# On the optimal constants in the two-sided Stechkin inequalities

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MIDS

# An exercise

$$\text{Given } x = (2.4, -42, 0, -\pi, 100) \in \mathbb{R}^5.$$

What is the closest vector with at most two non-zero entries?

# An exercise

Given $x = (2.4, -42, 0, -\pi, 100) \in \mathbb{R}^5$.

What is the closest vector with at most two non-zero entries?

It is $(\quad 0, -42, 0, \quad 0, 100)$.

Drop all but the largest coefficients.

# Our publication

T. J. and T. Ullrich
On the optimal constants in the two-sided Stechkin inequalities,
J. Approx. Theory 269:105607, 2021.

Tino Ullrich

# Best $n$-term approximation

$H$ a Hilbert space, separable and infinite-dimensional

$\Longrightarrow x = \sum_{e \in S} \underbrace{\langle x | e \rangle}_{=:x_k} e$ for all $x \in H$ with $S$ a countable ONB

$\Sigma_n = \{x \in H \mid \text{at most } n \text{ summands } \neq 0\}$

$E(x, \Sigma_n)_H = \inf_{y \in \Sigma_n} \|x - y\|_H$

Given $x$, what is the rate of decay of $E(x, \Sigma_n)_H$ in $n$?

# Finite-dimensional example

$$\text{Given } x = (2.4, -42, 0, -\pi, 100) \in \mathbb{R}^5.$$

$$\Sigma_n = \left\{ x \in \mathbb{R}^5 \;\middle|\; x \text{ has at most } n \text{ nonzero entries} \right\}$$

$$E(x, \Sigma_2) = \|x - (\quad 0, -42, 0, \quad 0, 100)\|_2.$$

Approximation error is computed from small entries.

# Approximation spaces

Pietsch 1981: approximation spaces

$$\|x\|_{\mathcal{A}_r^\alpha(H,\Sigma)} := \begin{cases} \left(\sum_{n=1}^\infty n^{\alpha r} E(x,\Sigma_{n-1})_H^r \frac{1}{n}\right)^{1/r}, & r < \infty, \\ \sup_{n\in\mathbb{N}} n^\alpha E(x,\Sigma_{n-1})_H, & r = \infty \end{cases}$$

Largest coefficients are best.

Through $H \cong \ell_2$, $q = 2\alpha + 1$, and $r = (\alpha + \frac{1}{2})^{-1}$, we arrive at expressions like $\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{k=n}^\infty (x_k^*)^q\right)^{\frac{1}{q}}$.

# Strong discrete Stechkin inequality

Let $c_1(q)$ and $C_1(q) > 0$ be infimal such that

$$\frac{1}{c_1(q)} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \leq \sum_{n=1}^{\infty} a_n \leq C_1(q) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}.$$

- Copson 1928: $C_1(q) \leq q^{\frac{1}{q}}$
- Стечкин 1951: $C_1(2) \leq \frac{2}{\sqrt{3}}$
- Стечкин 1955: $c_1(2) \leq 2$ (no proof), $c_1(2) \leq \frac{\pi}{2}$ (conjecture)
- Pietsch 1981: $c_1(2) \leq 2(2+\sqrt{2})$ and $C_1(2) \leq 6$

# Problem statement

What are the infimal constants $c, C > 0$ such that

$$\frac{1}{c} \bullet \quad \leq \quad \blacksquare \quad \leq C \quad \bullet \quad ?$$

- ▶ $\sum_{n=1}^{\infty} a_n$, $\quad \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$
- ▷ $\sup_{n\in\mathbb{N}} n a_n$, $\quad \sup_{n\in\mathbb{N}} n \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$
- ▷ $\int_0^{\infty} f(t)\mathrm{d}t$, $\quad \int_0^{\infty} \left(\frac{1}{t} \int_t^{\infty} f(s)^q \mathrm{d}s\right)^{\frac{1}{q}} \mathrm{d}t$
- ▷ $\sup_{t>0} t f(t)$, $\quad \sup_{t>0} t \left(\frac{1}{t} \int_t^{\infty} f(s)^q \mathrm{d}s\right)^{\frac{1}{q}}$

# Weak and continuous Stechkin inequalities

$$\frac{1}{c_{1,\infty}(q)} \sup_{n\in\mathbb{N}} n\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}} \le \sup_{n\in\mathbb{N}} na_n \le C_{1,\infty}(q) \sup_{n\in\mathbb{N}} n\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$$

$$\frac{1}{\bar{c}_1(q)} \int_0^\infty \left(\frac{1}{t}\int_t^\infty f(s)^q \mathrm{d}s\right)^{\frac{1}{q}} \mathrm{d}t \le \int_0^\infty f(t)\mathrm{d}t \overset{\textbf{HLP}}{\le} \bar{C}_1(q) \int_0^\infty \left(\frac{1}{t}\int_t^\infty f(s)^q \mathrm{d}s\right)^{\frac{1}{q}} \mathrm{d}t$$

$$\frac{1}{\bar{c}_{1,\infty}(q)} \sup_{t>0} t\left(\frac{1}{t}\int_t^\infty f(s)^q \mathrm{d}s\right)^{\frac{1}{q}} \le \sup_{t>0} tf(t) \le \bar{C}_{1,\infty}(q) \sup_{t>0} t\left(\frac{1}{t}\int_t^\infty f(s)^q \mathrm{d}s\right)^{\frac{1}{q}}$$

# Strong discrete Stechkin inequality: Geometric viewpoint

Consider

$$\frac{1}{c_1(2)} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} a_n \leq C_1(2) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}}.$$

