How random permutations create waves

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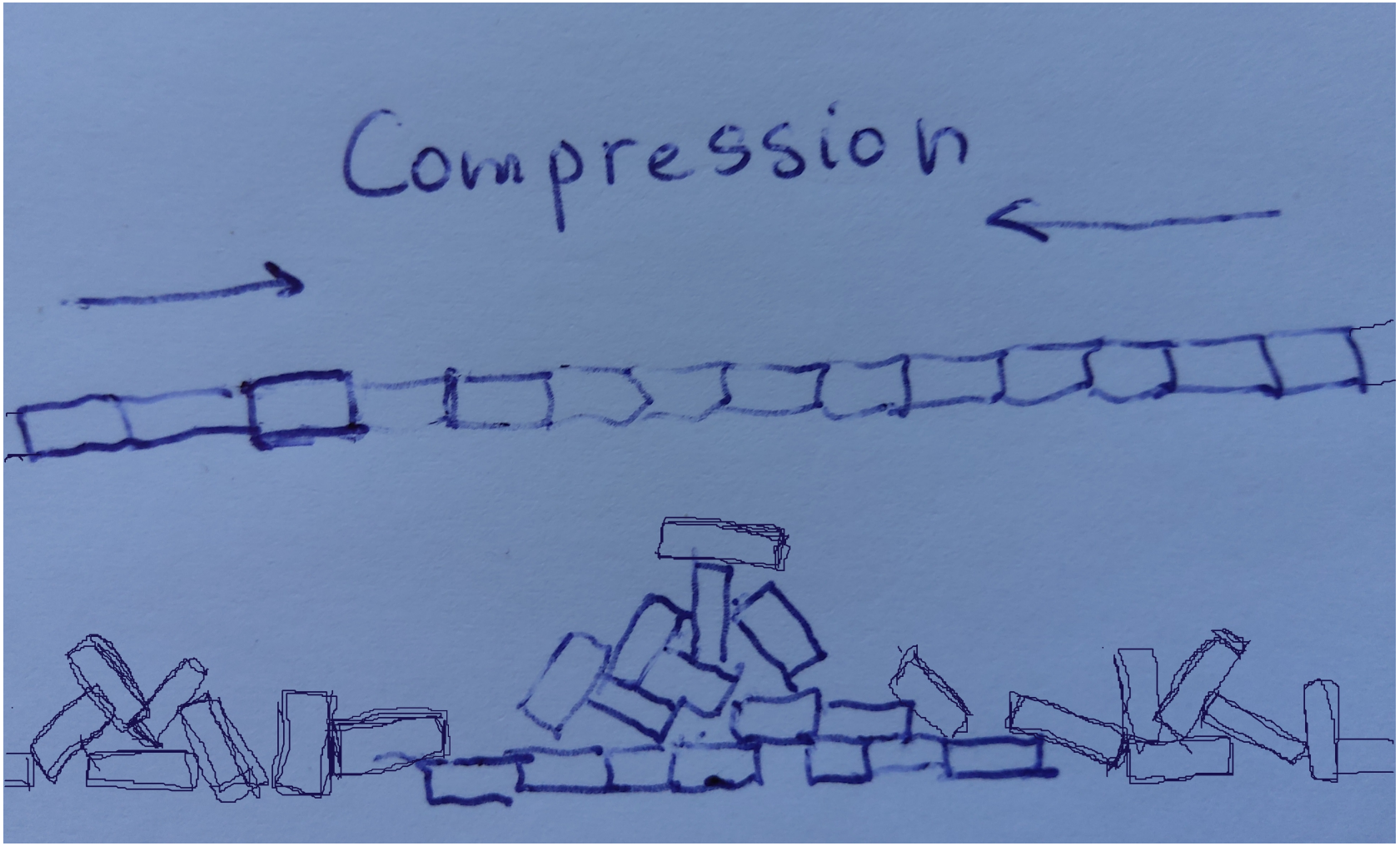
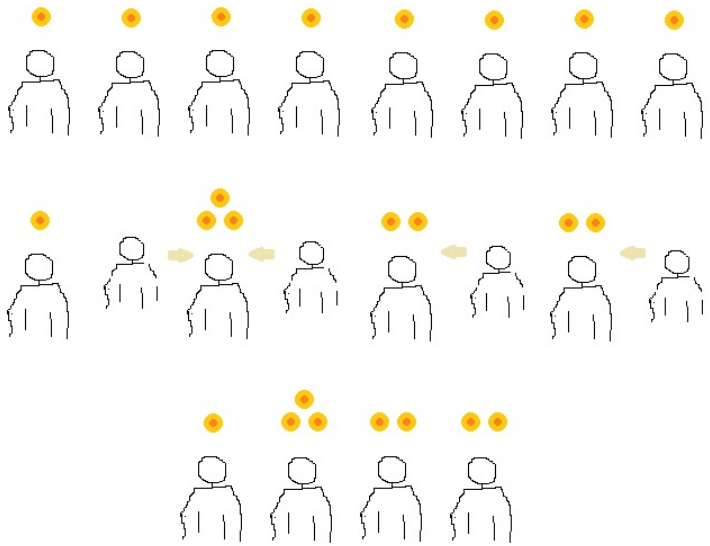
(Exponential) stochastic compression



