To prepare this lecture I used the following textbooks that I recommend as additional reading:

- S. Mac Lane, Homology, Springer, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 114.
- S. Mac Lane, Categories for the working mathematician, Springer Graduate Texts in Mathematics 5.

Additional references on specific topics are given in the text.

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0 Introduction

Homological algebra is a method that plays an important role in many areas of mathematics. It associates to a mathematical object, such as a topological space, an algebra or a group, a family \((X_n)_{n \in \mathbb{Z}}\) of abelian groups (possibly with additional structures) and a family of group homomorphisms \(d_n : X_n \rightarrow X_{n-1}\) that satisfy \(d_n \circ d_{n+1} = 0\) for all \(n \in \mathbb{Z}\). This condition ensures that \(\text{im}(d_{n+1}) \subset \ker(d_n)\) is a subgroup and allows one to consider the factor groups \(H_n(X) = \ker(d_n)/\text{im}(d_{n+1})\) for \(n \in \mathbb{Z}\). These factor groups are called the homologies of the object and encode relevant information such as the number of connected component of a topological space, the centre of an algebra or the invariants of a group representation. There is also a dual concept, cohomology, where one has a family \((X^n)_{n \in \mathbb{Z}}\) of abelian groups and a family \((d^n)_{n \in \mathbb{Z}}\) of group homomorphisms \(d^n : X^n \rightarrow X^{n+1}\) with \(d^n \circ d^{n-1} = 0\) for all \(n \in \mathbb{Z}\). The factor groups \(H^n(X) = \ker(d^n)/\text{im}(d^{n-1})\) are called the cohomologies of \(X\).

(Co)homologies arise in algebraic topology, such as (co)homologies of simplicial complexes, CW-complexes or, most generally, topological spaces, in algebra, where one considers (co)homologies of algebras, groups, bimodules over algebras, group representations and representations of Lie algebras, and in geometry, where one considers (co)homologies associated with differential forms on smooth manifolds, homologies associated with symplectic manifolds and homologies for specific problems such as intersection of smooth curves on surfaces.

Some of the reasons why homological methods are so widespread and useful are the following. Firstly, (co)homologies are based on linear structures, namely modules over rings and certain generalisations thereof. This makes them much more computable and accessible than non-linear structures. Modules over rings include vector spaces, abelian groups and representations of groups and algebras and are a very versatile and general concept of linearity.

Secondly, there are infinitely many (co)homologies associated with a mathematical object. This allows them to contain enough information about the mathematical object, and this information is organised in an efficient way. (Co)homologies for different values of \(n\) can be computed independently from each other. In many cases, all (co)homologies \(H_n(X)\) for \(n < 0\) vanish, and the complexity of the (co)homologies and their interpretation grows with increasing \(n\). If one requires only little information, it is therefore often sufficient to compute the lowest few homologies. In some cases, there are algorithms that compute homologies efficiently directly from the data that describes the mathematical structure.

Third, homology theories are very general and flexible and can be formulated abstractly, which allow one to apply them to many mathematical problems. There is a good understanding of what data is needed to define a (co)homology theory, and this data can be largely characterised in terms of combinatorics and very canonical constructions. This allows one to easily find new applications and to treat them systematically. Since they are largely formulated in the language of categories and functors, they allow one to make contact between different parts of mathematics and to adapt methods and insights to other contexts. In fact, the wish to have a unified framework for different versions of (co)homology theories was an important motivation in the development of category theory.
1 Algebraic background

1.1 Categories, functors and natural transformations

(Co)homology theories relate different mathematical structures such as topological spaces, algebras and groups to modules over certain rings and structure preserving maps such as continuous maps, algebra homomorphisms and group homomorphisms to module homomorphisms. The mathematical concepts that describe such a relation between different mathematical structures are categories and functors. These concepts not only simplify and unify the description of different (co)homology theories, but they are required for a systematic investigation of (co)homology theories and for a deeper understanding of their structure.

Definition 1.1.1: A category $\mathcal{C}$ consists of:
- a class $\text{Ob}\mathcal{C}$ of objects,
- for each pair of objects $X, Y \in \text{Ob}\mathcal{C}$ a set $\text{Hom}_\mathcal{C}(X, Y)$ of morphisms,
- for each triple of objects $X, Y, Z$ a composition map $\circ : \text{Hom}_\mathcal{C}(Y, Z) \times \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z),$

such that the following axioms are satisfied:

(C1) The sets $\text{Hom}_\mathcal{C}(X, Y)$ of morphisms are pairwise disjoint,

(C2) The composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$ for all morphisms $h \in \text{Hom}_\mathcal{C}(W, X), g \in \text{Hom}_\mathcal{C}(X, Y), f \in \text{Hom}_\mathcal{C}(Y, Z),$

(C3) For every object $X$ there is a morphism $1_X \in \text{Hom}_\mathcal{C}(X, X)$ with $1_X \circ f = f$ and $g \circ 1_X = g$ for all $f \in \text{Hom}_\mathcal{C}(W, X), g \in \text{Hom}_\mathcal{C}(X, Y).$

The morphisms $1_X$ are called identity morphisms.

Instead of $f \in \text{Hom}_\mathcal{C}(X, Y),$ we also write $f : X \to Y.$ The object $X$ is called the source of $f,$ and the object $Y$ the target of $f.$ A morphism $f : X \to X$ is called an endomorphism.

A morphism $f : X \to Y$ is called an isomorphism, if there is a morphism $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y.$ In this case, we call the objects $X$ and $Y$ isomorphic.

Example 1.1.2:

1. The category Set: the objects of Set are sets, and the morphisms are maps $f : X \to Y.$ The composition is the composition of maps and the identity morphisms are the identity maps. Isomorphisms are bijective maps.

Note that the definition of a category requires that morphisms between two objects in a category form a set, but not the objects. Requiring that the objects of a category form a set would force one to consider sets of sets when defining the category Set, which leads to a contradiction. A category whose objects form a set is called a small category.

2. The category Top of topological spaces. Objects are topological spaces, morphisms $f : X \to Y$ are continuous maps, isomorphisms are homeomorphisms.
3. The category $\text{Top}^*$ of **pointed topological spaces**: Objects are pairs $(X, x)$ of a topological space $X$ and a point $x \in X$, morphisms $f : (X, x) \to (Y, y)$ are continuous maps $f : X \to Y$ with $f(x) = y$.

4. The category $\text{Top}(2)$ of **pairs of topological spaces**: Objects are pairs $(X, A)$ of a topological space $X$ and a subspace $A \subset X$, morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ with $f(A) \subset B$. Isomorphisms are homeomorphisms $f : X \to Y$ with $f(A) = B$.

5. Many examples of categories we will use in the following are categories of algebraic structures. This includes the following:
   - the category $\text{Vect}_F$ of vector spaces over a field $F$:
     objects: vector spaces over $F$, morphisms: $F$-linear maps,
   - the category $\text{Vect}_F^{\text{fin}}$ of finite dimensional vector spaces over a field $F$:
     objects: vector spaces over $F$, morphisms: $F$-linear maps,
   - the category $\text{Grp}$ of groups:
     objects: groups, morphisms: group homomorphisms,
   - the category $\text{Ab}$ of abelian groups:
     objects: abelian groups, morphisms: group homomorphisms,
   - the category $\text{Ring}$ of rings:
     objects: rings, morphisms: ring homomorphisms,
   - the category $\text{URing}$ of unital rings:
     objects: unital rings, morphisms: unital ring homomorphisms,
   - the category $\text{Field}$ of fields:
     objects: fields, morphisms: field monomorphisms,
   - the category $\text{Alg}_F$ of algebras over a field $F$:
     objects: algebras over $F$, morphisms: algebra homomorphisms,
   - the categories $R\text{-Mod}$ and $\text{Mod-}R$ of left and right modules over a ring $R$:
     objects: $R$-left or right modules, morphisms: $R$-left or right module homomorphisms,
   - the category $R\text{-Mod-}S$ of $(R, S)$-bimodules:
     objects: $(R, S)$-bimodules, morphisms: $(R, S)$-bimodule homomorphisms.

In all of the categories in Example 1.1.2 the morphisms are maps. A category for which this is the case is called a **concrete category**. A category that is not concrete is given in Exercise 1. Other important examples and basic constructions for categories are the following.

**Example 1.1.3:**

1. A small category $C$ in which all morphisms are isomorphisms is called a **groupoid**.

2. A category with a single object $X$ is a **monoid**, and a groupoid $C$ with a single object $X$ is a group. Group elements are identified with endomorphisms $f : X \to X$ and the composition of morphisms is the group multiplication. More generally, for any object $X$ in a groupoid $C$, the set $\text{End}_C(X) = \text{Hom}_C(X, X)$ with the composition $\circ : \text{End}_C(X) \times \text{End}_C(X) \to \text{End}_C(X)$ is a group.
3. For every category \( C \), one has an **opposite category** \( C^{op} \), which has the same objects as \( C \), whose morphisms are given by \( \mathrm{Hom}_{C^{op}}(X,Y) = \mathrm{Hom}_C(Y,X) \) and in which the order of the composition is reversed.

4. The **Cartesian product** of categories \( C, D \) is the category \( C \times D \) with pairs \((C,D)\) of objects in \( C \) and \( D \) as objects, with \( \mathrm{Hom}_{C \times D}((C,D),(C',D')) = \mathrm{Hom}_C(C,C') \times \mathrm{Hom}_D(D,D') \) and the composition of morphisms \((h,k) \circ (f,g) = (h \circ f, k \circ g)\).

5. A **subcategory** of a category \( C \) is a category \( D \), such that \( \mathrm{Ob}(D) \subset \mathrm{Ob}(C) \) is a subclass, \( \mathrm{Hom}_D(D,D') \subset \mathrm{Hom}_C(D,D') \) for all objects \( D, D' \) in \( D \) and the composition of morphisms of \( D \) coincides with their composition in \( C \). A subcategory \( D \) of \( C \) is called a **full subcategory** if \( \mathrm{Hom}_D(D,D') = \mathrm{Hom}_C(D,D') \) for all objects \( D, D' \) in \( D \).

6. **Quotient categories**: Let \( C \) be a category with an equivalence relation \( \sim_{X,Y} \) on each morphism set \( \mathrm{Hom}_C(X,Y) \) that is compatible with the composition of morphisms: \( f \sim_{X,Y} g \) and \( h \sim_{Y,Z} k \) in implies \( h \circ f \sim_{X,Z} k \circ g \).

Then one obtains a category \( C' \), the **quotient category** of \( C \), with the same objects as \( C \) and equivalence classes of morphisms in \( C \) as morphisms.

The composition of morphisms in \( C' \) is given by \( [h] \circ [f] = [h \circ f] \), and the identity morphisms by \([1_X]\). Isomorphisms in \( C' \) are equivalence classes of morphisms \( f \in \mathrm{Hom}_C(X,Y) \) for which there exists a morphism \( g \in \mathrm{Hom}_C(Y,X) \) with \( f \circ g \sim_{Y,X} 1_Y \) and \( g \circ f \sim_{X,X} 1_X \).

The construction in the last example plays a fundamental role in classification problems, in particular in the context of topological spaces. Classifying the objects of a category \( C \) usually means classifying them up to isomorphism, i.e. giving a list of objects in \( C \) such that every object in \( C \) is isomorphic to exactly one object in this list. While this is possible in some contexts - for the category \( \mathrm{Vect}^{fin}_F \) of finite dimensional vector spaces over \( F \), the list contains the vector spaces \( F^n \) with \( n \in \mathbb{N}_0 \) - it is often too difficult to solve this problem in full generality. In this case, it is sometimes simpler to consider instead a quotient category \( C' \) and to attempt a partial classification. If two objects are isomorphic in \( C \), they are by definition isomorphic in \( C' \) since for any objects \( X, Y \) in \( C \) and any isomorphism \( f : X \to Y \) with inverse \( g : Y \to X \), one has \([g] \circ [f] = [g \circ f] = [1_X]\) and \([f] \circ [g] = [f \circ g] = [1_Y]\). However, the converse does not hold in general - the category \( C' \) yields a weaker classification result than \( C \).

To relate different categories, one must not only relate their objects but also their morphisms, in a way that is compatible with source and target objects, the composition of morphisms and the identity morphisms. This leads to the concept of a **functor**.

**Definition 1.1.4**: Let \( C, D \) be categories. A **functor** \( F : C \to D \) consists of:

- an assignment of an object \( F(C) \) in \( D \) to every object \( C \) in \( C \),
- for each pair of objects \( C, C' \) in \( C \), a map

\[
\mathrm{Hom}_C(C,C') \to \mathrm{Hom}_D(F(C),F(C')) , \quad f \mapsto F(f),
\]

that is compatible with the composition of morphisms and with the identity morphisms

\[
F(g \circ f) = F(g) \circ F(f) \quad \forall f \in \mathrm{Hom}_C(C,C'), g \in \mathrm{Hom}_C(C',C'') \\
F(1_C) = 1_{F(C)} \quad \forall C \in \mathrm{Ob} C.
\]
A functor \( F : \mathcal{C} \to \mathcal{C} \) is called an endofunctor. A functor \( F : \mathcal{C}^{\text{op}} \to \mathcal{D} \) is sometimes called a contravariant functor from \( \mathcal{C} \) to \( \mathcal{D} \). The composite of two functors \( F : \mathcal{B} \to \mathcal{C}, G : \mathcal{C} \to \mathcal{D} \) is the functor \( GF : \mathcal{B} \to \mathcal{D} \) given by the assignment \( B \mapsto GF(B) \) for all objects \( B \) in \( \mathcal{B} \) and the maps \( \text{Hom}_{\mathcal{B}}(B, B') \to \text{Hom}_{\mathcal{D}}(GF(B), GF(B')) \), \( f \mapsto G(F(f)) \).

**Example 1.1.5:**

1. For any category \( \mathcal{C} \), identity functor \( \text{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \), that assigns each object and morphism in \( \mathcal{C} \) to itself is an endofunctor of \( \mathcal{C} \).

2. The functor \( \text{Vect}_{\mathbb{F}} \to \text{Ab} \) that assigns to each vector space the underlying abelian group and to each linear map the associated group homomorphism, and the functors \( \text{Vect}_{\mathbb{F}} \to \text{Set}, \text{Ring} \to \text{Set}, \text{Grp} \to \text{Set}, \text{Top} \to \text{Set} \) etc that assign to each vector space, ring, group, topological space the underlying set and to each morphism the underlying map are functors. A functor of this type is called forgetful functor.

3. The functor \( * : \text{Vect}_{\mathbb{F}} \to \text{Vect}_{\mathbb{F}}^{\text{op}} \), which assigns to a vector space \( V \) its dual \( V^* \) and to a linear map \( f : V \to W \) its adjoint \( f^* : W^* \to V^* \), \( \alpha \mapsto \alpha \circ f \).

4. If \( \mathcal{C} \) is a groupoid with a single object \( X \), i.e. a group \( G = (\text{End}_{\mathcal{C}}(X), \circ) \), then a functor \( F : \mathcal{C} \to \text{Set} \) is a group action of \( G \) on the set \( F(X) \) and a functor \( F : \mathcal{C} \to \text{Vect}_{\mathbb{F}} \) a representation of the group \( G \) on the vector space \( F(X) \).

5. The **Hom-functors**: Let \( \mathcal{C} \) be a category and \( C \) an object in \( \mathcal{C} \).

   The functor \( \text{Hom}(\mathcal{C}, -) : \mathcal{C} \to \text{Set} \) assigns to an object \( C' \) the set \( \text{Hom}_{\mathcal{C}}(C, C') \) and to a morphism \( f : C' \to C'' \) the map \( \text{Hom}_{\mathcal{C}}(C, f) : \text{Hom}_{\mathcal{C}}(C, C') \to \text{Hom}_{\mathcal{C}}(C, C''), g \mapsto f \circ g \).

   The functor \( \text{Hom}(-, C) : \mathcal{C}^{\text{op}} \to \text{Set} \) assigns to an object \( C' \) the set \( \text{Hom}_{\mathcal{C}}(C', C) \) and to a morphism \( f : C' \to C'' \) the map \( \text{Hom}(f, C) : \text{Hom}_{\mathcal{C}}(C', C) \to \text{Hom}_{\mathcal{C}}(C'', C) \), \( g \mapsto g \circ f \).

6. The functor \( \pi_0 : \text{Top} \to \text{Set} \) that assigns to a topological space \( X \) the set \( \pi_0(X) = \{ P(x) \mid x \in X \} \) of its path components \( P(x) \) and to a continuous map \( f : X \to Y \) the map \( \pi_0(f) : \pi_0(X) \to \pi_0(Y) \), \( P(x) \mapsto P(f(x)) \).

7. The **fundamental group** defines a functor \( \pi_1 : \text{Top}^{\ast} \to \text{Grp} \) that assigns to a pointed topological space \( (x, X) \) its fundamental group \( \pi_1(x, X) \) and to a morphism \( f : (x, X) \to (y, Y) \) of pointed topological spaces the group homomorphism \( \pi_1(f) : \pi_1(x, X) \to \pi_1(y, Y) \), \( [\gamma] \mapsto [f \circ \gamma] \).

It will become apparent in the following that it is not sufficient to consider functors between different categories but one also needs a structure that relates different functors. As a functor \( F : \mathcal{C} \to \mathcal{D} \) involves maps between the sets \( \text{Hom}_{\mathcal{C}}(C, C') \) and \( \text{Hom}_{\mathcal{D}}(F(C), F(C')) \), a structure that relates two functors \( F, G : \mathcal{C} \to \mathcal{D} \) must in particular relate the sets \( \text{Hom}_{\mathcal{D}}(F(C), F(C')) \) and \( \text{Hom}_{\mathcal{D}}(G(C), G(C')) \). The simplest way to do this is to assign to each object \( C \) in \( \mathcal{C} \) a morphism \( \eta_C : F(C) \to G(C) \) in \( \mathcal{D} \). By requiring that this assignment is compatible with the morphisms \( F(f) : F(C) \to G(C') \) and \( G(f) : G(C) \to G(C') \) for all morphisms \( f : C \to C' \) in \( \mathcal{C} \), one obtains the notion of a natural transformation.

**Definition 1.1.6**: A natural transformation \( \eta : F \to G \) between functors \( F, G : \mathcal{C} \to \mathcal{D} \) is an assignment of a morphism \( \eta_C : F(C) \to G(C) \) in \( \mathcal{D} \) to every object \( C \) in \( \mathcal{C} \) such that the following diagram commutes for all morphisms \( f : C \to C' \) in \( \mathcal{C} \).
A natural isomorphism is a natural transformation $\eta : F \to G$, for which all morphisms $\eta_X : F(X) \to G(X)$ are isomorphisms. Two functors that are related by a natural isomorphism are called naturally isomorphic.

Example 1.1.7:

1. For any functor $F : C \to D$ the identity natural transformation $\text{id}_F : F \to F$ with component morphisms $(\text{id}_F)_X = 1_{F(X)} : F(X) \to F(X)$ is a natural isomorphism.

2. Consider the functors $\text{id} : \text{Vect} \to \text{Vect}$ and $** : \text{Vect} \to \text{Vect}$. Then there is a canonical natural transformation $\text{id} \to **$, whose component morphisms $\eta_V : V \to V^{**}$ assign to a vector $v \in V$ the unique vector $v^{**} \in V^{**}$ with $\eta^{**}(\alpha) = \alpha(v)$ for all $\alpha \in V^*$.

3. Consider the category $\text{CRing}$ of commutative unital rings and unital ring homomorphisms and the category $\text{Grp}$ of groups and group homomorphisms.

Let $F : \text{CRing} \to \text{Grp}$ the functor that assigns to a commutative unital ring $k$ the group $GL_n(k)$ of invertible $n \times n$-matrices with entries in $k$ and to a unital ring homomorphism $f : k \to l$ the group homomorphism $GL_n(f) : GL_n(k) \to GL_n(l)$, $M = (m_{ij})_{i,j=1,...,n} \mapsto f(M) = (f(m_{ij}))_{i,j=1,...,n}$.

Let $G : \text{CRing} \to \text{Grp}$ be the functor that assigns to a commutative unital ring $k$ the group $G(k) = k^\times = \{ r \in k : \exists s \in k : r \cdot r' = r' \cdot r = 1 \}$ of units and to a unital ring homomorphism $f : k \to l$ the induced group homomorphism $G(f) = f|_{k^\times} : k^\times \to l^\times$.

The determinant defines a natural transformation $\det : F \to G$ with component morphisms $\det_k : GL_n(k) \to k^\times$, since the following diagram commutes for every unital ring homomorphism $f : k \to l$

\[
\begin{array}{ccc}
GL_n(k) & \xrightarrow{\det_k} & k^\times \\
| & \downarrow & | \\
GL_n(f) & \xrightarrow{f|_{k^\times}} & l^\times.
\end{array}
\]

Remark 1.1.8: For any small category $C$ and category $D$, the functors $C \to D$ and natural transformations between them form a category $\text{Fun}(C, D)$, the functor category. The composite of two natural transformations $\eta : F \to G$ and $\kappa : G \to H$ is the natural transformation $\kappa \circ \eta : F \to H$ with component morphisms $(\kappa \circ \eta)_X = \kappa_X \circ \eta_X : F(X) \to H(X)$ and the identity morphisms are the identity natural transformations $1_F = \text{id}_F : F \to F$.

The notions of natural transformations and natural isomorphisms are particularly important as they allow one to generalise the notion of an inverse map and of a bijection to functors. While it is possible to define an inverse of a functor $F : C \to D$ as a functor $G : D \to C$ with
$GF = \text{id}_C$ and $FG = \text{id}_D$, it turns out that this is too strict. There are very few non-trivial examples of functors with an inverse. A more useful generalisation is obtained by weakening this requirement. Instead of requiring $FG = \text{id}_D$ and $GF = \text{id}_C$, one requires only that these functors are *naturally isomorphic* to the identity functors. This leads to the concept of an equivalence of categories.

**Definition 1.1.9:** A functor $F : C \to D$ is called an **equivalence of categories** if there is a functor $G : D \to C$ and natural isomorphisms $\kappa : GF \to \text{id}_C$ and $\eta : FG \to \text{id}_D$. In this case, the categories $C$ and $D$ are called **equivalent**.

Sometimes it is easier to use a more direct characterisation of an equivalences of categories in terms of its behaviour on objects and morphisms. The proof of the following lemma makes use of the axiom of choice and an be found for instance in [K], Chapter XI, Prop XI.1.5.

**Lemma 1.1.10:** A functor $F : C \to D$ is an equivalence of categories if and only if it is:

1. **essentially surjective:**
   for every object $D$ in $D$ there is an object $C$ of $C$ such that $D$ is isomorphic to $F(C)$.

2. **fully faithful:**
   all maps $\text{Hom}_C(C, C') \to \text{Hom}_D(F(C), F(C'))$, $f \mapsto F(f)$ are bijections.

**Example 1.1.11:**

1. The category $\text{Vect}^{\text{fin}}_F$ of finite-dimensional vector spaces over $\mathbb{F}$ is equivalent to the category $\mathcal{C}$, whose objects are non-negative integers $n \in \mathbb{N}_0$, whose morphisms $f : n \to m$ are $m \times n$-matrices with entries in $\mathbb{F}$ and with the matrix multiplication as composition of morphisms.

2. The category $\text{Set}^{\text{fin}}$ of finite sets is equivalent to the category $\text{Ord}^{\text{fin}}$, whose objects are finite ordinal numbers $\underline{n} = \{0, 1, ..., n - 1\}$ for all $n \in \mathbb{N}_0$ and whose morphisms $f : m \to n$ are maps $f : \{0, 1, ..., m - 1\} \to \{0, 1, ..., n - 1\}$ with the composition of maps as the composition of morphisms.

Many concepts and constructions from topological or algebraic settings can be generalised straightforwardly to categories. This is true whenever it is possible to characterise them in terms of *universal properties* involving only the *morphisms* in the category. In particular, there is a concept of categorical product and coproduct that generalise cartesian products and disjoint unions of sets and products and sums of topological spaces.

**Definition 1.1.12:** Let $C$ be a category and $(C_i)_{i \in I}$ a family of objects in $C$.

1. A **product** of the family $(C_i)_{i \in I}$ is an object $\Pi_{i \in I} C_i$ in $C$ together with a family of morphisms $\pi_i : \Pi_{i \in I} C_i \to C_i$, such that for all families of morphisms $f_i : W \to C_i$ there is a unique morphism $f : W \to \Pi_{i \in I} C_i$ such that the diagram

\[
\begin{array}{ccc}
W & \xrightarrow{f} & \Pi_{i \in I} C_i \\
\downarrow{f_i} & & \downarrow{\pi_i} \\
C_i
\end{array}
\]

commutes for all $i \in I$. This is called the **universal property** of the product.
2. A **coproduct** of the family \((C_i)_{i \in I}\) is an object \(\Pi_{i \in I} C_i\) in \(\mathcal{C}\) with a family \((\iota_i)_{i \in I}\) of morphisms \(\iota_i : C_i \to \Pi_{i \in I} C_i\), such that for every family \((f_i)_{i \in I}\) of morphisms \(f_i : C_i \to Y\) there is a unique morphism \(f : \Pi_{i \in I} C_i \to Y\) such that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\exists f} & \Pi_{i \in I} C_i \\
\downarrow{f_i} & & \downarrow{\iota_i} \\
C_i & & \\
\end{array}
\]

commutes for all \(i \in I\). This is called the **universal property** of the coproduct.

