular, then $q_1$ is never equal to $\pm 1$ and, therefore, remains inside one of the areas with $\pm q_1 > 1$ or $-1 < q_1 < 1$ in configuration space. However, a gap occurs at $q_1 = \pm 1$ for the range of $q_2$ when the energy $E$ is larger than $E_0$.

On the whole, the compact area between these gaps at $q_1 = \pm 1$ is defined as the planetary neighborhood in configuration space and the secondary mass (moon)
can escape from it through the contour lines between \( L_1 \) and \( L_2 \). Several zero-velocity lines are shown for different energies in Figure 5.1. The red one depicts the contour line for energy less than \( E_0 \), the green one for energy equal to \( E_0 \) and the blue one for energy greater than \( E_0 \). The secondary mass with such energies, can only be located in the correspondingly shaded areas.

![Figure 5.1: Zero-velocity lines for different energies](image)

5.2 Volume form on the energy surface

In order to calculate the volume of the flow, which pass from outside through one of the dividing surfaces at \( L_1 \) or \( L_2 \) and leave the planetary neighborhood after a while, one can consider the following 4-form on the energy surface \((E \text{ is a regular value of } H)\) given in Waalkens et al. [2005].

\[
\delta(E - H) \, dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2
\]  

(5.2)

As a result of the conservation of phase space volume, for almost every trajectory which enters the planetary neighborhood (compact subset) through \( S^4_{DS}(L_j) \) \((j = 1, 2)\) there exists a time, after that the trajectory has left the neighborhood. To make calculations easier, one can use polar coordinates \( r \in \mathbb{R}^+, \varphi \in [0, 2\pi] \) for the velocity \( \dot{q} \) (5.1)

\[
\begin{align*}
r \cos(\varphi) &:= p_1 + q_2, & r \sin(\varphi) &:= p_2 - q_1, \\
p_1 &= r \cos(\varphi) - q_2, & p_2 &= r \sin(\varphi) + q_1.
\end{align*}
\]

Lemma 5.1 (statement by Waalkens et al. [2005]). Using these coordinates, the volume form (5.2) becomes

\[
dq_1 \wedge dq_2 \wedge d\varphi,
\]  

(5.3)
after integration with respect to $r$ from 0 to infinity.

**Proof by author.** One can calculate
\[
dp_1 = \cos(\varphi) \, dr - r \sin(\varphi) \, d\varphi - dq_2,
\]
\[
dp_2 = \sin(\varphi) \, dr + r \cos(\varphi) \, d\varphi + dq_1.
\]

Using these equations result in
\[
dp_1 \wedge dp_2 = r \cos(\varphi)^2 \, dr \wedge d\varphi + \cos(\varphi) \, dr \wedge dq_1
\]
\[
- r \sin(\varphi)^2 \, d\varphi \wedge dr - r \sin(\varphi) \, d\varphi \wedge dq_1
\]
\[
- \sin(\varphi) \, dq_2 \wedge dr - r \cos(\varphi) \, dq_2 \wedge d\varphi - dq_2 \wedge dq_1.
\]

After applying these results to the volume form (5.2), it changes into
\[
\delta(E - H) \, dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2 = r \delta(E - H) \cos(\varphi)^2 \, dq_1 \wedge dq_2 \wedge dr \wedge d\varphi
\]
\[
- r \delta(E - H) \sin(\varphi)^2 \, dq_1 \wedge dq_2 \wedge d\varphi \wedge dr
\]
\[
= r \delta(E - H) \, dq_1 \wedge dq_2 \wedge dr \wedge d\varphi.
\]

The Hamiltonian in polar coordinates is
\[
H(r, \varphi, q_1, q_2) = \frac{1}{2} r^2 - \frac{3}{2} q_1^2 - \frac{3}{\|q\|}
\]

Define $g_{\varphi, q_1, q_2}(r) := E - H(r, \varphi, q_1, q_2)$ and choose $(\varphi, q_1, q_2)$ properly so that the square root below is well-defined, then
\[
g_{\varphi, q_1, q_2}(r) = 0 \text{ for } r_1 = \sqrt{2 \left( E + \frac{3}{2} q_1^2 + \frac{3}{\|q\|} \right)}, \text{ in which } r_1 > 0.
\]

The transformation of Dirac distributions where $g \in C^1(\mathbb{R}, \mathbb{R})$ is given by
\[
\delta(g(r)) = \sum_{j=1}^n \frac{\delta(r - r_j)}{|g'(r_j)|}, \text{ here } r_j \text{ denote the isolated roots of } g.
\]

Using the derivative $g'_{\varphi, q_1, q_2}(r) = -r$, that formula becomes
\[
\delta(g_{\varphi, q_1, q_2}(r)) = \frac{\delta(r - r_1)}{|g'(r_1)|} = \frac{\delta(r - r_1)}{r_1}.
\]

The equation above applied to the integral yields
\[
\int_0^\infty \delta(g_{\varphi, q_1, q_2}(r)) \, r \, dr = \frac{r_1}{r_1} = 1.
\]

Altogether, after integration with respect to $r$ the volume form $r \delta(E - H) \, dq_1 \wedge dq_2 \wedge dr \wedge d\varphi$ changes into
\[
\int_0^\infty r \delta(E - H) \, dq_1 \wedge dq_2 \wedge dr \wedge d\varphi = dq_1 \wedge dq_2 \wedge d\varphi
\]
which is the asserted equation in polar coordinates. \qed
The 3-form \((dq_1 \wedge dq_2 \wedge d\varphi)|_{\tilde{\Sigma}_E}\) is a volume form on \(\tilde{\Sigma}_E := \{(r, \varphi, q_1, q_2) \in \Sigma_E \mid r > 0\}\) and, hence, induces a measure. Therefore, \((\varphi, q_1, q_2)\) are coordinates on \(\tilde{\Sigma}_E\) (independent of \(r\)).

5.3 Volume of escaping flow

Now the volume of the flow which enters the planetary neighborhood through \(S_{DS}^4(L_1)\) and exit it through \(S_{DS}^4(L_2)\) can be calculated.

The dividing surfaces \(S_{DS}^4(L_1)\) and \(S_{DS}^4(L_2)\) are 4-dimensional submanifold of the energy surface. For every point \(x \in S_{DS}^4(L_j)\) there exist a neighborhood \(V(x) \subseteq S_{DS}^4(L_j)\) and a function with regular value \(0\) which fulfills \(V(x) \cap S_{DS}^4(L_j) = G_j^{-1}(0)\) (see Definition 2.34 in Knauf [2012]).

Now \(S_{DS}^4(L_1)\) is used as the first, \(S_{DS}^4(L_2)\) as the second Poincaré surface. Choose \(u_1 \in S_{DS}^4(L_1)\) with \(\{H, G_1\}(u_1) \neq 0\), so that there exists a time \(t_1 = t_1(u_1) > 0\) fulfilling

\[
G_j : V(x) \to \mathbb{R}
\]

which fulfills \(V(x) \cap S_{DS}^4(L_j) = G_j^{-1}(0)\) (see Definition 2.34 in Knauf [2012]).

Choose \(u_1 \in S_{DS}^4(L_1)\) with \(\{H, G_1\}(u_1) \neq 0\), so that there exists a time \(t_1 = t_1(u_1) > 0\) fulfilling

\[
u_2 := \Phi_{t_1}(u_1) \in S_{DS}^4(L_2)\text{ and }\{H, G_2\}(u_2) \neq 0.
\]

According to Theorem 10.17 in Knauf [2012], \(G_j\) is not constant on the orbit of \(\Phi_t(u_j)\), because the Poincaré brackets are not equal to zero.

**Lemma 5.2** (by author). There exist \((2n - 2) = 4\)-dimensional hyper surfaces

\[
u_1 \in U_1 \subseteq S_{DS}^4(L_1)\text{ and }\nu_2 \in U_2 \subseteq S_{DS}^4(L_2)
\]

as well as a unique continuously differentiable map

\[
T : U_1 \to \mathbb{R}\text{ with }T(u_1) = t_1\text{ and }\Phi_{T(x)}(\nu_2) \in U_2\text{ for all }\nu_1 \in U_1.
\]

That is

\[
G_2(\Phi_{T(x)}(\nu_2)) = 0\text{ for all }\nu_1 \in U_1.
\]

So after \(T_1\), the flow (which started at the first Poincaré surface) reaches the second Poincaré surface \((T_1\text{ is the so-called passage time})\).

