Banach gradient flows for various families of knot energies

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MIDS

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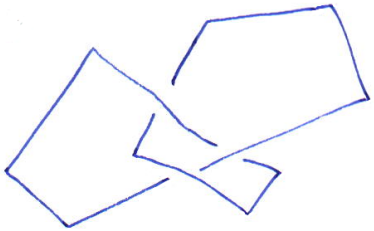
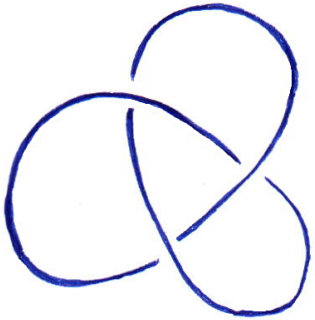
Introduction
- Knots
- Knot Energies
- Gradient Flows of Knot Energies

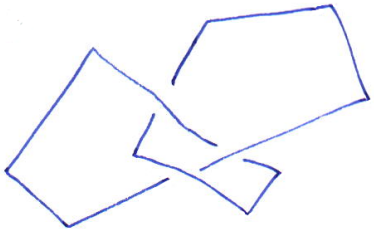
Gradient Flows in Banach Spaces and Main Theorem

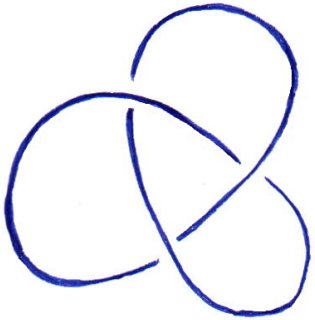
Proof of the Main Theorem
- From Gradient Flows to Curves of Maximal Slope
- Energy Dissipation Inequality
- Minimising Movement
- Curve of Maximal Slope





A knot is an equivalence class of embeddings $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$, i.e. topological homeomorphisms onto their images.





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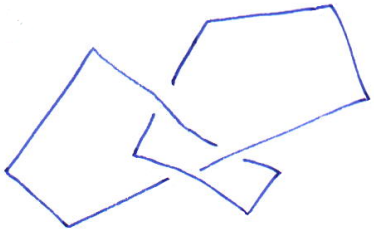
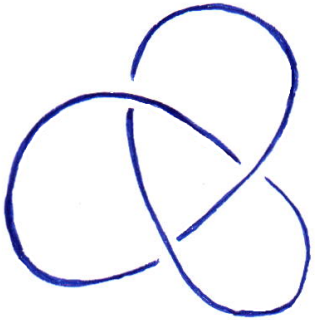
Two emdbeddings, called knotted curves, are identified if there exists an ambiant isotopy of $\mathbb{R}^3$ transforming one into the other.
Broadly speaking, no cutting, self-intersecting, or pull-tight phenomena of the curves are allowed.





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Broadly speaking, no cutting, self-intersecting, or pull-tight phenomena of the curves are allowed.
Questions in knot theory

- How to classify of all knots?
- How to easily assign an arbitrary knotted curve to the knot class it represents?

Knot Energies

introduced by Shinji Fukuhara [Fuk88] as an analytical tool used to deform a given knotted curve into a *standard form*

