Spectral flow for skew-adjoint Fredholm operators

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Plan

- Review of classical spectral flow
- $\mathbb{Z}_2$-valued spectral flow
- Application to a topological insulator (Kitaev chain)
Review of spectral flow

\( \mathcal{H} \) separable Hilbert space and \( \mathbb{B}(\mathcal{H}) \) bounded operators

\( T \in \mathbb{B}(\mathcal{H}) \) Fredholm \( \iff \) Ker\((T)\), Ker\((T^*)\) finite dimensional

\( T = T^* \) Fredholm \( \iff \) \( 0 \not\in \sigma_{\text{ess}}(T) \)

\( \mathbb{F}_{\text{sa}} = \{ T = T^* \text{ Fredholm} \} \) has 3 components which contract to

\[
\begin{align*}
\mathbb{F}^*_{\text{sa}} &= \{ T \in \mathbb{F}_{\text{sa}} \mid \sigma_{\text{ess}}(T) = \{-1, 1\} \} \\
\mathbb{F}^+_{\text{sa}} &= \{ T \in \mathbb{F}_{\text{sa}} \mid \sigma_{\text{ess}}(T) = \{1\} \} \\
\mathbb{F}^-_{\text{sa}} &= \{ T \in \mathbb{F}_{\text{sa}} \mid \sigma_{\text{ess}}(T) = \{-1\} \}
\end{align*}
\]

Theorem (Atiyah-Singer 1969)

*Homotopy groups of* \( \mathbb{F}^*_{\text{sa}} \) are \( \pi_{2n}(\mathbb{F}^*_{\text{sa}}) = 0 \) and \( \pi_{2n+1}(\mathbb{F}^*_{\text{sa}}) = \mathbb{Z} \)

**Aim**: spectral flow calculates \( \pi_1(\mathbb{F}^*_{\text{sa}}) \)
Intuitive notion of spectral flow

Given path $t \in [0, 1] \mapsto T_t = (T_t)^*$ of self-adjoint Fredholms on $\mathcal{H}$

Counting of eigenvalues passing 0 works if path analytic (APS)
For continuous paths need to go to "generic position", or:
Phillips’ analytic approach (1996)

∃ finite partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ of $[0, 1]$ and $a_n < 0 < b_n$ with $t \in [t_{n-1}, t_n] \mapsto \chi(T_t \in [a_n, b_n])$ continuous. Set:

$$\text{SF}(t \in [0, 1] \mapsto T_t) = \sum_{n=1}^{N} \text{Tr}_\mathcal{H} \left( \chi(T_{t_{n-1}} \in [a_n, 0]) - \chi(T_{t_n} \in [a_n, 0]) \right)$$
**Theorem (Phillips 1996)**

\[ \text{SF}(t \in [0, 1] \mapsto T_t) \text{ independent of partition and } a_n < 0 < b_n. \]

*It is a homotopy invariant when end points are kept fixed.*

*It satisfies concatenation and normalization:*

\[ \text{SF}(t \in [0, 1] \mapsto T + (1 - 2t)P) = -\dim(P) \text{ for } TP = P \]

**Theorem (Lesch 2004)**

Homotopy invariance, concatenation, normalization characterize SF

**Theorem (Perera 1993, Phillips 1996)**

SF on loops establishes isomorphism \( \pi_1(\mathbb{F}_{sa}^*) = \mathbb{Z} \)


Let \( T_1 = U^*T_0U \) invertible with \( U \) unitary and \([U, T_0] \) compact

\[ \text{SF}(t \in [0, 1] \mapsto (1-t)T_0 + tT_1) = -\text{Ind}(PUP|_{\mathcal{P}\mathcal{H}}), \ P = \chi(T_0 > 0) \]
Example: Laughlin argument 1981

Theorem (Macris 2002, De Nittis, S-B 2014)

$H$ disordered Harper-like operator on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^L$ with $\mu \in \text{gap}$

$H_\alpha$ Hamiltonian with extra flux $\alpha \in [0, 1]$ through 1 cell of $\mathbb{Z}^2$

Then $T_\alpha = H_\alpha - \mu \in \mathbb{F}^{sa}$ and with $P = \chi(H_\alpha \leq \mu)$, $U = \frac{X_1 + iX_2}{|X_1 + iX_2|}$

$\text{SF}\left(\alpha \in [0, 1] \mapsto H_\alpha \text{ through } \mu\right) = -\text{Ind}(PUP) = -\text{Ch}(P)$
Basics on skew-adjoint Fredholm operators

$\mathcal{H}_\mathbb{R}$ real Hilbert space with complexification $\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{R} \oplus i\mathcal{H}_\mathbb{R}$