**Proof by author.** This prove uses the implicit function theorem, let \(F(x, t) := G_2 \circ \Phi(x, t)\) for \((x, t) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}\), then

\[
DF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial t}\right).
\]
For \((u_1, t_1)\) is \(F(u_1, t_1) = G_2(u_2) = 0\) and one has to ensure that

\[
\frac{\partial F}{\partial t}(u_1, t_1) = \frac{\partial (G_2 \circ \Phi)}{\partial t}(u_1, t_1) \neq 0.
\]

Then there exists a neighborhood \(U_1\) of \(u_1\) and a unique continuously differentiable function \(T: U_1 \to \mathbb{R}\), with

\[
T(u_1) = t_1 \text{ and } F(x, T(x)) = G_2(\Phi_{T(x)}(x)) = 0 \text{ for all } x \in U_1.
\]

The claim \(\frac{\partial (G_2 \circ \Phi)}{\partial t}(u_1, t_1) \neq 0\) is true if and only if \(\{H, G_2\}(u_2) \neq 0\) according to Theorem 10.17 in Knauf [2012].

\[\square\]

Darboux’s Theorem below is proven in Knauf [2012] Theorem 10.31.

**Theorem 5.3** (Darboux’s Theorem). Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold, then for every point \(x_0 \in M\) there exists a map \((U, \varphi)\) at \(x_0\) with coordinates \(\varphi = (p_1, \ldots, p_n, q_1, \ldots, q_n)\), so that \(\omega\big|_U = \sum_{j=1}^{n} dq_j \wedge dp_j\).

**Proof by Knauf [2012].** The theorem is proven, if one shows the following statement in local maps around \(x_0\) instead: For two neighborhoods \(U_0 \subseteq \mathbb{R}^{2n}\) and \(U_1 \subseteq \mathbb{R}^{2n}\) of zero with corresponding symplectic forms \(\omega_0, \omega_1\) there exist two neighborhoods \(V_0 \subseteq U_0, V_1 \subseteq U_1\) of zero and a diffeomorphism \(F: V_0 \to V_1\) with \(\omega_0|_{V_0} = F^* (\omega_1|_{V_1})\).

By using the linear case of Darboux’s Theorem, the equation \(\omega_0(0) = \omega_1(0)\) holds after a linear change of the basis. The diffeomorphism \(F\) will result as the flow \(F: [0, 1] \times V_0 \to V_1\) of the ODE

\[
\frac{d}{dt} F_t = X_t(F_t) \text{ with initial value } F_0 = \text{id}. \tag{5.4}
\]

The vector field \(X_t\) will be defined uniquely below by \(\omega_t := t \omega_1 + (1 - t) \omega_0\) for \(t \in [0, 1]\) and depends on the time \(t\).

After using \(F_0^* \omega_0 = \omega_0\) as well as Theorem B.34 from Knauf [2012], the following equation must be true in order to get \(F_t^* \omega_t = \omega_0\) for all \(t \in [0, 1]\).

\[
\frac{d}{dt} F_t^* \omega_t = F_t^* \left( L_{X_t} \omega_t + \frac{d}{dt} \omega_t \right) = F_t^* \left( L_{X_t} \omega_t + \omega_1 - \omega_0 \right) = 0 \tag{5.5}
\]

Since \(\omega_j\) are symplectic and, thus, closed \((d\omega_j = 0, j = 0, 1)\), the same applies to \(\omega_1 - \omega_0\). This difference can be written as an exact form \(\omega_1 - \omega_0 = - d \theta\) locally on a ball \(B(0)\) around zero by using Poincaré’s Lemma B.35 in Knauf [2012]. Here \(\theta\) denotes an appropriate 1-form on \(B(0)\).

The Lie derivative \(L_X \omega\) of a \(k\)-form \(\omega\) with respect to a vector field \(X\) on \(M\) is a \(k\)-form itself

\[
L_X \omega := i_X \, d\omega + d i_X \omega.
\]
In which
\[ i_X \omega \left( X^{(1)}, \ldots, X^{(k-1)} \right) := \omega \left( X, X^{(1)}, \ldots, X^{(k-1)} \right) \]
denotes the inner product (a \((k - 1)\)-form) of \(X\) and \(\omega\). The form \(\omega_t\) is closed (\(d\omega_t = 0\)), hence, the claim (5.5) becomes
\[ F^*_t \left( L_X \omega_t + \omega_0 - \omega_1 \right) = F^*_t \left( d(i_X \omega_t - d\theta) \right) = F^*_t \left( d(i_X \omega_t - \theta) \right) \]

Additionally, \(\omega_t(0)\) is non-degenerated for all \(t \in [0, 1]\), because \(\omega_0(0) = \omega_1(0)\) and \(\omega_2(0) = \omega_0(0)\). Therefore, \(\omega_t\) is a symplectic form (closed and non-degenerated) on a neighborhood of zero for all \(t \in [0, 1]\). As a direct result of the non-degeneracy, equation (5.6) yields an uniquely defined vector field \(X_t\) for all \(t \in [0, 1]\) by \(i_X \omega_t = \theta\) on this neighborhood.

By using an additional term \(df\), one can obtain \(\theta(0) = 0\), that is \(X_t(0) = 0\) for all \(t\). Hence, the flow of (5.4) with \(F_t(0) = 0\) exists locally on a neighborhood \(V_t\) of zero for all \(t \in [0, 1]\). Altogether, with the vector field \(X_t\) defined above, the equation \(\frac{d}{dt} F^*_t \omega_t = 0\) and, thus, \(F^*_t \omega_t = \omega_0\) is true. Setting \(\tilde{F} = F_t\) proves the statement.

Lemma 5.2 yields the existence of \((2n - 2) = 4\)-dimensional manifolds \(U_1\), \(U_2\) and of a map \(T: U_1 \to \mathbb{R}\) for the symplectic manifold \((M, \omega) = (\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}), \omega)\).

**Theorem 5.4** (statement by Waalkens et al. [2005]). The volume of the \((2n - 1) = 5\)-dimensional manifold
\[ Z := \{ \Phi_t(x) \mid x \in U_1, t \in [0, T(x)] \} \subset H^{-1}(E) \]
with boundary can be calculated as
\[ \sigma(Z) = \int_Z \sigma = \int_{U_1} T(x) \frac{\omega|_{U_1}}{\Gamma^{(n-1)}}. \]

**Here** \(T: U_1 \subseteq S_0^{1}(L_1) \to \mathbb{R}\) denotes passage time from Lemma 5.2.

That is the integral of the passage time \(T(x)\) over the front surface \(U_1\) of \(Z\). The manifold \(Z\) consists of every orbit with initial point in \(U_1 \subseteq S_0^{1}(L_1)\) with time from 0 to \(T(x)\) and end point in \(U_2 \subseteq S_0^{1}(L_2)\). In particular, every initial point in \(U_1\) whose flow traverse through the planetary neighborhood from the left to the right, see Figure 5.2.

**Proof by author.** For every \(x_0 \in M\) with \(dH(x_0) \neq 0\) exists a symplectic map \((\tilde{Z}, \varphi)\) around \(x_0\), given by
\[ \varphi: \tilde{Z} \to \mathbb{R}^{2n} \]
\[ \varphi(u) = (q_1(u), q_2(u), \ldots, q_n(u), H|_{\tilde{Z}}(u) - H(x_{0}), p_2(u), \ldots, p_n(u)) \]
according to Theorem 5.2.19 from Abraham and Marsden [1987]. Additionally, \( \varphi(x_0) = (0, 0) \),

\[
\varphi(Z) = I \times W \subset \mathbb{R} \times \mathbb{R}^{2n-1} \quad (I \text{ open interval, } W \text{ open})
\]

and \( \varphi^{-1}|_{I \times \{w\}} \) is an integral curve of \( X_H \) for every \( w \in W \) with parameter \( q_1 \).

That is, \( \varphi^{-1}|_{I \times \{w\}} \) is a 1-dimensional submanifold within \( Z \), parametrized by \( q_1 \).