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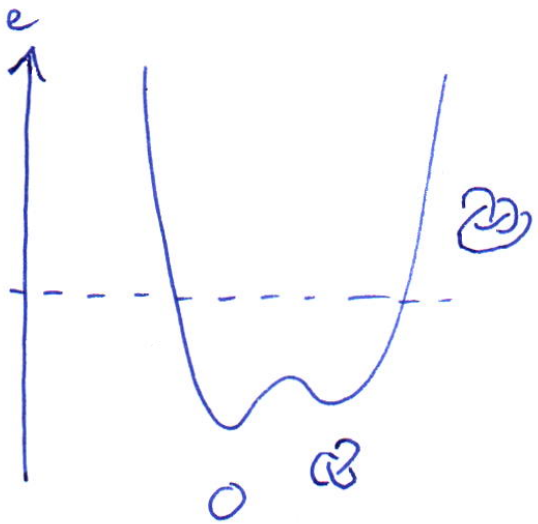
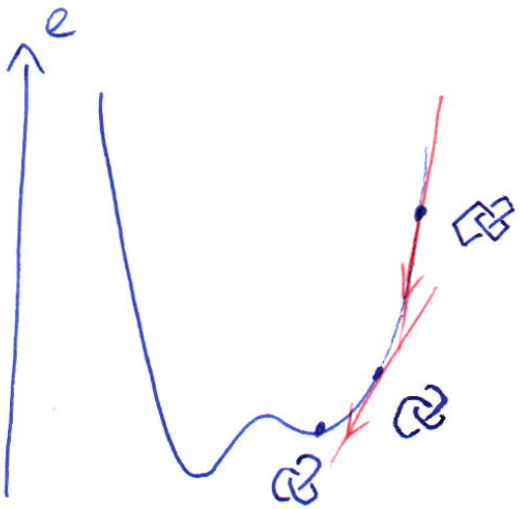
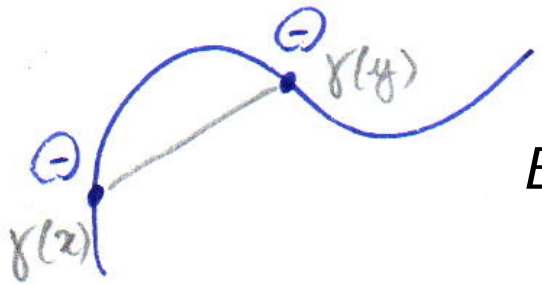


Classification problem: only finitely many knots below energy threshold?



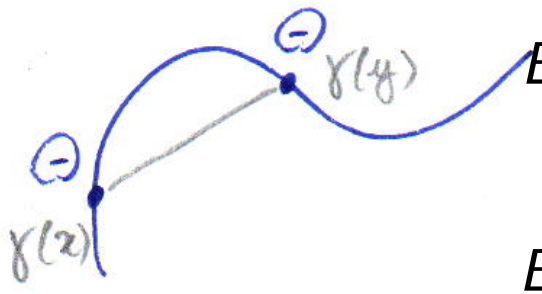
Compare standard forms instead of arbitrary curves

Jun O'Hara [O'H91] introduces the *Möbius energy*



$$E(\gamma) = \int_0^1 \left(\lim_{\varepsilon \downarrow 0} \int_{x+\varepsilon}^{1+x-\varepsilon} \frac{\mathrm{d}y}{|\gamma(x) - \gamma(y)|^2} - \frac{2}{\varepsilon} \right) \mathrm{d}x$$

Jun O'Hara [O'H91] introduces the *Möbius energy* and in [O'H92a] the two-parameter family of knot energies, called *O'Hara energies*



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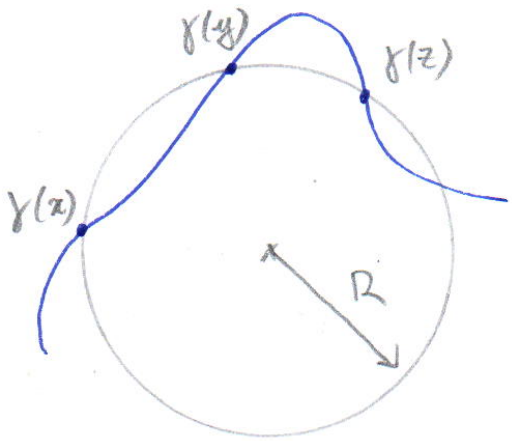
$$E(\gamma) = \iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{1}{|\gamma(x)-\gamma(y)|^2} - \frac{1}{|x-y|^2} \mathrm{d}x\mathrm{d}y - 2$$

$$E^{\alpha,p}(\gamma) = \iint_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{1}{|\gamma(x)-\gamma(y)|^\alpha} - \frac{1}{|x-y|^\alpha} \right)^p |\gamma'(x)||\gamma'(y)| \mathrm{d}x\mathrm{d}y$$