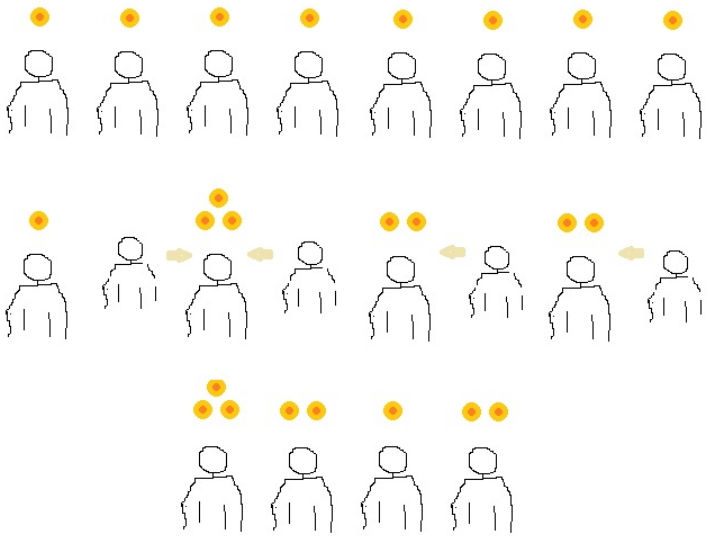
Such type of systems can model the following phenomena: movement of lithospheric plates; ice jam formation; donation pyramid; merging of cells in biological organisms; etc.

“Financial” model

We consider a simple donation pyramid model, the chain of citizens, where every second citizen gives her/his coin to a randomly chosen left or right neighbor, and, after that, leaves the chain. The remaining citizens take the places of the leaving citizens, ordered or with random permutations (disordered). The process is repeated many times. We call the process exponential stochastic compression. The main goal is to analyze the limit distribution of citizens by the number of coins in the ordered and disordered cases.



Ordered



Disordered

Density of citizens with the same income

Let ρ_i be the density of citizens with i coins.

Theorem

i) After N steps of the exponential ordered stochastic compression, the densities ρ_i are

$$\rho_i = \frac{1}{4^N} \begin{cases} i, & i = 1, ..., 2^N, \\ 2^{N+1} - i, & i = 2^N + 1, ..., 2^{N+1} - 1, \\ 0, & i \geqslant 2^{N+1}. \end{cases}$$

ii) After N steps of the exponential disordered stochastic compression, the densities ρ_i satisfy

$$\sum_{i=1}^{\infty} \rho_i z^i = \underbrace{P \circ ... \circ P}_{N}(z), \quad z \in \mathbb{C},$$

where $P(z) = \frac{1}{4}(z + 2z^2 + z^3)$.