Since the flow is unique, the following equation is true for \( t \in I \)

\[
\varphi^{-1}|_{I \times \{w\}}(t) = \Phi_t(\varphi^{-1}(\{0\} \times \{w\})).
\]

Using these coordinates on \( \tilde{Z} \) yields

\[
\omega|_{\tilde{Z}} = dq_1 \wedge dH|_{\tilde{Z}} + \sum_{j=2}^{n} dq_j \wedge dp_j
\]
as well as \( \{q_1, H|_{\tilde{Z}}\} = \{q_1, p_1\} = dq_1(X_H) \equiv 1 \). In particular, the time \( t \) is equal to \( q_1 \), which is the canonical conjugated coordinate to \( H|_{\tilde{Z}} \). Let \( x_0 \in H^{-1}(E) \) and choose the neighborhood \( \tilde{Z} \) so that \( \overline{dH(z)} \neq 0 \) for all \( z \in \tilde{Z} \), then Theorem 3.4.12 in Abraham and Marsden [1987] defines a \((2n-1)\)-form \( \sigma \) by

\[
\left(\frac{-1}{n!}\right)^{\binom{n}{2}} \omega|_{\tilde{Z}}^{\wedge n} = dq_1 \wedge dq_2 \wedge \ldots \wedge dq_n \wedge dH|_{\tilde{Z}} \wedge dp_2 \wedge \ldots \wedge dp_n
\]

\[
= (-1)^{(n-1)} \sigma \wedge dH|_{\tilde{Z}}.
\]

What is more, \( \sigma \) is a volume form on \( H^{-1}(E) \).

Scale \( U_1 \) down until it is a subset of \( \tilde{Z} \) and let \( x_0 \in U_1 \). With Theorem 3.3.21 from Abraham and Marsden [1987] one gets \( \{q_i, q_j\} = \{p_i, p_j\} \equiv 0 \) and \( \{q_i, p_j\} \equiv \delta_{i,j} \) \((i, j = 1, \ldots, n)\) on \( \tilde{Z} \). This also remains true, if the coordinates are restricted to \( U_1 \subset \tilde{Z} \). Then

\[
\omega|_{U_1} = \sum_{j=2}^{n} dq_j|_{U_1} \wedge dp_j|_{U_1}
\]

and the following equation is true for the cylinder’s volume

\[
\sigma(Z) = \int_Z \sigma = \int_{U_1} T(x) \left(\omega|_{U_1}\right)^{(n-1)}.
\]

Since \( \sigma \) is \( H \)-invariant, \( \Phi^*_t \sigma = \sigma \) holds and with Proposition B.27 in Knauf [2012], the volume of \( Z \) is equal to the integral over the front surface (for Hill’s Lunar Problem is \( n = 3 \)).
With the symmetry (4.3), the volume of the flow which passes from \( U_1 \subseteq S^4_{DS}(L_2) \) to \( U_2 \subseteq S^4_{DS}(L_1) \) is equal to (5.7). Altogether, the escaping flow, which enters the planetary neighborhood through \( U_1 \subseteq S^4_{DS}(L_1) \) and leaves it through \( U_2 \subseteq S^4_{DS}(L_2) \) and the one which enters the planetary neighborhood through \( U_1 \subseteq S^4_{DS}(L_2) \) and leaves it through \( U_2 \subseteq S^4_{DS}(L_1) \) is according to Waalkens et al. [2005] given by

\[
2 \int_{U_1} T(x) \left( \omega_{|U_1} \right)^{(n-1)}.
\]

Figure 5.2: Volume of escaping flow, projected onto the \( q_1 - q_2 \)-plane

### 5.4 Properties of the undisturbed system

Hill’s Lunar Problem can be interpreted as an undisturbed ODE system

\[
H^{(0)}: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}
\]

\[
H^{(0)}(p, q) = \frac{1}{2} \|p\|^2 + p_1 q_2 - p_2 q_1 - q_1^2 + \frac{1}{2} \left( q_2^2 + q_3^2 \right)
\]

(5.8)

with an additional perturbation

\[
H^{(1)}: \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \rightarrow \mathbb{R}
\]

\[
H^{(1)}(p, q) = -\frac{3}{\|q\|}.
\]
The corresponding vector fields are
\[
X^{(0)}(p, q) = \begin{pmatrix}
p_2 + 2q_1 \\
-p_1 - q_2 \\
p_1 + q_2 \\
p_2 - q_1 \\
p_1 + q_3 \\
p_2 - q_3
\end{pmatrix}, \quad X^{(1)}(p, q) = \begin{pmatrix}
-\frac{2q_1}{|q|^3} \\
-\frac{2q_2}{|q|^3} \\
-\frac{2q_3}{|q|^3} \\
0 \\
0 \\
0
\end{pmatrix}.
\]

\[\text{Lemma 5.5 (by author).} \] The flow of the Hamiltonian

\[H_\lambda(p, q) := H^{(0)}(p, q) + \lambda H^{(1)}(p, q)\]

with \(\lambda > 0\) is equivalent to the one of Hill’s Lunar Problem (4.1).

\[\text{Proof by author.} \] Using the homogeneity

\[
\lambda^{2/3}E \equiv \lambda^{2/3}H(p, q) = H^{(0)} \left( \lambda^{1/3}p, \lambda^{1/3}q \right) + \lambda H^{(1)} \left( \lambda^{1/3}p, \lambda^{1/3}q \right).
\]

That is

\[
\lambda^{2/3}E \equiv \lambda^{2/3}H(p, q) = H_\lambda \left( \lambda^{1/3}p, \lambda^{1/3}q \right)
\]

and one gets

\[
\Sigma_E(H) := \left\{ (p, q) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \mid H(p, q) \equiv E \right\}
\]

\[
= \left\{ (p, q) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \mid H_\lambda \left( \lambda^{1/3}p, \lambda^{1/3}q \right) \equiv \lambda^{2/3}E \right\}
\]

\[
= \Sigma_{\lambda^{2/3}E}(H_\lambda).
\]

On the whole, it does not matter whether one analyzes \(H_\lambda\) on its energy plane \(E\) or \(H\) on the energy plane \(\lambda^{-2/3}E\). In particular with the diffeomorphism \(\varphi(x) := \lambda^{1/3}x\), the following is true

\[
X_\lambda \circ \varphi = \varphi \circ X.
\]

Hence, the flows are equivalent. \(\square\)

From now one, the disturbed Hamiltonian \(H_\lambda\) will be analyzed. The flow of the undisturbed Hamiltonian (5.8) and its ODE

\[
\left(\dot{p}^{(0)}(t), \dot{q}^{(0)}(t)\right)^T = X^{(0)}(p^{(0)}(t), q^{(0)}(t))^T
\]
can be computed for all $t \in \mathbb{R}$ as follows

$$
\begin{align*}
    p_1^{(0)}(t) &= 3t(p_2(0) + q_1(0)) - \cos(t)(p_1(0) + q_2(0)) \\
        &\quad - \sin(t)(2p_2(0) + q_1(0)) + 2p_1(0) + q_2(0), \\
    p_2^{(0)}(t) &= \cos(t)(2p_2(0) + q_1(0)) - \sin(t)(p_1(0) + q_2(0)) - p_2(0) - q_1(0), \\
    p_3^{(0)}(t) &= \cos(t)p_3(0) - \sin(t)q_3(0), \\
    q_1^{(0)}(t) &= -\cos(t)(2p_2(0) + q_1(0)) + \sin(t)(p_1(0) + q_2(0)) \\
        &\quad + 2p_2(0) + 2q_1(0), \\
    q_2^{(0)}(t) &= -3t(p_2(0) + q_1(0)) + 2\sin(t)(2p_2(0) + q_1(0)) \\
        &\quad + 2\cos(t)(p_1(0) + q_2(0)) - 2p_1(0) - q_2(0), \\
    q_3^{(0)}(t) &= \cos(t)q_3(0) + \sin(t)p_3(0).
\end{align*}
$$

The special case $p_3^{(0)}(t) = 0 = q_3^{(0)}(t)$ is due to the vector field $X^{(0)}$ invariant

$$
\frac{d p_3}{dt} = -q_3^{(0)}, \quad \frac{d q_3}{dt} = p_3^{(0)}.
$$

From now on and without loss of generality, the motion is considered to be restricted to a plane with $p_3^{(0)}(t) = 0 = q_3^{(0)}(t)$.