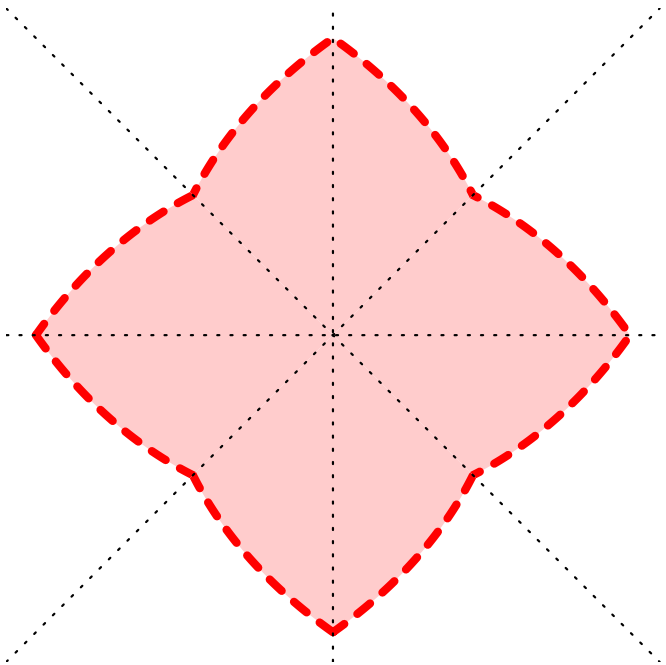
for finite sequences $(a_1, \ldots, a_N, 0, 0, \ldots)$:

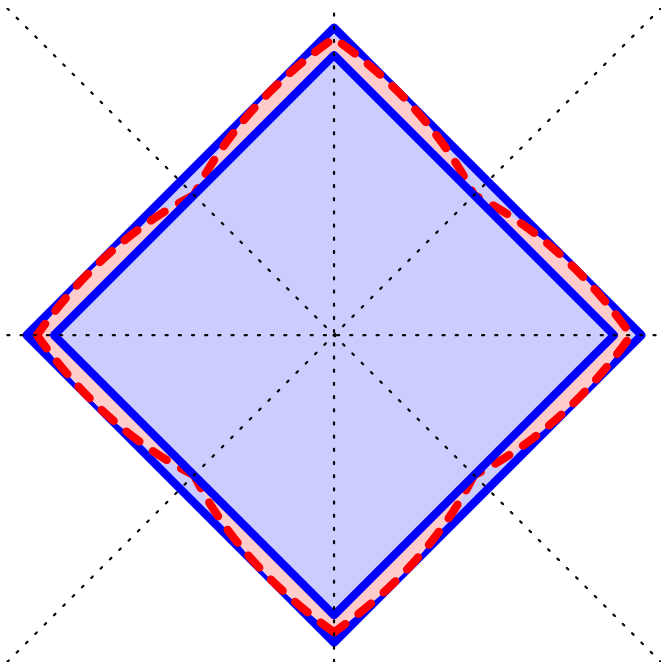
$$\frac{1}{c_1(2,N)} \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} a_k^2 \right)^{\frac{1}{2}} \leq \sum_{n=1}^{N} a_n \leq C_1(2,N) \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} a_k^2 \right)^{\frac{1}{2}}.$$

Take $N = 2$.

# Optimal containment of starshaped sets

$$\frac{1}{c}B_1 \subset B_2 \subset CB_1$$
$$\iff \forall\, x \in \mathbb{R}^2 :$$
$$\frac{1}{c}\sum_{n=1}^{2}\left(\frac{1}{n}\sum_{k=n}^{2}(x_k^*)^2\right)^{\frac{1}{2}} \le \sum_{n=1}^{2}|x_n| \le C\sum_{n=1}^{2}\left(\frac{1}{n}\sum_{k=n}^{2}(x_k^*)^2\right)^{\frac{1}{2}}$$