Densities in the disordered case

For the analysis of the densities in the disordered case, it is convenient to introduce

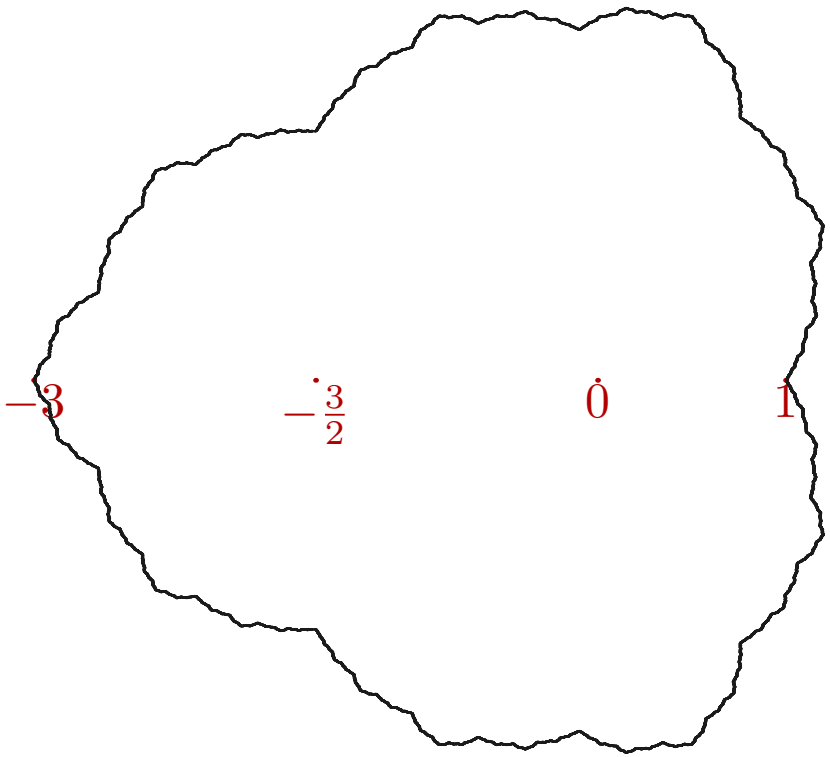
$$\Phi(z) = \lim_{N\to\infty} 4^N \underbrace{P \circ \ldots \circ P}_{N}(z),$$

which satisfy the Schröder-type functional equation

$$\Phi(P(z)) = \frac{1}{4}\Phi(z), \quad \Phi'(0) = 1.$$

The function $\Phi(z)$ is analytic in the domain $\mathscr{J}_P$, which is the filled Julia set for the polynomial P.
Roughly speaking, the filled Julia set is the set, where the polynomial iterations remain bounded, or tend to an attraction point. Often, this set has a fractal structure.

Julia set for $P(z) = \frac{1}{4}(z + 2z^2 + z^3)$



(a)

(b)

Figure: (a) Julia set for $P(z) = \frac{1}{4}(z + 2z^2 + z^3)$, it is the boundary of the filled Julia set $\mathscr{J}_P$; (b) The lake near my residence, where I came up with the correct polynomial $\frac{1}{4}(z + 2z^2 + z^3)$.

Relative densities ρ_i/ρ_1

Corollary

For the densities ρ_i in the exponential ordered (OSC) and disordered (DSC) stochastic compression models, the following identities hold

$$\mathrm{OSC}: \quad \frac{\rho_i}{\rho_1} = i, \quad N > \log_2 i; \qquad \mathrm{DSC}: \quad \frac{\rho_i}{\rho_1} \to \varphi_i, \quad N \to \infty,$$

where φ_i are given by $\Phi(z) = \sum_{i=1}^{+\infty} \varphi_i z^i$, and N is the number of compression steps.

i	1	2	3	4	5	6	7	8
φ_i	1.00000	2.66667	3.91111	5.55344	7.05507	8.26885	9.86538	11.41518

Table: Tailor coefficients of $\Phi(z)$.

Numerical simulations

i	1	2	3	4	5	6	7	8
N_i	$6.55 \cdot 10^8$	$13.1 \cdot 10^8$	$19.65 \cdot 10^8$	$26.2 \cdot 10^8$	$19.65 \cdot 10^8$	$13.1 \cdot 10^8$	$6.5 \cdot 10^8$	0
$\frac{N_i}{N_1}$	1.00000	2.00018	3.00015	4.00041	3.00003	2.00021	1.00001	0

(a) OSC, 2 steps

i	1	2	3	4	5	6	7	8
N_i	2413	4972	7463	10172	12322	14996	17561	19885
$\frac{N_i}{N_1}$	1.00000	2.06051	3.09283	4.21550	5.10651	6.21467	7.27766	8.24078