**Remark 1.1.13:** Products or coproducts do not necessarily exist for a given family of objects \((C_i)_{i \in I}\) in a category \(\mathcal{C}\), but if they exist, they are **unique up to unique isomorphism**:

If \((\Pi_{i \in I} C_i, (\pi_i)_{i \in I})\) and \((\Pi'_{i \in I} C'_i, (\pi'_i)_{i \in I})\) are two products for a family of objects \((C_i)_{i \in I}\) in \(\mathcal{C}\), then there is a unique morphism \(\pi' : \Pi'_{i \in I} C_i \to \Pi_{i \in I} C_i\) with \(\pi_i \circ \pi' = \pi'_i\) for all \(i \in I\), and this morphism is an isomorphism.

This follows directly from the universal property of the products: By the universal property of the product \(\Pi_{i \in I} C_i\) applied to the family of morphisms \(\pi'_i : \Pi'_{i \in I} C_i \to C_i\), there is a unique morphism \(\pi' : \Pi'_{i \in I} C_i \to \Pi_{i \in I} C_i\) such that \(\pi_i \circ \pi' = \pi'_i\) for all \(i \in I\). Similarly, the universal property of \(\Pi'_{i \in I} C_i\) implies that for the family of morphisms \(\pi_i : \Pi_{i \in I} C_i \to C_i\) there is a unique morphism \(\pi : \Pi_{i \in I} C_i \to \Pi'_{i \in I} C_i\) with \(\pi'_i \circ \pi = \pi_i\) for all \(i \in I\). It follows that \(\pi' \circ \pi : \Pi_{i \in I} C_i \to \Pi_{i \in I} C_i\) is a morphism with \(\pi_i \circ \pi \circ \pi' = \pi'_i \circ \pi = \pi_i\) for all \(i \in I\). Since the identity morphism on \(\Pi_{i \in I} C_i\) is another morphism with this property, the uniqueness implies \(\pi' \circ \pi = 1_{\Pi_{i \in I} C_i}\). By the same argument one obtains \(\pi \circ \pi' = 1_{\Pi'_{i \in I} C_i}\) and hence \(\pi'\) is an isomorphism with inverse \(\pi\).

![Diagram](image)

**Example 1.1.14:**

1. The cartesian product of sets is a product in Set, and the disjoint union of sets is a coproduct in Set. The product of topological spaces is a product in Top and the topological sum is a coproduct in Top. In Set and Top, products and coproducts exist for all families of objects.

2. The direct sum of vector spaces is a coproduct and the direct product of vector spaces a product in Vect. More generally, direct sums and products of \(R\)-left (right) modules over a unital ring \(R\) are coproducts and products in R-Mod (Mod-R). Again, products and coproducts exist for all families of objects in R-Mod (Mod-R).

3. The wedge sum is a coproduct in the category Top\(^*\) of pointed topological spaces. It exists for all families of pointed topological spaces.

4. The direct product of groups is a product in Grp and the free product of groups is a coproduct in Grp. They exist for all families of groups.
In particular, we can consider categorical products and coproducts over empty index sets \( I \). By definition, a categorical product for an empty family of objects is an object \( T = \prod_\emptyset \) such that for every object \( C \) in \( \mathcal{C} \) there is a unique morphism \( t_C : C \to T \). (This is the morphism associated to the empty family of morphisms from \( C \) to the objects in the empty family by the universal property of the product.) Similarly, a coproduct over an empty index set \( I \) is an object \( I := \coprod_\emptyset \) in \( \mathcal{C} \) such that for every object \( C \) in \( \mathcal{C} \), there is a unique morphism \( i_C : I \to C \).

Such objects are called, respectively, terminal and initial objects in \( \mathcal{C} \).

Initial and terminal objects do not exist in every category \( \mathcal{C} \), but if they exist they are unique up to unique isomorphism by the universal property of the products and coproducts.

An object that is both, terminal and initial, is called a zero object. If it exists, it is unique up to unique isomorphism by the universal property of the products and coproducts.

**Definition 1.1.15:** Let \( \mathcal{C} \) be a category. An object \( X \) in a category \( \mathcal{C} \) is called:

1. A **final** or terminal object in \( \mathcal{C} \) is an object \( T \) in \( \mathcal{C} \) such that for every object \( C \) in \( \mathcal{C} \) there is a unique morphism \( t_C : C \to T \).
2. A cofinal or initial object in \( \mathcal{C} \) is an object \( I \) in \( \mathcal{C} \) such that for every object \( C \) in \( \mathcal{C} \) there is a unique morphism \( i_C : I \to C \),
3. A null object or zero object in \( \mathcal{C} \) is an object \( 0 \) in \( \mathcal{C} \) that is both final and initial: for every object \( C \) in \( \mathcal{C} \) there are a unique morphisms \( t_C : C \to 0 \) and \( i_C : 0 \to C \).
4. If \( \mathcal{C} \) has a zero object, then the morphism \( 0 = i_{C'} \circ t_C : C \to C' \) between objects \( C, C' \) in \( \mathcal{C} \) is called the trivial morphism or zero morphism from \( C \) to \( C' \).

**Example 1.1.16:**

1. The empty set is an initial object in Set and the empty topological space an initial object in Top. Any set with one element is a final object in Set and any one point space an initial object in Top. The categories Set and Top do not have null objects.
2. The null vector space \( \{0\} \) is a null object in the category Vect_{\mathbb{F}}. More generally, for any ring \( R \), the trivial \( R \)-module \( \{0\} \) is a null object in \( \mathbb{R} \)-Mod (Mod-\( \mathbb{R} \)).
3. The trivial group \( G = \{e\} \) is a null object in Grp and in Ab.
4. The ring \( \mathbb{Z} \) is an initial object in the category URing, since for every unital ring \( R \), there is exactly one ring homomorphism \( f : \mathbb{Z} \to R \), namely the one determined by \( f(0) = 0_R \) and \( f(1) = 1_R \). The zero ring \( R = \{0\} \) with \( 0 = 1 \) is a final object in URing, but not an initial one. The category URing has no zero object.
5. The category Field does not have initial or final objects. As any ring homomorphism \( f : \mathbb{F} \to \mathbb{K} \) between fields is injective, an initial object \( \mathbb{F} \) in Field would be a subfield of any other field, and every field would be a subfield of a final object \( \mathbb{F} \). This would imply \( \text{char}(\mathbb{F}) = \text{char}(\mathbb{K}) \) for all other fields \( \mathbb{K} \), a contradiction.
1.2 Modules over rings

Modules over unital rings are one of the essential ingredients of (co)homology theories. They are useful because they unify different algebraic structures such as abelian groups, commutative rings, vector spaces over fields and representations of groups and algebras, which are all used to define different versions of (co)homology theories. By working with modules, we can relate these different notions of (co)homologies and treat them in a common framework. It also becomes apparent which properties are universal and which depend on the choice of the underlying ring.

In this section we summarise the basic constructions and results for modules over rings. Unless stated otherwise all rings are assumed to be unital as well, i.e. to map multiplicative units to multiplicative units.

Definition 1.2.1: Let \( R \) be a ring.

1. A **(left) module** over \( R \) or an **\( R \)-(left) module** is an abelian group \( (M, +) \) together with a map \( \triangleright : R \times M \to M, (r, m) \mapsto r \triangleright m \), the **structure map** that satisfies for all \( m, m' \in M \) and \( r, r' \in R \):
   \[
   r \triangleright (m + m') = r \triangleright m + r \triangleright m' \quad \quad (r + r') \triangleright m = r \triangleright m + r' \triangleright m \\
   (r \cdot r') \triangleright m = r \triangleright (r' \triangleright m) \quad \quad 1 \triangleright m = m.
   \]

2. A **morphism of \( R \)-modules** or an **\( R \)-linear map** from an \( R \)-module \( (M, +_M, \triangleright_M) \) to an \( R \)-module \( (N, +_N, \triangleright_N) \) is a group homomorphism \( \phi : (M, +_M) \to (N, +_N) \) that satisfies
   \[
   \phi(r \triangleright_M m) = r \triangleright_N \phi(m) \quad \forall m \in M, r \in R.
   \]
   A bijective \( R \)-module morphism \( f : M \to N \) is called a \( R \)-module **isomorphism**, and one writes \( M \cong N \). The set of \( R \)-module morphisms \( \phi : M \to N \) is denoted \( \text{Hom}_R(M, N) \).

Remark 1.2.2: Let \( R, S \) be rings.

1. A **right module** over \( R \) is a left module over the ring \( R^{op} \) with the opposite multiplication \( r \cdot_{op} s = s \cdot r \) and a morphism of \( R \)-right modules is a morphism of \( R^{op} \)-left modules.
   Equivalently, we can define a right module over \( R \) as an abelian group \( (M, +) \) together with a map \( \langle : M \times R \to M, (m, r) \mapsto m \triangleleft r \), such that for all \( m, m' \in M \) and \( r, r' \in R \):
   \[
   (m + m') \triangleleft r = m \triangleleft r + m' \triangleleft r \quad \quad m \triangleleft (r + r') = m \triangleleft r + m \triangleleft r' \\
   m \triangleleft (r \cdot r') = (m \triangleleft r) \triangleleft r' \quad \quad m \triangleleft 1 = m.
   \]
   If \( R \) is commutative, left and right modules over \( R \) coincide.

2. An **\( (R,S) \)-bimodule** is an abelian group \( (M, +) \) with an \( R \)-left module structure \( \triangleright : R \times M \to M \) and an \( S \)-right module structure \( \langle : M \times S \to M \) such that \( r \triangleright (m \triangleleft s) = (r \triangleright m) \triangleleft s \) for all \( r \in R, s \in S \) and \( m \in M \).

3. The left (right) modules over a ring \( R \) and the left (right) module morphisms form a category \( R\text{-Mod} \) (\( \text{Mod-}R = R^{op}\text{-Mod} \)). Similarly, the \( (R,S) \) bimodules and \( (R,S) \)-bimodule morphisms form a category \( R\text{-Mod-S} \).
Example 1.2.3:  Let $R, S$ be rings.

1. Every abelian group $(M,+)$ has a unique $\mathbb{Z}$-module structure determined by $0 \triangleright m = 0$ and $1 \triangleright m = m$ for all $m \in M$. The additivity of the structure map in the first argument determines the $\mathbb{Z}$-module structure uniquely since for all $m \in M$ and $n \in \mathbb{N}$ one has

\[
0 \triangleright m = (0 + 0) \triangleright m = 0 \triangleright m + 0 \triangleright m \quad \Rightarrow \quad 0 \triangleright m = 0 \\
\triangleright m = (1 + ... + 1) \triangleright m = 1 \triangleright m + ... + 1 \triangleright m = m + ... + m \\
0 = 0 \triangleright m = (n - n) \triangleright m = m + ... + m + (-n) \triangleright m \quad \Rightarrow \quad (-n) \triangleright m = -(m + ... + m).
\]

2. Modules over a field $\mathbb{F}$ are $\mathbb{F}$-vector spaces and $\mathbb{F}$-module morphisms are $\mathbb{F}$-linear maps.

3. $R$ is a left module over itself with $\triangleright : R \times R \to R$, $r \triangleright r' = r \cdot r'$ and a right module over itself with $\triangleleft : R \times R \to R$, $r' \triangleleft r = r' \cdot r$. This gives $R$ the structure of an $(R, R)$-bimodule.

4. For any set $X$ and $R$-module $M$, the set $\text{Map}(X, M)$ of maps $f : X \to M$ has a canonical $R$-left module structure given by

\[
(f + g)(x) = f(x) + g(x), \quad (r \triangleright f)(x) = r \triangleright f(x) \quad \forall x \in X, f, g : X \to M, r \in R.
\]

5. For any $S$-module $M$ and $R$-module $N$, the set $\text{Hom}_\mathbb{Z}(M, N)$ of group homomorphisms $f : M \to N$ has a canonical $R$-left module and $S$-right module structure given by

\[
(f + g)(m) = f(m) + g(m) \quad (r \triangleright f)(m) = r \triangleright f(m) \quad (f \triangleleft s)(m) = f(s \triangleright m)
\]

for all $m \in M$, $r \in R$, $s \in S$ and $f, g \in \text{Hom}_\mathbb{Z}(M, N)$. This gives $\text{Hom}_\mathbb{Z}(M, N)$ the structure of an $(R, S)$-bimodule.

6. If $\phi : R \to S$ is a ring homomorphism, then every $S$-module $M$ becomes an $R$-module with structure map $\triangleright_R : R \times M \to M$, $r \triangleright m = \phi(r) \triangleright_S m$. This is called the pullback of the module structure along $\phi$.

In particular, there is a unique ring homomorphism $\phi : \mathbb{Z} \to S$ given by $\phi(0) = 0_S$ and $\phi(1) = 1_S$. The induced $\mathbb{Z}$-module structure on $M$ is precisely its abelian group structure.

7. Every $R$-module $M$ is a module over the endomorphism ring $\text{End}_R(M) = \text{Hom}_R(M, M)$ with the evaluation map $\triangleright : \text{End}_R(M) \times M \to M$, $(f, m) \mapsto f(m)$.

8. If $R$ is an algebra over a field $\mathbb{F}$, then an $R$-module $M$ is a representation of $R$: an $\mathbb{F}$-vector space $M$ together with an algebra homomorphism $\rho : R \to \text{End}_\mathbb{F}(M)$.

Morphisms of $R$-modules are homomorphisms of representations: $\mathbb{F}$-linear maps $\phi : M \to N$ with $\rho_N(r) \circ \phi = \phi \circ \rho_M(r)$ for all $r \in R$.

The scalar multiplication on $M$ is given by $\lambda m = (\lambda 1_R) \triangleright m$ for $\lambda \in \mathbb{F}$ and the algebra homomorphism $\rho : R \to \text{End}_\mathbb{F}(M)$ by $\rho(r)m = r \triangleright m$ for all $r \in R$ and $m \in M$. It is $\mathbb{F}$-linear since $\rho(\lambda r)m = (\lambda r) \triangleright m = (\lambda 1_R) \triangleright (r \triangleright m) = \lambda (r \triangleright m)$ for $\lambda \in \mathbb{F}$. Conversely, every algebra homomorphism $\rho : R \to \text{End}_\mathbb{F}(M)$ defines an $R$-module structure on $M$ by $r \triangleright m = \rho(r)m$ for all $r \in R$ and $m \in M$.

The concept of algebra representations is important, since it allows one to describe algebras in terms of vector spaces and linear maps and to use techniques from linear algebra to understand their structure. There is an analogous concept of group representations.
Definition 1.2.4: Let $G$ be a group.

1. A representation of $G$ over a field $\mathbb{F}$ is a vector space $M$ over $\mathbb{F}$ together with a group homomorphism $\rho : G \rightarrow \text{Aut}_\mathbb{F}(M)$ into the group of linear automorphisms of $M$.

2. A homomorphism of group representations from $(M, \rho_M)$ to $(N, \rho_N)$ is an $\mathbb{F}$-linear map $\phi : M \rightarrow N$ with $\rho_N(g) \circ \phi = \phi \circ \rho_M(g)$ for all $g \in G$.

Group representations are important for the same reasons as representations of algebras, namely that they allow one to investigate groups with methods from Linear Algebra. It is therefore desirable to also incorporate group representations in the picture and to view them as modules over suitable rings. The relevant rings are the so-called group rings.

Lemma 1.2.5: Let $G$ be a group with unit $e$, $R$ a ring and $R[G]$ the set of maps $f : G \rightarrow R$ with $f(g) = 0$ for almost all $g \in G$. Then $R[G]$ is a ring with the pointwise addition of maps and the convolution product $\star : R[G] \times R[G] \rightarrow R[G]$

$$f_1 \star f_2(g) = \sum_{g_1 \cdot g_2 = g} f_1(g_1) \cdot f_2(g_2) = \sum_{h \in G} f_1(h) \cdot f_2(h^{-1} \cdot g).$$

The unit element is the map $\delta_e : G \rightarrow R$ with $\delta_e(e) = 1_R$ and $\delta_e(g) = 0$ for $g \neq e$.

The ring $R[G]$ is called the group ring of $G$ over $R$. If $R = \mathbb{F}$ is a field, then $\mathbb{F}[G]$ is an algebra over $\mathbb{F}$ with the pointwise scalar multiplication and called the group algebra of $G$.

Proof: Exercise. \qed

Remark 1.2.6: Every map $f : G \rightarrow R$ with $f(g) = 0$ for almost all $g \in G$ can be expressed uniquely as a finite linear combination $f = \sum_{g \in G} f(g) \delta_g$, where $\delta_g : G \rightarrow R$ are the maps with $\delta_g(g) = 1_R$ and $\delta_g(h) = 0_R$ for $g \neq h$. Their convolution product takes the form $\delta_g \star \delta_h = \delta_{gh}$.

With the notation $f = \sum_{g \in G} r_g \delta_g$ instead of $f = \sum_{g \in G} r_g \delta_g$, one then has

$$(\sum_{g \in G} r_g \delta_g) \star (\sum_{h \in G} s_h \delta_h) = \sum_{g,h \in G} r_g s_h \delta_{gh}.$$}

The notion of the group ring allows us to view group representations over a field $\mathbb{F}$ as modules over the group algebra $\mathbb{F}[G]$. In this case, the algebra homomorphism $\rho : \mathbb{F}[G] \rightarrow \text{End}_\mathbb{F}(M)$ from Example 1.2.3 restricts to a group homomorphism $\rho : G \rightarrow \text{Aut}_\mathbb{F}(M)$. This follows because the elements $g \in \mathbb{F}[G]$ have multiplicative inverses and $\rho(g^{-1}) \circ \rho(g) = \rho(g^{-1} \cdot g) = \rho(e) = \text{id}_M$.

Example 1.2.7: Let $G$ be a group and $\mathbb{F}$ a field. Then modules over the group algebra $\mathbb{F}[G]$ are the representations of $G$ and homomorphisms of $\mathbb{F}[G]$-modules are the homomorphisms of group representations.

After unifying known algebraic concepts into the notion of a module over a ring, we now generalise the basic constructions for vector spaces - linear subspaces, quotients, direct sums, products and tensor products - to this setting. This leads to the notions of submodules, quotients, direct sums, products and tensor products of modules, which are direct analogues of the corresponding concepts for vector spaces.
Definition 1.2.8: Let $R$ be a ring and $M$ a module over $R$. A submodule of $M$ is a subgroup $N \subseteq M$ that is closed under the operation of $R$: $r \triangleright n \in N$ for all $r \in R$ and $n \in N$.

Example 1.2.9:

1. For any $R$-module $M$, the trivial module $\{0\} \subseteq M$ and $M \subseteq M$ are submodules. All other submodules are called proper submodules.

2. For any module morphism $\phi : M \to N$, the kernel $\ker(\phi) = \{m \in M \mid \phi(m) = 0\} \subseteq M$ and the image $\im(\phi) = \{\phi(m) \mid m \in M\} \subseteq N$ are submodules.

3. If $M$ is an abelian group, i.e. a $\mathbb{Z}$-module, then submodules of $M$ are precisely the subgroups of $M$.

4. Submodules of modules over a field $F$ are linear subspaces.

5. Submodules of a ring $R$ as a left (right) module over itself are its left (right) ideals.

With the notion of a submodule, we can also generalise the notion of a quotient to modules. Any submodule $N \subseteq M$ of an $R$-module $M$ is a subgroup of the abelian group $M$. Consequently, the factor group $M/N$, whose elements are the cosets $mN = \{m + n \mid n \in N\}$, is an abelian group with addition $(mN) + (m'N) = (m+m')N$, and the canonical surjection $\pi : M \to M/N$, $m \mapsto mN$ is a group homomorphism. It is then natural to define an $R$-module structure on $M/N$ in such a way that the canonical surjection becomes an $R$-module morphism, i.e. to set $r \triangleright (mN) := (r \triangleright m)N$ for all $r \in R$ and $m \in M$.

Definition 1.2.10: Let $M$ be a module over a ring $R$ and $N \subseteq M$ a submodule. The quotient module $M/N$ is the factor group $M/N$ with the canonical $R$-module structure

$$\triangleright : R \times M/N \to M/N, \quad r \triangleright (mN) \mapsto (r \triangleright m)N.$$ 

Remark 1.2.11:

1. The quotient module structure on $M/N$ is the unique $R$-module structure on the abelian group $M/N$ with the following universal property:

The canonical surjection $\pi : M \to M/N$ is an $R$-module morphism. For any module morphism $\phi : M \to M'$ with $N \subseteq \ker(\phi)$, there is a unique module morphism $\bar{\phi} : M/N \to M'$ such that the following diagram commutes

$$\begin{array}{ccc}
M & \xrightarrow{\phi} & M' \\
\pi \downarrow & & \downarrow \exists \bar{\phi} \\
M/N & & 
\end{array}$$

2. If $\phi : M \to N$ is a morphism of $R$-modules, then we have a canonical isomorphism of $R$-modules $\phi : M/\ker(\phi) \to \im(\phi)$, $m\ker(\phi) \mapsto \phi(m)$.

3. If $M$ is a module over $R$ with submodules $U \subseteq V \subseteq M$ then $V/U$ is a submodule of $M/U$, and there is a canonical $R$-module isomorphism $(M/U)/(V/U) \to M/V$. 

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The concepts of a direct sum and a product of vector spaces also have direct generalisations to modules. This follows because their construction does not make use of the fact that a field is commutative and of the existence of multiplicative inverses. By replacing the scalar multiplication by the structure map of a module, we obtain the generalisations of these concepts for modules over general rings.

**Definition 1.2.12:** Let $R$ be a ring and $(M_i)_{i \in I}$ a family of modules over $R$ indexed by a set $I$. Then the **direct sum** $\bigoplus_{i \in I} M_i$ and the **product** $\prod_{i \in I} M_i$ are the sets

\[
\bigoplus_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i, m_i = 0 \text{ for almost all } i \in I\}
\]

\[
\prod_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i\}
\]

with the $R$-module structures given by

\[(m_i)_{i \in I} + (m'_i)_{i \in I} := (m_i + m'_i)_{i \in I}\]

\[r \cdot (m_i)_{i \in I} := (r \cdot m_i)_{i \in I}\]

**Lemma 1.2.13:** The direct product and the direct sum of modules are products and coproducts in the category $R$-Mod. More precisely:

1. **Universal property of direct sums:**
   The direct sum module structure is the unique $R$-module structure on $\bigoplus_{i \in I} M_i$ for which all inclusions $\iota_i : M_i \to M$, $m \mapsto (\delta_j m)_{j \in I}$ are module morphisms.

   For a family $(\phi_i)_{i \in I}$ of module morphisms $\phi_i : M_i \to N$ there is a unique module morphism $\phi : \bigoplus_{i \in I} M_i \to N$ such that the following diagram commutes for all $i \in I$

   \[
   \begin{array}{c}
   M_i \\
   \downarrow \phi_i \\
   \bigoplus_{j \in I} M_j \\
   \end{array}
   \]

   \[
   \begin{array}{c}
   N \\
   \downarrow \exists \phi \\
   \bigoplus_{j \in I} M_j \\
   \end{array}
   \]

2. **Universal property of products:**
   The product module structure is the unique $R$-module structure on $\prod_{i \in I} M_i$ for which all projection maps $\pi_i : \prod_{i \in I} M_i \to M_i$, $(m_j)_{j \in I} \mapsto m_i$ are module morphisms.

   For a family $(\psi_i)_{i \in I}$ of module morphisms $\psi_i : L \to M_i$ there is a unique module morphism $\psi : L \to \prod_{i \in I} M_i$ such that the following diagram commutes for all $i \in I$

   \[
   \begin{array}{c}
   M_i \\
   \downarrow \pi_i \\
   \prod_{j \in I} M_j \\
   \end{array}
   \]

   \[
   \begin{array}{c}
   L \\
   \downarrow \exists \psi \\
   \prod_{j \in I} M_j \\
   \end{array}
   \]

While the four basic constructions for modules are straightforward generalisations of the corresponding constructions for vector spaces, there is a fundamental difference between vector spaces and modules over general rings, namely the existence of bases and of complements. While every vector space has a basis and every linear subspace has a complement, this does not hold for general modules. Although there are always generating sets, there need not be a *linearly independent* generating set. In contrast to vector spaces general modules can therefore not be described in terms of bases but are characterised by presentations.
Definition 1.2.14: Let $R$ be a ring, $M$ an $R$-module and $A \subset M$ a subset.

1. The **submodule** $\langle A \rangle_M$ **generated by** $A$ is the smallest submodule of $M$ containing $A$

   $$\langle A \rangle_M = \bigcap_{N \subset M \text{ submodule, } A \subset N} N = \{ \Sigma_{a \in A} r_a \triangleright a : r_a \in R, r_a = 0 \text{ for almost all } a \in A \}.$$**

2. The subset $A \subset M$ is called a **generating set** of $M$ if $\langle A \rangle_M = M$. It is called a **basis** of $M$ if it is a generating set and **linearly independent**: $\Sigma_{a \in A} r_a \triangleright a = 0$ with $r_a \in R$ and $r_a = 0$ for almost all $a \in A$ implies $r_a = 0$ for all $a \in A$.

3. An $R$-module with a finite generating set is called **finitely generated**. An $R$-module with a generating set that contains only one element is called **cyclic**. An $R$-module with a basis is called **free**.

4. The **free** $R$-module generated by a set $A$ is the direct sum $\langle A \rangle_R = \oplus_{a \in A} R$. Equivalently, it can be characterised as the set $\langle A \rangle_R = \{ f : A \to R : f(a) = 0 \text{ for almost all } a \in A \}$ with the canonical $R$-module structure

   $$(f + g)(a) = f(a) + g(a) \quad (r \triangleright f)(a) = r \cdot f(a) \quad \forall f, g \in \langle A \rangle_R, r \in R, a \in A.$$**

   The maps $\delta_a : A \to R$ with $\delta_a(a) = 1_R$ and $\delta_a(a') = 0_R$ for $a' \neq a$ form a basis of $\langle A \rangle_R$, since every map $f : A \to R$ with $f(a) = 0$ for almost all $a \in A$ can be expressed as a finite $R$-linear combination $f = \sum_{a \in A} f(a) \triangleright \delta_a$.