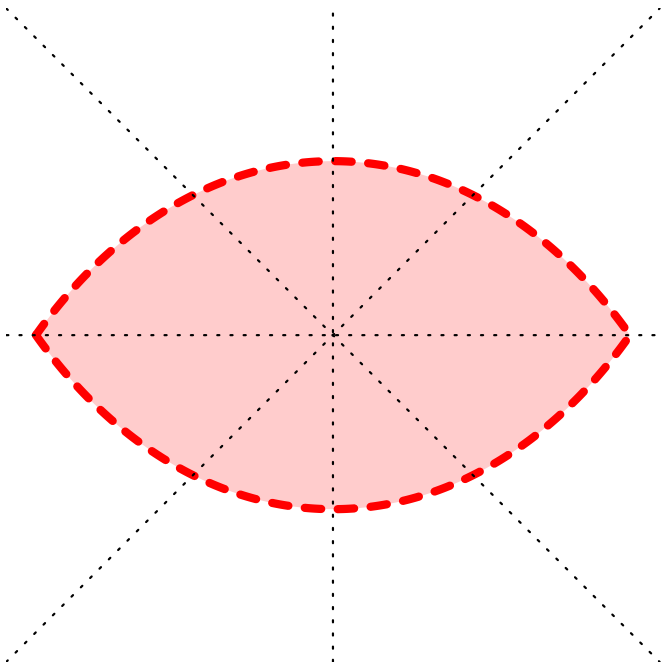


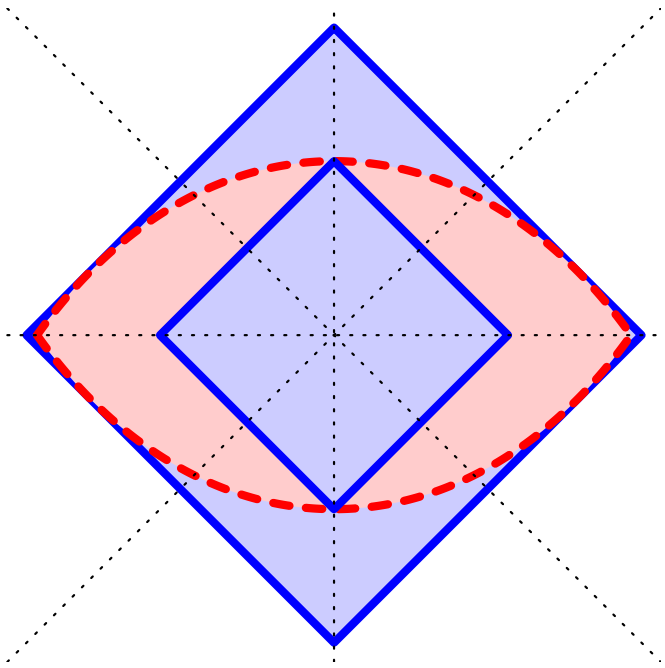
# Optimal containment of convex bodies

$$\frac{1}{c}B_1 \subset B_2 \subset CB_1$$

$$\iff \forall\, x \in \mathbb{R}^2 :$$

$$\frac{1}{c}\sum_{n=1}^{2}\left(\frac{1}{n}\sum_{k=n}^{2} \quad x_k \quad {}^2\right)^{\frac{1}{2}} \leq \sum_{n=1}^{2}|x_n| \leq C\sum_{n=1}^{2}\left(\frac{1}{n}\sum_{k=n}^{2} \quad x_k \quad {}^2\right)^{\frac{1}{2}}$$





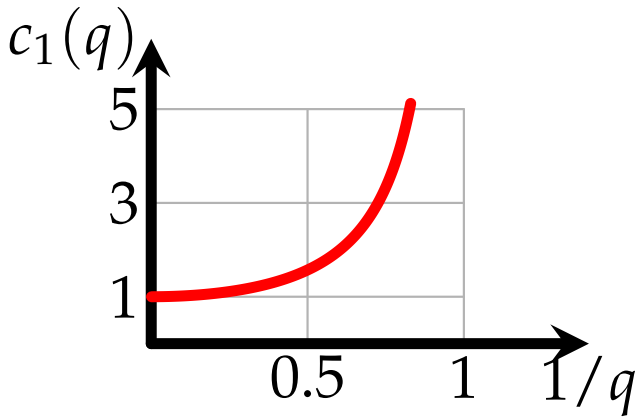
# What are the optimal constants?

Consider

$$\frac{1}{c_1(2)} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} a_n \leq C_1(2) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}}.$$

For $c_1(2)$:

- sequence of finite-dimensional approximations
- maximize convex function over simplex
- control function values at vertices
- arrive at $c_1(2) = \frac{\pi}{2}$
- Bennett 1988: $c_1(q) = \frac{\pi}{q \sin\left(\frac{\pi}{q}\right)}$



# Upper bounds on $C_1(2)$

**Theorem (J., Ullrich 2021)**

*Let $(b_n)_{n\in\mathbb{N}_0}$ be a strictly monotonically increasing sequence with $b_0 = 0$ and $\sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$. Then for all $(a_n)_{n\in\mathbb{N}} \in \ell_2$, we have*

$$\sum_{n=1}^{\infty} a_n \leq C_b \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}}$$

*with* $C_b = \sup \left\{ \sqrt{n(b_n - b_{n-1})^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2}} \;\middle|\; n \in \mathbb{N} \right\}$.

Proof: Cauchy–Schwarz inequality

# $C_1(2) \leq 1.1086983$

In

$$C_b = \sup\left\{\sqrt{n(b_n - b_{n-1})^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2}} \,\middle|\, n \in \mathbb{N}\right\},$$

take $b_n = (n(n+1))^p$.

- Copson 1928: $C_1(2) \leq \sqrt{2} \approx 1.414$
- Левин, Стечкин 1948: $C_1(2) \leq \frac{2}{\sqrt{3}} \approx 1.1547$ $p = 1$
- J., Ullrich 2021: $C_1(2) \leq \sqrt{\frac{e \ln(2)}{\sqrt{2}}} \approx 1.1542$ "optimal" $p$
- J., Ullrich 2021: $1.10516 \leq C_1(2) \leq 1.1086983$
- De Bruijn 1958: $C_1(2) \approx 1.1064957714$

# $C_1(2) \leq 1.1086983$

$$C_b = \sup\left\{ \sqrt{n(b_n - b_{n-1})^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2}} \;\middle|\; n \in \mathbb{N} \right\}$$

Choose $b_n = (n(n+1))^p$ with $p \in (\frac{1}{4}, 1]$.

# $C_1(2) \leq 1.1086983$

$$C_b = \sup\left\{ \sqrt{n(b_n - b_{n-1})^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2}} \,\middle|\, n \in \mathbb{N} \right\}$$

Choose $b_n = (n(n+1))^p$ with $p \in (\frac{1}{4}, 1]$. Then

$$A_n := n(b_n - b_{n-1})^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2} \leq \left( \frac{(n+1)^p - (n-1)^p}{n^{p-1}} \right)^2 \frac{1}{4p-1}$$

and the rhs is monotonically decreasing in $n$.