(b) OSC, 8 steps

i	1	2	3	4	5	6	7	8
N_i	$6.5 \cdot 10^8$	$16.2 \cdot 10^8$	$20 \cdot 10^8$	$22 \cdot 10^8$	$19.2 \cdot 10^8$	$11.5 \cdot 10^8$	$6.1 \cdot 10^8$	$2.3 \cdot 10^8$
$\frac{N_i}{N_1}$	1.00000	2.48012	3.05856	3.40276	2.94165	1.75454	0.94146	0.36279

(c) DSC, 2 steps

i	1	2	3	4	5	6	7	8
N_i	2508	6571	9462	13741	17573	20134	24709	28157
$\frac{N_i}{N_1}$	1.00000	2.62002	3.77273	5.47887	7.00678	8.02791	9.85207	11.22687

(d) DSC, 8 steps

Table: The quantities N_i of citizens with i coins in a segment of the chain of length $2^{26} \cdot 5^4 \approx 42 \cdot 10^9$ after 2 and 8 steps of ordered and disordered stochastic compression.

Near constancy oscillations and 146 percents

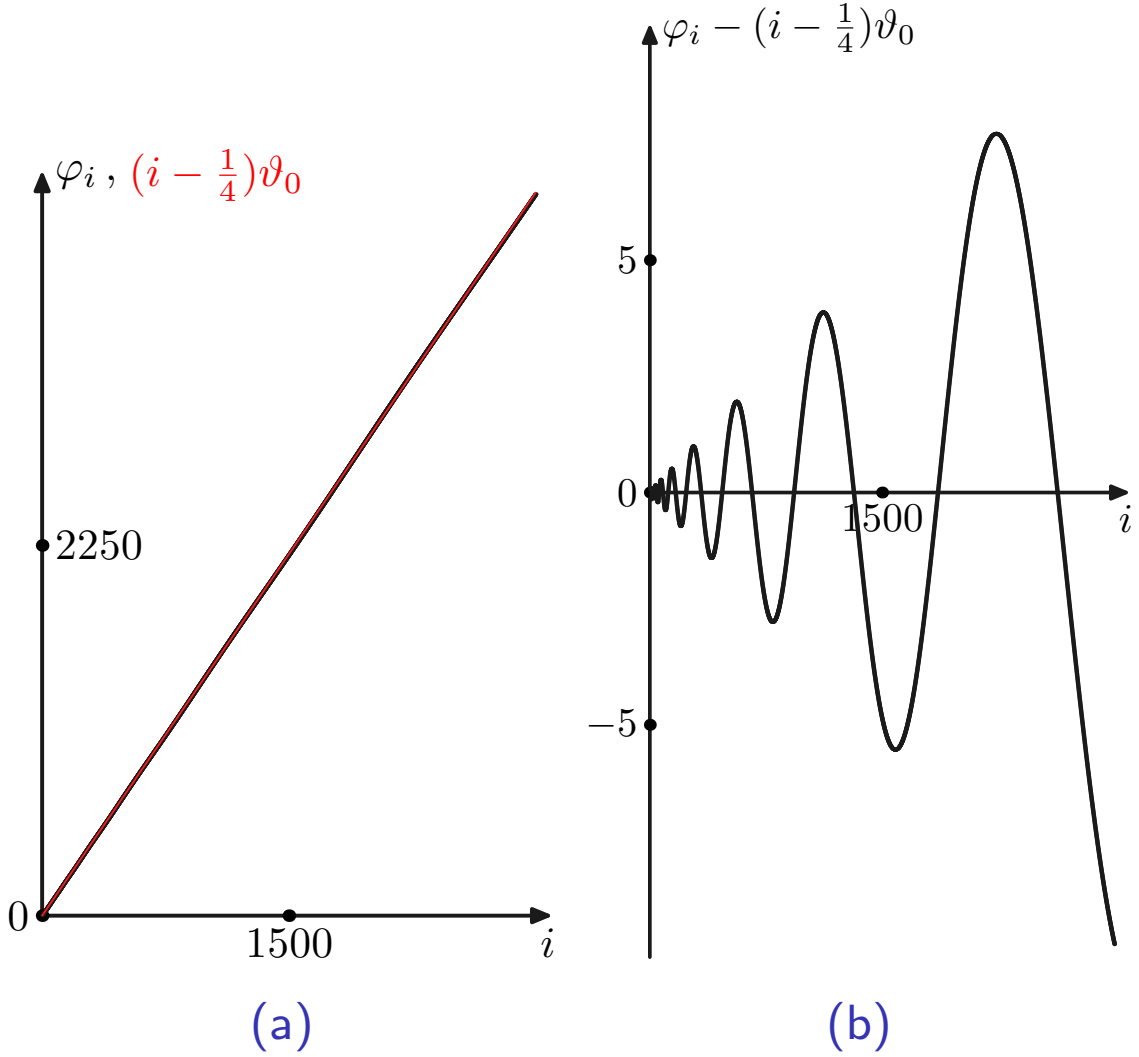


Figure: Tailor coefficients φ_i (black curve) and their approximation $(i-\frac{1}{4})\vartheta_0$ (red curve) are plotted in (a), the difference $\varphi_i-(i-\frac{1}{4})\vartheta_0$ is plotted in (b). Here $\vartheta_0 = 1.464910...$

Exact formula for φ_n and approximations

Exact recurrent formula:

$$\varphi_1 = 1, \quad \varphi_{n+1} = \frac{4}{1-4^{-n}} \sum_{k=0}^{n-1} 4^{k-n} \binom{2(n-k)}{k+1} \varphi_{n-k}, \quad n \geqslant 1.$$

Approximation based on Fourier coefficients θ_m of 1-periodic modified Karlin-McGregor function $K(z) = \Phi(\Pi(2^z))4^z$:

$$\varphi_n \approx \varphi_n^M := (-1)^n \sum_{m=-M}^{M} \left(\binom{\frac{2\pi \mathbf{i} m - \ln 4}{\ln 2}}{n} + \right.$$
$$\left. + \frac{5(2\pi \mathbf{i} m - \ln 4)}{8 \ln 2} \binom{1 + \frac{2\pi \mathbf{i} m - \ln 4}{\ln 2}}{n} \right) \theta_m,$$

where $\mathbf{i} = \sqrt{-1}$ and

$$\binom{x}{n} = \frac{x(x-1)(x-2)\ldots(x-n+1)}{n!} \approx \frac{(-1)^n}{\Gamma(-x) n^{x+1}}, \quad n \to +\infty.$$

Approximations are really good!