5. For a subset $B \subset \langle A \rangle_R$, we denote by $\langle A|B\rangle_R$ the quotient module

   $$\langle A|B\rangle_R = \langle A \rangle_R / \langle B \rangle_{\langle A \rangle_R}$$

   of the free $R$-module generated by $A$ with respect to its submodule generated by $B$. If $M = \langle A|B\rangle_R$, then $\langle A|B\rangle_R$ is called a **presentation** of $M$, the elements of $A$ are called **generators** and the elements of $B$ **relations**.

Remark 1.2.15:

1. Every module has a presentation $M = \langle A|B\rangle_R$. One can choose $A = M$ and $B = \ker(\tau)$ for the $R$-module homomorphism $\tau : \langle M \rangle_R \to M$, $\Sigma_{m \in M} r_m \triangleright \delta_m \mapsto \Sigma_{m \in M} r_m \triangleright m$. However, this presentation is not very useful in practice. One usually looks for presentations that have as few generators and relations as possible.

2. Presentations of modules are characterised by a **universal property**:

   For any $R$-module $M$ and any map $\phi : A \to M$, there is a unique $R$-module homomorphism $\phi' : \langle A \rangle_R \to M$ with $\phi'|_A = \phi$. If $B \subset \ker(\phi')$, then by the universal property of the quotient there is a unique map $\phi'' : \langle A|B\rangle_R \to M$ with $\phi'' \circ \pi = \phi'$, where $\pi : \langle A \rangle_R \to \langle A|B\rangle_R$ is the canonical surjection.

3. If $R$ is a commutative ring and $M$ a free $R$-module, then all bases of $M$ have the same cardinality. This number is called **rank** of $M$ and denoted $\text{rk}(M)$. This notion makes no sense for non-commutative rings since one can have $R^n \cong R^m$ as $R$-modules for $n \neq m$. 

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Example 1.2.16:

1. Every ring $R$ is a cyclic free module as a left or right module over itself: $R = \langle 1_R \rangle_R$.

2. If $F$ is a field, then every module over $F$ is free, since a module over $F$ is a vector space, and every vector space has a basis. The cyclic $F$-modules are precisely the one-dimensional vector spaces over $F$.

3. If $M$ is a free module over a principal ideal domain $R$, then every submodule $U \subset M$ is free with $\text{rk}(U) \leq \text{rk}(M)$. (For a proof, see [JS].)

4. The $\mathbb{Z}$-module $M = \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$ cyclic, since $\{1\}$ is a generating set, but it is not free. Any generating set of $\mathbb{Z}/n\mathbb{Z}$ must contain at least one element $\bar{k} \neq \bar{0}$, but $n \bar{1} = \bar{k} + \ldots + \bar{k} = n \bar{k} = \bar{0}$, and hence the generating set cannot be linearly independent. A presentation of the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is given by $\langle A \mid B \rangle_\mathbb{Z} = \langle 1 \mid n \rangle$.

For vector spaces, an important consequence of the existence of bases is the existence of a complement for any linear subspace $U \subset V$ - a linear subspace $W \subset V$ with $V = U \oplus W$. Such a complement can be constructed by completing a basis of $U$ to a basis of $V$ and taking $W$ as the span of those basis elements that are not contained in the basis of $U$. As modules over general rings do not need to have bases, this construction does not generalise to rings.

Indeed, there are many examples of submodules without complements. Consider for instance the submodule $n\mathbb{Z} \subset \mathbb{Z}$ of the ring $\mathbb{Z}$ as a module over itself for $n \geq 2$. As $1 \notin n\mathbb{Z}$, any complement $M$ of $n\mathbb{Z}$ would need to contain the element $1 \in \mathbb{Z}$ and hence be equal to $\mathbb{Z}$ since $1 \in M$ implies $n = n \cdot 1 \in M$ for all $n \in \mathbb{Z}$. The same argument shows that a proper submodule of a cyclic module can never have a complement. In many applications, one needs practical criteria to determine if a given submodule has a complement. A sufficient one is the following.

Lemma 1.2.17: Let $R$ be a ring and $M$ a module over $R$.

1. If $\phi : M \to F$ is a surjective $R$-module morphism into a free $R$-module $F$, then there is an $R$-module morphism $\psi : F \to M$ with $\phi \circ \psi = \text{id}_F$ and $M \cong \text{im}(\psi) \oplus \ker(\phi)$.
   One says that $\psi$ splits the module morphism $\phi : M \to F$.

2. If $N \subset M$ is a submodule such that $M/N$ is free, then there is a submodule $P \subset M$ with $P \cong M/N$ and $M \cong N \oplus P$.

Proof:

1. Choose a basis $B$ of $F$ and for every $b \in B$ an element $m_b \in \phi^{-1}(b) \subset M$. Define the $R$-module morphism $\psi : F \to M$ by $\psi(b) = m_b$ and $R$-linear extension to $F$. Then we have $m = \psi \circ \phi(m) + (m - \psi \circ \phi(m))$ for all $m \in M$ with $\psi \circ \phi(m) \in \text{im}(\psi)$ and $m - \phi \circ \psi(m) \in \ker(\phi)$, since $\phi \circ \psi = \text{id}_F$ implies $\phi(m - \psi \circ \phi(m)) = \phi(m) - (\phi \circ \psi)(\phi(m)) = \phi(m) - \phi(m) = 0$. As $\phi \circ \psi = \text{id}_F$, we have $\ker(\phi) \cap \text{im}(\psi) = \{0\}$ and hence $M = \ker(\phi) \oplus \text{im}(\psi)$.

2. By 1. there is a $R$-module morphism $\psi : M/N \to M$ which splits the surjective module morphism $\pi : M \to M/N$ and hence $M \cong \ker(\pi) \oplus \text{im}(\psi) \cong N \oplus \text{im}(\psi)$. The $R$-module morphism $\pi|_{\text{im}(\psi)} : \text{im}(\psi) \to M/N$ is surjective by definition and injective since $\pi \circ \psi = \text{id}_{M/N}$, hence an isomorphism. \qed
The fact that there are $R$-modules $M$ without bases is closely related to the presence of elements $m \in M$ for which there is an $r \in R \setminus \{0\}$ with $r \cdot m = 0$, the so-called torsion elements. It is clear that an element of a basis can never be a torsion element, and under certain assumptions on the ring, this holds for all non-zero elements of a free module.

**Definition 1.2.18:** Let $R$ be a ring and $M$ an $R$-module. An element $m \in M$ is called a **torsion element** if there is an $r \in R \setminus \{0\}$ with $r \cdot m = 0$. The set of torsion elements in $M$ is denoted $\text{Tor}_R(M)$. The $R$-module $M$ is called **torsion free** if $\text{Tor}_R(M) = 0$.

**Example 1.2.19:**

1. Any free module $M$ over an integral domain $R$ is torsion free.
   This follows because every torsion element $m \in M$ can be expressed as a finite linear combination $m = \sum_{i \in I} r_i \cdot m_i$ of basis elements $m_i$. The condition $r \cdot m = \sum_{i \in I} r(r_i) \cdot m_i = 0$ for $r \in R \setminus \{0\}$ then implies $r r_i = 0$ for all $i \in I$. Because $R$ has no zero divisors and $r \in R \setminus \{0\}$, it follows that $r_i = 0$ for all $i \in I$ and hence $m = 0$.

2. For a commutative ring $k$ as a module over itself, the torsion elements are precisely the zero divisors of $k$. This implies that every integral domain $R$ as a module over itself is torsion free. In particular, this holds for $\mathbb{Z}$, for any field $\mathbb{F}$ and for the ring $I[X]$ of polynomials over any integral domain $I$.

3. In the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$, every element is a torsion element since $n \cdot \bar{k} = \bar{n} \cdot \bar{k} = \bar{0}$ for all $k \in \mathbb{Z}$. The ring $\mathbb{Z}/n\mathbb{Z}$ as a module over itself is torsion free if and only if $n$ is a prime:

   $\text{Tor}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ \quad $\text{Tor}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = \{\bar{k} \mid k \in \mathbb{Z}, \gcd(k, n) > 1\}$.

One may expect that the set of torsion elements in an $R$-module $M$ is a submodule of $M$. However, this need not hold in general if $R$ is non-commutative or has zero divisors. However, if $R$ is an integral domain, the torsion elements form a submodule and by taking the quotient with respect to this submodule, one obtains a module that is torsion free.

**Lemma 1.2.20:** If $M$ is a module over an integral domain $R$ then $\text{Tor}_R(M) \subset M$ is a submodule and the module $M/\text{Tor}_R(M)$ is torsion free.

**Proof:**

Let $m, m' \in \text{Tor}_R(M)$ torsion elements and $r, r' \in R \setminus \{0\}$ with $r \cdot m = r' \cdot m' = 0$. Then $(r \cdot r') \cdot (m + m') = r' \cdot (r \cdot m) + r \cdot (r' \cdot m') = 0$. As $R$ is an integral domain, $r \cdot r' \neq 0$ and hence $m + m' \in \text{Tor}_R(M)$. Similarly, for all $s \in R$, one has $s \cdot (s \cdot m) = (r \cdot s) \cdot m = s \cdot (r \cdot m) = 0$, which implies $s \cdot m \in \text{Tor}_R(M)$, and hence $\text{Tor}_R(M) \subset M$ is a submodule. If $[m] \in M/\text{Tor}_R(M)$ is a torsion element, then there is an $r \in R \setminus \{0\}$ with $r \cdot [m] = [r \cdot m] = 0$. This implies $r \cdot m \in \text{Tor}_R(M)$, and there is an $r' \in R \setminus \{0\}$ with $r' \cdot (r \cdot m) = (r' \cdot r) \cdot m = 0$. As $R$ is an integral domain, one has $r \cdot r' \neq 0$, which implies $m \in \text{Tor}_R(M)$ and $[m] = 0$. \hfill $\Box$

Since $\text{Tor}_R(M) \subset M$ is a submodule and $M/\text{Tor}_R(M)$ is torsion free for any integral domain $R$, it is natural to ask if the torsion submodule $\text{Tor}_R(M)$ has a complement, an $R$-module $N$ with $M \cong \text{Tor}_R(M) \oplus N$. A sufficient condition that ensures the existence of such a complement is that $R$ is a principal ideal domain and $M$ is finitely generated. In this case, the classification
theorem for finitely generated modules over principal ideal domains allows one to identify the torsion elements and their complement. In particular, this applies to finitely generated abelian groups, the finitely generated modules over the principal ideal domain \( \mathbb{Z} \).

**Lemma 1.2.21:** Let \( R \) be a principal ideal domain. Then every finitely generated \( R \)-module \( M \) is of the form \( M \cong \text{Tor}_R(M) \oplus R^n \) with a unique \( n \in \mathbb{N}_0 \). In particular, every finitely generated torsion free \( R \)-module is free.

**Proof:**
This follows from the classification theorem for finitely generated modules over principal ideal domain, which states that every such module is of the form \( M \cong R^n \times R/q_1R \times \ldots \times R/q_mR \) with prime powers \( q_1, \ldots, q_m \in \mathbb{R} \) and \( n \in \mathbb{N}_0 \). Every element \( m \in R/q_1R \times \ldots \times R/q_lR \) is a torsion element since \( (q_1 \cdots q_m) \cdot m = 0 \). This shows that \( \text{Tor}_R(M) \cong R/q_1R \times \ldots \times R/q_mR \).

Conversely, any element \( m \in R^n \times R/q_1R \times \ldots \times R/q_mR \) is a sum \( m = \nu_1(m_1) + \nu_2(m_2) \) with \( m_1 \in R^n \) and \( m_2 \in R/q_1R \times \ldots \times R/q_mR \), where \( \nu_1 : R^n \to R^n \times R/q_1R \times \ldots \times R/q_mR \) and \( \nu_2 : R/q_1R \times \ldots \times R/q_mR \to R^n \times R/q_1R \times \ldots \times R/q_mR \) denote the inclusion maps for the direct sum. Then \( 0 = r \cdot (\nu_1(m_1) + \nu_2(m_2)) = \nu_1(r \cdot m_1) + \nu_2(r \cdot m_2) \) for \( r \in R \setminus \{0\} \) implies \( r \cdot m_1 = 0 \) and \( r \cdot m_2 = 0 \). As \( R^n \) is a free \( R \)-module, it is torsion free by Example 4.4.7. 1. and the first condition implies \( m_1 = 0 \). This shows that \( \text{Tor}_R(M) \subset R/q_1R \times \ldots \times R/q_mR \).

\( \square \)

After generalising four basic constructions for vector spaces to modules over rings and discussing the existence of bases, we will now focus on the last essential construction, namely tensor products. The construction of the tensor product for modules over rings is similar to the one for vector spaces. It is obtained as a quotient of a free module generated by the cartesian products of the underlying sets with respect to certain relations. However, if \( R \) is non-commutative, we have to consider a left and a right module over a ring \( R \) to form a tensor product, and the result is not an \( R \)-module but only an abelian group.

**Definition 1.2.22:** Let \( R \) be a ring, \( M \) an \( R \)-right module and \( N \) an \( R \)-left module. The tensor product \( M \otimes_R N \) is the abelian group generated by the set \( M \times N \) with relations

\[
(m, n) + (m', n) - (m + m', n), \quad (m, n) + (m, n') - (m, n + n'), \\
(m \cdot r, n) - (m, r \cdot n) \quad \forall m, m' \in M, n, n' \in N, r \in R.
\]

We denote by \( m \otimes n = \tau(m, n) \) the images of the elements \((m, n) \in M \times N \) under the map \( \tau = \pi \circ \iota : M \times N \to M \otimes_R N \), where \( \iota : M \times N \to (M \times N)_\mathbb{Z} \), \((m, n) \to (m, n)\) is the canonical inclusion and \( \pi : (M \times N)_\mathbb{Z} \to M \otimes_R N \) the canonical surjection.

**Remark 1.2.23:**

1. The set \( \{m \otimes n : m \in M, n \in N\} \) generates \( M \otimes_R N \), since the elements \((m, n) \) generate the free \( \mathbb{Z} \)-module \((M \times N)_\mathbb{Z}\) and the \( \mathbb{Z} \)-module morphism \( \pi : (M \times N)_\mathbb{Z} \to M \otimes_R N \) is surjective. The relations in Definition 1.2.22 induce the following identities in \( M \otimes_R N \):

\[
(m + m') \otimes n = m \otimes n + m' \otimes n, \quad m \otimes (n + n') = m \otimes n + m \otimes n', \\
(m \cdot r) \otimes n = m \otimes (r \cdot n) \quad \forall m, m' \in M, n, n' \in N, r \in R.
\]

\(^{1}\)Note that the product under consideration is a product over a finite index set and hence by Definition 1.2.12 it coincides with the direct sum.
2. If $M$ is an $R$-right-module and $N$ an $(R,S)$-bimodule, then $M \otimes_R N$ has a canonical $S$-right module structure given by $(m \otimes n) \triangleleft s := m \otimes (n \triangleleft s)$. Similarly, if $M$ is a $(Q,R)$-bimodule and $N$ an $R$-left module then $M \otimes_R N$ has a canonical $Q$-left module structure given by $q \triangleright (m \otimes n) := (q \triangleright m) \otimes n$.

3. As every left module over a commutative ring $R$ is an $(R,R)$-bimodule, it follows from 2. that the tensor product $M \otimes_R N$ of modules over a commutative ring $R$ has a canonical $(R,R)$-bimodule structure, given by

\[ r \triangleright (m \otimes n) = (r \triangleright m) \otimes n = (m \triangleleft r) \otimes n = m \otimes (r \triangleright n) = m \otimes (n \triangleleft r) = (m \otimes n) \triangleleft r. \]

4. As every module over a ring $R$ is an abelian group and hence a $(Z,Z)$-bimodule, it is always possible to tensor two $R$-modules over the ring $Z$. In this case, the last relation in Definition 1.2.22 is a consequence of the first two.

**Example 1.2.24:**

1. If $R = \mathbb{F}$ is a field, the tensor product of $R$-modules is the tensor product of vector spaces.
2. For any ring $R$ and $R^k := R \oplus R \oplus \ldots \oplus R$, one has $R^m \otimes R^n \cong R^{mn}$ (Exercise).
3. If $R$ is commutative and $R[X,Y]$ the polynomial ring over $R$ in two variables $X,Y$, then $R[X] \otimes_R R[Y] \cong R[X,Y]$.
4. The tensor product of the abelian groups $Z/nZ$ and $Z/mZ$ for $n,m \in \mathbb{N}$ is given by

\[ Z/nZ \otimes_Z Z/mZ \cong Z/\gcd(m,n)Z. \]

5. One has $Z/nZ \otimes_Z Q \cong 0$. More generally, if $R$ is an integral domain with associated quotient field $Q(R)$ and $M$ an $R$-module, then $\text{Tor}_R(M) \otimes_R Q(R) \cong 0$. This follows because for every torsion element $m \in M$ there is an $r \in R \setminus \{0\}$ with $r \triangleright m = 0$, and this implies

\[ m \otimes q = m \otimes (r \cdot q/r) = m \otimes (r \triangleright (q/r)) = (m \triangleleft r) \otimes q/r = 0 \otimes q/r = 0 \quad \forall q \in Q. \]

Just as submodules, quotients, direct sums and products of modules, tensor products of $R$-modules can be characterised by a universal property. As tensor products are defined in terms of a presentation, this universal property is obtained by applying the one in Remark 1.2.15 to the relations in Definition 1.2.22. The special form of these relations allows one to characterise the universal property in terms of bilinear maps $M \times N \to A$ into abelian groups $A$.

**Definition 1.2.25:** Let $R$ be a ring, $M$ an $R$-right module and $N$ an $R$-left module. A map $f : M \times N \to A$ into an abelian group $A$ is called $R$-bilinear if

\[
\begin{align*}
    f(m + m', n) & = f(m, n) + f(m', n), \\
    f(m \triangleleft r, n) & = f(m, r \triangleright n) \\
    f(m, n + n') & = f(m, n) + f(m, n'), \\
    f(m, n + n') & = f(m, n) + f(m, n'), \\
    \forall m, m' \in M, n, n' \in N, r \in R.
\end{align*}
\]

**Lemma 1.2.26:** Let $R$ be a ring, $M$ an $R$-right module and $N$ an $R$-left module. Then the tensor product $M \otimes_R N$ has the following universal property:
The map \( \tau : M \times N \to M \otimes_R N \), \((m, n) \mapsto m \otimes n\) is \( R \)-bilinear, and for any \( R \)-bilinear map \( f : M \times N \to A \), there is a unique group homomorphism \( f' : M \otimes_R N \to A \) with \( f' \circ \tau = f \):

\[
M \times N \xrightarrow{f} A \\
M \otimes_R N.
\]

**Proof:**

The first statement holds by definition, since the conditions in Definition 1.2.25 are the defining relations of the tensor product. To define \( f' \), note that \((M \times N)_Z\) is a free abelian group, and hence there is a unique group homomorphism \( f'' : (M \times N)_Z \to A \) with \( f''|_{M \times N} = f \). This is equivalent to the condition \( f'' \circ \iota = f \), where \( \iota : M \times N \to (M \times N)_Z \) is the canonical inclusion. The submodule \( U \subseteq (M \times n)_Z \) spanned by the relations of the tensor product is contained in the kernel of \( f'' \) by \( R \)-bilinearity of \( f \):

\[
f''((m + m', n) - (m, n) - (m', n)) = f(m + m', n) - f(m, n) - f(m', n) = 0
\]

and hence there is a unique group homomorphism \( f' : M \otimes_R N \to A \) with \( f' \circ \pi = f'' \), where \( \pi : (M \times N)_Z \to M \otimes_R N \) is the canonical surjection. This implies \( f' \circ \tau = f' \circ \pi \circ \iota = f'' \circ \iota = f \).

The uniqueness of \( f' \) follows directly from the fact that \( \tau \) is surjective. \( \Box \)

As the construction of the tensor product of modules is very similar to the tensor product of vector spaces, it has a number of properties that generalise the properties of tensor products of vector spaces. They are direct consequences of its definition and its universal property.

**Lemma 1.2.27:** Let \( R, S \) be rings, \( I \) an index set, \( M, M_i \) \( R \)-right modules, \( N, N_i \) \( R \)-left modules for all \( i \in I \), \( P \) a \((R, S)\)-bimodule and \( Q \) an \( S \)-left module. Then:

1. **tensor products with the trivial module:** \( 0 \otimes_R N \cong M \otimes_R 0 \cong 0 \),
2. **tensor product with the underlying ring:** \( M \otimes_R R \cong M, R \otimes_R N \cong N \),
3. **direct sums:** \( (\oplus_{i \in I} M_i) \otimes_R N \cong \oplus_{i \in I} M_i \otimes_R N, M \otimes_R (\oplus_{i \in I} N_i) \cong \oplus_{i \in I} M \otimes_R N_i \),
4. **associativity:** \( (M \otimes_R P) \otimes_S Q \cong M \otimes_R (P \otimes_S Q) \).

**Proof:**

1. This follows directly from the relations of the tensor product in Remark 1.2.23 which imply \( 0 \otimes n = (0 \otimes 0) \otimes n = 0 \otimes 0 \otimes n = 0 \otimes 0 \otimes 0 = 0 \) for all \( n \in N \) and \( m \otimes 0 = 0 \) for all \( m \in M \).

2. We consider the group homomorphism \( \phi : M \to M \otimes_R R, m \mapsto m \otimes 1 \). The group homomorphism \( \psi : M \otimes_R R \to M, m \otimes r \mapsto m \otimes r \) is an inverse of \( \phi \), since \( \psi \circ \phi(m) = m \otimes 1 = m \) and \( \phi \circ \psi(m \otimes r) = (m \otimes r) \otimes 1 = m \otimes (r \otimes 1) = m \otimes r \). The proof for \( R \otimes_R N \cong N \) is analogous.

3. Consider the group homomorphisms \( \phi_i : M_i \otimes_R N \to (\oplus_{i \in I} M_i) \otimes_R N, \phi_i(m_i \otimes n) = i(m_i) \otimes n, \) where \( i_i : M_i \to \oplus_{i \in I} M_i \) is the canonical inclusion. By the universal property of the direct sum this defines a unique group homomorphism \( \phi : \oplus_{i \in I} M_i \otimes_R N \to (\oplus_{i \in I} M_i) \otimes_R N \) with \( \phi \circ \iota' = \phi_i \) for the inclusion maps \( \iota_i : M_i \otimes_R N \to \oplus_{i \in I} M_i \otimes_R N \). This group homomorphism has an inverse \( \psi : (\oplus_{i \in I} M_i) \otimes_R N \to \oplus_{i \in I} M_i \otimes_R N \) given by \( \psi(\iota_i(m_i) \otimes n) = i_i'(m_i \otimes n) \) and hence is an isomorphism. The proof for the other identity is analogous.
4. A group isomorphism \( \phi : (M \otimes_R P) \otimes_S Q \rightarrow M \otimes_R (P \otimes_S Q) \) is given by
\[
\phi((m \otimes p) \otimes q) = m \otimes (p \otimes q) \quad \forall m \in M, p \in P, q \in Q.
\]
\[\square\]

It remains to investigate the interaction of tensor products over \( R \) with \( R \)-linear maps. One finds a similar pattern as for the tensor product of vector spaces. The universal property of the tensor product over \( R \) allows one to form the product \( \phi \otimes \psi : M \otimes_R N \rightarrow M' \otimes_R N' \) of an \( R \)-left module morphism \( \phi : M \rightarrow M' \) and an \( R \)-right module morphism \( \psi : N \rightarrow N' \). The result is a morphism of abelian groups. As this construction is compatible with the identity maps and the composition of module morphisms, we obtain a functor \( \otimes : \text{Mod-}R \times \text{R-Mod} \rightarrow \text{Ab} \).

**Theorem 1.2.28:** Let \( R \) be a ring.

1. The tensor product of \( R \)-modules defines a functor \( \otimes : \text{Mod-}R \times \text{R-Mod} \rightarrow \text{Ab} \) that assigns to an element \((M, N) \in \text{Mod-}R \times \text{R-Mod}\) the abelian group \( M \otimes_R N \) and to a pair of module morphisms \((\phi, \psi) : (M, N) \rightarrow (M', N')\) the unique group homomorphism \( \phi \otimes \psi : M \otimes_R N \rightarrow M' \otimes_R N' \) for which the following diagram commutes
\[
\begin{array}{ccc}
M \times N & \xrightarrow{\phi \times \psi} & M' \times N' \\
\downarrow{\tau} & & \downarrow{\tau'} \\
M \otimes_R N & \xrightarrow{\phi \otimes \psi} & M' \otimes_R N'.
\end{array}
\]

2. For each \( R \)-right module \( M \), this defines an functor \( M \otimes - : \text{R-Mod} \rightarrow \text{Ab} \) and for each \( R \)-left module \( N \) a functor \(- \otimes N : \text{Mod-}R \rightarrow \text{Ab}\).

3. If \( R \) is commutative, this defines a functor \( \otimes : \text{R-Mod} \times \text{R-Mod} \rightarrow \text{R-Mod} \) and functors \( M \otimes - , - \otimes N : \text{R-Mod} \rightarrow \text{R-Mod} \).