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and the rhs is monotonically decreasing in $n$. So

$$C_b \leq \max\left\{ \max_{n=1}^{N-1} \sqrt{A_n}, \frac{(N+1)^p - (N-1)^p}{N^{p-1}} \frac{1}{\sqrt{4p-1}} \right\}$$

$\ldots N = 100$ and $p = 0.88$ give $C_1(2) \leq 1.1086983$

# $1.10516 \leq C_1(2)$

Numerically compute the optimal bound on sequences of fixed finite length:

$$\text{Minimize} \qquad \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^2\right)^{\frac{1}{2}}$$

$$\text{s.t.} \qquad a_n \geq 0, \quad a_n \downarrow 0, \quad a_{N+1} = 0, \quad \sum_{n=1}^{\infty} a_n = 1$$

# $1.10516 \leq C_1(2)$

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# $1.10516 \leq C_1(2)$

$$\text{Minimize} \qquad \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} a_k^2 \right)^{\frac{1}{2}}$$

$$\text{s.t.} \qquad a_n \geq 0, \quad \sum_{n=1}^{N} a_n = 1$$

$$\Delta'_N := \left\{ (a_1, \ldots, a_N) \,\middle|\, a_n \geq 0, \quad \sum_{n=1}^{N} a_n = 1 \right\}$$

$$B_{n,N} := \left\{ (a_1, \ldots, a_N) \,\middle|\, \left( \frac{1}{n} \sum_{k=n}^{N} a_k^2 \right)^{\frac{1}{2}} \leq 1 \right\}$$

# $1.10516 \leq C_1(2)$

[Boţ, Hendrich 2013] After initialization,

$$
(\forall n \geq 0)
\left\lfloor
\begin{array}{l}
p_{1,n} = P_{\Delta'_N}\big(x_n - \frac{\tau}{2}\sum_{i=1}^m v_{i,n}\big) \\
\qquad \text{\# project to feasible set} \\
w_{1,n} = 2p_{1,n} - x_n \\
\text{For } k = 1, \ldots, N \\
\left\lfloor
\begin{array}{l}
p_{2,k,n} = P_{B^\circ_{k,N}}\big(v_{k,n} + \frac{\sigma_k}{2} w_{1,n}\big) \\
\qquad \text{\# splitting of summands} \\
w_{2,k,n} = 2p_{2,k,n} - v_{k,n}
\end{array}
\right. \\
z_{1,n} = w_{1,n} - \frac{\tau}{2}\sum_{k=1}^m w_{2,k,n} \\
x_{n+1} = x_n + \lambda_n(z_{1,n} - p_{1,n}) \\
\text{For } k = 1, \ldots, N \\
\left\lfloor
\begin{array}{l}
z_{2,k,n} = w_{2,k,n} + \frac{\sigma_k}{2}(2z_{1,n} - w_{1,n}) \\
v_{k,n+1} = v_{k,n} + \lambda_n(z_{2,k,n} - p_{2,k,n}).
\end{array}
\right.
\end{array}
\right.
$$

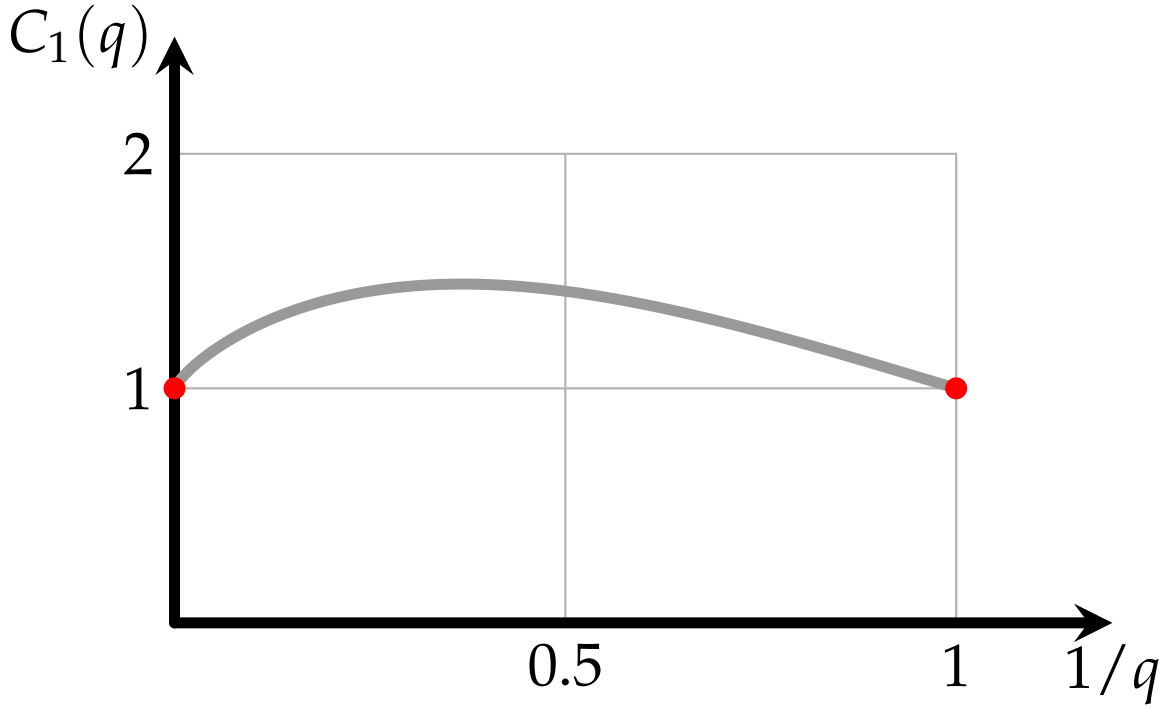
# $1.10516 \leq C_1(2)$

| $N$ | # iter. | $C_1(2) \geq$ |
|---|---|---|
| 2 | 35 | 1.03527 |
| 3 | 61 | 1.05291 |
| 4 | 98 | 1.06342 |
| 5 | 166 | 1.07037 |
| 6 | 249 | 1.07531 |
| 7 | 346 | 1.07900 |
| 8 | 454 | 1.08186 |
| 9 | 571 | 1.08415 |

| $N$ | # iter. | $C_1(2) \geq$ |
|---|---|---|
| 293 | 25822 | 1.10513 |
| 294 | 25678 | 1.10513 |
| 295 | 25956 | 1.10514 |
| 296 | 26239 | 1.10514 |
| 297 | 26095 | 1.10514 |
| 298 | 26375 | 1.10515 |
| 299 | 26659 | 1.10515 |
| 300 | 26515 | 1.10516 |