The flow $q_1^{(0)}(t)$ only contains periodic terms, therefore, it is bounded for all $t \in \mathbb{R}$. In contrast, $q_2^{(0)}(t)$ has a linear term in $t$, hence, $\left| q_2^{(0)}(t) \right|$ diverges towards infinity for $t \to \pm \infty$.

It is important to note, that $p$ is not equal to the velocity $\dot{q}$ anymore, because of the additional mixed terms $p_1q_2 - p_2q_1$. These terms are a result of the coordinate system’s rotation with a constant velocity $\omega \in \mathbb{R} \setminus \{0\}$ (see section 2.1).

**Remark 5.6.** Even though the disturbed flow shown in Figure 5.4 looks like the motion of a charged particle (set charge $e_0 = 1$) within a magnetic field $B$, the systems are not similar. According to Section 6.3.3 in Knauf [2012], the Hamiltonian which describes the latter dynamic is given by

$$
H_B(p, q) = \frac{1}{2}||p||^2 + \frac{1}{2} (B_{1,1}q_1 + B_{1,2}q_2)^2 + \frac{1}{2} (B_{2,1}q_1 + B_{2,2}q_2)^2 \\
- B_{1,1}p_1q_1 - B_{1,2}p_1q_2 - B_{2,1}p_2q_1 - B_{2,2}p_2q_2
$$

where $B = (B_{i,j})_{i,j=1,2} \in \text{Mat}(2 \times 2, \mathbb{R})$ antisymmetric. The system matrix

$$
\begin{pmatrix}
B_{1,1} & B_{2,1} \\
B_{1,2} & B_{2,2}
\end{pmatrix}
$$

has the Jordan form

$$
J_B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & i(B_{1,2} - B_{2,1}) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i(B_{1,2} - B_{2,1})
\end{pmatrix}.
$$
Figure 5.3: Orbit of the undisturbed Hamiltonian $H^{(0)}$ with initial point $(1.5, -1.5, 2, 0)^T$; time from 0 to 10

Figure 5.4: Numerical orbit of the disturbed Hamiltonian $H_{\lambda} = H^{(0)} + \lambda H^{(1)}$ with initial point $(1.5, -1.5, 2, 0)^T$; time from 0 to 10
Which leads to eigenvalues 0 (geometric multiplicity two) and ±i(B_{1,2} - B_{2,1}). They do not coincide with the ones of $A$ (5.9), therefore, the matrices are not similar.

The system matrix $A \in \text{Mat}(4, \mathbb{R})$ of the undisturbed Hamiltonian and its ODE (5.8) in the planar case

$$A := \begin{pmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \quad (5.9)$$

has eigenvalues 0 (with geometric multiplicity one) and ±i. In particular, every eigenvalue has vanishing real parts, which would otherwise lead to an exponential growth of the solutions.

One can define new variables for the undisturbed flow

$$x^{(0)}(t) := 2p_2^{(0)}(t) + q_1^{(0)}(t) = \cos(t)(2p_2(0) + q_1(0)) - \sin(t)(p_1(0) + q_2(0)), \quad (5.10)$$

$$|x^{(0)}(t)| \leq |2p_2(0) + q_1(0)| + |p_1(0) + q_2(0)| =: c_x^{(0)}, \quad (5.11)$$

$$y^{(0)}(t) := p_2^{(0)}(t) + q_1^{(0)}(t) = p_2(0) + q_1(0) =: c_y^{(0)}, \quad (5.12)$$

$$z^{(0)}(t) := 2p_1^{(0)}(t) + q_2^{(0)}(t) = 3t(p_2(0) + q_1(0)) + 2p_1(0) + q_2(0) =: 3c_y^{(0)}t + c_z^{(0)}. \quad (5.13)$$

Figure 5.5: Undisturbed coordinates $(x^{(0)}(t), y^{(0)}(t), z^{(0)}(t))$ within band $B_{(0)}^+$

The last coordinate $z^{(0)}$ is linear in $t$ and does not have a periodic term in contrast to $q_2^{(0)}$. Its derivative with respect to $t$ is $y^{(0)}(t)$ and constant for all time. Additionally, $y^{(0)}(t)t$ is equal to the linear term of $q_2^{(0)}(t)$, so the latter diverges towards plus or minus infinity for large time, depending on $\pm y^{(0)}(t) > 2\epsilon > 0$. With these properties a constant difference between the disturbed coordinates of $H_{\lambda} = H^{(0)} + \lambda H^{(1)}$

$$x(t) := 2p_2(t) + q_1(t), \quad (5.14)$$

$$y(t) := p_2(t) + q_1(t), \quad (5.15)$$

$$z(t) := 2p_1(t) + q_2(t) \quad (5.16)$$
and undisturbed coordinates \((x^{(0)}(t), y^{(0)}(t), z^{(0)}(t))\) will be calculated in Theorem 6.3. The first coordinate \(x^{(0)}\) is defined independent of the other two and bounded (5.11). The three undisturbed coordinates stay inside a band \(B^\pm(0) = B^\pm_0(p(0), q(0))\) if \(\pm y^{(0)}(t) \equiv \pm c^{(0)}_y > 2\epsilon > 0\) for all \(t \geq 0\) (see Figure 5.5)

\[
B^+(0) := [-c^0_x, c^0_x] \times \{c^0_y\} \times [c^0_z, \infty], \\
B^-(0) := [-c^0_x, c^0_x] \times \{c^0_y\} \times (-\infty, c^0_z].
\] (5.17)

Comparison of the undisturbed flow in Figure 5.3 with the disturbed one in Figure 5.4 suggests, that they might have similar dynamics for certain initial points. The general idea is to use a self consistent approach in order to prove that the disturbed flow has a similar behavior for large time \(t\) as the undisturbed (in Section 6.1). In particular, one can prove that \((x(t), y(t), z(t))\) with specific initial points remains inside a thickened beam and that the flow has a minimal velocity \(\pm \dot{q}_2(t) > |v|, v \in \mathbb{R} \setminus \{0\}\) and is bounded in \(q_1\)-direction as well (see Figure 5.6).

6 Flow for large time

The flow of Hill’s Lunar Problem cannot be calculated analytically. Nevertheless, one can obtain similar statements for the disturbed system as for the undisturbed for large time. In particular, the disturbed flow also diverges towards infinity in \(q_2\)-direction and is bounded in \(q_1\)-direction. Once again, the invariant planar case of Hill’s Lunar Problem is analyzed in this section.

At first, it is most obvious to use perturbation theory for ODE (see Section 3.5 in Thirring [1988]) in order to compute the difference between the disturbed coordinates \((x(t), y(t), z(t))\) and the undisturbed \((x^{(0)}(t), y^{(0)}(t), z^{(0)}(t))\).

Remark 6.1. To compute the time evolution of the disturbed coordinate \(z\) (5.16)
specified above, one has to evaluate the perturbation series
\[
\exp(tL_{H(0)+\lambda H(1)}) f = f(t) + \sum_{n \in \mathbb{N}} (-\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \left\{ H^{(1)}(t_1), \left\{ H^{(1)}(t_2), \ldots \left\{ H^{(1)}(t_n), f(t) \right\} \ldots \right\} \right\}
\]
with \(H^{(0)}\) and \(H^{(1)}\) from Lemma 5.5.

This series includes integrals over interlaced Poincaré brackets. Unfortunately, even the first integral with \(f(t) := z^{(0)}(t) = 2p_1^{(0)}(t) + q_2^{(0)}(t) = 3t(p_2(0) + q_1(0)) + 2p_1(0) + q_2(0)\) diverge towards infinity for large \(t\).

\[
\left\{ H^{(1)}(t_1), f(t) \right\} = 2 \frac{\partial H^{(1)}(t_1)}{\partial q_1(0)} - 3t \frac{\partial H^{(1)}(t_1)}{\partial p_1(0)} + 3t \frac{\partial H^{(1)}(t_1)}{\partial q_2(0)} - \frac{\partial H^{(1)}(t_1)}{\partial p_2(0)}
\]
\[
= \frac{3}{(q_1^2(t_1) + q_2^2(t_1))^{3/2}} (2q_1(t_1) + 3q_2(t_1)(t - t_1))
\]

With the claim \(q_2(t) \propto c_{2,1}t + c_{2,2}\) and \(|q_1(t)| < c_1\) this leads to

\[
\int_0^t \left\{ H^{(1)}(t_1), f(t) \right\} dt_1 \sim \int_0^t \frac{3}{|q_2(t_1)|^3} (3q_2(t_1)(t - t_1)) dt_1
\]
\[
\sim \int_0^t \frac{3}{|c_{2,1}t_1 + c_{2,2}|^3} (3c_{2,1}t_1 - 3c_{2,1}t_1^2) dt_1.
\]

The last equation diverges for \(t \to \infty\) and does not lead to a constant difference between \(z^{(0)}(t)\) and \(z(t)\) for all \(t > 0\).