Oscar Gonzalez and John Maddocks [GM99] introduce the *integral Menger curvature*



$$R^p(a,b,c) = 2^{-p}\frac{|(b-a)\wedge(c-a)|^p}{|b-a|^p|c-a|^p|b-c|^p}$$

$$\mathrm{intM}^p(\gamma) = \iiint_{(\mathbb{R}/\mathbb{Z})^3} \frac{|\gamma'(x)|\gamma'(y)||\gamma'(z)|}{R^p\big(\gamma(x),\gamma(y),\gamma(z)\big)}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Oscar Gonzalez and John Maddocks [GM99] introduce the *integral Menger curvature*
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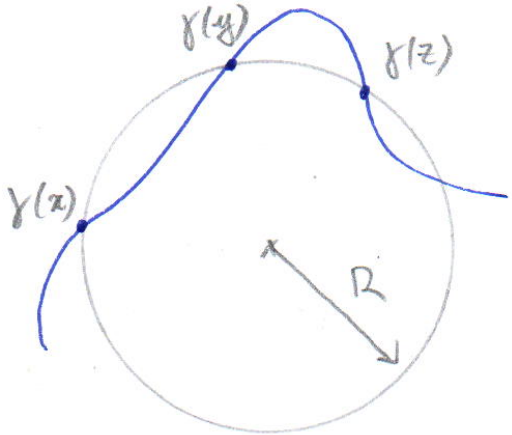

$$R^{p,q}(a,b,c) = \frac{|(b-a)\wedge(c-a)|^q}{|b-a|^p|c-a|^p|b-c|^p}$$

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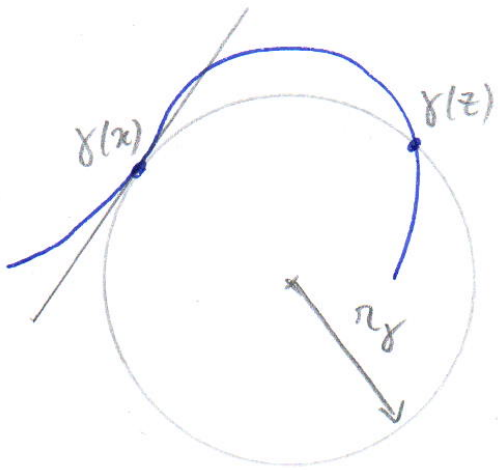
Paweł Strzelecki and Heiko von der Mosel [SvdM12] introduce *tangent-point energies*



$$r_\gamma^p(\gamma(x),\gamma(y)) = 2^{-p}\frac{|\gamma(x)-\gamma(y)|^{2p}}{\mathrm{dist}^p\big(\gamma(x)+\mathbb{R}\gamma'(x),\gamma(y)\big)}$$

$$\mathrm{TP}^p(\gamma) = \iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{|\gamma'(x)||\gamma'(y)|}{r^p\big(\gamma(x),\gamma(y)\big)}\mathrm{d}x\mathrm{d}y$$

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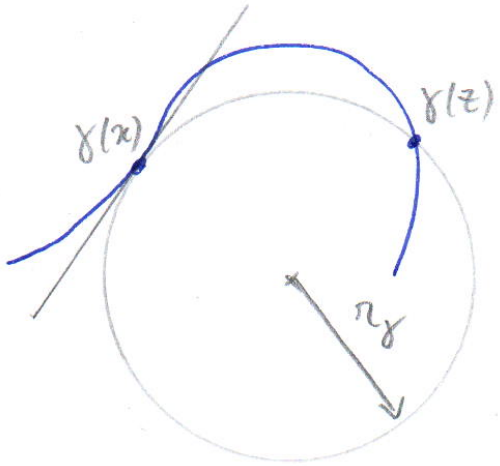

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Knot Energy [O'H03, Definition 1.1]

functional $\mathscr{E}$ on a topological space of knotted curves satisfying

- $\mathscr{E} \geq 0$
- If $\gamma_n \to \gamma$ and γ has double points, then $\mathscr{E}(\gamma_n) \to \infty$.

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All energies presented above are knot energies.

- $E^{\alpha,p}$, $\alpha \in (0,\infty)$, $p \in [1,\infty)$ with $2 \leq \alpha p < 2p+1$ [O'H92b]
- $\mathrm{intM}^{p,q}$, $q \in (1,\infty)$, $p \in (\frac{2}{3}q+1, q+\frac{2}{3})$ [BR15b]
- $\mathrm{TP}^{p,q}$, $q \in (1,\infty)$, $p \in (q+2, 2q+1)$ [BR15a]

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All these results rely on **bi-Lipschitz estimates** for injective and arclength parametrised curves.