**Proof:**
The map \( \tau' \circ (\phi \times \psi) : M \times N \rightarrow M' \otimes_R N' \) is \( R \)-bilinear, and by the universal property of the tensor product there exists a unique group homomorphism \( \phi \otimes \psi : M \otimes_R N \rightarrow M' \otimes_R N' \) with \((\phi \otimes \psi) \circ \tau = \tau' \circ (\phi \times \psi)\), or, equivalently, \((\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)\) for all \( m \in M, n \in N \). That this defines a functor \( \otimes : \text{Mod-}R \times \text{R-Mod} \rightarrow \text{Ab} \) follows from the fact the the following two diagrams commute
\[
\begin{array}{ccc}
M \times N & \xrightarrow{id_M \times id_N} & M \times N \\
\downarrow{\tau} & & \downarrow{\tau'} \\
M \otimes_R N & \xrightarrow{id_{M \otimes_R N}} & M \otimes_R N \\
M \times N & \xrightarrow{\phi \times \psi} & M' \times N' \\
\downarrow{\tau} & & \downarrow{\tau'} \\
M \otimes_R N & \xrightarrow{\phi \otimes \psi} & M' \otimes_R N' \\
M \times N & \xrightarrow{\phi' \times \psi'} & M'' \times N'' \\
\downarrow{\tau} & & \downarrow{\tau'} \\
M \otimes_R N & \xrightarrow{\phi' \otimes \psi'} & M'' \otimes_R N''.
\end{array}
\]
The functor \( M \otimes - : \text{R-Mod} \rightarrow \text{Ab} \) assigns to an \( R \)-left module \( N \) the abelian group \( M \otimes_R N \) and to an \( R \)-linear map \( \psi : N \rightarrow N' \) the group homomorphism \( \text{id}_M \otimes \psi : M \otimes_R N \rightarrow M \otimes_R N' \). The functor \(- \otimes N : \text{Mod-}R \rightarrow \text{Ab} \) assigns to an \( R \)-right module \( M \) the abelian group \( M \otimes_R N \) and to an \( R \)-linear map \( \psi : M \rightarrow M' \) the group homomorphism \( \psi \otimes \text{id}_N : M \otimes_R N \rightarrow M' \otimes_R N \).
\[\square\]
2 Examples of (co)homologies

In this section, we introduce examples of homology and cohomology theories and illustrate how they encode information about different mathematical objects such as topological spaces, simplicial complexes, bimodules over algebras, modules over group rings and representations of Lie algebras. The basic pattern of a homology theory is to associate to the mathematical object a family \((X_n)_{n \in \mathbb{N}_0}\) of modules over a ring and a family \((d_n)_{n \in \mathbb{N}_0}\) of module morphisms \(d : X_n \to X_{n-1}\) that satisfy the condition \(d_n \circ d_{n+1} = 0\) for all \(n \in \mathbb{N}_0\). This ensures that \(\text{im}(d_{n+1}) \subset \ker(d_n) \subset X_n\) are submodules, and one can form the quotient module \(\ker(d_n)/\text{im}(d_{n+1})\). These quotients are the homologies of the mathematical object under consideration and encode relevant information about it.

2.1 Singular and simplicial homologies of topological spaces

Historically, the first homology theories were homology theories of topological spaces. The wish to unify different notions of homology for topological spaces was one of the main motivations to develop an abstract formalism for homologies. The basic idea of topological homology theories is to characterise a topological space in terms of certain standard spaces or building blocks in \(\mathbb{R}^n\) that can be described in a mostly combinatorial way. These are the affine simplexes.

Definition 2.1.1: Let \((e_1, \ldots, e_n)\) be the standard basis of \(\mathbb{R}^n\) and set \(e_0 := 0 \in \mathbb{R}^n\).

1. An affine \(m\)-simplex \(\Delta \subset \mathbb{R}^n\) is the convex hull of \(m + 1\) points \(v_0, \ldots, v_m \in \mathbb{R}^n\)
\[
\Delta = \text{conv}(\{v_0, \ldots, v_m\}) = \{\Sigma_{i=0}^{m} \lambda_i v_i \mid 0 \leq \lambda_i \leq 1, \Sigma_{i=0}^{m} \lambda_i = 1\}.
\]
The \(k\)-simplexes \(\text{conv}(\{v_{i_0}, \ldots, v_{i_k}\})\) for subsets \(\{v_{i_0}, \ldots, v_{i_k}\} \subset \{v_0, \ldots, v_m\}\) with \(k + 1\) elements are called the \(k\)-faces of \(\Delta\). The points \(v_0, \ldots, v_m\) are called the vertices of \(\Delta\).

2. An ordered \(m\)-simplex is an affine \(m\)-simplex with an ordering of its vertices. We write \([v_0, \ldots, v_m]\) for the affine \(m\)-simplex \(\Delta = \text{conv}(v_0, \ldots, v_m)\) with ordering \(v_0 < v_1 < \ldots < v_m\).

3. For \(n \in \mathbb{N}_0\) the standard \(n\)-simplex \(\Delta^n \subset \mathbb{R}^n\) is the ordered \(n\)-simplex \([e_0, \ldots, e_n]\).

4. For \(n \in \mathbb{N}\) and \(i \in \{0, \ldots, n\}\) the \(i\)th face map is the affine linear map \(f_i^n : \Delta^{n-1} \to \Delta^n\)
\[
f_i^n(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i. \end{cases}
\]
that sends \(\Delta^{n-1} = [e_0, \ldots, e_{n-1}]\) to the \((n-1)\)-face \([e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n]\) opposite \(e_i\).

The ordering of an affine \(m\)-simplex is pictured by drawing an arrow on each 1-face that points from its vertex of lower order to its vertex of higher order. Note that the face maps respect the ordering of vertices in the standard \(n\)-simplexes. They omit vertices but do not change their ordering. Hence, the ordering of the vertices in the \((n-1)\)-face \(f_i^n(\Delta^{n-1}) \subset \Delta^n\) induced by the ordering of \(\Delta^{n-1}\) coincides with the one induced by the ordering of \(\Delta^n\).
The standard $n$-simplexes for $n = 0, 1, 2, 3$.

The basic idea of topological homology theories is to probe a topological space $X$ by mapping the standard $n$-simplexes continuously into $X$. To describe $X$ in terms of continuous maps $\sigma : \Delta^n \to X$ for $n \in \mathbb{N}_0$, one must decide which continuous maps $\sigma$ to consider - all of them or only a specific set of continuous maps that satisfy certain compatibility conditions. Different choices lead to different versions of homology. In the following, we focus on two main examples, namely singular and simplicial homology. The former admits all continuous maps $\sigma : \Delta^n \to X$, even very singular ones that map the entire simplex to a single point. The latter is based on collections of maps that are homeomorphisms onto their image when restricted to the interior of the standard $n$-simplex and satisfy certain matching conditions.

**Definition 2.1.2:** Let $k$ be a commutative ring and $X$ a topological space.

1. For $n \in \mathbb{N}_0$ a **singular $n$-simplex** is a continuous map $\sigma : \Delta^n \to X$.
2. The $k$-module $C_n(X, k)$ of **singular $n$-chains** is the free $k$-module generated by the set of singular $n$-simplexes for $n \in \mathbb{N}_0$ and the trivial module for $n < 0$:

   $$C_n(X, k) = \begin{cases} \langle \sigma : \Delta^n \to X \text{ continuous}\rangle_k & n \in \mathbb{N}_0 \\ 0 & n < 0. \end{cases}$$

3. The **singular boundary operator** $d_n : C_n(X, k) \to C_{n-1}(X, k)$ is the $k$-module morphism defined by $d_n = 0$ for $n \leq 0$ and

   $$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ f^n_i \quad \forall \sigma \in \text{Hom}_{\text{Top}}(\Delta^n, X), n \in \mathbb{N}.$$
For a 3-simplex $\sigma : \Delta^3 \to X$ the sign in front of the term $\sigma \circ f^3_i$ is given by the right hand rule. If one equips each 2-face of $\Delta^3$ with the orientation defined above and the fingers of the right hand follow this orientation, then the sign is $+1$ if the thumb of the right hand points out of $\Delta^3$ and $-1$ if it points inside $\Delta^3$.

The boundary operator is called boundary operator because it assigns to a singular $n$-simplex $\sigma : \Delta^n \to X$ the alternating sum of the singular $(n - 1)$-simplexes $\sigma \circ f^n_i : \Delta^{n-1} \to X$ that are obtained by restricting $\sigma$ to the $(n - 1)$-faces of $\Delta^n$, which form the boundary $\partial \Delta^n$. The signs in front of the terms $\sigma \circ f^n_i$ ensure that applying the boundary operator twice gives zero. This has a geometrical interpretation. Each $(n - 2)$-face of $\Delta^n$ is contained in the boundary of exactly two $(n - 1)$-faces. In one of them it is oriented parallel to the orientation of the $(n - 1)$-face, in the other against it. Hence, the two contributions have opposite signs and cancel. This encodes the fact that the boundary of the boundary of $\Delta^n$ is empty: one has $\partial \Delta^n = \cup_{i=0}^n f^n_i(\Delta^{n-1})$ and $\partial(\partial \Delta^n) = 0$. The algebraic counterpart of this geometrical argument is the following.
Lemma 2.1.3:

1. The face maps satisfy $f_i^n \circ f_{j-1}^{n-1} = f_j^n \circ f_{i-1}^{n-1}$ for all $0 \leq i < j \leq n$.

2. For any topological space $X$, the boundary operators $d_n : C_n(X, k) \to C_{n-1}(X, k)$ satisfy $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{N}$.

Proof:

1. As the face maps are affine linear, they are determined completely by their values on the vertices $e_0, ..., e_n$ of $\Delta^n$. It is therefore sufficient to check this relation on the vertices. We have for $0 \leq i < j - 1 \leq n$

   $$f_i^n \circ f_{j-1}^{n-1}(e_k) = \begin{cases}
f_i^n(e_k) & k < j - 1 \\
f_i^n(e_{k+1}) & k \geq j - 1 
\end{cases} = \begin{cases}
e_k & k < i \\
e_{k+1} & i \leq k < j - 1 \\
e_{k+2} & k \geq j - 1
\end{cases}$$

   and for $0 \leq i = j - 1 \leq n$

   $$f_j^n \circ f_{j-1}^{n-1}(e_k) = \begin{cases}
f_j^n(e_k) & k < j - 1 \\
f_j^n(e_{k+1}) & k \geq j - 1 
\end{cases} = \begin{cases}
e_k & k < j - 1 \\
e_{k+1} & k \geq j - 1
\end{cases}$$

2. Using these relations, we obtain for all continuous maps $\sigma : \Delta^n \to X$:

   $$d_{n-1} \circ d_n(\sigma) = d_{n-1} \left( \Sigma_{j=0}^n (-1)^j \sigma \circ f_j^n \right) = \Sigma_{j=0}^{n-1} \Sigma_{i=0}^n (-1)^{i+j} \sigma \circ f_j^n \circ f_i^{n-1}$$

   $$= \Sigma_{0 \leq i < j \leq n} (-1)^{i+j} \sigma \circ f_j^n \circ f_i^{n-1} + \Sigma_{0 \leq j < i < n} (-1)^{i+j} \sigma \circ f_j^n \circ f_i^{n-1}$$

   $$\overset{1}{=} \Sigma_{0 \leq i < j \leq n} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} + \Sigma_{0 \leq j < i < n} (-1)^{i+j} \sigma \circ f_j^n \circ f_i^{n-1}$$

   $$= \Sigma_{0 \leq i < j \leq n} (-1)^{i+j+1} \sigma \circ f_i^n \circ f_j^{n-1} + \Sigma_{0 \leq j < i < n} (-1)^{i+j} \sigma \circ f_j^n \circ f_i^{n-1}$$

   $$= \Sigma_{0 \leq i < j \leq n} (-1)^{i+j+1} \sigma \circ f_i^n \circ f_j^{n-1} + \Sigma_{0 \leq j < i < n} (-1)^{i+j} \sigma \circ f_j^n \circ f_i^{n-1} = 0.$$

As $d_n : C_n(X, k) \to C_{n-1}(X, k)$ is $k$-linear and the singular $n$-simplexes $\sigma : \Delta^n \to X$ generate $C_n(X, k)$, this proves the claim. \hfill \Box

Due to the relations $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{N}_0$, it follows that $\text{im}(d_{n+1}) \subset \ker(d_n)$ is a $k$-submodule. We can therefore take the quotient $\ker(d_n)/\text{im}(d_{n+1})$, which is called the $n$th singular homology of the topological space $X$ with values in $k$.

Definition 2.1.4: Let $k$ be a commutative ring and $X$ be a topological space.

1. Elements of $Z_n(X, k) := \ker(d_n) \subset C_n(X, k)$ are called singular $n$-cycles.

2. Elements of $B_n(X, k) := \text{im}(d_{n+1}) \subset Z_n(X, k)$ are called singular $n$-boundaries.

3. The $n$th singular homology of $X$ is the $k$-module

   $$H_n(X, k) = \frac{Z_n(X, k)}{B_n(X, k)}.$$
The $n$th homology counts the possibilities of combining singular $n$-simplexes in such a way that there is no boundary, up to those combinations that arise as the boundaries of $(n+1)$-simplexes. This can be viewed as a measure for the number of holes in the topological space. For each $(n + 1)$-simplex $\sigma : \Delta^{n+1} \to X$, the boundary $d_{n+1}(\sigma)$ is an $n$-cycle. If we remove a point $x$ in the interior of $\sigma(\Delta^{n+1})$ from $X$, the continuous map $\sigma$ is no longer defined, while $d_{n+1}(\sigma)$ still defines an $n$-cycle in $Z_n(X \setminus \{x\}, k)$. In this way, we have created an $n$-cycle that is not an $n$-boundary. To gain more insight into the interpretation of singular homology, we compute the first two singular homology groups and relate them to concepts from topology.

**Example 2.1.5:** Let $X$ be a topological space and $k$ a commutative ring. Then

$$H_0(X, k) = \frac{\langle X \rangle_k}{\langle \sigma(0) - \sigma(1) \mid \sigma : [0,1] \to X \text{ continuous} \rangle_k} = \bigoplus_{\pi_0(X)} k$$

where $\pi_0(X)$ is the set of path components of $X$.

**Proof:**
As $\Delta^0 = \{0\}$ and any map $\sigma : \{0\} \to X$ is continuous and takes exactly one value, we have

$$C_0(X, k) = \langle X \rangle_k \quad \text{and} \quad d_0 : C_0(X, k) \to \{0\}, \quad \sigma \mapsto 0$$

$$C_1(X, k) = \langle \sigma : [0,1] \to X \text{ continuous} \rangle_k \quad \text{and} \quad d_1 : C_1(X, k) \to C_0(X, k), \quad \sigma \mapsto \sigma(1) - \sigma(0).$$

This yields $Z_0(X, k) = C_0(X, k)$ and $B_0(X, k) = \langle \sigma(1) - \sigma(0) \mid \sigma : [0,1] \to X \text{ continuous} \rangle_k$. Hence, two points $x, y \in X$ are related by a 0-boundary if and only if there is a continuous map $\sigma : [0,1] \to X$ with $\sigma(0) = x$ and $\sigma(1) = y$. Such a map is a path from $x$ to $y$, and hence $x, y \in X$ are identified if and only if they are in the same path component of $X$.

Given this interpretation of $H_0(X, k)$, it is natural to expect that the first homology group $H_1(X, k)$ should be related to the fundamental group of a topological space $X$. A 1-chain is a $k$-linear combination of continuous maps $\sigma : [0,1] \to X$, or, equivalently, of paths in $X$. The identity $d_1(\sigma) = \sigma(1) - \sigma(0)$ implies that a singular 1-simplex $\sigma : [0,1] \to X$ is a 1-cycle if and only if $\sigma(0) = \sigma(1)$, i.e., the path $\sigma : [0,1] \to X$ is closed. One also expects that homotopies between paths with the same endpoints should be related to 2-simplexes.

However, there are essential differences between the fundamental group $\pi_1(x, X)$ and the first homology group. The group multiplication of $\pi_1(x, X)$ is induced by the concatenation of paths and in general not abelian, whereas the composition of 1-cycles is given by the addition in the abelian group $Z_n(X, k)$. For a collection of paths based at $x$ the associated product in the fundamental group $\pi_1(x, X)$ keeps track of the order in which the paths are composed, whereas the sum of their homology classes in $H_1(X, k)$ only takes into account how often each path in the collection is traversed with or against its orientation. As a consequence, the first homology group and the fundamental group cannot coincide in general.

We will show that for path-connected topological spaces the first homology group $H_1(X, \mathbb{Z})$ is the abelisation of the fundamental group $\pi_1(x, X)$. For this, recall that the commutator subgroup $[G, G]$ of a group $G$ is the normal subgroup of $G$ generated by the group commutators $[g, h] = ghg^{-1}h^{-1}$ of all elements $g, h \in G$ and that the factor group $G/[G, G]$ is abelian. It is called the abelisation of $G$ and often denoted $\text{Ab}(G)$. In fact, one can show that abelisation defines a functor $\text{Ab} : \text{Grp} \to \text{Ab}$ from the category $\text{Grp}$ of groups to the category $\text{Ab}$ of abelian groups.
Theorem 2.1.6: Let $k$ be a commutative ring, $X$ a path connected topological space, $x \in X$.

1. The map $\phi : \pi_1(x, X) \to H_1(X, k)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is a group homomorphism.
2. It induces an isomorphism $\phi : \text{Ab}(\pi_1(x, X)) \to H_1(X, \mathbb{Z})$, the Hurwicz isomorphism.

Proof:
1. We show that $\phi : \pi_1(x, X) \to H_1(X, k)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is well-defined.

Note first that any path $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1)$ is a singular 1-cycle, since we have $\Delta^1 = [0, 1]$ and $d_1(\gamma) = \gamma(1) - \gamma(0) = 0$. It remains to show that homotopic paths are related by a 1-boundary. Let $\gamma_1, \gamma_2 : [0, 1] \to X$ be paths with $\gamma_i(0) = \gamma_i(1) = x$ and $h : [0, 1] \times [0, 1] \to X$ a homotopy from $\gamma_1$ to $\gamma_2$. Then we have $h(0, t) = \gamma_1(t)$, $h(1, t) = \gamma_2(t)$ and $h(s, 0) = h(s, 1) = x$ for all $t, s \in [0, 1]$. The map $\sigma : \Delta^2 \to X$ defined by

$$\sigma(s, t) = h(s, \frac{s}{s+t}, s+t) \text{ for } (s, t) \neq (0, 0), \quad \sigma(0, 0) = x$$

is continuous since $h : [0, 1] \times [0, 1] \to X$ is continuous with $h(s, 0) = x$ for all $s \in [0, 1]$. By applying the boundary operator, we obtain $d_2(\sigma) = \sigma \circ f_0^2 - \sigma \circ f_1^2 + \sigma \circ f_2^2$ with

$$\sigma \circ f_0^2(t) = \sigma(1 - t, t) = x, \quad \sigma \circ f_1^2(t) = \sigma(0, t) = \gamma_2(t), \quad \sigma \circ f_2^2(t) = \sigma(t, 0) = \gamma_1(t).$$

Hence, $\sigma$ sends the face $[e_1, e_2]$ of $\Delta^2$ to $x$, the face $[e_0, e_2]$ to $\text{im}(\gamma_2)$ and the face $[e_0, e_1]$ to $\text{im}(\gamma_1)$. We have $d_2(\sigma) = \gamma_x - \gamma_2 + \gamma_1$ with the constant 1-cycle $\gamma_x : [0, 1] \to X$, $t \mapsto x$.

As $\gamma_x = d_2(\rho_x)$ with the constant singular 2-simplex $\rho_x : \Delta^2 \to X$, $(s, t) \mapsto x$, we have $0 = [\gamma_x]_{H_1} = [d_2(\sigma)]_{H_1} = [\gamma_1]_{H_1} - [\gamma_2]_{H_1}$. This shows that $[\gamma]_{H_1}$ depends only on the homotopy class of $\gamma$ and $\phi$ is well-defined.

2. We show that $\phi : \pi_1(x, X) \to H_1(X, k)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is a group homomorphism. For this, consider paths $\gamma_1, \gamma_2 : [0, 1] \to X$ with $\gamma_i(0) = \gamma_i(1) = x$ and $g : \Delta^2 \to [0, 1]$, $(s, t) \mapsto \frac{s}{2} + t$. By concatenating $\gamma_1$ and $\gamma_2$ and composing the resulting path with $g$ we obtain a 1-boundary:

$$d_2((\gamma_2 \ast \gamma_1) \circ g) = (\gamma_2 \ast \gamma_1) \circ g \circ f_0^2 - (\gamma_2 \ast \gamma_1) \circ g \circ f_1^2 + (\gamma_2 \ast \gamma_1) \circ g \circ f_2^2$$

with

$$(\gamma_2 \ast \gamma_1) \circ g \circ f_0^2(t) = \gamma_2 \ast \gamma_1(g(t, 0)) = \gamma_2 \ast \gamma_1(\frac{t}{2}) = \gamma_2(t)$$

$$(\gamma_2 \ast \gamma_1) \circ g \circ f_1^2(t) = \gamma_2 \ast \gamma_1(g(0, t)) = \gamma_2 \ast \gamma_1(t)$$

$$(\gamma_2 \ast \gamma_1) \circ g \circ f_2^2(t) = \gamma_2 \ast \gamma_1(g(t, 0)) = \gamma_2 \ast \gamma_1(\frac{t}{2}) = \gamma_1(t).$$

As $\gamma_1 + \gamma_2 - \gamma_2 \ast \gamma_1 = d_2((\gamma_2 \ast \gamma_1) \circ g)$ is a 1-boundary, we obtain

$$\phi([\gamma_1]_{\pi_1}) + \phi([\gamma_2]_{\pi_1}) = [\gamma_1]_{H_1} + [\gamma_2]_{H_1} = [\gamma_2 \ast \gamma_1]_{H_1} = \phi([\gamma_2 \ast \gamma_1]_{\pi_1}) = \phi([\gamma_2]_{\pi_1} \cdot [\gamma_1]_{\pi_1}).$$
As $H_1(X, k)$ is abelian, we have $[\pi_1(x, X), \pi_1(x, X)] \subset \ker(\phi)$ and obtain a a group homomorphism $\phi : \text{Ab}(\pi_1(x, X)) \to H_1(X, k)$.

3. Suppose now that $k = \mathbb{Z}$. We show that $\phi : \text{Ab}(\pi_1(x, X)) \to H_1(X, \mathbb{Z})$ is a group isomorphism by constructing its inverse.

We choose for every point $y \in X$ a path $\gamma^y : [0, 1] \to X$ with $\gamma^y(0) = y, \gamma^y(1) = x$. Then for any singular 1-simplex $\sigma : [0, 1] \to X$, the map $\sigma' = \gamma^\sigma(1) \ast \sigma \ast \gamma^\sigma(0) : [0, 1] \to X$ is a path with $\sigma'(0) = \sigma'(1) = x$.

As $C_1(X, \mathbb{Z})$ is the free abelian group generated by the singular 1-simplexes $\sigma : [0, 1] \to X$, this defines a group homomorphism

$$K : C_1(X, \mathbb{Z}) \to \text{Ab}(\pi_1(x, X)), \quad \sigma \mapsto [\gamma^\sigma(1) \ast \sigma \ast \gamma^\sigma(0)]_{\text{Ab}(\pi_1)}.$$

For any 2-simplex $\omega : \Delta^2 \to X$ we obtain

$$K(d_2 \omega) = K(\omega \circ f_2^0 - \omega \circ f_2^1 + \omega \circ f_2^2) = K(\omega \circ f_2^0) - K(\omega \circ f_2^1) + K(\omega \circ f_2^2)$$

$$= [\gamma^{\omega(0,1)} \ast (\omega \circ f_2^0) \ast \gamma^{\omega(1,0)}]_{\text{Ab}(\pi_1)} - [\gamma^{\omega(0,1)} \ast (\omega \circ f_2^1) \ast \gamma^{\omega(0,0)}]_{\text{Ab}(\pi_1)}$$

$$+ [\gamma^{\omega(1,0)} \ast (\omega \circ f_2^2) \ast \gamma^{\omega(0,0)}]_{\text{Ab}(\pi_1)},$$

$$= [\gamma^{\omega(0,0)} \ast (\omega \circ f_2^0) \ast \gamma^{\omega(1,1)} \ast (\omega \circ f_2^0) \ast \gamma^{\omega(1,0)} \ast (\omega \circ f_2^2) \ast \gamma^{\omega(0,0)}]_{\text{Ab}(\pi_1)}$$

$$= [\gamma^{\omega(0,0)} \ast (\omega \circ f_2^0) \ast \gamma^{\omega(1,0)}]_{\text{Ab}(\pi_1)} = [\gamma]_{\text{Ab}(\pi_1)},$$

where $\gamma : [0, 1] \to X$ is a loop with base point $x$ that circles the boundary $\partial \omega(\Delta^2) \subset X$ counterclockwise. As $\gamma$ is null homotopic, we have $K(d_2 \omega) = 0$. This implies $B_1(X, \mathbb{Z}) \subset \ker(K)$, and $K$ induces a group homomorphism $K : H_1(X, \mathbb{Z}) \to \text{Ab}(\pi_1(x, X))$.

For any path $\delta : [0, 1] \to X$ with $\delta(0) = \delta(1) = x$, we have

$$K \circ \phi([\delta]_{\text{Ab}(\pi_1)}) = [\gamma^x \ast \delta \ast \gamma^x]_{\text{Ab}(\pi_1)} = [\gamma^x]_{\text{Ab}(\pi_1)} - [\gamma^x]_{\text{Ab}(\pi_1)} + [\delta]_{\text{Ab}(\pi_1)} = [\delta]_{\text{Ab}(\pi_1)}$$

$$\phi \circ K([\delta]_{H_1}) = [\gamma^x \ast \delta \ast \gamma^x]_{H_1} = [\gamma^x]_{H_1} + [\gamma^x]_{H_1} + [\delta]_{H_1} = [\delta]_{H_1},$$

because $[\gamma^x]_{H_1} = -[\gamma^x]_{H_1}$. Hence $K = \text{Ab}(\phi)^{-1}$ and $\phi : \text{Ab}(\pi_1(x, X)) \to H_1(X, \mathbb{Z})$ is a group isomorphism. \hfill \Box
Remark 2.1.7: There are analogues of this statement for higher homology and homotopy groups, the Hurewicz theorem:

1. For any commutative ring \( k \), path connected topological space \( X \) and point \( x \in X \) there are group homomorphisms \( \phi_n : \pi_n(x, X) \to H_n(X, k) \) for all \( n \geq 2 \).