# Upper bounds on $C_1(q)$

$$\sum_{n=1}^{\infty} a_n \leq C_1(q) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}$$




Copson 1928

# Upper bounds on $C_1(q)$

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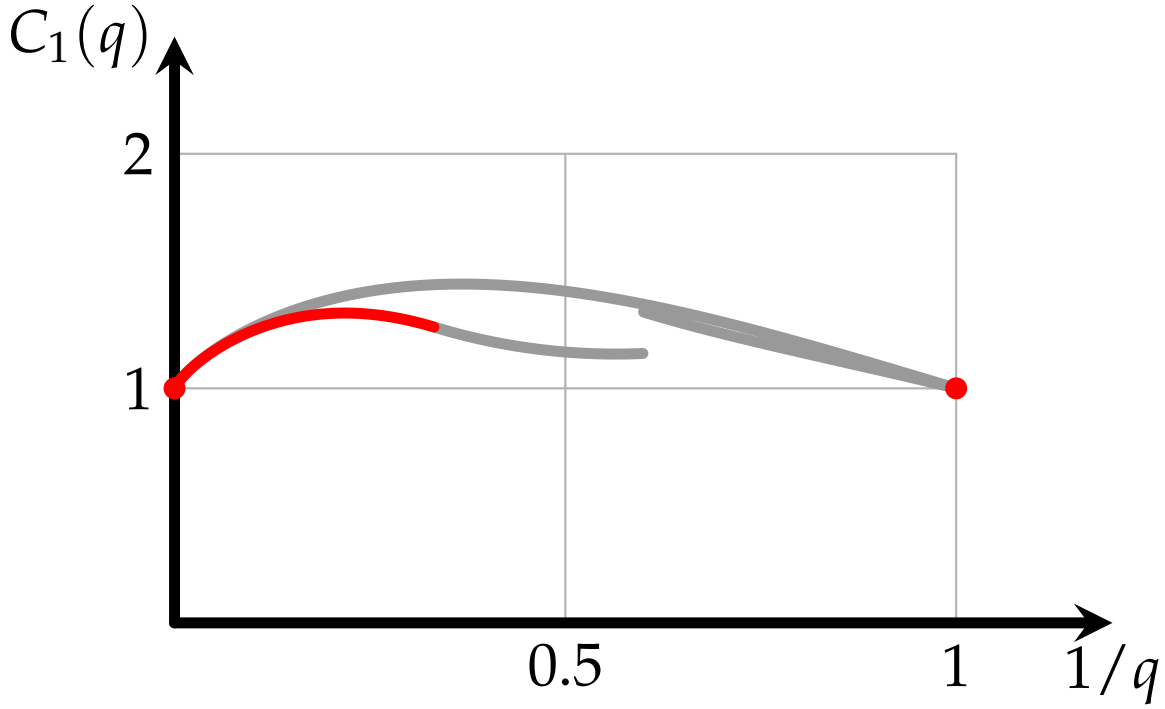


Левин, Стечкин 1948

# Upper bounds on $C_1(q)$

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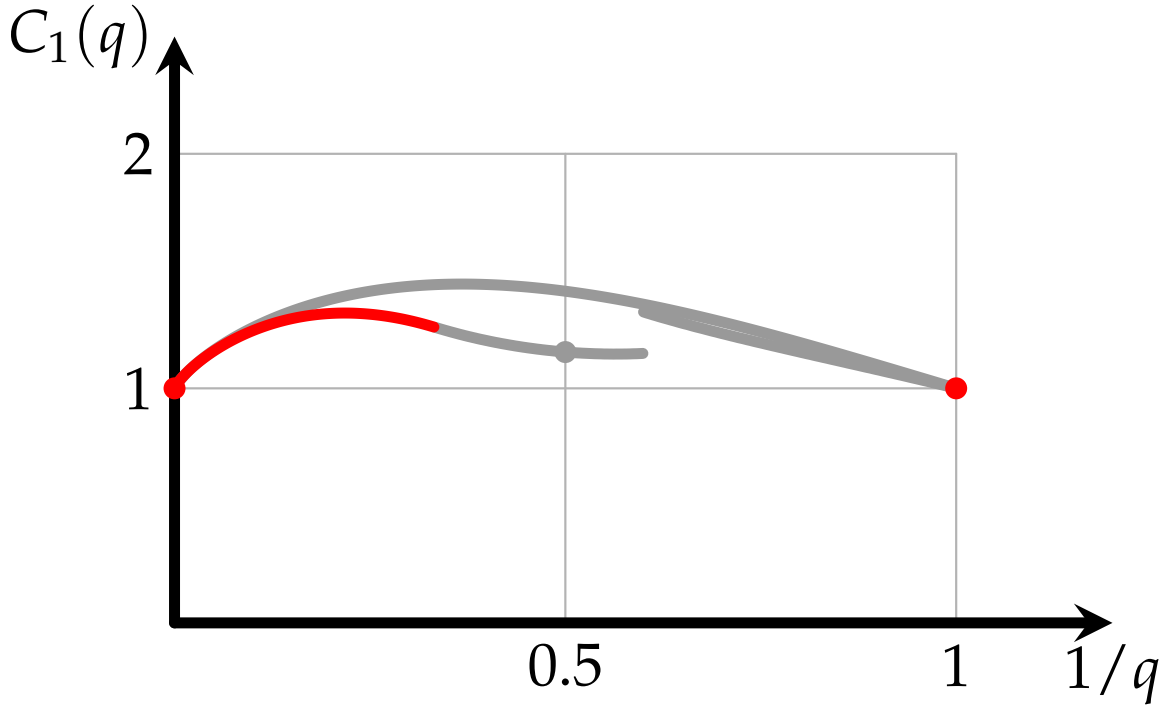