### 6.1 Duhamel principle

In a second attempt, one can use the Duhamel principle (Theorem 4.20 from Knauf [2012]) and the special form of eigenvalues of the system matrix \(A\). The initial value system \(\dot{x}(t) = A(t)x(t) + b(t), x(t_0) = x_0\) where the functions \(A : I \subset \mathbb{R} \to \text{Mat}(n, \mathbb{R})\) and \(b : I \to \mathbb{R}^n\) are continuous, can be solved easily if the homogeneous solution operator \(\Phi^{(0)}(t, t_0)\) is known.

\[
x(t) = \Phi^{(0)}(t, t_0)x_0 + \int_{t_0}^t \Phi^{(0)}(t, s)b(s) \, ds \quad (t \in I)
\]

With system matrix \(A\) (5.9) the operator \(\Phi^{(0)}(t, s)\) can be calculated as\(^8\)

\[
\Phi^{(0)}(t, s) = \begin{pmatrix}
2 - \cos(t-s) & 3(t-s) - 2\sin(t-s) & 3(t-s) - \sin(t-s) & 1 - \cos(t-s) \\
- \sin(t-s) & 2\cos(t-s) - 1 & \cos(t-s) - 1 & - \sin(t-s) \\
\sin(t-s) & 2 - 2\cos(t-s) & 2\cos(t-s) & \sin(t-s) \\
2\cos(t-s) - 2 & -3(t-s) + 4\sin(t-s) & -3(t-s) + 2\sin(t-s) & 2\cos(t-s) - 1
\end{pmatrix},
\]

\(^8\)The operator \(\Phi^{(0)}(t, s)\) fulfills \(\frac{\partial}{\partial t} \Phi^{(0)}(t, s) = A \Phi^{(0)}(t, s)\) and \(\Phi(s, s) = I\).
additionally, the perturbation is given by
\[ b(s) := \lambda \left( \begin{array}{c}
-3q_1(s) \\
-3q_2(s) \\
0
\end{array} \right). \]

After applying to the Duhamel principle, the solution of Hill’s Lunar Problem can be calculated with
\[ x(t) = \Phi(0)(t,0)x_0 + \lambda \int_0^t \Phi(0)(t,s)b(s)\,ds \]
\[ \begin{pmatrix}
p_1(t) \\
p_2(t) \\
q_1(t) \\
q_2(t)
\end{pmatrix} = \begin{pmatrix}
p_1(0) \\
p_2(0) \\
q_1(0) \\
q_2(0)
\end{pmatrix} + \lambda \int_0^t \begin{pmatrix}
(2 - \cos(t-s))q_1(s) + (3(t-s) - 2\sin(t-s))q_2(s) \\
-\sin(t-s)q_1(s) + \left(2\cos(t-s) - 1\right)q_2(s) \\
\sin(t-s)q_1(s) + \left(2 - 2\cos(t-s)\right)q_2(s) \\
(2\cos(t-s) - 2)q_1(s) + (-3(t-s) + 4\sin(t-s))q_2(s)
\end{pmatrix} ds. \tag{6.1} \]

### 6.2 Escaping initial points and minimal velocity

The undisturbed coordinates \( x^{(0)}(t), y^{(0)}(t) \) and \( z^{(0)}(t) \) (5.11)-(5.13) stay inside a band \( B_\pm^{(0)} \) (5.17) for all \( t > 0 \), if \( \pm y^{(0)} > 2\epsilon > 0 \). If the perturbation is small enough (that is \( \lambda > 0 \) small), then the disturbed coordinates \( x(t), y(t) \) and \( z(t) \) (5.14)-(5.16) with special initial points also remain inside a thickened beam \( B_\pm \) for a certain time. What is more, as long as they stay inside the beam, \( |q_2(t)| \) diverges towards infinity with a minimal velocity \( v \in \mathbb{R} \setminus \{0\} \).

**Theorem 6.2 (by author).** For all \( (p(0), q(0)) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \) with either
- \( (p_2(0) + q_1(0)) > 2\epsilon > 0 \)
- \(-2c_x + 3c_y - |v| > 0 \quad (|v| \neq 0) \)

or
- \( (p_2(0) + q_1(0)) < -2\epsilon < 0 \)
- \(-2c_x - 3c_y - |v| > 0 \quad (|v| \neq 0) \).

there exists a \( \lambda_0 > 0 \) such that for all \( \lambda \in [0, \lambda_0] \) the following estimations
\[ \mp \dot{q}_2(t) > |v| \text{ and } |q_1(t)| \leq C \quad (|v| \neq 0, C \neq 0) \]
are true for all \( t \in [0, T_0] \). Here \( T_0 = T_0(p(0), q(0), \lambda, C, |v|) > 0 \) is uniquely defined by

\[
T_0 := \sup \left\{ \tilde{T}_0 \in \mathbb{R}^+ \mid \begin{array}{l}
(x(t), y(t), z(t)) \in B^\pm \\
\text{for all } t \in [0, \tilde{T}_0]
\end{array} \right\} \tag{6.2}
\]

and \( B^\pm = B^\pm(p(0), q(0), \lambda, C, |v|) \) denote the beams (see Figure 5.6)

\[
B^+ := [-c_x^{(0)} + \lambda \delta_x, c_x^{(0)} + \lambda \delta_x] \times [c_y^{(0)} - \lambda \delta_y, c_y^{(0)} + \lambda \delta_y] \times (c_z^{(0)} - \lambda \delta_z(0), \infty],
\]

\[
B^- := [-c_x^{(0)} - \lambda \delta_x, c_x^{(0)} + \lambda \delta_x] \times [c_y^{(0)} - \lambda \delta_y, c_y^{(0)} + \lambda \delta_y] \times (-\infty, c_z^{(0)} + \lambda \delta_z(0)] .
\]

From (5.11), (5.12) and (5.13)

\[
c_z^{(0)} = |2p_2(0) + q_1(0)| + |p_1(0) + q_2(0)|, \quad c_y^{(0)} = p_2(0) + q_1(0),
\]

\[
c_x^{(0)} = 2p_1(0) + q_2(0),
\]

and define

\[
\delta_x := \frac{3}{|v|} \left( \frac{C}{2|q_2(0)|^2 + \frac{2}{|q_2(0)|}} \right), \quad \delta_y := \frac{3}{|v|} \left( \frac{1}{|q_2(0)|} \right),
\]

\[
\delta_z(0) := \frac{3\lambda}{|v|} \left( \frac{C}{|q_2(0)|^2 + \frac{3 + 3|q_2(0)|}{|v|}} - \frac{3}{|v|} \log \left( |q_2(0)| \right) \right).
\]

The initial point \( (x(0), y(0), z(0)) \) is inside \( B^\pm(p(0), q(0), \lambda, C, |v|) \) with positive distance to its border.