2. If \( k = \mathbb{Z} \) and \( X \) is \( (n-1) \)-connected, that is non-empty and path-connected with \( \pi_k(x, X) = \{ \{ \} \} \) for \( k \leq n-1 \), then \( \phi_n \) is a group isomorphism.

The Hurewicz theorem clarifies the geometrical interpretation of the singular homology groups \( H_n(X, \mathbb{Z}) \). For \( (n-1) \)-connected topological spaces it reduces the computation of \( \pi_n(x, X) \) to the computation of the \( n \)-th homology group \( H_n(X, \mathbb{Z}) \). For \( n > 1 \) no abelisation is required since the homotopy group \( \pi_n(x, X) \) is already abelian.

The Hurewicz theorem is useful for the computation of higher homotopy groups, since it relates them to homology groups, which are in general much simpler to compute than homotopy groups. However, the computation of singular homology groups is still difficult without a more detailed understanding of the properties of homologies. The main reason is that the \( n \)-th homology group \( H_n(X, k) \) is a quotient of a huge \( k \)-module, the \( k \)-module of singular \( n \)-cycles, by another huge \( k \)-module, the \( k \)-module of singular \( n \)-boundaries.

This suggests that one should obtain a simpler and more computable notion of homology by considering a smaller family of continuous maps \( \sigma : \Delta^n \to X \). Clearly, the images of the maps in this family should still cover \( X \). To be able to restrict the boundary operator to this family, one must impose that for each simplex \( \sigma : \Delta^n \to X \) in this family all simplexes \( \sigma \circ f_i^n : \Delta^{n-1} \to X \) are also contained in it. Finally, to work with a family that is as small as possible, it makes sense to impose that the simplexes \( \sigma : \Delta^n \to X \) are injective at least in the interior of \( \Delta^n \) and that the images of different \( n \)-simplexes overlap only along the images of \( k \)-simplexes for \( k < n \). Finally, the topology on \( X \) should be compatible with the topology induced by the simplexes in the family, i.e. be the final topology induced by them.

Definition 2.1.8: A (finite) \( \Delta \)-complex or semisimplicial complex is a topological space \( X \), together with a (finite) family \( \{ \sigma_\alpha \}_{\alpha \in I} \) of continuous maps \( \sigma_\alpha : \Delta^{n_\alpha} \to X \) such that:

(S1) The maps \( \sigma_\alpha|_{\Delta^{n_\alpha}} : \Delta^{n_\alpha} \to X \) are injective for all \( \alpha \in I \).

(S2) For every point \( x \in X \) there is a unique \( \alpha \in I \) with \( x \in \sigma_\alpha(\Delta^{n_\alpha}) \).

(S3) For every \( \alpha \in I \) and \( i \in \{0, ..., n_\alpha \} \) there is an \( \beta \in I \) with \( \sigma_\alpha \circ f_i^{n_\alpha} = \sigma_\beta : \Delta^{n_\alpha-1} \to X \).

(S4) The topology on \( X \) is the final topology induced by the family \( \{ \sigma_\alpha \}_{\alpha \in I} \):
   A subset \( A \subset X \) is open if and only if \( \sigma_\alpha^{-1}(A) \subset \Delta^{n_\alpha} \) is open for all \( \alpha \in I \).

A semisimplicial complex is called a simplicial complex if

(S5) For each \( \alpha \in I \) the images of the vertices of \( \Delta^{n_\alpha} \) under \( \sigma_\alpha \) are all distinct:
   \[ \sigma_\alpha(e_i) \neq \sigma_\alpha(e_j) \text{ for all } i \neq j \in \{0, ..., n_\alpha \} . \]

(S6) \( \{ \sigma_\alpha(e_0), ..., \sigma_\alpha(e_{n_\alpha}) \} = \{ \sigma_\beta(e_0), ..., \sigma_\beta(e_{n_\beta}) \} \) implies \( \alpha = \beta \).

A subcomplex of a (semi)simplicial complex \( (X, \{ \sigma_\alpha \}_{\alpha \in I}) \) is a subspace \( A \subset X \) together with a subset \( J \subset I \) such that \( (A, \{ \sigma_\alpha \}_{\alpha \in J}) \) is a (semi)simplicial complex.

A simplicial map \( f : (X, \{ \sigma_\alpha \}_{\alpha \in I}) \to (Y, \{ \tau_\beta \}_{\beta \in J}) \) between (semi)simplicial complexes is a continuous map \( f : X \to Y \) such that for each \( \alpha \in I \) there is a \( \beta \in J \) with \( f \circ \sigma_\alpha = \tau_\beta \).
Note that a given topological space can have many (semi)simplicial complex structures and that the notion of a simplicial complex is more restrictive than the one of a semisimplicial complex. Axiom (S5) forbids that the images distinct vertices of an n-simplex coincide, and condition (S6) forbids that the vertex sets of different simplexes coincide. This allows one to describe a simplicial complex in a purely combinatorial way. Every k-face in a simplicial complex is determined uniquely by its vertices, while this is not the case for semisimplicial complexes. The price one pays for this is that simplicial complexes usually require a larger number of simplexes and hence lead to lengthier computations. One can show that every semisimplicial complex can be transformed into a simplicial one by subdivision.

Simplicial \( n \)-chains, boundary operators, \( n \)-cycles, \( n \)-boundaries and homologies are obtained in the same way as their singular counterparts, by restricting attention to singular simplexes in the chosen family. This works because the second axiom ensures that the boundary operator \( d_n \) maps the submodule of \( \mathbb{Z}_n(X,k) \) that is generated by the \( n \)-simplexes in the family to the submodule of \( \mathbb{Z}_{n-1}(X,k) \) generated by the \((n-1)\)-simplexes in the family.

**Definition 2.1.9:** Let \( k \) be a commutative ring, \( \Delta = (X, \{ \sigma_\alpha \}_{\alpha \in I}) \) a semisimplicial complex.

1. The \( k \)-module of simplicial \( n \)-chains is the trivial \( k \) module for \( n < 0 \) and the free \( k \)-module \( C_n(\Delta, k) = \langle \{ \sigma_\alpha \mid \alpha \in I, n_\alpha = n \} \rangle_k \) for \( n \in \mathbb{N}_0 \).

2. The simplicial boundary operator \( d_n : C_n(\Delta, k) \to C_{n-1}(\Delta, k) \) is the \( k \)-module morphism defined by \( d_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha \circ f^n_i \forall n \in \mathbb{N}, \alpha \in I \) with \( n_\alpha = n \).

The simplicial boundary operators satisfy \( d_{n-1} \circ d_n = 0 \) for all \( n \in \mathbb{Z} \) by Lemma 2.1.3.

3. The \( k \)-modules of simplicial \( n \)-cycles and simplicial \( n \)-boundaries are the \( k \)-modules \( Z_n(\Delta, k) = \ker(d_n) \subset C_n(\Delta, k) \) and \( B_n(\Delta, k) = \im(d_{n+1}) \subset Z_n(\Delta, k) \).

4. The \( n \)th simplicial homology of \( \Delta \) with values in \( k \) is the quotient module

\[
H_n(\Delta, k) = \frac{Z_n(\Delta, k)}{B_n(\Delta, k)}.
\]

**Example 2.1.10:**

1. A semisimplicial structure on the circle \( S^1 \) is given by any continuous map \( \sigma : [0,1] \to S^1 \) with \( \sigma(0) = \sigma(1) = 1 \) and \( \sigma |_{(0,1)} : (0,1) \to S^1 \) injective and \( \rho : \{0\} \to S^1, 0 \mapsto 1 \).

As \( d_1(\sigma) = \sigma(1) - \sigma(0) = 0 \) and \( d_0(\rho) = 0 \), we have \( H_0(\Delta, k) \cong Z_0(\Delta, k) = \langle \rho \rangle_k = k \), \( H_1(\Delta, k) = Z_1(\Delta, k)/B_1(\Delta, k) = Z_1(\Delta, k) = \langle \sigma \rangle_k = k \) and \( H_n(\Delta, k) = \{0\} \) for all \( n > 1 \).
2. The torus is the quotient $T = [0, 1] \times [0, 1]/ \sim$ with respect to the equivalence relation $(x, 0) \sim (x, 1)$ and $(0, x) \sim (1, x)$ for all $x \in [0, 1]$. It has the structure of a semisimplicial complex with two 2-simplexes, three 1-simplexes and one 0-simplex. They are obtained by composing the canonical surjection $\pi : [0, 1] \times [0, 1] \to T$ with the affine linear maps

$$\rho : [e_0, e_1, e_2] \to [e_0, e_2, e_1 + e_2], \quad \sigma : [e_0, e_1, e_2] \to [e_0, e_1, e_1 + e_2], \quad p : [e_0] \to [e_0]$$

$$a : [e_0, e_1] \to [e_0, e_1], \quad b : [e_0, e_1] \to [e_0, e_2], \quad c : [e_0, e_1] \to [e_0, e_1 + e_2].$$

Setting $x' = \pi \circ x$ for $x \in \{p, a, b, c, \rho, \sigma\}$, we find that the $k$-modules of $n$-chains are given by $C_n(\Delta, k) = 0$ for $n \geq 3$ and

$$C_0(\Delta, k) = \langle p' \rangle \cong k, \quad C_1(\Delta, k) = \langle a', b', c' \rangle_k \cong k \oplus k \oplus k, \quad C_2(\Delta, k) = \langle \rho', \sigma' \rangle_k \cong k \oplus k.$$

The boundary operators are given by

$$d_0(p') = 0, \quad d_1(a') = d_1(b') = d_1(c') = p' - p' = 0, \quad d_2(\rho') = d_2(\sigma') = a' + b' - c',$$

and this implies

$$Z_0(\Delta, k) = \langle p' \rangle \cong k \quad \text{and} \quad B_0(\Delta, k) = 0$$

$$Z_1(\Delta, k) = \langle a', b', c' \rangle \cong k \oplus k \oplus k \quad \text{and} \quad B_1(\Delta, k) = \langle a' - b' - c' \rangle \cong k$$

$$Z_2(\Delta, k) = \langle \rho' - \sigma' \rangle \cong k \quad \text{and} \quad B_2(\Delta, k) = 0.$$

This yields the simplicial homologies $H_n(\Delta, k) = 0$ for $n > 2$ and

$$H_0(\Delta, k) \cong \langle p' \rangle_k \cong k,$$

$$H_1(\Delta, k) \cong \langle a', b', c' \rangle_k / \langle a' + b' - c' \rangle \cong \langle a', b' \rangle_k \cong k \oplus k,$$

$$H_2(\Delta, k) \cong \langle \rho' - \sigma' \rangle_k \cong k.$$

3. Real projective space $\mathbb{RP}^2$ is the quotient $\mathbb{RP}^2 = [0, 1] \times [0, 1]/ \sim$ with the equivalence relation $(x, 1) \sim (1 - x, 0)$ and $(0, x) \sim (1, 1 - x)$ for all $x \in [0, 1]$. It has a semisimplicial structure with two 2-simplexes, three 1-simplexes and two 0-simplexes which are obtained by composing the canonical surjection $\pi : [0, 1] \times [0, 1] \to \mathbb{RP}^2$ with the affine simplexes

$$\rho : [e_0, e_1, e_2] \to [e_0, e_2, e_1 + e_2], \quad \sigma : [e_0, e_1, e_2] \to [e_0, e_1, e_1 + e_2],$$

$$a : [e_0, e_1] \to [e_0, e_1], \quad b : [e_0, e_1] \to [e_0, e_2], \quad c : [e_0, e_1] \to [e_0, e_1 + e_2],$$

$$p : [e_0] \to [e_0], \quad q : [e_0] \to [e_1].$$
Remark 2.1.11: One can show that for any semisimplicial complex \( \Delta = (X, (\sigma_\alpha)_{\alpha \in I}) \), the simplicial homology of \( \Delta \) agrees with the singular homology of \( X \): \( H_n(X, k) \cong H_n(\Delta, k) \) for all \( n \in \mathbb{Z} \). This implies in particular that all semisimplicial complex structures on a topological space \( X \) yield the same simplicial homologies.

Setting \( x' = \pi \circ x \) for \( x \in \{ p, q, a, b, c, \rho, \sigma \} \) we have the \( k \)-modules of \( n \)-chains \( C_n(\Delta, k) = 0 \) for \( n \geq 3 \) and

\[
C_0(\Delta, k) = \langle p', q' \rangle \cong k \oplus k, \quad C_1(\Delta, k) = \langle a', b', c' \rangle_k \cong k \oplus k \oplus k, \quad C_2(\Delta, k) = \langle \rho', \sigma' \rangle_k \cong k \oplus k.
\]

The boundary operators are given by

\[
d_0(p') = d_0(q') = 0, \quad d_1(a') = d_1(b') = q' - p', \quad d_1(c') = p' - p' = 0, \\
d_2(\sigma') = a' - b' - c', \quad d_2(p') = -a' + b' - c',
\]

and this implies

\[
Z_0(\Delta, k) = \langle p', q' \rangle_k \cong k \oplus k, \\
B_0(\Delta, k) = \langle q' - p' \rangle \cong k, \\
Z_1(\Delta, k) = \langle a' - b', c' \rangle_k \cong k \oplus k, \\
B_1(\Delta, k) = \langle a' - b', c' \rangle_k \cong k \oplus k, \\
Z_2(\Delta, k) = \{ r(\sigma' + \rho') \mid 2r = 0 \} \cong \{ r \in k \mid 2r = 0 \}, \\
B_2(\Delta, k) = 0.
\]

The simplicial homologies are then given by \( H_n(\Delta, k) = 0 \) for \( n > 2 \) and

\[
H_0(\Delta, k) \cong \langle p', q' \rangle_k / \langle q' - p' \rangle_k \cong \langle p' \rangle \cong k, \\
H_1(\Delta, k) \cong \langle a' - b', c' \rangle_k / \langle a' - b' - c', a' - b' + c' \rangle \cong \langle c' \rangle_k / \langle 2c' \rangle \cong k / 2k \\
H_2(\Delta, k) \cong \{ r \in k \mid 2r = 0 \}.
\]

This shows that the homologies depend on the choice of the commutative ring \( k \). If \( k = \mathbb{F} \) is a field with \( \text{char}(\mathbb{F}) \neq 2 \), we have \( H_2(\Delta, \mathbb{F}) = H_1(\Delta, \mathbb{F}) = 0 \), for \( k = \mathbb{Z} \), we have \( H_1(\Delta, \mathbb{Z}) \cong \mathbb{Z} / 2 \mathbb{Z} \) and \( H_2(\Delta, \mathbb{Z}) = 0 \) whereas \( k = \mathbb{Z} / 2 \mathbb{Z} \) yields \( H_2(\Delta, \mathbb{Z} / 2 \mathbb{Z}) \cong H_1(\Delta, \mathbb{Z} / 2 \mathbb{Z}) \cong \mathbb{Z} / 2 \mathbb{Z} \).
There is also a dual version of singular and simplicial homology, called singular and simplicial cohomology. It is obtained from singular and simplicial homology theory with coefficients in a commutative ring $k$ by applying the functor $\text{Hom}_k(-, M) : k\text{-Mod}^{op} \to k\text{-Mod}$ that assigns to a $k$-module $L$ the $k$-module $\text{Hom}_k(L, M)$ of $k$-module morphisms from $L$ to $M$ and to a $k$-linear map $f : L \to L'$ the $k$-module morphism $\text{Hom}_k(f, M) : \text{Hom}_k(L', M) \to \text{Hom}_k(L, M)$, $g \mapsto g \circ f$.

As this functor reverses morphisms, the direction of the boundary operators is reversed as well.

Definition 2.1.12: Let $k$ be a commutative ring, $M$ a $k$-module and $X$ a topological space.

1. The $k$-module of **singular $n$-cochains** with coefficients in $M$ is the $k$-module $C^n(X, M) = \text{Hom}_k(C_n(X, k), M)$ of $k$-linear maps $f : C_n(X, k) \to M$ for $n \in \mathbb{Z}$.

2. The **singular coboundary operator** $d^n : C^n(X, M) \to C^{n+1}(X, M)$ is the $k$-module morphism defined by $d^n = 0$ for $n < 0$ and

$$d^n(\phi)(\sigma) = \phi(d_{n+1}(\sigma)) = \sum_{i=0}^{n+1} (-1)^i \phi(\sigma \circ f_i^{n+1}) \quad \forall \sigma : \Delta^{n+1} \to X \text{ continuous, } n \in \mathbb{N}_0.$$ 

The singular coboundary operators satisfy $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ by Lemma 2.1.3.

3. The $k$-modules of **singular $n$-cocycles** and **singular $n$-coboundaries** are the $k$-modules $Z^n(X, M) = \ker(d^n) \subset C^n(X, M)$ and $B^n(X, M) = \text{im}(d^{n-1}) \subset Z^n(X, M)$.

4. The $n$th **singular cohomology** of $X$ with coefficients in $M$ is the $k$-module

$$H^n(X, M) = \frac{Z^n(X, M)}{B^n(X, M)}.$$ 

Definition 2.1.13: Let $k$ be a commutative ring, $M$ a $k$-module and $\Delta = (X, \{\sigma_i\}_{i \in I})$ a semisimplicial complex.

1. The $k$-module of **simplicial $n$-cochains** with coefficients in $M$ is the trivial $k$-module for $n < 0$ and $k$-module $C^n(\Delta, M) = \text{Hom}_k(C_n(\Delta, k), M)$ for $n \in \mathbb{N}_0$.

2. The **simplicial coboundary operator** $d^n : C^n(\Delta, M) \to C^{n+1}(\Delta, M)$ is the $k$-module morphism defined by $d^n = 0$ for $n < 0$ and

$$d^n(\phi)(\sigma) = \phi(d_{n+1}(\sigma)) = \sum_{i=0}^{n+1} (-1)^i \phi(\sigma \circ f_i^{n+1}) \quad \forall \alpha \in I \text{ with } n_\alpha = n + 1, n \in \mathbb{N}_0.$$ 

The simplicial coboundary operators satisfy $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ by Lemma 2.1.3.

3. The $k$-modules of **simplicial $n$-cocycles** and **simplicial $n$-coboundaries** are the $k$-modules $Z^n(\Delta, M) = \ker(d^n) \subset C^n(\Delta, M)$ and $B^n(\Delta, M) = \text{im}(d^{n-1}) \subset Z^n(\Delta, M)$.

4. The $n$th **simplicial cohomology** of $\Delta$ with coefficients in $M$ is the $k$-module

$$H^n(\Delta, M) = \frac{Z^n(\Delta, M)}{B^n(\Delta, M)}.$$ 

The resulting cohomology theories, singular and simplicial cohomology, have a similar interpretation to the associated homology theories and contain similar information. However, in some cases it is advantageous to use cohomologies instead of homologies. They can be related more directly to smooth geometrical structures such as differential forms and they are often more well-behaved or easier to compute.
2.2 Hochschild homology and cohomology

We now consider homologies and cohomologies of unital algebras. In the following we suppose that all algebras and algebra homomorphisms are unital unless stated otherwise. To define homologies and cohomologies of algebras in a way that allows us to relate them to other homologies and cohomologies later, we work with a slightly more general notion of algebra, namely an algebra over a commutative ring $k$. The definition is analogous to that of an algebra over a field, only that the scalar multiplication is replaced by a $k$-module structure.

**Definition 2.2.1:** Let $k$ be a commutative ring.

1. An algebra over $k$ is a ring $(A, +, \cdot)$ with a $k$-module structure $\triangleright : k \times A \to A$, $(\lambda, a) \mapsto \lambda a$ that satisfies $(\lambda a) \cdot b = \lambda(a \cdot b) = a \cdot (\lambda b)$ for all $a, b \in A$ and $\lambda \in k$.

2. A morphism of $k$-algebras is a ring homomorphism that is also a morphism of $k$-modules.

**Example 2.2.2:**

1. An algebra over $\mathbb{Z}$ is a ring, and a homomorphism of $\mathbb{Z}$-algebras is a ring homomorphism. This follows because every ring $k$ has a unique $\mathbb{Z}$-module structure, namely its abelian group structure. The compatibility condition between this $\mathbb{Z}$-module structure and the multiplication follows from the distributive law.

2. For any group $G$ and any commutative ring $k$, the group ring $k[G]$ is an algebra over $k$ with $k$-module structure $(\lambda f)(g) := \lambda f(g)$ for all $f : G \to k$, $g \in G$ and $\lambda \in k$.

3. The ring $k[X]$ of polynomials with coefficients in a commutative ring $k$ is an algebra over $k$.

4. The ring Mat$(n, k)$ of $(n \times n)$-matrices with entries in a commutative ring $k$ is an algebra over $k$ with the matrix multiplication, matrix addition and simultaneous multiplication of all entries with elements of $k$.

5. For any commutative ring $k$ and any $k$-module $M$, the ring $\text{End}_k(M) = \text{Hom}_k(M, M)$ of $k$-module morphisms $\phi : M \to M$ is an algebra over $k$ with the $k$-module structure by pointwise multiplication $(\lambda \phi)(m) := \lambda \phi(m) = \phi(\lambda m)$ for all $\lambda \in k$, $m \in M$.

Left and right modules over a $k$-algebra $A$ are defined as left and right modules over the ring $A$. Just as in the case of an algebra over a field $\mathbb{F}$, every left module $M$ over $A$ inherits a $k$-module structure given by $\lambda m = (\lambda 1_A) \triangleright m$ for all $m \in M$. The same holds for right and bimodules. It also follows directly that every $A$-module homomorphism is $k$-linear.

With the notion of an algebra over a commutative ring and of a bimodule over such an algebra, we can now proceed to define homologies and cohomologies of a $k$-algebra $A$ with values in an $(A, A)$-bimodule $M$. For this we consider the following structures.

**Definition 2.2.3:** Let $A$ be an algebra over a commutative ring $k$ and $M$ an $(A, A)$-bimodule with structure maps $\triangleright : A \times M \to M$ and $\triangleleft : M \times A \to M$. Denote by $A^{\otimes n} = A \otimes_k \ldots \otimes_k A$ the $n$-fold tensor product of $A$ over $k$ with $A^{\otimes 0} := k$. 
Lemma 2.2.5: Let $d$ maps (co)boundary operators is zero. This is a consequence of the combinatorial properties of the boundary operators $n$ with structure maps

\[ \begin{align*}
\text{Definition 2.2.4:} & \quad \text{Let } M \text{ and } A \text{ by } \\
& \quad \text{the structures in Definition 2.2.3 will define Hochschild cohomology.}
\end{align*} \]

Instead of the $k$-modules of $M \otimes A$ we can also consider the $k$-module of $k$-linear maps $f : A^\otimes n \rightarrow M$. This leads to a dual version of Definition 2.2.3 that will define Hochschild cohomology.

**Definition 2.2.4:** Let $A$ be an algebra over a commutative ring $k$ and $M$ an $(A, A)$-bimodule with structure maps $\triangleright : A \times M \rightarrow M$ and $\triangleleft : M \times A \rightarrow M$. Denote by $A^\otimes n = A \otimes_k \ldots \otimes_k A$ the $n$-fold tensor product of $A$ over $k$ with $A^\otimes 0 := k$.

1. The $k$-module of $n$-cochains is

\[ C^n(A, M) := \begin{cases} 
\text{Hom}_k(A^{\otimes n}, M) & n \in \mathbb{N}_0 \\
0 & n < 0.
\end{cases} \]

2. The coboundary operators are the $k$-linear maps $d^n : C^n(A, M) \rightarrow C^{n+1}(A, M)$ given by $d^n = 0$ for $n < 0$ and $d^n = \Sigma_{i=0}^{n+1} (-1)^i d_i^n$ for $n \in \mathbb{N}_0$ with

\[ (d_i^n f)(a_0 \otimes \ldots \otimes a_n) = \begin{cases} 
a_0 \triangleright f(a_1 \otimes \ldots \otimes a_n) & i = 0 \\
f(a_0 \otimes \ldots \otimes a_{i-2} \otimes (a_{i-1} \cdot a_i) \otimes a_{i+1} \otimes \ldots \otimes a_n) & 1 \leq i \leq n \\
f(a_0 \otimes \ldots \otimes a_{n-1}) \triangleleft a_n & i = n + 1.
\end{cases} \]

Just as the singular and simplicial (co)boundary operators, the (co)boundary operators in Definitions 2.2.3 and 2.2.4 satisfy the condition that the composite of two subsequent (co)boundary operators is zero. This is a consequence of the combinatorial properties of the maps $d_i^n$ and $d^n_i$ in Definitions 2.2.3 and Definition 2.2.4.

**Lemma 2.2.5:** Let $k$ be a commutative ring, $A$ an algebra over $k$ that is a free $k$-module, and $M$ an $(A, A)$-bimodule.