Стечкин 1951

# Upper bounds on $C_1(q)$

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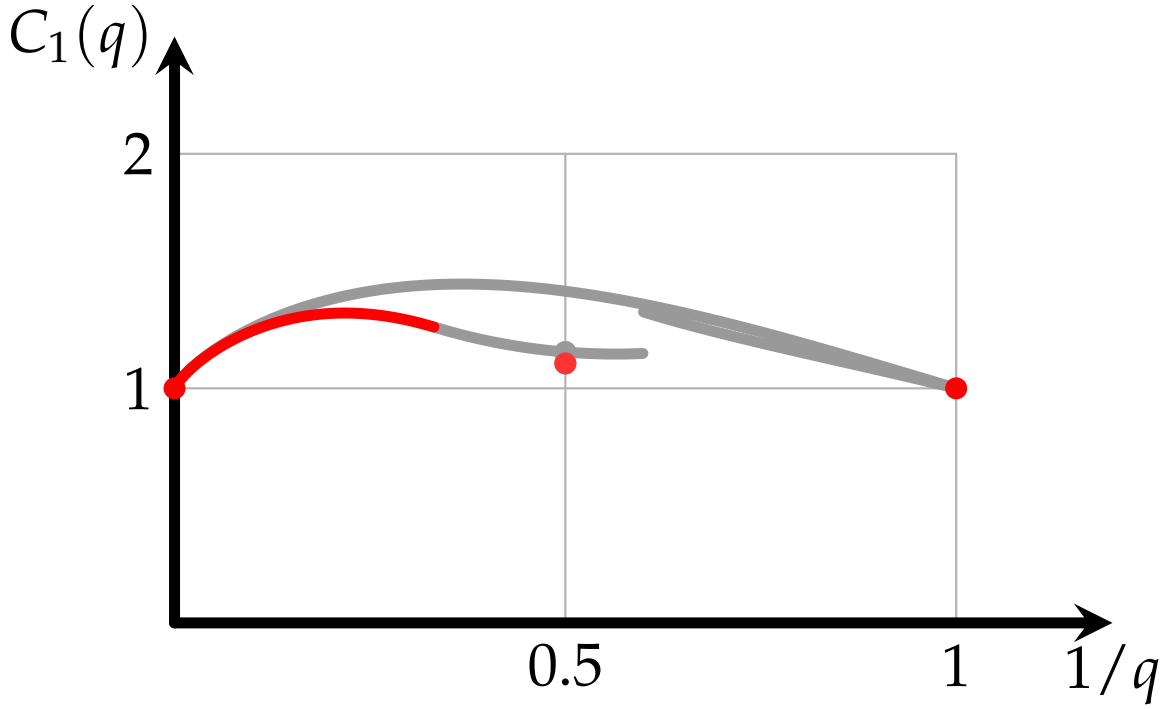


De Bruijn 1958

# Upper bounds on $C_1(q)$

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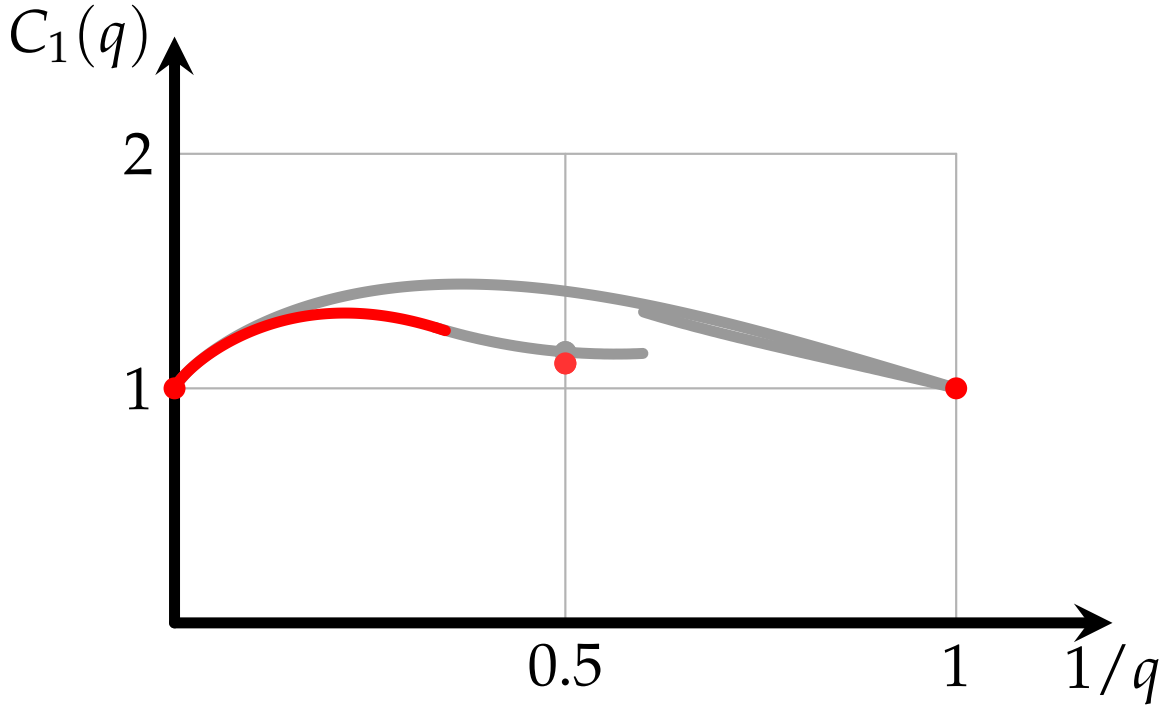