**Proof by author.** Using the definition (6.2), one gets

\[
x(t) = 2p_2(t) + q_1(t) = c_1 \in \begin{bmatrix} -c_x^{(0)} - \lambda \delta_x, c_x^{(0)} + \lambda \delta_x \end{bmatrix},
\]

\[
y(t) = p_2(t) + q_1(t) = c_2 \in \begin{bmatrix} c_y^{(0)} - \lambda \delta_y, c_y^{(0)} + \lambda \delta_y \end{bmatrix}
\]

\[
z(t) = 2p_1(t) + q_2(t) = c_3 \geq c_z^{(0)} - \lambda \delta_z(0), \quad z(t) = c_3 \leq c_z^{(0)} + \lambda \delta_z(0)
\]

as long as \( t \in [0, T_0] \). In particular,

\[
p_1 = \frac{1}{2}(c_3 - q_2), \quad p_2 = c_1 - c_2,
\]

\[
q_1 = 2c_2 - c_1, \quad q_2 = p_2 - q_1 = 2c_1 - 3c_2
\]

for \( t \in [0, T_0] \).

\[9\] The notation implies that \( (x(t), y(t), z(t)) \) remains inside \( B^\pm \) depending on the sign of \((p_2(0) + q_1(0))\).
This leads to the following inequalities for \( p_2, q_1 \) and \( q_2 \) if \( t \in [0, T_0] \)

\[
|p_2(t)| \leq |c_1| + |c_2| + \left| c_x^{(0)} \right| + \lambda \delta_x + \left| c_y^{(0)} \right| + \lambda \delta_y,
\]

\[\text{(6.3)}\]

\[
|q_1(t)| \leq 2|c_2| + |c_1| + 2 \left| c_y^{(0)} \right| + \left| c_x^{(0)} \right| + \lambda (2 \delta_y + \delta_x),
\]

\[\dot{q}_2(t) \leq 2 \left( c_x^{(0)} + \lambda \delta_x \right) - 3 \left( c_y^{(0)} - \lambda \delta_y \right),
\]

\[\dot{q}_2(t) \geq -2 \left( c_x^{(0)} + \lambda \delta_x \right) - 3 \left( c_y^{(0)} + \lambda \delta_y \right).
\]

\[\text{(6.4)}\]

\[\text{(6.5)}\]

Now the following limitations can be satisfied

\[
|q_1(t)| \leq C \iff \frac{1}{2 \delta_y + \delta_x} \left( C - 2 \left| c_y^{(0)} \right| - c_x^{(0)} \right) > \lambda,
\]

\[\text{(6.6)}\]

\[
\dot{q}_2(t) < -|v| \iff \frac{-2c_x^{(0)} + 3c_y^{(0)} - |v|}{2 \delta_x + 3 \delta_y} > \lambda \quad \text{in case } c_y^{(0)} > 2 \epsilon,
\]

\[\text{(6.7)}\]

\[
\dot{q}_2(t) > |v| \iff \frac{-2c_x^{(0)} - 3c_y^{(0)} - |v|}{2 \delta_x + 3 \delta_y} > \lambda \quad \text{in case } c_y^{(0)} < -2 \epsilon.
\]

\[\text{(6.8)}\]

Set \( C > 2 \left| c_y^{(0)} \right| + c_x^{(0)} \) and (6.6) can be fulfilled. With the second assumptions \(-2c_x^{(0)} + 3c_y^{(0)} - |v| > 0\) for \( \pm c_y^{(0)} > 2 \epsilon \) the inequalities for \( \dot{q}_2 \) can be fulfilled.

Now with appropriately chosen \( \lambda_0 \) inequalities (6.6) and (6.7) or (6.8) are fulfilled for every \( \lambda \in [0, \lambda_0] \).

With the initial points from Theorem 6.2, the flow cannot leave the beam in finite time.

**Theorem 6.3 (by author).** For all initial points \((x(0), y(0), z(0))\) and corresponding beam \( B^\pm(p(0), q(0), \lambda, C, |v|) \) similar to Theorem 6.2, there exists a \( \tilde{\lambda}_0 \in ]0, \lambda_0], \) such that for all \( \lambda \in ]0, \tilde{\lambda}_0 \] the flow stays inside that beam for all \( t > 0 \).

**Proof by author.** The Duhamel formula (6.1) yields the following equations for \( x(t), y(t) \) and \( z(t) \), let \( \lambda \in ]0, \lambda_0] \)

\[
2p_2(t) + q_1(t) = 2p_2^{(0)}(t) + q_1^{(0)}(t)
\]

\[
- 3 \lambda \int_0^t \frac{1}{||q(s)||^3} \left( - \sin(t - s)q_1(s) + 2 \cos(t - s)q_2(s) \right) \, ds,
\]

\[
p_2(t) + q_1(t) = p_2^{(0)}(t) + q_1^{(0)}(t) - 3 \lambda \int_0^t \frac{1}{||q(s)||^3} q_2(s) \, ds,
\]

\[
2p_1(t) + q_2(t) = 2p_1^{(0)}(t) + q_2^{(0)}(t)
\]

\[
- 3 \lambda \int_0^t \frac{1}{||q(s)||^3} \left( 2q_1(s) + 3(t - s)q_2(s) \right) \, ds.
\]
As long as \( t \in [0, T_0] \) is was shown in Theorem 6.2 that \( \mp q_2(t) \geq |v| \) (in particular, \( \mp q_2(t) \geq |v|t + |q_2(0)| \)) and \( |q_1(t)| \leq C \). The difference between \( x(t) \) and \( x^{(0)}(t) \) can be estimated using the Duhamel principle for \( t \in [0, T_0] \) as

\[
\left| 2p_2(t) + q_1(t) - 2p_2^{(0)}(t) - q_1^{(0)}(t) \right| \leq 3\lambda \int_0^t \frac{1}{|q_2(s)|^3} \left( |q_1(s)| + 2|q_2(s)| \right) \, ds
\]

\[
= 3\lambda \int_0^t \frac{|q_1(s)|}{|q_2(s)|^3} + \frac{2}{|q_2(s)|^2} \, ds
\]

\[
\leq 3\lambda \int_0^t \frac{C}{(|v|s + |q_2(0)|)^3} + \frac{2}{(|v|s + |q_2(0)|)^2} \, ds
\]

\[
= \frac{3\lambda}{|v|} \left( \frac{C}{2|q_2(0)|^2} - \frac{C}{2(|v|t + |q_2(0)|)^2} + \frac{2}{|q_2(0)|} - \frac{2}{|v|t + |q_2(0)|} \right)
\]

\[
\leq \frac{3\lambda}{|v|} \left( \frac{C}{2|q_2(0)|^2} - \frac{C}{2(|v|T_0 + |q_2(0)|)^2} + \frac{2}{|q_2(0)|} - \frac{2}{|v|T_0 + |q_2(0)|} \right)
\]

\[
\lim_{T_0 \to \infty} \frac{3\lambda}{|v|} \left( \frac{C}{2|q_2(0)|^2} + \frac{2}{|q_2(0)|} \right) = \lambda \delta_x. \tag{6.9}
\]

Secondly, the difference between \( y(t) \) and \( y^{(0)}(t) \) can be estimated for \( t \in [0, T_0] \) as

\[
\left| p_2(t) + q_1(t) - p_2^{(0)}(t) - q_1^{(0)}(t) \right| \leq 3\lambda \int_0^t \frac{1}{|q_2(s)|^2} \, ds
\]

\[
\leq 3\lambda \int_0^t \frac{1}{(|v|s + |q_2(0)|)^2} \, ds
\]

\[
= \frac{3\lambda}{|v|} \left( \frac{1}{|q_2(0)|} - \frac{1}{|v|t + |q_2(0)|} \right)
\]

\[
\leq \frac{3\lambda}{|v|} \left( \frac{1}{|q_2(0)|} - \frac{1}{|v|T_0 + |q_2(0)|} \right)
\]

\[
\lim_{T_0 \to \infty} \frac{3\lambda}{|v|} \left( \frac{1}{|q_2(0)|} \right) = \lambda \delta_y. \tag{6.10}
\]

Lastly, the difference between \( z(t) \) and \( z^{(0)}(t) \) is estimated for \( t \in [0, T_0] \) as

\[
\left| 2p_1(t) + q_2(t) - 2p_1^{(0)}(t) - q_2^{(0)} \right| \leq 3\lambda \int_0^t \frac{2C}{(|v|s + |q_2(0)|)^3}
\]

\[
+ \frac{3t}{(|v|s + |q_2(0)|)^2} + \frac{3s}{(|v|s + |q_2(0)|)^2} \, ds
\]
\[
\frac{3\lambda}{|v|} \left( \frac{C}{|q_2(0)|^2} - \frac{C}{(|v|t + |q_2(0)|)^2} + 3t \left( \frac{1}{|q_2(0)|} - \frac{1}{|v|t + |q_2(0)|} \right) \right) \\
+ \frac{3}{|v|} \left( \frac{|q_2(0)|}{|v|t + |q_2(0)|} + \log \left( \frac{|v|t + |q_2(0)|}{|q_2(0)|} \right) - 1 - \log \left( |q_2(0)| \right) \right) \\
\leq \frac{3\lambda}{|v|} \left( \frac{C}{|q_2(0)|^2} + 3t \frac{|q_2(0)|^2}{|q_2(0)|^2} + 3 + 3|q_2(0)| + \frac{3}{|v|} \log \left( |q_2(0)| \right) \right) \\
=: t\lambda\delta_z + \lambda\delta_z(0).
\]