$T \in \mathcal{B}(\mathcal{H}_\mathbb{R})$ extends to complex linear operator (e.g. for spectrum)

$T^* = -T$ skew-adjoint $\implies \sigma(T) = \overline{\sigma(T)} \subset i\mathbb{R}$

$T^* = -T$ Fredholm $\iff 0 \notin \sigma_{\text{ess}}(T)$

Theorem (Atiyah Singer 1969)

$F_{sk} = \{ T = -T^* \text{ Fredholm} \}$ has two connected components distinguished by: $\text{Ind}_2(T) = \dim(\text{Ker}(T)) \mod 2$

Homotopy groups satisfy $\pi_n(F_{sk}) = \pi_{n+8}(F_{sk})$ and are given by

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_n(F_{sk})$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
<td>$2\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
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**Aim:** define $\mathbb{Z}_2$-valued spectral flow calculates $\pi_1(F_{sa}^*)$

**Note:** $\text{SF}(t \in [0, 1] \mapsto T_t \in F_{sk}) = 0$
Start with example in $\mathcal{H}_\mathbb{R} = \mathbb{R}^2$

$$T_t = (2t - 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{T}_t = |2t - 1| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Spectra identical $\sigma(T_t) = \sigma(\tilde{T}_t) = \{(1 - 2t)i, (2t - 1)i\}$, but

$$\tilde{T}_t(s) = |2ts - 1| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{F}_{sk}$$

homotopy of paths with $\tilde{T}_t(1) = \tilde{T}_t$ and $\tilde{T}_t(0)$ constant

No such homotopy for $T_t$!

Defect in change of orientation of eigenfunctions:

$$T_1 = A^* T_0 A \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\text{sgn}(\det(A)) < 0$
**Definition**

\[ \dim(\mathcal{H}_\mathbb{R}) < \infty \] and \( T_0, T_1 \in \mathbb{F}_{sk} \) with nullity \( \dim(\mathcal{H}_\mathbb{R}) \mod 2 \)

If \( T_1 = A^* T_0 A \) for some invertible \( A \), then

\[ SF_2(T_0, T_1) = \text{sgn}(\det(A)) \in \mathbb{Z}_2 \]

Now similar as Phillips: path \( t \mapsto T_t \in \mathbb{F}_{sk} \) with \( \text{Ind}_2(T_t) = 0 \)
Definition of $\mathbb{Z}_2$-valued spectral flow

For $a > 0$ set $Q_a(t) = \chi(T_t \in (-ia, ia))$

$\exists$ finite partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ and $a_n > 0$

- $t \in [t_{n-1}, t_n] \mapsto Q_{a_n}(t)$ continuous and constant finite rank
- $\|Q_{a_n}(t) - Q_{a_n}(t')\| < \epsilon \ \forall \ t, t' \in [t_{n-1}, t_n]$
- $\|\pi(T_t) - \pi(T_{t'})\|_Q < \epsilon \ \forall \ t, t' \in [t_{n-1}, t_n]$

for some $\epsilon \leq \frac{1}{5}$

$V_n : \text{Ran}(Q_{a_n}(t_{n-1})) \to \text{Ran}(Q_{a_n}(t_n))$ orthogonal projection, namely $V_n v = Q_{a_n}(t_n)v$. Check: $V_n$ is a bijection.

Define $T_t^{(a)} = Q_a(t) T_t Q_a(t) + R_t$ with $R_t$ lifting kernel

\[
\text{SF}_2(t \in [0, 1] \mapsto T_t) = \sum_{n=1}^{N} \text{SF}_2(T_{t_{n-1}}^{(a_n)}, V_n^* T_{t_n}^{(a_n)} V_n) \mod 2
\]
Basic properties

Theorem

\( \text{SF}_2(t \in [0, 1] \mapsto T_t \in \mathbb{F}_{sk}) \) independent of partition and \( a_n > 0 \).

*It is a homotopy invariant when end points are kept fixed.*

*It satisfies concatenation.*

*It satisfies a normalization (later).*

\( \text{SF}_2 \) has characterizing properties of \( \text{SF} \), but *no "spectral flowing"*

Theorem

\( \text{SF}_2 \) *on loops establishes isomorphism* \( \pi_1(\mathbb{F}_{sk}) = \mathbb{Z}_2 \)
Reformulation

\[ J \in \mathcal{B}(\mathcal{H}_\mathbb{R}) \text{ complex structure} \iff J^* = -J \text{ and } J^2 = -1 \]

**Theorem**

\[ J_0, J_1 \text{ complex structures with } \| \pi(J_0) - \pi(J_1) \|_Q < 1. \text{ Then} \]
\[ \text{SF}_2(t \in [0, 1] \mapsto tJ_0 + (1-t)J_1 \in \mathbb{F}_{sk}) = \frac{1}{2} \dim(\ker(J_0 + J_1)) \mod 2 \]