The "right" topological spaces for these knot energies have been characterised as

- $W_{\mathrm{ir}}^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ for $E^{\alpha,p}$ [Bla12a]
- $W_{\mathrm{ir}}^{\frac{3p-2}{q}-1,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ for $\mathrm{intM}^{p,q}$ [BR15b]
- $W_{\mathrm{ir}}^{\frac{p-1}{q},q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ for $\mathrm{TP}^{p,q}$ [BR15a]

where $W_{\mathrm{ir}}^{1+s,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ is the subspace of $W^{1,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ with

$$\iint_{(\mathbb{R}/\mathbb{Z})^2}\left(\frac{|\gamma'(x)-\gamma'(y)|}{|x-y|_{\mathbb{R}/\mathbb{Z}}^s}\right)^\rho\frac{\mathrm{d}x\mathrm{d}y}{|x-y|_{\mathbb{R}/\mathbb{Z}}}<\infty.$$

and γ injectiv and regular, i.e. $|\gamma'|>0$.

Gradient Flows of Knot Energies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) = -\nabla\mathscr{E}[\Gamma(t)]$$

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L^2-gradient flows of knot energies

- Zheng-Xu He [He00] shows short-time existence of an L^2-gradient flow of the Möbius energy E.
- Simon Blatt [Bla12b] shows long-time existence of an L^2-gradient flow of E if starting sufficiently close to a local minimiser of E. Convergence to a critical point.
- Simon Blatt [Bla18] shows long-time existence of an L^2-gradient flow of $E^{\alpha,1} + \mathcal{L}$ (functional regularised with the length of γ). Convergence to a critical point.

Gradient Flows of Knot Energies

Hilbert space gradient flow in energy space

- Philipp Reiter and Henrik Schumacher [RS21] show short-time existence of a gradient flow of $E^{2,1}$ with respect to a metric related to the one of $W^{\frac{3}{2},2}$.
- Jan Knappmann, Henrik Schumacher, Daniel Steenebrügge, and Heiko von der Mosel [KSSvdM22] prove long-time existence of the gradient flow of $\mathrm{intM}^{p,2}$ in $W^{\frac{3}{2}p-2,2}$.
- Daniel Steenebrügge [Ste22] proves long-time existence of the gradient flow of $\mathrm{TP}^{p,2}$ in $W^{\frac{p-1}{2},2}$.
- Gradient flows in respective energy spaces can lead to numerical procedures that are substantially more **efficient** and **robust** in comparision to the methods for the L^2-gradient flow. [RS21]

Gradient flows in Banach spaces

$\mathcal{H}$ Hilbert space, $\phi \in C^1(\mathcal{H}, \mathbb{R})$, $u \in C^1((a,b), \mathcal{H})$.
A gradient flow is given by

$$u'(t) = -\nabla\phi[u(t)]$$

$$\nabla = \mathfrak{J}_{\mathcal{H}}^{-1} \circ D$$

where $\mathfrak{J}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^*$ is the Riesz isomorphism, and
$D : C^1(\mathcal{H}, \mathbb{R}) \to C^0(\mathcal{H}, \mathcal{H}^*)$ is the derivative operator.

Gradient flows in Banach spaces

$\mathcal{B}$ Banach space, duality mapping with weights $\frac{1}{\theta} + \frac{1}{\theta^*} = 1$:

$$\begin{aligned} \mathfrak{J}_{(\mathcal{B},\theta)} : &\mathcal{B} \to 2^{\mathcal{B}^*} \\ &x \mapsto \{\xi \in \mathcal{B}^* \mid \langle \xi, x \rangle_{\mathcal{B}^* \times \mathcal{B}} = \|x\|_{\mathcal{B}} \|\xi\|_{\mathcal{B}^*}, \|x\|_{\mathcal{B}}^{\theta} = \|\xi\|_{\mathcal{B}^*}^{\theta^*}\} \end{aligned}$$

Examples:

- If $\mathcal{B} = \mathcal{H}$, then $\mathfrak{J}_{(\mathcal{B},2)} = \mathfrak{J}_{\mathcal{H}}$.
- If $\mathcal{B} = L^p$, then $\mathfrak{J}_{(L^p,p)}(f) = |f|^{p-2} f \in L^q$.