Choose \( \tilde{\lambda}_0 := \min \left\{ \frac{3|c(y_0)|}{\delta_z}, \lambda_0 \right\} \) with \( \lambda_0 \) from Theorem 6.2, because this ensures with (6.11)

\[
z(t) > t \left( 3c_y(0) - \lambda\delta_z \right) + c_z(0) - \lambda\delta_z(0) \\
\geq c_z(0) - \lambda\delta_z(0) \text{ if } c_y(0) > 2\epsilon, \\
z(t) < t \left( 3c_y(0) + \lambda\delta_z \right) + c_z(0) + \lambda\delta_z(0) \\
\leq c_z(0) + \lambda\delta_z(0) \text{ if } c_y(0) < -2\epsilon.
\]

for all \( t \geq 0 \).

For all \( \lambda \in \left[ 0, \tilde{\lambda}_0 \right] \) with calculations (6.9)-(6.11) above, \( (x(t), y(t), z(t)) \) is a curve which remains inside \( B^\pm \) from Theorem 6.2 for all \( t \in [0, T_0] \) independent of \( T_0 \), therefore, \( T_0 = \infty \).

Altogether, Theorem 6.2 yield that the flow with according initial points inside one of the beams \( B^\pm \) has minimal velocity \( \mp \dot{q}_2 > |v|, v \in \mathbb{R}\setminus\{0\} \) and is bounded in \( q_2 \)-direction as long as the flow stays inside that beam. Now Theorem 6.3 states, that if the flow with these initial points has a minimal velocity and is bounded in \( q_1 \)-direction, it will stay inside the beam for all time \( t > 0 \).

So both theorems together result in the fact that the disturbed flow of the specified initial points drifts towards plus or minus infinity in \( q_2 \)-direction with a minimal velocity \( v \) and is bounded in \( q_1 \)-direction for all \( t > 0 \).

A direct result of these two theorems is that the secondary (moon) cannot drift towards the first primary (sun).

**Corollary 6.4** (by author). The flow with similar initial points as in Theorem 6.3 cannot drift towards the first primary for \( t > 0 \).

**Proof by author.** The Hamiltonian of Hill’s Lunar Problem was derived in Section 2.2 and due to the scaling \( \mu \to 0 \) at the end, the first primary’s location \( (-\mu^{-1/3}, 0, 0) \) diverged to minus infinity on the \( q_1 \)-axis.

According to Theorem 6.3, \( |q_1(t)| \) is bounded by \( C \) for all \( t > 0 \) and, therefore, cannot drift towards the primary. \( \square \)
6.3 Regularization of Hill’s Lunar Problem

A direct result of Theorem 6.2 is, that the flow does not reach infinity in finite time.

**Corollary 6.5** (by author). The flow of Hill’s Lunar Problem does not diverge towards infinity in finite time with the same assumptions as in Theorem 6.2.

**Proof by author.** With inequalities (6.4) and (6.5) one gets for all \( t \geq 0 \)

\[
\dot{q}_2(t) \leq 2 \left( c_x^{(0)} + \lambda \delta_x \right) - 3 \left( c_y^{(0)} - \lambda \delta_y \right) < -|v|,
\]

\[
\dot{q}_2(t) \geq -2 \left( c_x^{(0)} + \lambda \delta_x \right) - 3 \left( c_y^{(0)} + \lambda \delta_y \right) > |v| - 6c_y^{(0)}
\]

in case \( c_y^{(0)} > 2 \epsilon \) and for the case \( c_y^{(0)} < -2 \epsilon \)

\[
\dot{q}_2(t) \leq 2 \left( c_x^{(0)} + \lambda \delta_x \right) - 3 \left( c_y^{(0)} - \lambda \delta_y \right) < -|v| - 6c_y^{(0)};
\]

\[
\dot{q}_2(t) \geq -2 \left( c_x^{(0)} + \lambda \delta_x \right) - 3 \left( c_y^{(0)} + \lambda \delta_y \right) > |v|.
\]

This proves that \( q_2(t) \) does not diverge towards infinity in finite time \( t \geq 0 \). Inequality (6.6) shows that \( q_1(t) \) is bounded for all \( t \geq 0 \), hence, \( q(t) \) cannot reach infinity in finite time.

Inequality (6.3) yields that \( p_2(t) \) is bounded for all \( t \geq 0 \). The Hamiltonian from lemma 5.5 is a constant of motion

\[
E \equiv \frac{1}{2} \|p\|^2 + p_1q_2 - p_2q_1 - q_1^2 + \frac{1}{2}q_2^2 - \lambda \frac{3}{\|q\|}
\]

which means that \( p_1(t) \) also cannot reach infinity in finite time. 

\( \square \)

7 Traverse flow from Lagrangian points

The volume of escaping flow which starts on the dividing surface at \( L_1 \), passes through the planetary neighborhood and ends on the dividing surface at \( L_2 \) has been given in Theorem 5.4. Once again, the invariant planar case of Hill’s Lunar Problem is analyzed in this section.

**Theorem 7.1** (by author). It exists an energy \( E_1 \) such that for every \( E > E_1 \) there are initial points \( (p(0), q(0)) \) whose corresponding flow enter the planetary neighborhood in configuration space through the gap at \( L_1 \) and leaves it through the gap at \( L_2 \), see Figure 7.1.

**Proof by author.** Consider the planar case \( p_3(0) = 0 = q_3(0) \) and define for \( a = \alpha + 1 > \delta + 1 > 1 \) and \( b > \epsilon > 0 \) the box

\[
C := [-a, a] \times [\epsilon, b].
\]
For the desired dynamic choose $p_1(0) = -\tau$, $q_1(0) = -1 - \delta$, $q_2(0) = \tau > \epsilon$ and with the Duhamel principle (6.1) the following difference between the disturbed and undisturbed flow are true as long as $q(t) \in C$

\[
|q_1(t) - q_1^0(t)| \leq \int_0^t \frac{3\lambda}{\|q\|^2} |\sin(t-s)q_1(s) + (2 - 2\cos(t-s))q_2(s)| \, ds
\]
\[
\leq \frac{3\lambda}{\epsilon^3}(a + 4b)t,
\]
\[
|q_2(t) - q_2^0(t)| \leq \int_0^t \frac{3\lambda}{\|q\|^2} \left(2\cos(t-s) - 2q_1(s) + (-3(t-s) + 4\sin(t-s))q_2(s) \right) \, ds
\]
\[
\leq \frac{3\lambda}{\epsilon^3} \left(\frac{9}{2}bt^2 + 4(a + b)t \right).
\]

Find $t_0 > 0$ such that

\begin{align*}
q_1^0(t_0) &= a, \quad (7.1) \\
q_1(t_0) &> 1 + \beta > 1 \text{ with } \beta < \alpha, \quad (7.2) \\
q(t) &\in C \text{ for all } t \in [0, t_0]. \quad (7.3)
\end{align*}

Then with $E \equiv \frac{1}{2}p_2^2 + (1 + \delta)p_2 - (1 + \delta)^2 - \lambda \frac{3}{\sqrt{(1+\delta)^2 + \tau^2}}$ one gets $p_2(0) =$
\[-(1 + \delta) \pm \sqrt{3(1 + \delta)^2 + \lambda \frac{6}{\sqrt{(1 + \delta)^2 + \tau^2}} + 2E} \text{ (value of Hamiltonian given in Lemma 5.5) and}
\]
\[q_1^{(o)}(t) = -\cos(t)(2p_2(0) - 1 - \delta) + 2p_2(0) - 2 - 2\delta,\]
\[q_2^{(o)}(t) = -3t(p_2(0) - 1 - \delta) + 2\sin(t)(2p_2(0) - 1 - \delta) + \tau.\]