**Proof:** Both sides are homotopy invariants... \[ \square \]

**Theorem**

For above partition of path \( t \in [0, 1] \mapsto T_t, \text{ set } J_n = T_{t_n}\big| T_{t_n}^{-1} \)

Then
\[ \text{SF}_2(t \in [0, 1] \mapsto T_t) = \left( \sum_{n=1}^{N} \frac{1}{2} \dim(\ker(J_{n-1} + J_n)) \right) \mod 2 \]

For classical spectral flow similar with index of pairs of projections
### Theorem

**J complex structure**, \( O = (O^*)^{-1} \) orthogonal with \([O, J]\) compact

\[ SF_2(t \in [0, 1] \mapsto (1 - t)J + tO^*JO) = \dim \ker(P_{\mathcal{P}}|_{\mathcal{P}\mathcal{H}}) \text{ mod } 2 \]

**where** \( P = \chi(iJ > 0) \) **Hardy**

### Example:

\( \mathcal{H}_\mathbb{R} = L^2_\mathbb{R}(S^1) \otimes \mathbb{R}^2 \) and \( \mathcal{H}_\mathbb{C} = L^2_\mathbb{C}(S^1) \otimes \mathbb{C}^2 \)

Fourier \( \mathcal{F} : \mathcal{H}_\mathbb{C} \to \ell^2_\mathbb{C}(\mathbb{Z}) \otimes \mathbb{C}^2 \)

\( J = \mathcal{F}^*\hat{J}\mathcal{F} \) where \( \hat{J} = i \text{ sgn}(X) \otimes 1_2 + |0\rangle\langle 0| \otimes i\sigma_2 \)

\( O = (O(k))_{k \in S^1} \) fibered with \( 2 \times 2 \) rotation matrix \( O(k) \) by \( k \)

\( SF_2(t \in [0, 1] \mapsto (1 - t)J + tO^*JO) = 1 = \dim \ker(P_{\mathcal{P}}|_{\mathcal{P}\mathcal{H}_\mathbb{C}}) \)
Skew-adjoint Fredholm = gapped BdG

Fermionic quadratic Hamiltonian $\mathbf{H} = (a a^*)H^a(\mathbf{a})$ on $\mathcal{F}_-(\mathcal{H})$

BdG Hamiltonian $H \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ satisfies even PHS

$$K^* \overline{H} K = -H \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then Majorana representation:

$$H_{\text{Maj}} = C^* HC = - \overline{H_{\text{Maj}}} = i \ T \ , \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - i \\ 1 & i \end{pmatrix}$$

Then: $\overline{T} = T$ and $T^* = -T$ and

$T \in F_{\text{sk}} \iff 0$ in gap of $H$

Thus: paths of BdG’s have a $\mathbb{Z}_2$-valued spectral flow
Kitaev chain with flux (disorder suppressed)

Here $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ and with shift $S$ and $\mu \in \mathbb{R}$:

$$H = \frac{1}{2} \begin{pmatrix} S + S^* + 2\mu & i(S - S^*) \\ i(S - S^*) & -(S + S^* + 2\mu) \end{pmatrix}$$

$$= S_0 + S_0^* + \mu \mathbf{1} \otimes \sigma_3 , \quad S_0 = S \otimes \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

Insert flux: $H_\alpha = S_\alpha + S_\alpha^* + \mu \mathbf{1} \otimes \sigma_3$

$$S_\alpha = S_0 + |1\rangle \langle 0| \otimes \frac{1}{2} \begin{pmatrix} e^{-i\pi \alpha} - 1 & i(e^{-i\pi \alpha} - 1) \\ i(e^{i\pi \alpha} - 1) & -(e^{i\pi \alpha} - 1) \end{pmatrix}$$
Spectral flow and bound states at defect

**Proposition**

For $|\mu| < 1$,

\[
\text{SF}_2 (\alpha \in [0, 1] \mapsto H_\alpha) = 1
\]

Time-reversal symmetry $\sigma_3 \overline{H} \sigma_3 = H$, hence in CAZ Class BDI

Also holds for half flux: $\sigma_3 \overline{H}_{\frac{1}{2}} \sigma_3 = H_{\frac{1}{2}}$

**Proposition**

For $|\mu| < 1$, $H_{\frac{1}{2}}$ has odd number of evenly degenerate zero modes:

\[
\frac{1}{2} \dim \mathbb{C}(\Ker_{\mathbb{C}}(H_{\frac{1}{2}})) \mod 2 = 1
\]

**Proof:** Symmetry $\sigma(H_\alpha) = \sigma(H_{1-\alpha})$ and above Proposition