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Can formulate differential inclusion

$$\mathfrak{J}_{(\mathcal{B},\theta)}(u'(t)) \ni -D\phi[u(t)]$$

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If the geometry of $\mathcal{B}$ is "nice", then $\mathfrak{J}_{(\mathcal{B},\theta)} \colon \mathcal{B} \to \mathcal{B}^*$ is a homeomorphism, and the inclusion becomes an equation:

$$u'(t) = -\mathfrak{J}_{(\mathcal{B},\theta)}^{-1}(D\phi[u(t)]).$$

Total Energy ϕ

total energy

$1 < \kappa, \theta \quad 0 < \varepsilon \quad \mathscr{E} \in \{\mathrm{TP}^{p,q}, \mathrm{intM}^{p,q}, E^{\alpha,p}\} \quad s, \rho = s, \rho(\mathscr{E})$

$$\phi(\gamma) = \begin{cases} \mathscr{E}(\gamma) + \|\Sigma(\gamma)\|_{W^{s,\rho}}^{\kappa}, & \gamma \in W_{\mathrm{ir}}^{1+s+\varepsilon,\rho} \\ +\infty, & \text{else.} \end{cases}$$

where

logarithmic strain

$\Sigma : W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to W^{s,\rho}(\mathbb{R}/\mathbb{Z})$

$$\Sigma(\gamma)(u) = \log|\gamma'(u)|$$

used in [KSSvdM22] to control the parametrisation of the curves.

[MSvdM, Theorem 1.1]

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$\forall \gamma_0 \in W^{1+s+\varepsilon,\rho}_{\mathrm{ir}} \quad \exists \mathbf{\Gamma} \in C^1\left([0,\infty), W^{1+s+\varepsilon,\rho}\right)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Gamma}(t) = -\mathfrak{J}^{-1}_{(W^{1+s+\varepsilon,\rho},\theta)} D\phi[\mathbf{\Gamma}(t)]$$

$$\mathbf{\Gamma}(0) = \gamma_0$$

[MSvdM, Theorem 1.1]

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$\mathfrak{J}^{-1}_{(W^{1+s+\varepsilon,\rho},\theta)}$ is not Lipschitz-continuous, hence Picard-Lindelöf theory is not applicable.

Metric gradient flows

Instead: metric theory of minimising movements and curves of maximal slope as described in [AGS08]

Metric gradient flows

In Hilbert space: $u, E \in C^1$.

$$u'(t) = -\nabla E[u(t)]$$

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Cauchy-Schwarz inequality

$$\Leftrightarrow \quad \begin{aligned} \|u'(t)\| &= \|\nabla E[u(t)]\| \\ \langle u'(t), \nabla E[u(t)]\rangle_{\mathcal{H}} &= -\|u'(t)\| \, \|\nabla E[u(t)]\| \end{aligned}$$

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Young's inequality

$$\Leftrightarrow \langle u'(t), \nabla E[u(t)]\rangle_{\mathcal{H}} \leq -\frac{1}{2}\left\|u'(t)\right\|^2 - \frac{1}{2}\|\nabla E[u(t)]\|^2$$

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Chain rule

$$\Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}E\big(u(t)\big) \leq -\frac{1}{2}\left\|u'(t)\right\|^2 - \frac{1}{2}\|\nabla E[u(t)]\|^2$$

metric derivative

$$\left\|u'(t)\right\| \rightsquigarrow |u'|(t) := \lim_{s\to t} \frac{d\left(u(t), u(s)\right)}{|s-t|}$$

local slope

$$\|\nabla E[u]\| \rightsquigarrow |\partial E|\big(u\big) := \limsup_{v \xrightarrow{d} u} \frac{\big(E(u) - E(v)\big)^+}{d\big(v, u\big)}$$

upper gradient

$$\frac{\mathrm{d}}{\mathrm{d}t} E\big(u(t)\big) = \langle \nabla E[u(t)], u'(t)\rangle \rightsquigarrow \left|\frac{\mathrm{d}}{\mathrm{d}t} E\big(u(t)\big)\right| \leq |\partial E|\big(u(t)\big)|u'|(t)$$