1. The $k$-linear maps $d_i^n : C_i^n(A, M) \rightarrow C_{i-1}^{n+1}(A, M)$ from Definition 2.2.3 satisfy

\[ d_{i-1}^n \circ d_i^n = d_{i-1}^{n-1} \circ d_i^n \quad \forall 0 \leq i < j \leq n, \] \( (4) \)

and this implies $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

2. The $k$-linear maps $d_i^n : C_i^n(A, M) \rightarrow C_{i+1}^{n+1}(A, M)$ from Definition 2.2.4 satisfy

\[ d_{i+1}^n \circ d_i^n = d_{i+1}^{n+1} \circ d_i^n \quad \forall 0 \leq i \leq j \leq n, \] \( (5) \)

and this implies $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$.
Proof:
We prove the second part of the lemma. The proof of the first part is analogous. We compute for $0 < i < j < n$:

$$d_i^{n+1}(d_i^n(f)(a_0 \otimes \ldots \otimes a_{n+1})) = (d_i^n f)(a_0 \otimes \ldots \otimes a_{i-2} \otimes (a_{i-1}a_i) \otimes a_{i+1} \otimes \ldots \otimes a_{n+1})$$

$$= f(a_0 \otimes \ldots \otimes a_{i-2} \otimes (a_{i-1}a_i) \otimes a_{i+2} \otimes \ldots \otimes a_{n+1}) = (d_i^n f)(a_0 \otimes \ldots \otimes a_{i-1} \otimes (a_{i+1}) \otimes a_{i+2} \otimes \ldots \otimes a_{n+1})$$

$$= d_{i+1}^{n+1}(d_i^n f)(a_0 \otimes \ldots \otimes a_{n+1}),$$

for $0 < i < j < n$:

$$d_i^{n+1}(d_i^n(f)(a_0 \otimes \ldots \otimes a_{n+1})) = (d_i^n f)(a_0 \otimes \ldots \otimes a_{i-2} \otimes (a_{i-1}a_i) \otimes a_{i+1} \otimes \ldots \otimes a_{n+1})$$

$$= f(a_0 \otimes \ldots \otimes a_{i-2} \otimes (a_{i-1}a_i) \otimes a_{i+2} \otimes \ldots \otimes a_{n+1})$$

$$= d_{i+1}^{n+1}(d_i^n f)(a_0 \otimes \ldots \otimes a_{n+1}),$$

for $i = j = 0$:

$$d_0^{n+1}(d_0^n f(a_0 \otimes \ldots \otimes a_{n+1})) = a_0 \triangleright (d_0^n f)(a_1 \otimes \ldots \otimes a_{n+1}) = a_0 \triangleright (a_1 \triangleright (d_0^n f)(a_2 \otimes \ldots \otimes a_{n+1}))$$

$$= (a_0a_1) \triangleright f(a_2 \otimes \ldots \otimes a_{n+1}) = (d_0^{n+1} f)((a_0a_1) \otimes \ldots \otimes a_{n+1}) = d_0^{n+1}(d_0^n f)(a_0 \otimes \ldots \otimes a_{n+1}),$$

and for $0 < i < j < n$:

$$d_i^{n+1}(d_i^n f(a_0 \otimes \ldots \otimes a_{n+1})) = a_0 \triangleright (d_i^n f)(a_1 \otimes \ldots \otimes a_{n+1}) = a_0 \triangleright f(a_1 \otimes \ldots \otimes a_{j-i} \otimes (a_{j+1}a_{j+1}) \otimes \ldots \otimes a_{n+1})$$

$$= (d_i^{n+1} f)(a_0 \otimes \ldots \otimes a_{j-i} \otimes (a_{j+1}a_{j+1}) \otimes \ldots \otimes a_{n+1}) = d_{j+1}^{n+1}(d_i^n f)(a_0 \otimes \ldots \otimes a_{n+1}).$$

The computations for $i = j = n$ and $0 < i < j = n$ are analogous. These relations imply

$$d^{n+1} \circ d^n = \sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^{i+j} d_i^{n+1} \circ d_j^n$$

$$= \sum_{0 \leq i < j \leq n+2} (-1)^{i+j} d_i^{n+1} \circ d_j^n + \sum_{0 \leq i \leq j \leq n+1} (-1)^{i+j} d_i^{n+1} \circ d_j^n$$

$$= \sum_{0 \leq i < j \leq n+2} (-1)^{i+j} d_i^{n+1} \circ d_j^n + \sum_{0 \leq i \leq j \leq n+2} (-1)^{i+j} d_j^{n+1} \circ d_i^n$$

$$= \sum_{0 \leq i < j \leq n+2} (-1)^{i+j} (d_i^{n+1} \circ d_j^n - d_j^{n+1} \circ d_i^n) = 0.$$  \(\square\)

As the boundary operators $d_n : C_n(A,M) \rightarrow C_{n-1}(A,M)$ satisfy the relations $d_n \circ d_{n+1} = 0$, we have $d_n(\text{im}(d_{n+1})) = 0$ and hence $\text{im}(d_{n+1}) \subset \text{ker}(d_n) \subset C_n(A,M)$. This allows us to consider the quotient modules $\text{ker}(d_n)/\text{im}(d_{n+1})$. Similarly, the relations $d^{n} \circ d^{n-1} = 0$ for the coboundary operators implies that $\text{im}(d^{n+1}) \subset \text{ker}(d^n) \subset C^n(A,M)$ are submodules and allows us to form the quotient module $\text{ker}(d^n)/\text{im}(d^{n+1})$. These quotients are called, respectively, the Hochschild homologies and cohomologies of $A$ with coefficients in $M$.

**Definition 2.2.6:** Let $A$ be an algebra over a commutative ring $k$ and $M$ an $(A,A)$-bimodule with structure maps $\triangleright : A \times M \rightarrow M$ and $\triangleleft : M \times A \rightarrow M$.

- The $k$-module $Z_n(A,M) = \text{ker}(d_n) \subset C_n(A,M)$ is called the $k$-module of $n$-cycles and the submodule $B_n(A,M) = \text{im}(d_{n+1}) \subset Z_n(A,M)$ the $k$-module of $n$-boundaries.

- The $n$th Hochschild homology of $A$ with coefficients in $M$ is the quotient module

$$H_n(A,M) = \frac{Z_n(A,M)}{B_n(A,M)} = \frac{\text{ker}(d_n)}{\text{im}(d_{n+1})}.$$
The $k$-module $Z^n(A, M) = \ker(d^n) \subset C^n(A, M)$ is called $k$-module of \emph{n-cocycles} and the submodule $B^n(A, M) = \im(d^{n-1}) \subset Z^n(A, M)$ the $k$-module of \emph{n-coboundaries}.

The $n$th \textbf{Hochschild cohomology} of $A$ with coefficients in $M$ is the quotient module

$$H^n(A, M) = \frac{Z^n(A, M)}{B^n(A, M)} = \frac{\ker(d^n)}{\im(d^{n-1})}.$$ 

Hochschild (co)homologies of $A$ with coefficients in $M$ carry important information about the $k$-algebra $A$ and the $(A, A)$-bimodule $M$. As every $k$-algebra $A$ is in particular an $(A, A)$-bimodule over itself with respect to left and right multiplication, we can always consider the $(A, A)$-bimodule $M = A$ and extract information about the algebra $A$ itself. We will show that the zeroth Hochschild cohomology is the \textit{centre} of an $(A, A)$-bimodule $M$, the submodule of elements on which the left and right action of $A$ coincide.

To interpret the first Hochschild cohomology, we require the concept of a \textit{derivation}. The name \textit{derivation} is motivated by the fact that derivations generalise derivatives. To see this, consider the real algebra $C^n(U)$ of $n$-times continuously differentiable real functions on an open subset $U \subset \mathbb{R}^n$ with the pointwise addition, multiplication and multiplication by $\mathbb{R}$. As the product of a $C^n$-function and a $C^{n-1}$-function is again a $C^n$-function, we can view the algebra $C^{n-1}(U)$ as a bimodule over $C^n(U)$ with $f \triangleright g = g \triangleleft f = f \cdot g$ for all $f \in C^n(U)$ and $g \in C^{n-1}(U)$.

The derivative $' : C^n(U) \to C^{n-1}(U)$, $f \mapsto f'$ is $\mathbb{R}$-linear and satisfies the \textit{Leibniz identity}: $(f \cdot g)' = f \cdot g' + f' \cdot g$ for all $f, g \in C^k(U)$. By replacing the real algebra $C^n(U)$ with an algebra over a commutative ring $k$ and $C^{n-1}(U)$ with a general $(A, A)$-bimodule $M$, we obtain the following definition.

\begin{definition}
Let $A$ be an algebra over a commutative ring $k$ and $M$ an $(A, A)$-bimodule with structure maps $\triangleright : A \times M \to M$ and $\triangleleft : M \times A \to M$.

1. A \textbf{derivation} on $A$ with values in $M$ is a $k$-linear map $f : A \to M$ that satisfies $f(ab) = f(a)b + a \triangleright f(b)$ for all $a, b \in A$. The $k$-module of derivations $f : A \to M$ is denoted $\text{Der}(A, M)$.

2. A derivation on $A$ with values in $M$ is called an \textbf{inner derivation} if it is of the form $f_m : A \to M$, $a \mapsto a \triangleright m - m \triangleleft a$ for some $m \in M$. The $k$-module of inner derivations $f : A \to M$ is denoted $\text{InnDer}(A, M)$.
\end{definition}

By computing the first two Hochschild cohomologies from Definition \ref{def:Hochschild} we can relate them to, respectively, the centre of a bimodule $M$ and to the derivations on $A$ with coefficients in $M$. An analogous computation can be performed for the Hochschild homologies (Exercise \ref{ex:42}).

\begin{lemma}
Let $A$ be an algebra over a commutative ring $k$ and $M$ an $(A, A)$-bimodule with structure maps $\triangleright : A \times M \to M$ and $\triangleleft : M \times A \to M$.

- The first two Hochschild cohomologies of $A$ with coefficients in $M$ are given by

$$H^0(A, M) = Z_A(M)$$

$$H^1(A, M) = \frac{\text{Der}(A, M)}{\text{InnDer}(A, M)}.$$ 

where $Z_A(M) = \{m \in M \mid a \triangleright m = m \triangleleft a \forall a \in A\}$ is called the \textbf{centre} of $M$.
\end{lemma}
• For $M = A$ as an $(A, A)$-bimodule over itself with left and right multiplication we have

$$H^0(A, A) = Z(A) \quad \quad H^1(A, M) = \frac{\text{Der}(A, A)}{\text{InnDer}(A, A)}.$$  

where $Z(A) = \{a \in A \mid ab = ba \forall b \in A\}$ is the centre of $A$.

**Proof:**

As $\phi : \text{Hom}_k(k, M) \to M, f \mapsto f(1)$ is a $k$-linear isomorphism, $C^0(A, M) = \text{Hom}_k(k, M) \cong M$. With this identification, the first two coboundary operators from Definition 2.2.4 are given by

$$d^0 : M \to \text{Hom}_k(A, M), \quad m \mapsto f_m \quad \quad f_m(a) = a \triangleright m - m \triangleleft a$$

$$d^1 : \text{Hom}_k(A, M) \to \text{Hom}_k(A^\otimes 2, M), \quad (d^1 f)(a \otimes b) = a \triangleright f(b) - f(ab) + f(a) \triangleleft b,$$

and we obtain

$$\ker(d^0) = \{m \in M \mid a \triangleright m = m \triangleleft a \forall a \in A\} = Z_A(M)$$

$$\ker(d^1) = \{f : A \to M \mid a \triangleright f(b) - f(ab) + f(a) \triangleleft b = 0 \forall a, b \in A\} = \text{Der}(A, M)$$

$$\text{im}(d^0) = \{f_m : A \to M, a \mapsto a \triangleright m - m \triangleleft a \mid m \in M\} = \text{InnDer}(A, M).$$

This shows that the Hochschild cohomology $H^0(A, M)$ measures the failure of the module $M$ to be commutative with respect to the left and right $A$-module structures on $M$. In case of an algebra $M = A$ as a bimodule over itself, it measures the failure of $A$ to be commutative. If we consider a ring $A$, viewed as an algebra over $\mathbb{Z}$, and an $A$-module $N$, then the abelian group $M = \text{End}_\mathbb{Z}(N)$ of $\mathbb{Z}$-module endomorphisms $f : N \to N$ becomes an $(A, A)$-bimodule by Example 2.2.5 with the bimodule structure $(a \triangleright f)(n) = a \triangleright f(n)$ and $(f \triangleleft a)(n) = f(a \triangleright n)$. In this case, $H^0(A, M)$ is the subgroup of $A$-module endomorphisms $f : N \to N$.

The first Hochschild cohomology $H^1(A, M)$ counts the derivations on $A$ with values in $M$, up to inner derivations. If $M$ is commutative with respect to the left and right action of $A$, we have $M = Z_A(M)$ and $\text{InnDer}(A, M) = \{0\}$. In this case, $H^1(A, M)$ counts the derivations on $A$ with values in $M$. If $M = A$ as an $(A, A)$-bimodule over itself, then derivations are $k$-linear maps $f : A \to A$ with $f(ab) = af(b) - f(a)b$ and inner derivations are precisely the commutator maps $f_b : A \to A, a \mapsto [a, b] = ab - ba$. By the Leibniz rule, every commutator map is a derivation. Hence, the Hochschild cohomology $H^1(A, A)$ counts the non-trivial derivations on $A$, that is the derivations that are not commutator maps.

### 2.3 Group cohomology

In this section we investigate cohomologies of groups. Given a commutative ring $k$ and a group $G$, we can consider the group algebra $k[G]$, which is an algebra over $k$. This allows us to define homologies and cohomologies of groups as Hochschild homologies of group algebras $k[G]$. However, compared to general algebras over $k$, group algebras are much easier to handle, due to the following simplifications:

• For all $n \in \mathbb{N}_0$ the map $\phi : k[G^{\times n}] \to k[G]^\otimes n, \lambda(g_1, ..., g_n) \mapsto \lambda g_1 \otimes ... \otimes g_n$ is an isomorphism of $k$-modules.
• For all $n \in \mathbb{N}_0$ and $k[G]$-modules $M$, the map $\psi : \text{Map}(G, M) \to \text{Hom}_k(k[G], M)$ that extends a map $f : G \to M$ to a $k$-linear map $f' : k[G] \to M$ is an isomorphism of $k$-modules.

• Every $k[G]$-left module $M$ becomes a $(k[G], k[G])$-bimodule with the trivial $k[G]$-right module structure $\triangleleft : M \times k[G] \to M$, $m \triangleleft g = m$.

• Similarly, every $k$-module $M$ becomes a $k[G]$-module with the trivial $k[G]$-left module structure $\triangleright : k[G] \times M \to M$, $g \triangleright m = m$.

The first two points lead to technical simplifications. They allow us to describe Hochschild homologies and cohomologies of group algebras $k[G]$ in terms of group products $G^\times n$ and maps $f : G^\times n \to M$ instead of tensor products $k[G]^\otimes n$ and $k$-linear maps $f : k[G]^\otimes n \to M$.

The third and fourth point are more fundamental because they allow us to consider bimodules that are (partly) trivial. For a general $k$-algebra $A$, a trivial $A$-module structure on a $k$-module $M$ can be defined as a $A$-module structure $\triangleright : A \times M \to M$ with $a \triangleright m = \epsilon(a)m$ for all $m \in M$ with a map $\epsilon : A \to k$. In order to define an $A$-module structure on $M$, the map $\epsilon$ needs to be an algebra homomorphism, and such an algebra homomorphism is called an augmentation map. For $A = k[G]$ this augmentation map is given by $\epsilon : \Sigma g \in G \lambda g \mapsto \Sigma g \in G \lambda_g$. However, for a general $k$-algebras $A$, it is not guaranteed that there is basis of $A$ such that products of basis elements are basis elements. One therefore cannot define an augmentation map in this way and the existence of augmentation maps is not guaranteed.

Taking advantage of the simplifications for group algebras above, one defines group (co)homologies as Hochschild (co)homologies of the group algebra $k[G]$ with coefficients in a $k[G]$-left module $M$ that is equipped with the trivial $k[G]$-right module structure. Because the cohomologies have a simpler interpretation and are more useful in practice, we restrict attention to the group cohomologies.

**Definition 2.3.1:** Let $k$ be a commutative ring, $G$ a group and $M$ a left module over the group ring $k[G]$ with structure map $\triangleright : k[G] \times M \to M$. Denote by $G^\times n = G \times ... \times G$ the $n$-fold product of $G$ with $G^\times 0 := \{1\}$.

1. The $k$-module of $n$-cochains is

$$C^n(G, M) = \begin{cases} \text{Map}(G^\times n, M) & n \in \mathbb{N}_0 \\ 0 & n \leq 0 \end{cases}$$

2. The coboundary operators are the $k$-linear maps $d^n : C^n(G, M) \to C^{n+1}(G, M)$ given by $d^n = 0$ for $n < 0$ and $d^n = \Sigma_{i=0}^{n+1}(-1)^i d^n_i$ for $n \in \mathbb{N}_0$ with

$$(d^n_i f)(g_0, ..., g_n) = \begin{cases} g_0 \triangleright f(g_1, ..., g_n) & i = 0 \\ f(g_0, ..., g_{i-2}, g_{i-1} g_i, g_{i+1}, ..., g_n) & 1 \leq i \leq n \\ f(g_0, ..., g_{n-1}) & i = n + 1. \end{cases}$$

They satisfy $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ by Lemma [2.2.5].

3. The $k$-module of $n$-coboundaries is the submodule $Z^n(G, M) = \ker(d^n) \subset C^n(G, M)$ and the $k$-module of $n$-coboundaries the submodule $B^n(G, M) = \text{im}(d^{n-1}) \subset Z^n(G, M)$. 

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4. The $n$th group cohomology of $G$ with coefficients in $M$ is the $k$-module

$$H^n(G, M) = \frac{Z^n(G, M)}{B^n(G, M)} = \frac{\ker(d^n)}{\text{im}(d^{n-1})}.$$ 

By adapting the results fromLemma 2.2.8 to the situation at hand, we obtain a characterisation of the first two group cohomologies $H^0(G, M)$ and $H^1(G, M)$ in terms of the centre $Z_{k[G]}(M)$ and in terms of derivations on $k[G]$. The only difference is that these notions become simpler and have a more direct interpretation. As the $k[G]$-right module structure is chosen to be trivial, the centre $Z_{k[G]}(M)$ is given by the $k$-submodule $M^G$ of invariants.

**Definition 2.3.2:** Let $G$ be a group, $k$ a commutative ring and $M$ a $k[G]$-module with structure map $\triangleright : k[G] \times M \to M$. The $k$-submodule of invariants is the $k$-submodule

$$M^G = \{ m \in M \mid g \triangleright m = m \forall g \in G \}.$$ 

Similarly, the notion of a derivation and of an inner derivation simplify for group algebras because the $k[G]$-right module structure is chosen to be trivial and because we can characterise $k$-linear maps $k[G] \to M$ in terms of maps $G \to M$. This yields the following definition.

**Definition 2.3.3:** Let $G$ be a group, $k$ a commutative ring and $M$ a $k[G]$-module with structure map $\triangleright : k[G] \times M \to M$.

1. A derivation on $G$ with values in $M$ is a map $f : G \to M$ with $f(gh) = f(g) + g \triangleright f(h)$ for all $g, h \in G$. The $k$-module of derivations $f : G \to M$ is denoted $\text{Der}(G, M)$.

2. An inner derivation on $G$ with values in $M$ is a derivation of the form $f_m : G \to M$, $g \mapsto g \triangleright m - m$ for some $m \in M$. The $k$-module of inner derivations $f : G \to M$ is denoted $\text{InnDer}(G, M)$.

Using these definitions and specialising the results in Lemma 2.2.8 to the case of group algebras $k[G]$ and to modules with trivial $k[G]$-right module structures, we obtain the group cohomology counterpart of Lemma 2.2.8. It characterises the group cohomologies $H^0(M, G)$ and $H^1(M, G)$ in terms of, respectively, centres and derivations.

**Corollary 2.3.4:** Let $k$ be a commutative ring, $G$ a group and $M$ a $k[G]$-module with structure map $\triangleright : k[G] \times M \to M$. Then

- The first two group cohomologies of $G$ with coefficients in $M$ are given by
  $$H^0(G, M) = M^G \quad H^1(G, M) = \frac{\text{Der}(G, M)}{\text{InnDer}(G, M)}.$$ 

- In particular, we have for a trivial $k[G]$-module $M$
  $$H^0(G, M) = M \quad H^1(G, M) = \text{Hom}_{\text{Grp}}(G, M).$$ 

As group cohomologies are a special example of Hochschild cohomologies, they contain the same type of information. However, the simplifications for group algebras $k[G]$ make them much simpler to compute and to interpret. We illustrate this by computing and interpreting the second cohomology group $H^2(G, M)$. This requires the concept of a group extension.
Definition 2.3.5:

1. An extension of a group $G$ by a group $M$ is a triple $(E, \iota, \pi)$ of a group $E$, an injective group homomorphism $\iota : M \to E$ and a surjective group homomorphism $\pi : E \to G$ with $\ker(\pi) = \mathrm{im}(\iota)$.

2. An extension $(E, \iota, \pi)$ of $G$ by $M$ is called central if $\iota(M) \subset E$ is central in $E$:
   $$\iota(m) \cdot e = e \cdot \iota(m)$$
   for all $m \in M$ and $e \in E$.

3. A morphism of group extensions from $(E, \iota, \pi)$ to $(E', \iota', \pi')$ is a group homomorphism $f : E \to E'$ with $f \circ \iota = \iota'$ and $\pi' \circ f = \pi$. An isomorphism of group extensions is a bijective morphism of group extensions.

If $(E, \pi, \iota)$ is a group extension of $G$ by $M$, then we have $G \cong E/\ker(\pi)$ by the surjectivity of $\pi$ and $\ker(\pi) = \mathrm{im}(\iota) \cong M$ by injectivity of $\iota$. Hence, a group extension of $G$ by $M$ is a group $E$ that contains $M$ as a normal subgroup and satisfies $G \cong E/M$. We will now show that group extensions arise from 2-cocycles $f : G \times G \to M$, and 2-cocycles that are related by 2-coboundaries define isomorphic group extensions. Central extensions of $G$ by $M$ arise from trivial $G$-modules $M$.

Theorem 2.3.6: Let $G$ be a group and $M$ an abelian group.

1. Isomorphism classes of extensions of $G$ by $M$ are in bijection with pairs $(\triangleright, [f])$ of a $\mathbb{Z}[G]$-module structure $\triangleright : \mathbb{Z}[G] \times M \to M$ on $M$ and an element $[f] \in H^3(G, M)$.

2. Isomorphism classes of central extensions of $G$ by $M$ are in bijection with elements of $H^3(G, M)$, where $M$ is equipped with the trivial $\mathbb{Z}[G]$-module structure.

Proof:

1. Let $\triangleright : \mathbb{Z}[G] \times M \to M$ be a $\mathbb{Z}[G]$-module structure on $M$. By Definition 2.3.1, the coboundary operators $d^1 : C^1(G, M) \to C^2(G, M)$ and $d^2 : C^2(G, M) \to C^3(G, M)$ are given by
   $$d^1(F)(g, h) = g \triangleright F(h) - F(gh) + F(g)$$
   $$d^2(f)(g, h, k) = g \triangleright f(h, k) - f(gh, k) + f(g, hk) - f(g, h)$$
   for all maps $F : G \to M$ and $f : G \times 2 \to M$. Hence, a 2-cocycle is a map $f : G \times G \to M$ with
   $$g \triangleright f(h, k) - f(gh, k) + f(g, hk) - f(g, h) = 0 \quad \forall g, h, k \in G, \quad (6)$$
   and a 2-coboundary is a map $f : G \times G \to M$ of the form
   $$f : G \times G \to M, \quad (g, h) \mapsto g \triangleright F(h) - F(gh) + F(g). \quad (7)$$

1.1. We show that every 2-cocycle $f : G \times G \to M$ gives rise to a group extension of $G$ by $M$:

Every 2-cocycle $f : G \times G \to M$ defines a group structure $\cdot_f$ on $M \times G$ given by
   $$(m, g) \cdot_f (m', g') = (m + g \triangleright m' + f(g, g') - g \triangleright f(1, 1), gg'). \quad (8)$$
   The associativity of $\cdot_f$ follows directly from the 2-cocycle condition $[f]$. To prove that $(0, 1)$ is a unit element and that inverses are given by $(m, g)^{-1} = (-g^{-1} \triangleright m - f(g^{-1}, g) + g^{-1} \triangleright f(1, 1), g^{-1})$, we note that every 2-cocycle $f$ satisfies the conditions
   $$f(g, 1) = g \triangleright f(1, 1) \quad f(1, g) = f(1, 1) \quad g \triangleright f(g^{-1}, g) = f(g, g^{-1}) - g \triangleright f(1, 1) + f(1, 1) \quad (9)$$
for all \( g \in G \) (Exercise). A short computation then shows that the inclusion map \( \iota : M \to M \times G, m \mapsto (m, 1) \) and projection map \( \pi : M \times G \to G, (m, g) \mapsto g \) are group homomorphisms with respect to \( \cdot_f \) and the group structures on \( M \) and \( G \). As we have \( \ker(\pi) = \im(\iota) \), it follows that \( (M \times G, \cdot_f) \) is an extension of \( G \) by \( M \).