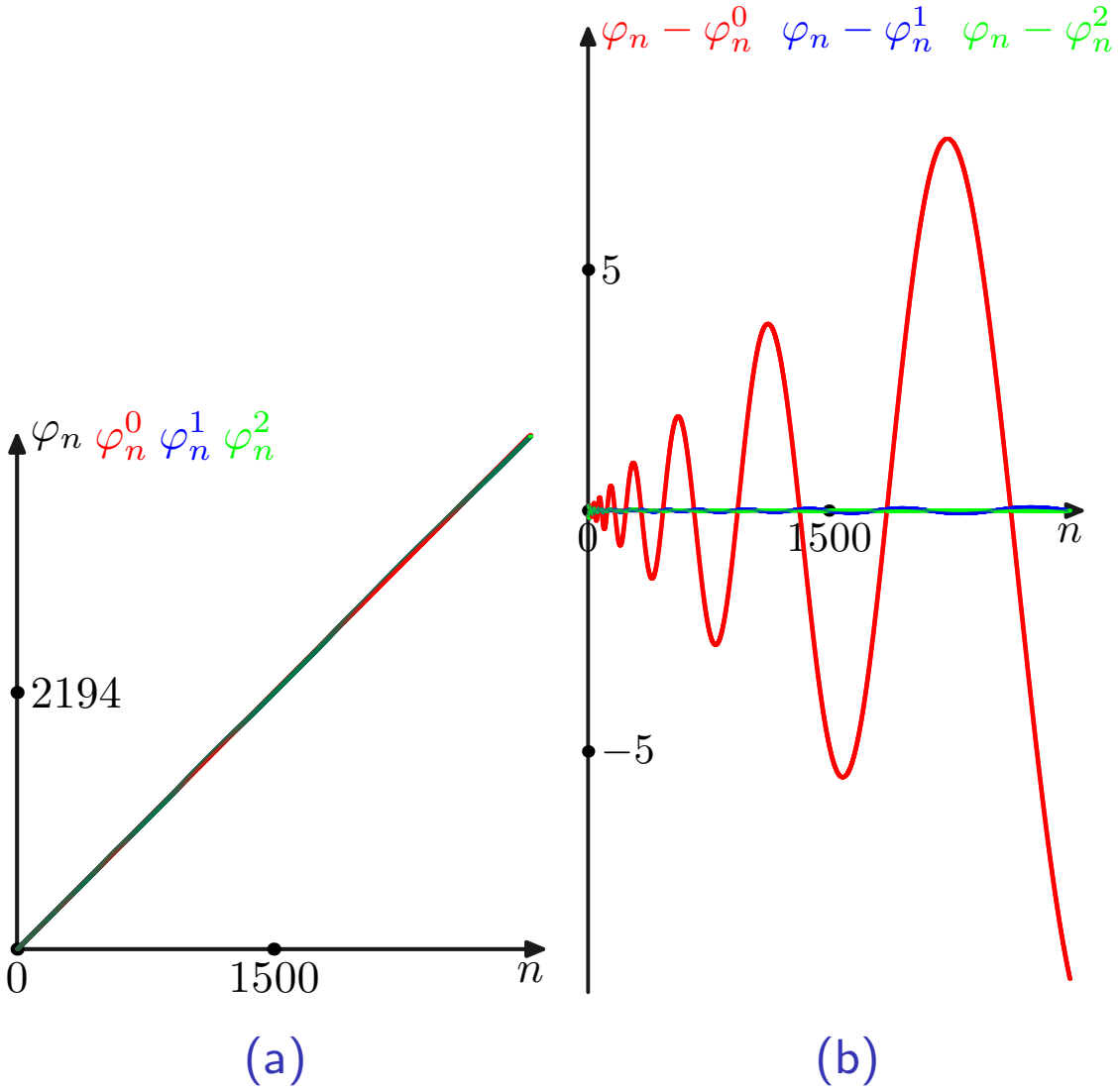


Figure: Tailor coefficients φ_n (black curve) and their approximations φ_n^M (red, blue, and green curves) for $M = 0, 1, 2$ are plotted in (a), the differences $\varphi_n^M - \varphi_n$ (red, blue, and green curves) are plotted in (b).

Rescaled densities

It is natural to define

$$\rho(x) = \lim_{N\to\infty} 2^N \rho_{[x2^N]}, \quad x \in [0,\infty),$$

where ρ_i are the densities after N steps of compression, and $[a]$ denotes an integer part of the real number a.

Theorem

For the ordered case, we have

$$\rho(x) = \begin{cases} x, & x \in [0,1], \\ 2-x, & x \in [1,2], \\ 0, & x \geqslant 2. \end{cases}$$

For the disordered case, we have

$$\rho(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathbf{i}tx}\Pi(\mathbf{i}t)dt,$$

where $\mathbf{i} = \sqrt{-1}$ *and* $\Pi(z)$ *is an entire function satisfying Poincaré-type functional equation*