Energy Dissipation Inequality: gradient flow

$$\frac{\mathrm{d}}{\mathrm{d}t} E\big(u(t)\big) \leq -\frac{1}{2}\left\|u'(t)\right\|^2 - \frac{1}{2}\|\nabla E[u(t)]\|^2$$

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Energy Dissipation Inequality: curve of maximal slope

$$\frac{\mathrm{d}}{\mathrm{d}t} E\big(u(t)\big) \leq -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial E|^2\big(u(t)\big)$$

metric derivative

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Energy Dissipation Inequality: integral

$$E\big(u(t)\big) + \frac{1}{2}\int_s^t |u'|^2(r)\mathrm{d}r + \frac{1}{2}\int_s^t |\partial E|^2\big(u(r)\big)\mathrm{d}r \leq E\big(u(s)\big)$$

In order to solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Gamma}(t) = -\mathfrak{J}^{-1}_{(W^{1+s+\varepsilon,\rho},\theta)} D\phi[\mathbf{\Gamma}(t)]$$

$$\mathbf{\Gamma}(0) = \gamma_0$$

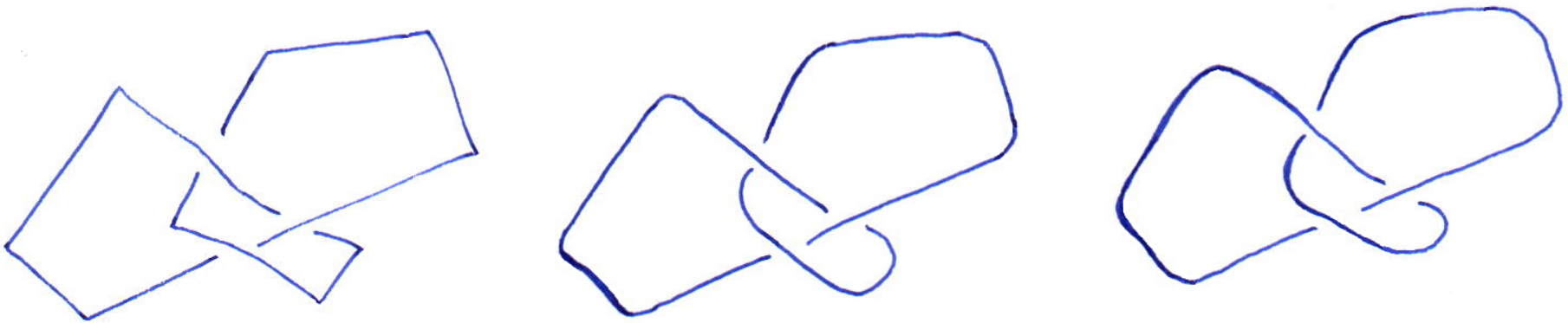
find a solution of

$$\phi\big(\Gamma(t)\big) + \frac{1}{2}\int_s^t |\Gamma'|^2(r)\mathrm{d}r + \frac{1}{2}\int_s^t |\partial\phi|^2\big(\Gamma(r)\big)\mathrm{d}r \leq \phi\big(\Gamma(s)\big)$$

Existence of Curves of Maximal Slope

via minimising movements [AGS08, Chapter 2 and 3]

step 1: fix *step size* $\tau > 0$; solve iterative minimisation scheme



$\mathbf{\Gamma}_\tau(0)$ $\mathbf{\Gamma}_\tau(\tau)$ $\mathbf{\Gamma}_\tau(2\tau)$

$$\Phi_{n,\tau}(\gamma) = \phi(\gamma) + \frac{1}{2\tau}\|\gamma - \gamma_{n-1}\|^2_{W^{1+s+\varepsilon,\rho}}$$

$$\gamma_n \in \operatorname{argmin}\Phi_{n,\tau}$$

obtain $\mathbf{\Gamma}_\tau$ piecewise constant: $\mathbf{\Gamma}_\tau(t) := \gamma_n,\ t \in ((n-1)\tau, n\tau]$