One has to claim (in particular the negative solution of \(p_2(0)\) is not appropriate)
\[2p_2(0) - 1 - \delta > 0 \quad (7.4)\]
in order to ensure \(q_1^{(o)}(t) > 0\) for \(0 < t < \pi\). Then with (7.1)
\[t_0 := \arccos\left(\frac{2p_2(0) - 3 - 2\delta - \alpha}{2p_2(0) - 1 - \delta}\right) \in [0, \pi].\]

The time \(t_0\) must be well-defined, so
\[
\frac{2p_2(0) - 3 - 2\delta - \alpha}{2p_2(0) - 1 - \delta} \in [-1, 1].
\]

That is
\[
\alpha > -2 - \delta \text{ and } p_2(0) > 1 + \frac{3}{4}\delta + \frac{1}{4}\alpha \quad (7.5)
\]
\[
\iff E > \left(\frac{1}{2} + \frac{7}{4}\delta + \frac{1}{4}\alpha\right)^2 - \frac{3}{2}(1 + \delta)^2 - \lambda \frac{3}{\sqrt{(1 + \delta)^2 + \tau^2}}.
\]

With (7.5), the inequality (7.4) is automatically true. Additionally, for (7.2)
\[q_1(t_0) \geq q_1^{(o)}(t_0) - \frac{3\lambda}{e^3}(a + 4b)t_0 > 1 + \beta\]
\[
\iff t_0 < \frac{(\alpha - \beta)e^3}{3\lambda(a + 4b)} \quad (7.6)
\]
Lastly (7.3) for all \(t \in [0, t_0]\) and \(p_2(0) > 1 + \delta\) (true due to equation 7.5 and \(\alpha > \delta\))
\[q_2(t) \leq q_2^{(o)}(t) + \frac{3\lambda}{e^3}\left(\frac{9}{2}bt^2 + 4(a + b)t\right)\]
\[< 2\sin(t_0)(2p_2(0) - 1 - \delta) + \tau + \frac{3\lambda}{e^3}\left(\frac{9}{2}bt_0^2 + 4(a + b)t_0\right) \leq b, \quad (7.7)\]
\[q_2(t) \geq q_2^{(o)}(t) - \frac{3\lambda}{e^3}\left(\frac{9}{2}bt^2 + 4(a + b)t\right)\]
\[> -3t_0(p_2(0) - 1 - \delta) + \tau - \frac{3\lambda}{e^3}\left(\frac{9}{2}bt_0^2 + 4(a + b)t_0\right) \geq \epsilon. \quad (7.8)\]
Now for the limit $E$ to infinity, the initial point $p_2(0)$ diverges towards infinity and $t_0$ converges towards 0 from above. Therefore, the upper estimation of $q_2(t)$ converges for $t_0 \searrow 0$ towards $\tau$ from above and the lower estimation converges towards $\tau$ from below in the same limit.

Altogether, there exists a smallest energy $E_1 > E > \frac{3}{2} (2 + \frac{7}{4} \delta + \frac{1}{4} \alpha)^2 - \frac{3}{2} (1 + \delta)^2 - \lambda \sqrt{(1 + \delta)^2 + \tau^2}$, so that for every $E \geq E_1$ the inequalities (7.6), (7.7) and (7.8) are true.

The difference between $q_2(0)(t)$ and $q_2(t)$ given by (7.7) and (7.8) is also shown in Figure 7.1. Since the flow is continuous, there exists a neighborhood of the calculated initial point

$$
\left( -\tau, -(1 + \delta) + \sqrt{3(1 + \delta)^2 + \lambda \frac{6}{\sqrt{(1 + \delta)^2 + \tau^2}} + 2E, -1 - \delta, \tau } \right),
$$

so that the flow of every initial point within that neighborhood traverse through the gaps at $L_1$ and $L_2$ in configuration space.

The dividing surfaces $S^4_{\text{DS}}(L_j)$ are defined explicitly for the Birkhoff normal form $\mathcal{H}_s$ of Hill’s Lunar Problem in canonical coordinates $(P, Q)$. Nevertheless, the initial point above might lie inside the area resulting from inverse normal transformation of $S^4_{\text{DS}}(L_j)$ (see Waalkens et al. [2005]), with adequate selections of $\alpha > \delta > 0$, $\tau > \epsilon > 0$ and $0 < \beta < \alpha$ and sufficient accuracy.
8 Summary

The Hamiltonian of Hill’s Lunar Problem was derived from the restricted Three-Body Problem in Section 2. The latter is a limiting case of the full Three-Body Problem, in which both primaries rotate around their center of mass with constant velocity \( \omega \in \mathbb{R}\setminus\{0\} \) and it describes the motion of a small secondary mass (moon). One could then introduce co-rotating coordinates, in which the primaries remain at the same point for all time; the second primary (earth) with mass \( \mu \) at \((1-\mu, 0, 0)^T\) and the first primary (sun) with mass \(1-\mu\) at \((-\mu, 0, 0)^T\). In order to define Hill’s Lunar Problem, the limit for the second primary’s mass \( \mu \to 0 \) has to be considered. As a result, the heavy primary tends to minus infinity on the \( q_1 \)-axis.

Since the linearization of a Hamiltonian vector field at an equilibrium is infinitesimal symplectic, its eigenvalues \( \pm i \omega_j, \omega_j > 0 \) have to be imaginary in order to ensure that the equilibrium is Liapunov-stable. With such an equilibrium and pairwise different frequencies \( \omega_j \) that do not satisfy a resonance relation of order \( s \) or smaller, a Hamiltonian \( H \in \mathbb{C}^{s+1} \) can be reduced to a Birkhoff normal form \( \mathcal{H}_s \) of degree \( s \) as described in Section 3. That means, \( \mathcal{H}_s \) is a polynomial of degree \( \lfloor \frac{s}{2} \rfloor \) in the action coordinates \( J_j := \frac{1}{2} \left( P_j^2 + Q_j^2 \right) \). Furthermore, a Birkhoff-like normal form for Hamiltonians with an equilibrium of saddle-center-...-center type could also be derived, only depended on \( I = P_1 Q_1 \) and \( J_j \) under similar circumstances.

The Birkhoff-like normal form of degree 2 of Hills’ Lunar Problem was analyzed in Section 4. Several submanifolds of the energy surfaces were defined, which were used to describe the process of escaping the planetary neighborhood. In particular, the flow has to cross the dividing surfaces in order to enter or exit this compact neighborhood. The planar case has been visualized in certain figures, which show the projection of the dynamic onto the \( P_j - Q_j \)-planes.

In Section 5, a volume form on the energy surface was computed at first. Afterwards, one could prove a formula for the flow’s volume, which crosses the planetary neighborhood from one dividing surface to the other. It is given by the integral of the passage time over the front surface of a certain subset, which consists of such escaping orbits.

New coordinates for the undisturbed flow, consisting of sums of \( p \) and \( q \), were introduced which stay inside a half-open set \( B_{\{0\}}^\pm \) for all \( t > 0 \). The same coordinates could be defined for the disturbed Hamiltonian and its dynamic compared to the undisturbed ones in Section 6. A thickened half-open set \( B^\pm \) was defined for specific initial points, so that the disturbed coordinates stay inside it for all \( t > 0 \). As a result, the disturbed flow is bounded in \( q_1 \)-direction, therefore, cannot fly towards the heavy primary, and has a minimal velocity \( \mp \dot{q}_2(t) > |v|, v \in \mathbb{R}\setminus\{0\} \) for all time \( t > 0 \).

Lastly, the existence of an energy \( E_1 \) was shown in Section 7, so that for every energy \( E \) larger than \( E_1 \) there exists initial points whose flow crosses the planetary neighborhood in the configuration space from the gap at \( L_1 \) to the one at \( L_2 \).
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Friedrich-Alexander-University, Erlangen (Germany)
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REFERENCES

• References available on request
Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe, dass alle Stellen der Arbeit, die wörtlich oder sinngemäß aus anderen Quellen übernommen wurden, als solche kenntlich gemacht sind und dass die Arbeit in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegt wurde.

Erlangen, den 25. September 2013