$$P(\Pi(z)) = \Pi(2z), \quad \Pi'(0) = 1, \quad z \in \mathbb{C}.$$

Rescaled densities, plots

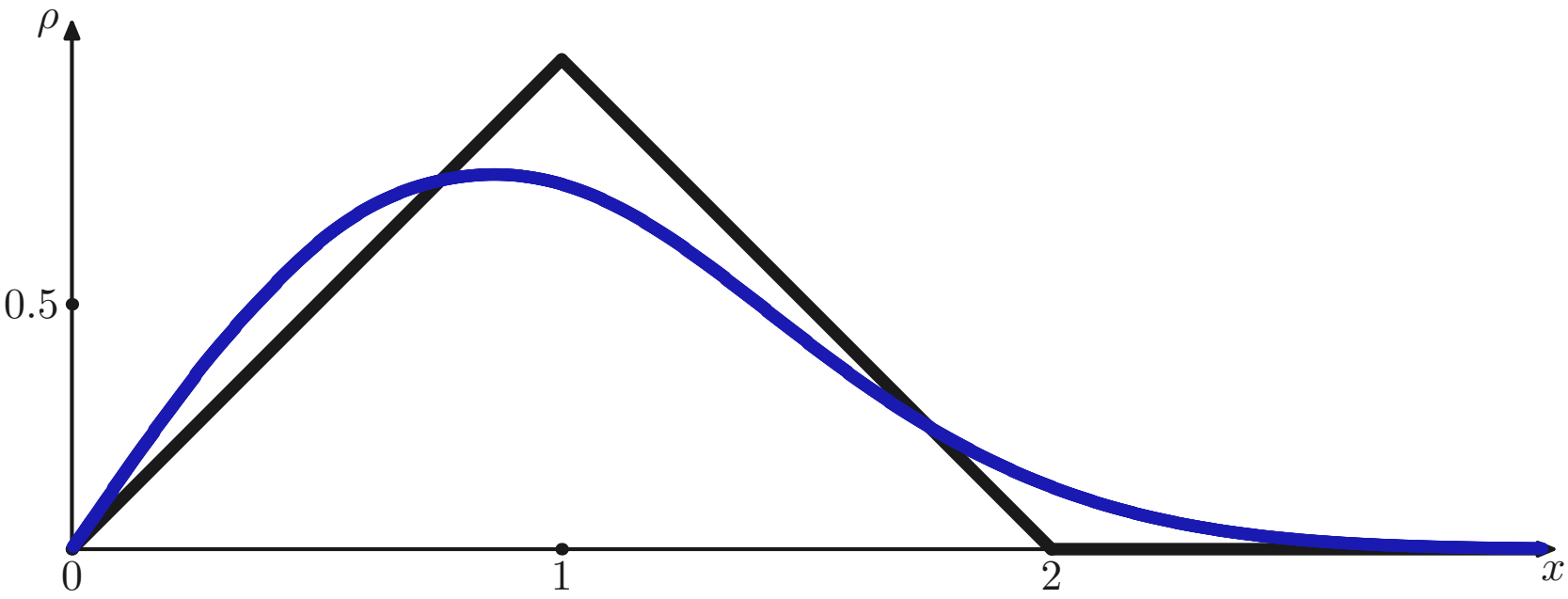


Figure: Rescaled densities $\rho(x)$ in the ordered (black curve) and disordered (blue curve) cases are plotted. For the computation of points of blue curve, 9 iterations of $P(z) = (z + 2z^2 + z^3)/4$ are used.

$$\Pi(\mathbf{i}t) = \lim_{N\to\infty} \underbrace{P \circ \ldots \circ P}_{N}(1 + \frac{\mathbf{i}t}{2^N})$$

Rescaled density in the disordered case, asymptotic

Theorem

The following asymptotic holds

$$\rho(x) = x\sigma(\log_2 x) + o(x), \quad x \to 0,$$

where σ is 1-periodic continuous function defined by the formula

$$\sigma(x) = 2^{-x}\rho(2^x) + \frac{2^{-x}}{4\pi}\int_{-\infty}^{+\infty} H(-\mathbf{i}t2^{x+1})(2\Pi(\mathbf{i}t)^2 + \Pi(\mathbf{i}t)^3)dt$$

for $x \in \mathbb{R}$, where the entire function H satisfies

$$H(z) = \sum_{n=2}^{\infty} \frac{z^n}{n!(2^{n-1}-1)}, \quad z \in \mathbb{C}.$$

The convergence of the integral in the second identity above is ensured by the facts that $\Pi(\mathbf{i}t) = O(|t|^{-2})$ and $H(\mathbf{i}t) = O(|t|\ln|t|)$ for large by modulus $t \in \mathbb{R}$.