Minimising Movement

step 2: limit $\tau \to 0$.
piecewise constant $\mathbf{\Gamma}_\tau$ satisfy discrete EDI

$$\phi(\mathbf{\Gamma}_\tau(t)) + \frac{1}{2}\int_0^t |\mathbf{\Gamma}_\tau'|^2 \mathrm{d}r + \frac{1}{2}\int_0^t |\mathfrak{D}_\tau \phi|^2 \mathrm{d}r \leq \phi(\underbrace{\mathbf{\Gamma}_\tau(0)}_{=\gamma_0})$$

Using dEDI and Arzelà-Ascoli: passing to the limit $\tau \to 0$ (subsequence)
there exists a *minimising movement* $\mathbf{\Gamma} \in AC\left([0,\infty), W^{1+s+\varepsilon}\right)$

$$\lim_{k\to\infty} \mathbf{\Gamma}_{\tau_k}(t) = \mathbf{\Gamma}(t)$$

Curve of Maximal Slope

step 3: show that $\mathbf{\Gamma}$ satisfies EDI.

$$\phi(\mathbf{\Gamma}_\tau(t)) + \frac{1}{2}\int_0^t |\mathbf{\Gamma}'_\tau|^2 \mathrm{d}r + \frac{1}{2}\int_0^t |\mathfrak{D}_\tau \phi|^2 \mathrm{d}r \leq \phi(\gamma_0)$$

passing to the limit $\tau_k \to 0$ in dEDI:

$$\phi(\mathbf{\Gamma}(t)) + \frac{1}{2}\int_0^t |\mathbf{\Gamma}'|^2(r)\mathrm{d}r + \frac{1}{2}\int_0^t |\partial^- \phi|^2(\mathbf{\Gamma}(r))\mathrm{d}r \leq \phi(\gamma_0)$$

$$|\partial^- \phi|(\gamma) = \inf\{\liminf_{k\to\infty} |\partial\phi|(\gamma_k) \mid \gamma_k \rightharpoonup \gamma, \sup_{k\in\mathbb{N}} \phi(\gamma_k) < \infty\}$$

Almost EDI

$\mathscr{E} \in \{\mathrm{TP}^{p,q}, \mathrm{intM}^{p,q}, E^{\alpha,p}\}$ satisfies

- (E1) $\mathscr{E}$ is continuously Fréchet-differentiable on $W^{1+s,\rho}_{\mathrm{ir}}$
- (E2) $\mathscr{E}$ is weakly lower semicontinuous on $W^{1+s,\rho}_{\mathrm{ir}}$
- (E3) $\mathscr{E}$ satisfies uniform bi-Lipschitz estimates:

$$\frac{1}{c_1} \leq |\gamma'| \leq c_1 \quad \& \quad \mathscr{E}(\gamma) \leq c_2 \quad \Rightarrow \quad \mathrm{BiLip}(\gamma) \geq C(c_1, c_2, \mathscr{E})$$

where $\mathrm{BiLip}(\gamma) := \inf_{\substack{x,y \in \mathbb{R}/\mathbb{Z} \\ x \neq y}} \frac{|\gamma(y)-\gamma(x)|}{|x-y|_{\mathbb{R}/\mathbb{Z}}}$.

[BR15b], [BR15a], [O'H92b], [MSvdM, Theorem 5.1]

[MSvdM, Theorem 3.2]

$1 < \kappa, \theta \quad 0 < \varepsilon \quad \mathscr{E}$ satisfies (E1)-(E3) $\quad \mathscr{E} \geq 0 \quad s, \rho = s, \rho(\mathscr{E})$

$\forall \gamma_0 \in W^{1+s+\varepsilon,\rho}_{\mathrm{ir}} \quad \exists \mathbf{\Gamma} \in C^1\left([0,\infty), W^{1+s+\varepsilon,\rho}\right)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Gamma}(t) = -\tilde{\mathfrak{J}}^{-1}_{(W^{1+s+\varepsilon,\rho},\theta)} D\phi[\mathbf{\Gamma}(t)]$$

$$\mathbf{\Gamma}(0) = \gamma_0$$

$$\Phi_{n,\tau}(\gamma) = \phi(\gamma) + \frac{1}{2\tau} \|\gamma - \gamma_{n-1}\|^2_{W^{1+s+\varepsilon,\rho}}$$