1.2. We show that the group extensions for two 2-cocycles \( f_1, f_2 : G \times G \to M \) are isomorphic if \( f_1 - f_2 \) is a 2-coboundary:

Let \( f_1, f_2 : G \times G \to M \) be 2-cocycles that are related by a 2-coboundary. Then there is a map \( F : G \to M \) with \( f_1(g, h) - f_2(g, h) = g \triangleright F(h) - F(gh) + F(g) \) for all \( g, h \in G \). This implies in particular that \( f_1(1, 1) - f_2(1, 1) = F(1) \). We consider the associated extension groups \( E_i = (M \times G, \cdot_{f_i}) \) and show that the map

\[
\phi : (M \times G, \cdot_{f_1}) \to (M \times G, \cdot_{f_2}), \quad (m, g) \mapsto (m + F(g) - F(1), g)
\]

is a group isomorphism. This follows by a direct computation from \( \phi \)

\[
\phi(m, g) \cdot_{f_2} \phi(m', g') = (m + F(g) - F(1), g) \cdot_{f_2} (m' + F(g') - F(1), g')
\]

\[
= (m + F(g) - F(1) + g \triangleright (m' + F(g') - F(1)) + f_2(g, g') - g \triangleright f_2(1, 1), gg')
\]

\[
= \phi(m + g \triangleright m' + f_1(g, g') - g \triangleright f_1(1, 1), gg')
\]

\[
= \phi((m, g) \cdot_{f_1} (m', g')).
\]

The group homomorphism \( \phi \) is invertible with inverse \( \phi^{-1} : (m, g) \mapsto (m - F(g) + F(1), g) \), and we have for all \( g \in G \) and \( m \in M \)

\[
\phi \circ \iota(m) = \phi(m, 1) = (m + F(1) - F(1), 1) = (m, 1) = \iota(m)
\]

\[
\pi \circ \phi(m, g) = \pi(m + F(g) - F(1), g) = g = \pi(g, m).
\]

This shows that \( \phi \) is an isomorphism of group extensions from \((E_1, \iota, \pi)\) to \((E_2, \iota, \pi)\). Hence, every element \([f] \in H^2(G, M)\) defines an isomorphism class of extensions of \( G \) by \( M \).

1.3 We show that every group extension \((E, \iota, \pi)\) of \( G \) by \( M \) defines a \( \mathbb{Z}[G]\)-module structure on \( M \) and an element of \( H^2(G, M) \):
Again by injectivity of \( \iota \) it follows that \( \triangleright : G \times M \to M \) defines a \( \mathbb{Z}[G] \)-module structure on \( M \). From the associativity of the multiplication in \( E \) we obtain for all \( g, h, k \in G \)

\[
\iota(g \triangleright f(h, k)) = \sigma(g)\iota(f(h, k))\sigma(g)^{-1} = \sigma(g)\sigma(h)\sigma(k)\sigma(hk)^{-1}\sigma(g)^{-1}
\]

\[
= \iota(f(g, h))\sigma(g)\sigma(h)\sigma(k)\sigma(hk)^{-1}\sigma(g)^{-1} = \iota(f(g, h))\iota(f(h, k))\sigma(g)\sigma(hk)\sigma(hk)^{-1}\sigma(g)^{-1}
\]

\[
= \iota(f(g, h))\iota((f(h, g))\iota((f(h, k))\iota((f(h, k)))^{-1} = \iota(f(g, h) + f(g, k) - f(h, k)).
\]

Using again that \( \iota \) is injective, we see that \( f \) is a 2-cocycle. Hence, we have shown that every extension of \( G \) by \( M \) defines a \( \mathbb{Z}[G] \)-module structure \( \triangleright \) on \( M \) and a 2-cocycle \( f : G \times G \to M \).

1.4. We investigate how the \( \mathbb{Z}[G] \)-module structure \( \triangleright : \mathbb{Z}[G] \times M \to M \) and the 2-cocycle \( f : G \times G \to M \) depend on the choice of the map \( \sigma : G \to E \) in 1.3:

Let \( \sigma_1, \sigma_2 : G \to E \) be maps with \( \pi \circ \sigma_i = \text{id}_G \) and \( f_i : G \times G \to M \) the associated 2-cocycles. Then we have \( \pi(\sigma_2(g)\sigma_1(g)^{-1}) = gg^{-1} = 1 \), which implies \( \sigma_2(g)\sigma_1(g)^{-1} \in \iota(M) = \ker(\pi) \) for all \( g \in G \). As \( \iota \) is injective, this defines a map

\[
F : G \to M \quad \text{with} \quad \iota(F(g)) = \sigma_2(g)\sigma_1(g)^{-1}.
\]

Using this definition and formula \([10]\) for the \( \mathbb{Z}[G] \)-module structure, we obtain

\[
\iota(g \triangleright_1 m) = \sigma_1(g)\iota(m)\sigma_1(g)^{-1} = (\sigma_1(g)\sigma_2(g)^{-1})(\sigma_2(g)\iota(m)\sigma_2(g)^{-1})(\sigma_2(g)\sigma_1(g)^{-1})
\]

\[
= \iota(F(g))\iota(g \triangleright_2 m)\iota(F(g))^{-1} = \iota(F(g) + g \triangleright_2 m - F(g)) = \iota(g \triangleright_2 m)
\]

and hence the \( \mathbb{Z}[G] \)-module structure on \( M \) does not depend on the choice of \( \sigma \). A direct computation using the definitions, the associativity of the multiplication in \( E \) and the fact that \( \iota(M) \subset E \) is normal then shows that the 2-cocycles \( f_i : G \to M \) are related by a 2-coboundary

\[
\iota(f_2(g, h)) = \sigma_2(g)\sigma_2(h)\sigma_2(gh)^{-1} = \iota(F(g))\sigma_1(g)\sigma_2(h)\sigma_2(gh)^{-1}
\]

\[
= \iota(F(g))\sigma_1(g)\iota(F(h))\sigma_1(h)\sigma_2(gh)^{-1} = \iota(F(g))\iota(g \triangleright F(h))\sigma_1(g)\sigma_1(h)\sigma_2(gh)^{-1}
\]

\[
= \iota(F(g))\iota(g \triangleright F(h))\sigma_1(g)\sigma_1(h)\sigma_1(gh)^{-1}\sigma_2(gh)\sigma_1(gh)^{-1}
\]

\[
= \iota(F(g))\iota(g \triangleright F(h))\iota(f_1(g, h))\iota(F(gh))^{-1} = \iota(g \triangleright F(h) - F(gh) + F(g) + f_1(g, h)).
\]

As \( \iota \) is injective, this implies \( f_2(g, h) - f_1(g, h) = g \triangleright F(h) - F(gh) - F(g) \) for all \( g, h \in G \). It follows that different choices of \( \sigma \) define the same cohomology class \([f_1] = [f_2] \in H^2(G, M)\).

1.5. We show that isomorphic group extensions \((E, \pi, \iota)\) and \((E', \pi', \iota')\) define the same \( \mathbb{Z}[G] \)-module structure on \( M \) and the same element of \( H^2(G, M) \):

Let \((E, \pi, \iota)\) and \((E', \pi', \iota')\) be isomorphic group extensions. Then there is a group isomorphism \( \phi : E \to E' \) with \( \pi \circ \iota = \iota' \) and \( \pi' \circ \phi = \pi \). For any map \( \sigma : G \to E \) with \( \pi \circ \sigma = \text{id}_G \), the map \( \sigma' = \phi \circ \sigma : G \to E' \) satisfies \( \pi' \circ \sigma' = \pi' \circ \phi \circ \sigma = \pi \circ \sigma = \text{id}_G \). This implies for the \( \mathbb{Z}[G] \)-module structure \( \triangleright' : G \times M \to M \) and the 2-cocycle \( f' : G \times G \to M \) given by \( \sigma' \)

\[
\iota'(g \triangleright' m) = \sigma'(g)\iota'(m)\sigma'(g)^{-1} = (\phi \circ \sigma)(g) \cdot (\phi \circ \iota)(m) \cdot (\phi \circ \sigma)(g)^{-1} = \phi(\sigma(g))\iota(m)\sigma(g)^{-1}
\]

\[
= \phi \circ \iota(g \triangleright m) = \iota'(g \triangleright m)
\]

\[
\iota'(f'(g, h)) = \sigma'(g)\sigma'(h)\sigma'(gh)^{-1} = (\phi \circ \sigma)(g)(\phi \circ \sigma)(h)(\phi \circ \sigma)(gh)^{-1} = \phi(\sigma(g))\sigma(h)\sigma(gh)^{-1}
\]

\[
= \phi \circ \iota(f(g, h)) = \iota'(f(g, h))
\]

As \( \iota' = \phi \circ \iota \) is the composite of injective maps, it is injective. It follows that the \( \mathbb{Z}[G] \)-module structures and cocycles defined by \( \sigma \) and \( \sigma' \) are equal, and so are the corresponding elements.
of $H^2(G, M)$. As $\phi : E \to E'$ is invertible, this proves that the isomorphic group extensions $(E, \iota, \pi)$ and $(E', \iota', \pi')$ define the same element of $H^2(G, M)$.

2. If $\triangleright : \mathbb{Z}[G] \times M \to M$ is the trivial $\mathbb{Z}[G]$-module structure, then the multiplication on $M \times G$ from (11) takes the form

$$\tag{11} (m, g) \cdot f (m', g') = (m + m' + f(g, g') - f(1,1), gg').$$

This implies with (9)

$$\tag{9} (m, 1) \cdot f (m', g') = (m + m' + f(1, g') - f(1,1), g') = (m + m', g')
= (m + m' + f(g, 1) - f(1,1), g') = (m', g') \cdot f (m, 1)$$

and hence $M \times \{1\}$ is central in $(M \times G, \cdot_f)$. Conversely, if $(E, \iota, \pi)$ is a central extension of $G$ by $M$, then $\mathbb{Z}[G]$-module structure on $M$ defined in (10) satisfies

$$\iota(g \triangleright m) = \iota(g)\iota(m)(\sigma(g)^{-1} = \sigma(g)\sigma(g)^{-1} \iota(m) = \iota(m)$$

By injectivity of $\iota$, this implies $g \triangleright m = m$ for all $g \in G$ and $m \in M$. \hfill \square

**Example 2.3.7:** We determine the central extensions of $\mathbb{Z}/2\mathbb{Z}$ by an abelian group $M$ and the cohomologies $H^1(\mathbb{Z}/2\mathbb{Z}, M)$ and $H^2(\mathbb{Z}/2\mathbb{Z}, M)$ for an abelian group $M$ with the trivial $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-module structure.

By Corollary 2.3.4, the first cohomology $H^1(\mathbb{Z}/2\mathbb{Z}, M)$ is the $\mathbb{Z}$-module of group homomorphisms $f : \mathbb{Z}/2\mathbb{Z} \to M$. As any such group homomorphism $\phi$ is determined uniquely by $f(\bar{1}) \in M$ and must satisfy $2f(\bar{1}) = f(\bar{1} + \bar{1}) = f(\bar{0}) = 0$, we have

$$H^1(\mathbb{Z}/2\mathbb{Z}, M) = \{ m \in M \mid 2m = 0 \}.$$ 

A map $f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to M$ is a 2-cocycle if and only if for all $\bar{p}, \bar{q}, \bar{r} \in \mathbb{Z}/2\mathbb{Z}$

$$f(\bar{q}, \bar{r}) - f(\bar{p} + \bar{q}, \bar{r}) + f(\bar{p}, \bar{q} + \bar{r}) - f(\bar{p}, \bar{q}) = 0.$$ 

A short computation shows that this holds if and only if $f(\bar{1}, \bar{0}) = f(\bar{0}, \bar{1}) = f(\bar{0}, \bar{0})$. This shows that 2-cocycles are determined by a pair of elements $f(\bar{0}, \bar{0})$, $f(\bar{1}, \bar{1}) \in M$, and we have $\mathbb{Z}^2(\mathbb{Z}/2\mathbb{Z}, M) \cong M \oplus M$. For a map $F : \mathbb{Z}/2\mathbb{Z} \to M$ we have

$$d_1(F)(\bar{p}, \bar{q}) = F(\bar{p}) - F(\bar{p} + \bar{q}) + F(\bar{q}),$$

which implies $d_1(F)(\bar{1}, \bar{1}) = 2F(\bar{1}) = F(\bar{0})$ and $d_1(F)(\bar{0}, \bar{1}) = d_1(F)(\bar{1}, \bar{0}) = F(\bar{0})$. Hence, there is an element $F : \mathbb{Z}/2\mathbb{Z} \to M$ with $f = d_1(F)$ if and only if $f(\bar{1}, \bar{1}) = f(\bar{0}, \bar{0})$ is contained in the subgroup $2M \subset M$, and we obtain

$$H^2(\mathbb{Z}/2\mathbb{Z}, M) \cong M/2M.$$ 

- If $M = \mathbb{Z}$, we have $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \{0\}$ and $H^2(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. There are two isomorphism classes of central extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$. The one of the trivial extension $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ corresponds to even values of $f(\bar{1}, \bar{1}) + f(\bar{0}, \bar{0})$. The other class of cohomologies $f(\bar{1}, \bar{1}) + f(\bar{0}, \bar{0})$ is given by the set $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with group multiplication

$$(z, \bar{p}) \cdot (z', \bar{p}') = \begin{cases} (z + z' + 1, \bar{0}) & \bar{p} = \bar{p}' = \bar{1} \\ (z + z', \bar{p} + \bar{p}') & \text{else.} \end{cases}$$
If \( M = \mathbb{Z}/n\mathbb{Z} \) with odd \( n \in \mathbb{N} \), we have again \( H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \{0\} \). However, in this case
\( 2(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \) and hence \( H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \{0\} \) as well. Up to isomorphisms, there is only one central extension of \( \mathbb{Z}/2\mathbb{Z} \) by \( \mathbb{Z}/n\mathbb{Z} \), namely the abelian group \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

If \( M = \mathbb{Z}/2^k\mathbb{Z} \) for \( k \in \mathbb{N} \) we have \( H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2^k\mathbb{Z}) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2^k\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \). There are two group homomorphisms \( \phi : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \), namely \( \phi = 0 \) and the one with \( \phi(\bar{1}) = 2^{k-1} \).

Up to isomorphisms, there are two central extensions of \( \mathbb{Z}/2\mathbb{Z} \) by \( \mathbb{Z}/2^k\mathbb{Z} \), the direct product \( \mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and the set \( \mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) with multiplication:

\[
(q, p) \cdot (q', p') = \begin{cases} 
(q + q' + \bar{1}, 0) & \text{if } p = p' = \bar{1} \\
(q + q', p + p') & \text{else.}
\end{cases}
\]

By the classification theorem for finite abelian groups, up to isomorphisms there are only two abelian groups of order \( 2^{k+1} \), which contain \( \mathbb{Z}/2^k\mathbb{Z} \) as a subgroup, namely the group \( \mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and the group \( \mathbb{Z}/2^{k+1}\mathbb{Z} \). Hence, the non-trivial central extension is isomorphic to \( \mathbb{Z}/2^{k+1}\mathbb{Z} \).

(Exercise: Find a group isomorphism.)

This example illustrates that group cohomologies contain non-trivial information about groups that would be difficult to access otherwise. However, the procedure used to compute the group cohomologies \( H^2(\mathbb{Z}/2\mathbb{Z}, M) \) is too complicated for groups with more elements. We will later develop more efficient methods for the computation of group cohomologies that allow us to treat more complicated examples.

### 2.4 Lie algebra cohomology

In this section, we consider cohomologies of Lie algebras. Finite-dimensional Lie algebras can be viewed as a infinitesimal counterparts of Lie groups, smooth manifolds with a group structure such that the group multiplication and inversion are smooth maps. The Lie algebra \( \mathfrak{g} = \text{Lie} \, G \) of a Lie group \( G \) is the tangent space of \( G \) in the unit element. Many questions surrounding the classification of Lie groups and their representation theory can be addressed by investigating the associated questions for their Lie algebras, which are more accessible.

Important examples of finite-dimensional Lie groups are matrix Lie groups, closed subgroups of the matrix groups \( \text{GL}(n, \mathbb{C}) \) or \( \text{GL}(n, \mathbb{R}) \) of invertible \( n \times n \)-matrices with entries in \( \mathbb{R} \) or \( \mathbb{C} \). The associated Lie algebras are called matrix Lie algebras. They are linear subspaces of the vector spaces \( \mathfrak{gl}(n, \mathbb{R}) \) or \( \mathfrak{gl}(n, \mathbb{C}) \) of \( n \times n \)-matrices with entries in \( \mathbb{R} \) or \( \mathbb{C} \). The underlying matrix Lie groups or certain subgroups thereof are obtained by exponentiating the matrix Lie algebras.

**Definition 2.4.1:** Let \( \mathbb{F} \) be a field.

1. A **Lie algebra** over \( \mathbb{F} \) is a vector space \( \mathfrak{g} \) over \( \mathbb{F} \) together with an \( \mathbb{F} \)-bilinear map \( [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \), the **Lie bracket** that satisfies:
   (L1) **antisymmetry:** \( [x, x] = 0 \) for all \( x \in \mathfrak{g} \).
   (L2) **Jacobi identity:** \( [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \) for all \( x, y, z \in \mathfrak{g} \).

2. A **Lie subalgebra** \( \mathfrak{h} \subseteq \mathfrak{g} \) is a linear subspace \( \mathfrak{h} \subseteq \mathfrak{g} \) that is a Lie algebra with the restriction of the Lie bracket on \( \mathfrak{g} \), i.e. a linear subspace \( \mathfrak{h} \subseteq \mathfrak{g} \) with \( [x, y] \in \mathfrak{h} \) for all \( x, y \in \mathfrak{h} \).
3. A morphism of Lie algebras is a $F$-linear map $f : \mathfrak{g} \to \mathfrak{h}$ with $[f(x), f(y)]_h = f([x, y]_g)$ for all $x, y \in \mathfrak{g}$. An isomorphism of Lie algebras is a bijective morphism of Lie algebras.

The category of Lie algebras over $F$ and Lie algebra morphisms is denoted $\text{Liealg}_F$.

The Lie bracket of a Lie algebra $\mathfrak{g}$ can be viewed as the infinitesimal counterpart of the group multiplication of a Lie group $G$. The antisymmetry of the Lie bracket encodes the fact that $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$ for all elements $g, h \in G$, and the Jacobi identity is the infinitesimal counterpart of the associativity of the group multiplication. Note that the Jacobi identity implies that a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is in general non-associative.

Example 2.4.2:

1. Every vector space $V$ over $F$ becomes a Lie algebra over $F$ with the trivial Lie bracket $[\cdot, \cdot] = 0 : V \times V \to V, v \mapsto 0$. A Lie algebra with a trivial Lie bracket is called abelian.

2. If $A$ is an associative (not necessarily unital) algebra over $F$, then $A$ is a Lie algebra with the commutator bracket $[\cdot, \cdot] : A \times A \to A, (a, b) \mapsto [a, b] = a \cdot b - b \cdot a$. This holds in particular for the algebra $\text{End}_F(V)$ of linear endomorphisms of an $F$-vector space $V$.

3. For any algebra $A$ over $F$ the $F$-vector space $\text{Der}(A, A) \subset \text{Hom}_F(A, A)$ of derivations on $A$ is a Lie subalgebra of the Lie algebra $\text{End}_F(A)$ with the commutator bracket.

4. Any matrix algebra $\mathfrak{gl}(n, F)$ of $n \times n$-matrices with entries in $F$ is a Lie algebra with the commutator bracket. The linear subspaces

$$\mathfrak{sl}(n, F) = \{ M \in \text{Mat}(n, F) \mid \text{tr} (M) = 0 \}$$
$$\mathfrak{o}(n, F) = \{ M \in \text{Mat}(n, F) \mid M^T = -M \}$$
$$\mathfrak{so}(n, F) = \{ M \in \text{Mat}(n, F) \mid M^T = -M, \text{tr} (M) = 0 \}$$
$$\mathfrak{c}(n, F) = \{ M \in \text{Mat}(n, F) \mid M_{ij} = 0 \text{ for } i \neq j \}$$
$$\mathfrak{t}_+(n, F) = \{ M \in \text{Mat}(n, F) \mid M_{ij} = 0 \text{ for } i > j \}$$
$$\mathfrak{t}_-(n, F) = \{ M \in \text{Mat}(n, F) \mid M_{ij} = 0 \text{ for } i < j \}$$

of traceless, antisymmetric, diagonal and upper and lower triangular matrices are Lie subalgebras of $\mathfrak{gl}(n, F)$.

5. The matrix algebras

$$\mathfrak{u}(n, \mathbb{C}) = \{ M \in \text{Mat}(n, \mathbb{C}) \mid M^\dagger = -M \}$$
$$\mathfrak{su}(n, \mathbb{C}) = \{ M \in \text{Mat}(n, \mathbb{C}) \mid M^\dagger = -M, \text{tr} (M) = 0 \}$$

of antihermitian and traceless antihermitian matrices are Lie subalgebras of $\mathfrak{gl}(n, \mathbb{C})$.

6. Ado’s Theorem states that any finite-dimensional Lie algebra $\mathfrak{g}$ over a field $F$ of characteristic $\text{char}(F) = 0$ is isomorphic to a Lie subalgebra of a matrix algebra $\mathfrak{gl}(n, F)$.

Just as for groups and algebras, it is advantageous to describe a Lie algebra in terms of representations. Lie algebra representations are Lie algebra homomorphisms into the algebra of endomorphisms of a vector space with the commutator bracket. If the vector space is finite-dimensional, then this amounts to a description in terms of matrices. This allows one to apply results and methods from linear algebra to describe and classify finite-dimensional Lie algebras.
Definition 2.4.3: Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$.

1. A representation of $\mathfrak{g}$ is a vector space $M$ over $\mathbb{F}$ together with a Lie algebra morphism $\rho : \mathfrak{g} \to \text{End}_{\mathbb{F}}(M)$, where $\text{End}_{\mathbb{F}}(M)$ is equipped with the commutator bracket.
2. A homomorphism of Lie algebra representations from $\rho : \mathfrak{g} \to \text{End}_{\mathbb{F}}(M)$ to $\rho' : \mathfrak{g} \to \text{End}_{\mathbb{F}}(M')$ is an $\mathbb{F}$-linear map $\phi : M \to M'$ with $\rho'(x) \circ \phi = \phi \circ \rho(x)$ for $x \in \mathfrak{g}$. An isomorphism of Lie algebra representations is a bijective morphism of Lie algebra representations.

The category of representations of $\mathfrak{g}$ and morphisms of $\mathfrak{g}$-representations is denoted $\text{Rep}(\mathfrak{g})$.

Example 2.4.4:

1. Every Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ has a trivial representation $\rho = 0 : \mathfrak{g} \to \text{End}_{\mathbb{F}}(M)$, $x \mapsto 0$ on any vector space $M$ over $\mathbb{F}$.
2. Every Lie algebra $\mathfrak{g}$ has a representation on itself, the adjoint representation $\rho = \text{ad} : \mathfrak{g} \to \text{End}_{\mathbb{F}}(\mathfrak{g})$, $x \mapsto \text{ad}_x$ with $\text{ad}_x(y) = [x, y]$ for all $y \in \mathfrak{g}$.
3. Any Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{F})$ has a representation $\rho : \mathfrak{h} \to \text{End}_{\mathbb{F}}(\mathbb{F}^n)$, $M \mapsto \phi_M$ with $\phi_M(v) = M \cdot v$.

Just as a representation of a group $G$ on a vector space over $\mathbb{F}$ can be viewed as a module over the group algebra $\mathbb{F}[G]$, we can view a representation of a Lie algebra $\mathfrak{g}$ as a module over a certain algebra, the universal enveloping algebra $U(\mathfrak{g})$.

Definition 2.4.5: Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$.

1. The tensor algebra $T(\mathfrak{g})$ is the $\mathbb{F}$-vector space $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$ with the multiplication
   $$(x_1 \otimes \ldots \otimes x_k) \cdot (y_1 \otimes \ldots \otimes y_l) = x_1 \otimes \ldots \otimes x_k \otimes y_1 \otimes \ldots \otimes y_l \quad \forall x_i, y_j \in \mathfrak{g}.$$
2. The universal enveloping algebra $U(\mathfrak{g})$ is the quotient
   $$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$$
   of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal $I = (x \otimes y - y \otimes x - [x, y])$ generated by the elements $x \otimes y - y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$.

Proposition 2.4.6: (Universal property of the universal enveloping algebra)

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ with universal enveloping algebra $U(\mathfrak{g})$ and $\iota : \mathfrak{g} \cong \mathfrak{g}^{\otimes 1} \to U(\mathfrak{g})$ the canonical inclusion of $\mathfrak{g}$ into $U(\mathfrak{g})$.

Then $\iota$ is a Lie algebra homomorphism, and for every Lie algebra homomorphism $\phi : \mathfrak{g} \to A$ into an algebra $A$ with the commutator bracket, there is a unique algebra homomorphism $\phi' : U(\mathfrak{g}) \to A$ with $\phi' \circ \iota = \phi$.

Proof:

By definition of $U(\mathfrak{g})$ we have
$$\iota(x)\iota(y) - \iota(y)\iota(x) = (x+I)\otimes(y+I) - (y+I)\otimes(x+I) = x \otimes y - y \otimes x + I = [x, y] + I = \iota([x, y]).$$
By the universal property of the tensor algebra $T(g)$, any $F$-linear map $\phi : g \to A$ induces a unique algebra homomorphism $\phi'' : T(g) \to A$, $x_1 \otimes \ldots \otimes x_k \mapsto \phi(x_1) \cdots \phi(x_k)$ with $\phi''|_g = \phi$. If $\phi$ is a Lie algebra homomorphism, we have 

$$\phi''(x \otimes y - y \otimes x - [x, y]) = \phi''(x)\phi''(y) - \phi''(y)\phi''(x) - \phi''([x, y]) = [\phi(x), \phi(y)] - \phi([x, y]) = 0,$$

which implies $I \subset \ker(\phi'')$. Hence, we obtain a unique algebra homomorphism $\phi' : U(g) \to A$ with $\phi' \circ \iota = \phi''|_g = \phi$.