Plot of $\sigma(x)$

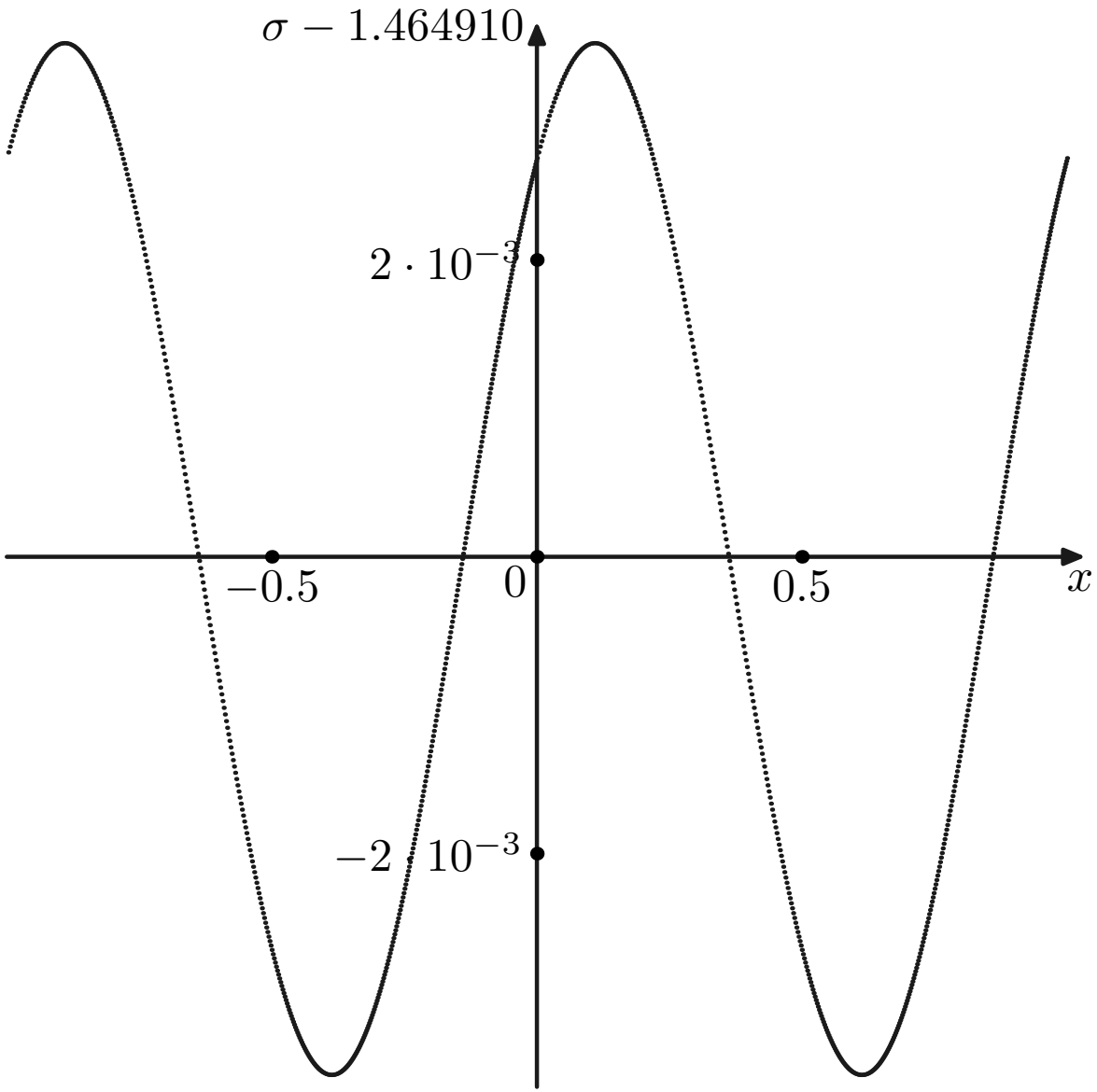


Figure: The 1-periodic function $\sigma(x) - 1.464910$.

Average of $\sigma(x)$ over its period

There are a couple of identities for the average of $\sigma(x)$:

$$\int_0^1 \sigma(x)dx = \frac{-1}{\ln 2}\int_{-\infty}^{0} t(2\Pi(t)^2 + \Pi(t)^3)dt =$$
$$\frac{-1}{\ln 2}\int_0^{+\infty} t\,\mathrm{Re}(2\Pi(\mathbf{i}t)^2 + \Pi(\mathbf{i}t)^3)dt =$$
$$\frac{-2}{\pi \ln 2}\int_0^{+\infty} (\ln t)t\,\mathrm{Im}(2\Pi(\mathbf{i}t)^2 + \Pi(\mathbf{i}t)^3)dt.$$

The convergence of improper integrals is due to $\Pi(t) = O(t^{-2})$ for large t with $\mathrm{Re}\, t \leqslant 0$. The best rate of convergence is shown by using the last formula. All the formulas show $\int_0^1 \sigma(x)dx = 1.464910....$