Solve minimisation scheme using direct method of calculus of variation:

- Φ is coercive
- sublevel-sets of Φ are weakly relatively compact
- Φ is weakly lower semicontinuous

Φ is weakly lower semicontinuous

on $W^{1+s+\varepsilon,\rho}_{\mathrm{ir}}$

$$\begin{aligned}\Phi_{n,\tau}(\gamma) &= \phi(\gamma) + \frac{1}{2\tau} d^2\big(\gamma, \gamma_{n-1}\big) \\ &= \mathscr{E}(\gamma) + \|\Sigma(\gamma)\|^{\kappa}_{W^{s,\rho}} + \frac{1}{2\tau}\|\gamma - \gamma_{n-1}\|^2_{W^{1+s+\varepsilon,\rho}}\end{aligned}$$

this is weakly lower semicontinuous.

Φ is weakly lower semicontinuous

However, if $\gamma \notin W_{\mathrm{ir}}^{1+s+\varepsilon,\rho}$, then $\Phi_{n,\tau}(\gamma) = \infty$.

It remains to show that if

$$\sup_{k\in\mathbb{N}} \phi\big(\tilde{\gamma}_k\big) =: M < \infty \quad \text{and} \quad \tilde{\gamma}_k \rightharpoonup \tilde{\gamma}$$

then

$$\tilde{\gamma} \in W_{\mathrm{ir}}^{1+s+\varepsilon,\rho}.$$

Φ is weakly lower semicontinuous

Since $\phi(\tilde{\gamma}_k) \leq M$

- $\tilde{\gamma}_k \in W^{1+s,\rho}_{\mathrm{ir}}$ by definition of ϕ
- hence $\|\Sigma(\tilde{\gamma}_k)\|^{\kappa}_{W^{s,\rho}} \leq M$

by embedding $W^{s,\rho} \hookrightarrow C^0$

$$\sup_{k\in\mathbb{N}} \sup_{x\in\mathbb{R}/\mathbb{Z}} |\log|\tilde{\gamma}_k'(x)|| < \infty$$

thus $(\tilde{\gamma}_k)_{k\in\mathbb{N}}$ is uniformly regular:

$$\frac{1}{c_1} \leq |\tilde{\gamma}_k'(x)| \leq c_1$$

In particular, since $W^{1+s,\rho} \hookrightarrow C^1$ embeds compactly, $\tilde{\gamma}$ is regular.

Φ is weakly lower semicontinuous

Furthermore, $\phi(\tilde{\gamma}_k) \leq M$ implies that

$$\sup_{k \in \mathbb{N}} \mathscr{E}(\tilde{\gamma}_k) < \infty$$

Recall

(E3) $\mathscr{E}$ satisfies uniform bi-Lipschitz estimates:

$$\frac{1}{c_1} \leq |\gamma'| \leq c_1 \quad \& \quad \mathscr{E}(\gamma) \leq c_2 \quad \Rightarrow \quad \mathrm{BiLip}(\gamma) \geq C(c_1, c_2, \mathscr{E})$$

Therefore,

$$\inf_{k \in \mathbb{N}} \mathrm{BiLip}(\tilde{\gamma}_k) > 0$$

Thus $\tilde{\gamma}$ is injective.

The local slope of ϕ is

$$|\partial\phi|(\gamma) = \|D\phi[\gamma]\|_{(W^{1+s+\varepsilon,\rho})^*}$$

Recall

$$\phi(\gamma) = \mathscr{E}(\gamma) + \|\Sigma(\gamma)\|^{\kappa}_{W^{s,\rho}}$$

Since ϕ is C^1 in $W^{1+s,\rho}$ and $W^{1+s+\varepsilon,\rho} \hookrightarrow W^{1+s,\rho}$ embeds compactly, $\|D\phi[\gamma]\|_{(W^{1+s+\varepsilon})^*}$ is weakly sequentially continuous. Therefore, $|\partial^- \phi| = |\partial\phi|$. By [AGS08, Theorem 2.3.3], $\mathbf{\Gamma}$ is a curve of maximal slope.

Thank you for your attention.

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