Gao 2011

# Upper bounds on $C_1(q)$

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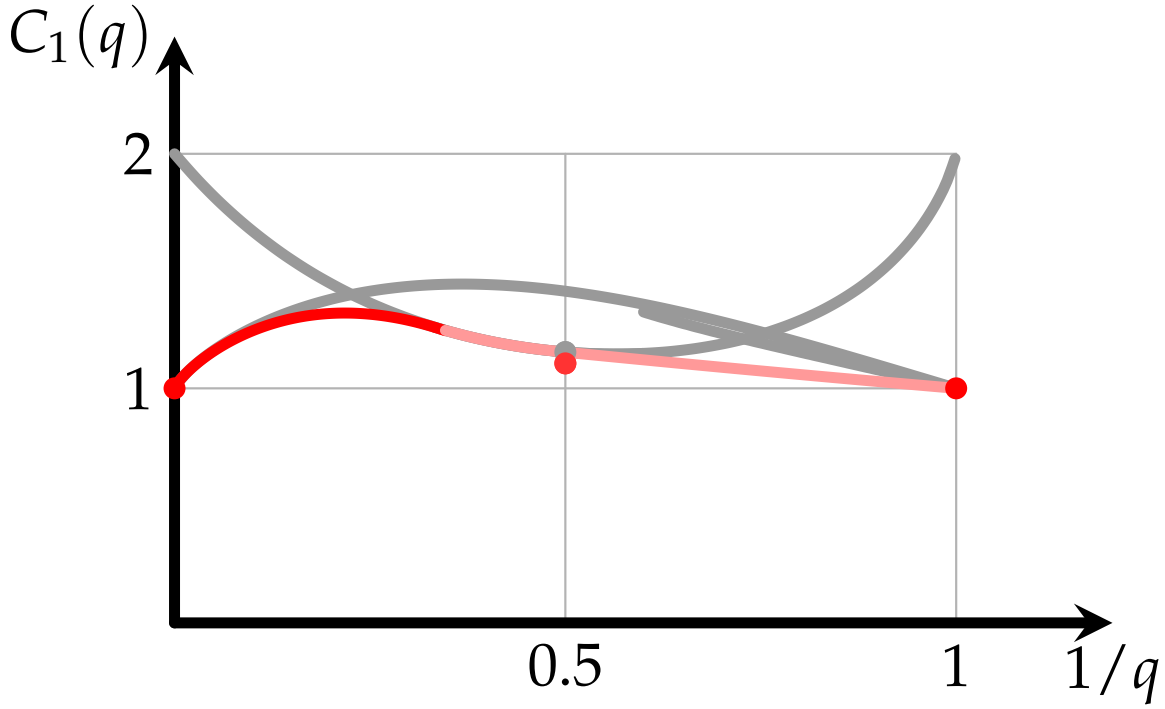


J., Ullrich 2021

# Weak Stechkin inequality

- ▷ $\sum_{n=1}^{\infty} a_n$, $\quad \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$
- ▶ $\sup_{n\in\mathbb{N}} n a_n$, $\quad \sup_{n\in\mathbb{N}} n \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$
- ▷ $\int_0^{\infty} f(t)\mathrm{d}t$, $\quad \int_0^{\infty} \left(\frac{1}{t} \int_t^{\infty} f(s)^q \mathrm{d}s\right)^{\frac{1}{q}} \mathrm{d}t$
- ▷ $\sup_{t>0} t f(t)$, $\quad \sup_{t>0} t \left(\frac{1}{t} \int_t^{\infty} f(s)^q \mathrm{d}s\right)^{\frac{1}{q}}$

# Weak Stechkin inequality

$$\frac{1}{c_{1,\infty}(q)} \sup_{n\in\mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \leq \sup_{n\in\mathbb{N}} n a_n \leq C_{1,\infty}(q) \sup_{n\in\mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}$$

- Pietsch 1981: There exist constants $c_{1,\infty}(q)$ and $C_{1,\infty}(q)$.

# Weak Stechkin inequality

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- Pietsch 1981: There exist constants $c_{1,\infty}(q)$ and $C_{1,\infty}(q)$.
- Temlyakov 1986: $n\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}} \le \sum_{n=1}^{\infty} a_n$

# Weak Stechkin inequality

$$\frac{1}{c_{1,\infty}(q)} \sup_{n\in\mathbb{N}} n\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}} \le \sup_{n\in\mathbb{N}} na_n \le C_{1,\infty}(q) \sup_{n\in\mathbb{N}} n\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$$

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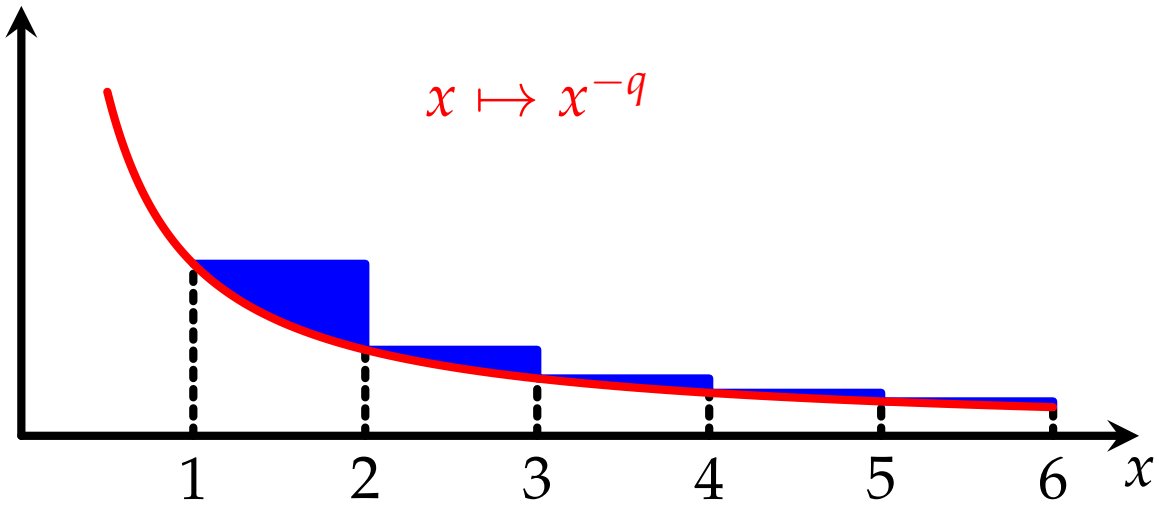
- Foucart, Rauhut 2013:
  $n\left(\frac{1}{n}\sum_{k=n+1}^{\infty} a_k^q\right)^{\frac{1}{q}} \le (q-1)^{-1/q} \sup_{n\in\mathbb{N}} na_n$

# Euler–Mascheroni constant




$$\text{blue area} = \zeta(q) - \frac{1}{q-1} \xrightarrow{q \to 1} 0.57721\ldots$$

$$\zeta(q) \geq 0.57721\ldots + \frac{1}{q-1}$$

# Weak Stechkin inequality

$$\frac{1}{c_{1,\infty}(q)} \sup_{n\in\mathbb{N}} n \left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}} \le \sup_{n\in\mathbb{N}} n a_n \le C_{1,\infty}(q) \sup_{n\in\mathbb{N}} n \left(\frac{1}{n}\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$$

**Theorem (J., Ullrich 2021)**

$$c_{1,\infty}(2) = \zeta(q)^{1/q}, \qquad C_{1,\infty}(2) = \left(\frac{1}{q}\right)^{\frac{1}{q}} \left(1 - \frac{1}{q}\right)^{1-\frac{1}{q}}$$

# Proof of $c_{1,\infty}(q) = \zeta(q)^{1/q}$

- The supremum

$$\sup\left\{\sup_{n\in\mathbb{N}} n^{1-\frac{1}{q}}\left(\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}} \,\middle|\, a_1 \geq a_2 \geq \ldots \geq 0, \sup_{n\in\mathbb{N}} na_n = 1\right\}.$$

is attained at $(\frac{1}{n})_{n\in\mathbb{N}}$.

.

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- So it evaluates to $\sup_{n\in\mathbb{N}} n^{1-\frac{1}{q}}\left(\sum_{k=n}^{\infty}\frac{1}{k^q}\right)^{\frac{1}{q}}$

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-

$$\sum_{k=n}^{\infty} k^{-q} = n^{-q} + \sum_{k=n+1}^{\infty} k^{-q}$$

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-

$$\sum_{k=n}^{\infty} k^{-q} = n^{-q} + \sum_{k=n+1}^{\infty} k^{-q} \leq n^{-q} + \int_n^{\infty} x^{-q}\mathrm{d}x = n^{-q} + \frac{n^{-q+1}}{q-1}$$

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# Proof of $c_{1,\infty}(q) = \zeta(q)^{1/q}$

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$$\implies \quad n^{1-\frac{1}{q}}\left(\sum_{k=n}^{\infty} k^{-q}\right)^{\frac{1}{q}} \leq \left(n^{-1} + \frac{1}{q-1}\right)^{\frac{1}{q}} \qquad .$$

# Proof of $c_{1,\infty}(q) = \zeta(q)^{1/q}$

- The supremum

$$\sup\left\{\sup_{n\in\mathbb{N}} n^{1-\frac{1}{q}}\left(\sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}} \,\middle|\, a_1 \geq a_2 \geq \ldots \geq 0, \sup_{n\in\mathbb{N}} na_n = 1\right\}.$$

is attained at $(\frac{1}{n})_{n\in\mathbb{N}}$.

- So it evaluates to $\sup_{n\in\mathbb{N}} n^{1-\frac{1}{q}}\left(\sum_{k=n}^{\infty}\frac{1}{k^q}\right)^{\frac{1}{q}} = \zeta(q)^{\frac{1}{q}}$.
- $\frac{1}{2} + \frac{1}{q-1} \leq \zeta(q)$ and

$$\sum_{k=n}^{\infty} k^{-q} = n^{-q} + \sum_{k=n+1}^{\infty} k^{-q} \leq n^{-q} + \int_n^{\infty} x^{-q}\mathrm{d}x = n^{-q} + \frac{n^{-q+1}}{q-1}$$

$$\implies \quad n^{1-\frac{1}{q}}\left(\sum_{k=n}^{\infty} k^{-q}\right)^{\frac{1}{q}} \leq \left(n^{-1} + \frac{1}{q-1}\right)^{\frac{1}{q}} \leq \zeta(q)^{\frac{1}{q}}.$$

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**Thank you for your attention.**