As a representation of a Lie algebra $g$ over $F$ on an $F$-vector space $V$ is a Lie algebra homomorphism $\rho : g \to \text{End}_F(V)$, where $\text{End}_F(V)$ is equipped with the commutator bracket, it follows that $\rho$ extends to an algebra homomorphism $\rho' : U(g) \to \text{End}_F(V)$ with $\rho' \circ \iota = \phi$ or, equivalently, to an $U(g)$-module structure on $V$ with $\iota(x) \triangleright v = \rho(x)v$ for all $x \in g$ and $v \in V$. Conversely, for any $U(g)$-module $V$, we obtain a Lie algebra homomorphism $\rho : g \to \text{End}_F(V)$ given by $\rho(x)v = \iota(x) \triangleright v$ for all $x \in g$ and $v \in V$.

**Corollary 2.4.7:** For any Lie algebra $g$ and vector space $V$ over $F$, representations of $g$ on $V$ are in bijection with $U(g)$-module structures on $V$.

With the concept of a Lie algebra and a Lie algebra representation, we can define homologies or cohomologies of Lie algebras. As the latter are often simpler to compute and more well-behaved in the infinite-dimensional case, we focus on cohomologies. The definition is very similar to the one of group cohomology, only that the module over the group algebra $k[G]$ is replaced by a representation of a Lie algebra and the coboundary operators take a different form. The deeper reason for this is that any Lie group representation defines a representation of the associated Lie algebra. Hence, the structures for a Lie algebra can be obtained from the ones for Lie groups by differentiating in the unit element.

**Definition 2.4.8:** Let $g$ be a Lie algebra over $F$ and $\rho : g \to \text{End}_F(M)$ a representation of $g$. Denote by $\text{Hom}_F(\Lambda^n g, M)$ the vector space of alternating $n$-linear maps $\omega : g^n \to M$.

1. The $k$-module of $n$-cochains is

$$C^n(g, M) = \begin{cases} 
\text{Hom}_F(\Lambda^n g^*, M) & n \in \mathbb{N}_0 \\
0 & n \leq 0.
\end{cases}$$

2. The coboundary operators are the $F$-linear maps $d^n : C^n(g, M) \to C^{n+1}(g, M)$ given by $d^n = 0$ for $n < 0$ and

$$(d^n f)(x_0, \ldots, x_n) = \sum_{i=0}^n (-1)^i \rho(x_i)f(x_0, \ldots, \widehat{x_i}, \ldots, x_n) + \sum_{0 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n)$$

for $n \in \mathbb{N}_0$. They satisfy $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ (Exercise).

3. The $F$-vector spaces of $n$-cocycles and of $n$-coboundaries are the linear subspaces $Z^n(g, M) = \ker(d^n) \subset C^n(g, M)$ and $B^n(g, M) = \text{im}(d^{n-1}) \subset Z^n(g, M)$.

4. The $n$th Lie algebra cohomology of $g$ with coefficients in $M$ is the quotient space $H^n(g, M) = \frac{Z^n(g, M)}{B^n(g, M)} = \frac{\ker(d^n)}{\text{im}(d^{n-1})}$.
The interpretation of the first two cohomologies for Lie algebras are similar to the ones for groups and algebras. The Lie algebra cohomology \( H^0(g, M) \) describes the invariants of the representation of \( g \), and the first cohomology \( H^1(g, M) \) the derivations modulo inner derivations. The only difference is that the concepts of an invariant and of a derivation are the infinitesimal versions of the ones for a group.

**Definition 2.4.9:**
Let \( g \) be a Lie algebra over \( \mathbb{F} \) and \( \rho : g \to \text{End}_\mathbb{F}(M) \) a representation of \( g \) on \( M \).

1. An invariant of the representation \( \rho \) is an element \( m \in M \) with \( \rho(x)m = 0 \) for all \( x \in g \). The vector space of invariants of \( \rho \) is denoted \( M^g \).

2. A derivation on \( g \) with values in \( M \) is a linear map \( f : g \to M \) that satisfies \( f([x, y]) = \rho(x)f(y) - \rho(y)f(x) \) for all \( x, y \in g \). The vector space of derivations \( f : g \to M \) is denoted \( \text{Der}(g, M) \).

3. An inner derivation on \( g \) with values in \( M \) is a derivation of the form \( f_m : g \to M, x \mapsto \rho(x)m \) for some \( m \in M \). The vector space of inner derivations \( f : g \to M \) is denoted \( \text{InnDer}(g, M) \).

Given the concepts of an invariant and an (inner) derivation, we can derive and interpret the first two cohomology groups for a Lie algebra \( g \) and a representation \( \rho : g \to \text{End}_\mathbb{F}(M) \). The computation and the result is fully analogous to the one for groups.

**Lemma 2.4.10:** Let \( g \) be a Lie algebra over \( \mathbb{F} \) and \( \rho : g \to \text{End}_\mathbb{F}(M) \) a representation of \( g \).

- The first two Lie algebra cohomologies of \( g \) with coefficients in \( M \) are given by
  \[
  H^0(g, M) = M^g \
  H^1(g, M) = \frac{\text{Der}(g, M)}{\text{InnDer}(g, M)}.
  \]

- In particular, we have for the trivial representation \( \rho = 0 : g \to \text{End}_\mathbb{F}(M), x \mapsto 0 \)
  \[
  H^0(G, M) = M \
  H^1(G, M) = \text{Hom}_{\text{Liealg}_\mathbb{F}}(g, M) = \{ f : g \to M | f([x, y]) = 0 \ \forall x, y \in g \}.
  \]

**Proof:**
The first non-trivial coboundary operators are given by

\[
\begin{align*}
  d^0 : C^0(g, M) &= M \to C^1(g, M), \ m \mapsto f_m \quad \text{with} \quad f_m(x) = \rho(x)m \\
  d^1 : C^1(g, M) &= C^2(g, M) \quad \text{with} \quad d^1(f)(x, y) = \rho(x)f(y) - \rho(y)f(x) - f([x, y]),
\end{align*}
\]
and this implies

\[
\begin{align*}
  \ker(d^0) &= \{ m \in M \mid \rho(x)m = 0 \ \forall x \in g \} = M^g \\
  \text{im}(d^0) &= \{ f : g \to M \mid \exists m \in M : f(x) = \rho(x)m \} = \text{InnDer}(g, M) \\
  \ker(d^1) &= \{ f : g \to M \mid f([x, y]) = \rho(x)f(y) - \rho(y)f(x) \ \forall x, y \in g \} = \text{Der}(g, M).
\end{align*}
\]
If \( \rho : g \to M \) is a trivial representation, then \( \rho(x)m = 0 \) for all \( x \in g \) and \( m \in M \), and the condition on a derivation reduces to \( f([x, y]) = 0 \) for all \( x, y \in g \), which states that \( f : g \to M \) is a Lie algebra morphism. \( \square \)
The similarities between group and Lie algebra cohomologies extend also to higher cohomologies. The only difference is that the concepts that describe these group cohomologies have to be adapted to Lie algebras by replacing modules over group rings by Lie algebra representations, group multiplications by Lie brackets and group homomorphisms by Lie algebra homomorphisms. If one applies this procedure to the concept of a group extension from Definition 2.3.5, one obtains the following definition.

Definition 2.4.11: Let \( \mathfrak{g}, \mathfrak{h} \) be Lie algebras over \( F \).

1. A Lie algebra extension of \( \mathfrak{g} \) by \( \mathfrak{h} \) is a triple \((\mathfrak{e}, \iota, \pi)\) of a Lie algebra \( \mathfrak{e} \) over \( F \) together with an injective Lie algebra morphism \( \iota: \mathfrak{h} \to \mathfrak{e} \) and a surjective Lie algebra morphism \( \pi: \mathfrak{e} \to \mathfrak{g} \) such that \( \ker(\pi) = \operatorname{im}(\iota) \).

2. An extension \((\mathfrak{e}, \iota, \pi)\) of \( \mathfrak{g} \) by \( \mathfrak{h} \) is called central if \( [x, y] = 0 \) for all \( x \in \mathfrak{h} \) and \( y \in \mathfrak{e} \).

3. A morphism of Lie algebra extensions from \((\mathfrak{e}, \iota, \pi)\) to \((\mathfrak{e}', \iota', \pi')\) is a Lie algebra morphism \( f: \mathfrak{e} \to \mathfrak{e}' \) with \( \phi \circ \iota = \iota' \) and \( \pi' \circ f = \pi \). An isomorphism of Lie algebra extensions is a bijective morphism of Lie algebra extensions.

Given the concept of a Lie algebra extension, we can show that the cohomology \( H^2(\mathfrak{g}, M) \) classifies isomorphism classes of extensions of \( \mathfrak{g} \) by \( M \) with the trivial Lie algebra structure. The result and its proof is in complete analogy to the one for groups, only that some of the computations simplify because the structures under consideration are linear in the Lie algebra case. We leave the proof of the following theorem as an exercise for the reader.

Theorem 2.4.12: Let \( \mathfrak{g} \) be a Lie algebra and \( M \) a vector space over \( F \).

1. Isomorphism classes of extensions of \( \mathfrak{g} \) by the abelian Lie algebra \( M \) are in bijection with pairs \((\rho, [f])\) of a representation \( \rho: \mathfrak{g} \to \operatorname{End}_F(M) \) and element \([f] \in H^2(\mathfrak{g}, M)\).

2. Isomorphism classes of central extensions of \( \mathfrak{g} \) by \( M \) are in bijection with elements of \( H^2(\mathfrak{g}, M) \), where \( M \) is equipped with the abelian Lie algebra structure.

Proof: Let \( \rho: \mathfrak{g} \to \operatorname{End}_F(M) \) be a representation of \( \mathfrak{g} \) on \( M \). By Definition 2.4.8, the coboundary operators \( d^1: C^1(\mathfrak{g}, M) \to C^2(\mathfrak{g}, M) \) and \( d^2: C^2(\mathfrak{g}, M) \to C^3(\mathfrak{g}, M) \) are given by

\[
d^1(F)(x, y) = \rho(x)F(y) - \rho(y)F(x) - F([x, y])
\]

\[
d^2(f)(x, y, x) = \rho(x)f(y, z) - \rho(y)f(x, z) + \rho(z)f(x, y) - f([x, y], z) + f([x, z], y) - f([y, z], x)
\]

for all linear maps \( F: \mathfrak{g} \to M \) and alternating linear maps \( f: \mathfrak{g} \times \mathfrak{g} \to M \). Hence, a 2-cocycle is an alternating bilinear map \( f: \mathfrak{g} \times \mathfrak{g} \to M \) with

\[
\rho(x)f(y, z) - \rho(y)f(x, z) + \rho(z)f(x, y) = -f([x, y], z) + f([x, z], y) - f([y, z], x), \quad (12)
\]

for all \( x, y, z \in \mathfrak{g} \), and a 2-boundary is an alternating bilinear map \( f: \mathfrak{g} \times \mathfrak{g} \to M \) of the form

\[
f : \mathfrak{g} \times \mathfrak{g} \to M, \quad (x, y) \mapsto \rho(x) \triangleright F(y) - \rho(y)F(x) - F([x, y]). \quad (13)
\]

1.1 We show that every 2-cocycle \( f: \mathfrak{g} \times \mathfrak{g} \to M \) gives rise to a group extension of \( \mathfrak{g} \) by \( M \).
Every 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to M$ defines a Lie bracket $[,]_f$ on $M \oplus \mathfrak{g}$ given by
\[\left[(m, x), (m', x')\right]_f = (\rho(x)m' - \rho(x')m - f(x, x'), [x, x']).\] (14)

The bilinearity and antisymmetry is obvious and the Jacobi identity follows directly from the 2-cocycle condition \([12]\). A short computation shows that the inclusion map $\iota : M \to M \oplus \mathfrak{g}$, $m \mapsto (m, 0)$ and projection map $\pi : M \oplus \mathfrak{g} \to \mathfrak{g}$, $(m, x) \mapsto x$ are Lie algebra homomorphisms with respect to $[,]_f$ and the Lie algebra structures on $M$ and $\mathfrak{g}$. As we have $\ker(\pi) = \text{im}(\iota)$, it follows that $(M \oplus \mathfrak{g}, [\ , \ ]_f)$ is an extension of $\mathfrak{g}$ by the abelian Lie algebra $M$.

1.2. We show that 2-cocycles that are related by a 2-coboundary define isomorphic extensions.

Suppose that $f_1, f_2 : \mathfrak{g} \times \mathfrak{g} \to M$ are 2-cocycles such that $f_2 - f_1$ is a 2-coboundary: $f_2(x, y) - f_1(x, y) = \rho(x)F(y) - \rho(y)F(x) - F([x, y])$ for some linear map $F : \mathfrak{g} \to M$ and $x, y \in \mathfrak{g}$. We consider the associated extension Lie algebras $\mathfrak{e}_i = (M \oplus \mathfrak{g}, [\ , \ ]_{f_i})$ and show that
\[\phi : (M \oplus \mathfrak{g}, [\ , \ ]_{f_1}) \to (M \oplus \mathfrak{g}, [\ , \ ]_{f_2}), \ (m, x) \mapsto (m + F(x), x)\]
is a Lie algebra isomorphism. This follows by a direct computation from (14)
\[\phi(m, x, \phi(m', x')) = (m + F(x), m' + F(x'), x')\]
\[= (\rho(x)m' + \rho(x)F(x') - \rho(x')m - \rho(x')F(x) - f_2(x, x'), [x, x']).\]
\[= (\rho(x)m' - \rho(x')m + F([x, x']) - f_1(x, x'), [x, x']) = \phi(([m, x], (m', x')])_{f_1}).\]

As the Lie algebra homomorphism $\phi$ is invertible with inverse
\[\phi^{-1} : M \oplus \mathfrak{g} \to M \oplus \mathfrak{g}, \ (m, x) \mapsto (m - F(x), x)\]
and we have for all $x \in \mathfrak{g}$ and $m \in M$
\[\phi \circ \iota(m) = \phi(m, 0) = (m, 0) = \iota(m) \quad \pi \circ \phi(m, x) = \pi(m + F(x), x) = x = \pi(m, x),\]
this shows that the $\phi$ is an isomorphism of Lie algebra extensions from $(\mathfrak{e}_1, \iota, \pi)$ to $(\mathfrak{e}_2, \iota, \pi)$. Hence, every element $[f] \in H^2(\mathfrak{g}, M)$ defines an isomorphism class of extensions of $\mathfrak{g}$ by $M$.

1.3. We show that an extension $(\mathfrak{e}, \iota, \pi)$ be an extension of $\mathfrak{g}$ by $M$ defines a representation $\rho : \mathfrak{g} \to \text{End}_F(M)$ and a 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to M$. If $(\mathfrak{e}, \iota, \pi)$ is an extension of $\mathfrak{g}$ by $M$, then $\iota(M) = \ker(\pi) \subset \mathfrak{e}$ is an abelian Lie subalgebra isomorphic to $M$. Because the Lie algebra homomorphism $\pi : \mathfrak{e} \to \mathfrak{g}$ is surjective, we can choose an element $\sigma(x) \in \pi^{-1}(x)$ for each $x \in \mathfrak{g}$ and obtain a map $\sigma : \mathfrak{g} \to \mathfrak{e}$ with $\pi \circ \sigma = \text{id}_\mathfrak{g}$. For all $m \in M$ and $x \in \mathfrak{e}$, we have $\pi([x, \iota(m)]) = [\pi(x), \pi \iota(m)] = [\pi(x), 0] = 0$ and hence $[x, \iota(m)] = \iota(M)$ for all $x \in \mathfrak{e}$, $m \in M$. As $\pi : \mathfrak{e} \to \mathfrak{g}$ is a Lie algebra morphism with $\pi \circ \sigma = \text{id}_\mathfrak{g}$, we also have $\pi([\sigma(x), \sigma(y)] - \sigma([x, y]) = [x, y] - [x, y] = 0$ and hence $[\sigma(x), \sigma(y)] - \sigma([x, y]) \in \iota(M) = \ker(\pi)$ for all $x, y \in \mathfrak{e}$. As $\iota : M \to \mathfrak{e}$ is injective, this defines two maps
\[\rho : \mathfrak{g} \to \text{End}_F(M) \quad \text{with} \quad \iota(\rho(x)m) = [\sigma(x), \iota(m)]\]
\[= [\sigma(x), \iota(\rho(y)m)] - [\sigma(y), \iota(\rho(x)m)] = [\sigma(x), \iota(\rho(y)m)] = [\sigma([x, y]), \iota(m)] + [\iota(f(x, x')), \iota(m)] \]
\[= \iota(\rho([x, y]), m) + \iota([f(x, x'), m]) = \iota(\rho([x, x'])), m),\]

Clearly, $\rho$ is linear and $f$ is alternating and bilinear. To show that $\rho$ is a representation of $\mathfrak{g}$ on $M$, we compute
\[\iota(\rho(x)\rho(y)m - \rho(y)\rho(x)m) = [\sigma(x), \iota(\rho(y)m)] - [\sigma(y), \iota(\rho(x)m)] = [\sigma(x), \iota(\rho(y)m)] = [\sigma([x, y]), \iota(m)] + [\iota(f(x, x')), \iota(m)] \]
\[= \iota(\rho([x, y]), m) + \iota([f(x, x'), m]) = \iota(\rho([x, x'])), m),\]
where we used first the definition of $\rho$, then the Jacobi identity in $\mathfrak{e}$ and the definition of $f$ to pass to the third line and the fact that $M$ is abelian in the last line. As $\iota$ is injective, this shows that $\rho : \mathfrak{g} \to \text{End}_\mathbb{F}(M)$ is a representation of $\mathfrak{g}$ on $M$. From the Jacobi identity in $\mathfrak{e}$, we obtain

$$\iota(\rho(x)f(y,z) - \rho(y)f(x,z) + \rho(z)f(x,y))$$

$$= [\sigma(x), \iota \circ f(y,z)] - [\sigma(y), \iota \circ f(x,z)] + [\sigma(z), \iota \circ f(x,y)]$$

$$= [\sigma(x), [\sigma(y), \sigma(z)]] - [\sigma(x), \sigma([y,z])] + [\sigma(y), [\sigma(x), \sigma(z)]] + [\sigma(y), \sigma([x,z])] - [\sigma(y), [\sigma(x), \sigma(y)]] + [\sigma(z), [\sigma(x), \sigma(y)]]$$

$$+ [\sigma(y), \sigma([x,z])] + [\sigma(z), [\sigma(x), \sigma(y)]] - [\sigma(z), [\sigma(x), \sigma(y)]]$$

$$= -[\sigma(x), \sigma([y,z])] + [\sigma(y), \sigma([x,z])] - [\sigma([y,z]), [\sigma(x), \sigma(y)] + [\sigma([x,z]), [\sigma(x), \sigma(y)] + [\sigma([y,z]), [\sigma(x), \sigma(y)]$$

$$= -f([x,y,z]) + f([x,z],y) - f(y,z,x)$$

Using again that $\iota$ is injective and comparing with (12), we see that $f$ is a 2-cocycle. Hence, we have shown that every extension of $\mathfrak{g}$ by $M$ defines a representation of $\mathfrak{g}$ on $M$ and a 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to M$.

1.4. We investigate how the representation and the 2-cocycle in 1.3 depend on $\sigma : \mathfrak{g} \to \mathfrak{c}$.

Let $\sigma_1, \sigma_2 : \mathfrak{g} \to \mathfrak{c}$ two maps with $\pi \circ \sigma_i = \text{id}_\mathfrak{g}$, let $\rho_i : \mathfrak{g} \to \text{End}_\mathbb{F}(M)$ by the associated representations and $f_i : \mathfrak{g} \times \mathfrak{g} \to M$ the associated 2-cocycles defined by (15). Then we have $\pi(\sigma_2(x) - \sigma_1(x)) = x - x = 0$, which implies $\sigma_2(x) - \sigma_1(x) \in \iota(M) = \ker(\pi)$ for all $x \in \mathfrak{g}$. As $\iota$ is injective, this defines a map

$$F : \mathfrak{g} \to M,$$ with $\iota(F(x)) = \sigma_2(x) - \sigma_1(x)$.

Then we have from (15)

$$\iota(\rho_2(x)m) = [\sigma_2(x), \iota(m)] = [\sigma_1(x) + \iota(F(x)), \iota(m)] = [\sigma_1(x), \iota(m)] + [\iota(F(x)), \iota(m)]$$

$$= [\sigma_1(x), \iota(m)] = \iota(\rho_1(x)m),$$

where we used the fact that $F(x) \in M$ and $M$ is abelian. As $\iota$ is injective, this implies $\rho_1 = \rho_2 =: \rho$. By a direct computation using the definitions, the fact that $M$ is abelian and the Jacobi identity in $\mathfrak{c}$ for all $x \in \mathfrak{c}$ and $m \in M$, we can relate this map to the 2-cocycles

$$\iota(f_2(x,y)) = [\sigma_2(x), \sigma_2(y)] - \sigma_2([x,y])$$

$$= [\sigma_1(x) + \iota \circ F(x), \sigma_1(y) + \iota \circ F(y)] - \sigma_1([x,y]) - \iota \circ F([x,y])$$

$$= [\sigma_1(x), \sigma_1(y)] - \sigma_1([x,y]) + [\sigma_1(x), \iota \circ F(y)] - [\sigma_1(y), \iota \circ F(y)] - \iota \circ F([x,y])$$

$$= \iota(f_1(x,y) + \rho(F(y) - \rho(y)F(x) - F([x,y])))$$

As $\iota$ is injective, this shows that the 2-cocycles $f_i : \mathfrak{g} \times \mathfrak{g} \to M$ are related by a coboundary: $f_2(x,y) - f_1(x,y) = \rho(x)F(y) - \rho(y)F(x) - F([x,y])$ for all $x, y \in \mathfrak{g}$. It follows that different choices of $\sigma$ define the same cohomology class $[f_1] = [f_2] \in H^2(\mathfrak{g}, M)$.

1.5 We show that isomorphic extensions define the same representations of $\mathfrak{g}$ and the same elements in $H^2(\mathfrak{g}, M)$.

If $(\mathfrak{c}, \iota, \pi)$ and $(\mathfrak{c}', \iota', \pi')$ are isomorphic extensions, then there is a Lie algebra isomorphism $\phi : \mathfrak{c} \to \mathfrak{c}'$ with $\phi \circ \iota = \iota'$ and $\pi' \circ \phi = \pi$. For any map $\sigma : \mathfrak{g} \to \mathfrak{c}$ with $\pi \circ \sigma = \text{id}_\mathfrak{g}$, the map $\sigma' = \phi \circ \sigma : \mathfrak{g} \to \mathfrak{c}'$ satisfies $\pi' \circ \sigma' = \pi \circ \phi \circ \sigma = \pi \circ \sigma = \text{id}_\mathfrak{g}$. This implies for the representation
\[\rho' : g \to \text{End}_F(M)\] and the 2-cocycle \(f' : g \times g \to M\) defined by \(\sigma'\)

\[\ell'(\rho'(x)m) = [\sigma'(x), \ell'(m)] = [\phi \circ \sigma(x), \phi \circ \ell(m)] = \phi([\sigma(x), \ell(m)]) = \phi \circ \ell(\rho(x)m) = \ell'(\rho(x)m)\]

\[\ell'(f'(x, y)) = [\sigma'(x), \sigma'(y)] - \sigma'([x, y]) = [\phi \circ \sigma(x), \phi \circ \sigma(y)] - \phi \circ \sigma([x, y])\]

As \(\ell' = \phi \circ \ell\) is the composite of injective maps, it is injective. It follows that the representations and cocycles defined by \(\sigma\) and \(\sigma'\) are equal, and so are the corresponding elements of \(H^2(g, M)\). As \(\phi : \mathfrak{c} \to \mathfrak{c}'\) is invertible, this proves that the isomorphic Lie algebra extensions \((\mathfrak{c}, \iota, \pi)\) and \((\mathfrak{c}', \iota', \pi')\) define the same element of \(H^2(g, M)\).

2. If \(\rho : g \to \text{End}_F(M), x \mapsto 0\) is a trivial representation, then the Lie bracket on \(M \oplus g\) from [14] takes the form \([[(m, x), (m', x')]} = (f(x, x'), [x, x'])\) for all \(x, x' \in g\) and \(m, m' \in M\).

This implies \([[m, 0], (m', x')]} = (f(0, x'), [0, x']) = f(0, 0) = 0\), and hence the extension \(M \oplus g\) is central. Conversely, if \((\mathfrak{c}, \iota, \pi)\) is a central extension of \(g\) by \(M\), then the representation \(\rho : g \to \text{End}_F(M)\) from [15] satisfies \(\iota(\rho(x)m) = [\sigma(x), \iota(m)] = 0\), and by injectivity of \(\iota\), this implies \(\rho(x)m = 0\) for all \(x \in g\) and \(m \in M\).

\[\square\]

2.5 Summary and questions

In this section, we encountered homologies and cohomologies for different mathematical objects, namely topological spaces, algebras and bimodules over algebras, groups and modules over group rings as well as Lie algebras and Lie algebra representations. Although the mathematical objects under consideration were very different, the associated (co)homology theories have strong structural similarities and in many cases also similar interpretations.

The general pattern of these homology theories is that we associated to the mathematical object under investigation a family \((X_n)_{n \in \mathbb{N}_0}\) of modules over a ring, the modules of \(n\)-cycles, and a family \((d_n)_{n \in \mathbb{N}_0}\) of module homomorphisms \(d_n : X_n \to X_{n-1}\), the boundary operators that satisfy the conditions \(d_{n-1} \circ d_n = 0\). This condition ensured that \(\text{im}(d_{n+1}) \subset \text{ker}(d_n)\) and allowed us to define the homologies of the mathematical object under consideration as the quotients \(H_n = \text{ker}(d_n)/\text{im}(d_{n+1})\). The homologies contained information about the underlying mathematical object. The patterns for cohomology theories were similar, only that the direction of the coboundary operators is reversed with respect to the boundary operators.

There are many other examples of homology and cohomology theories such as deRham-cohomology of smooth manifolds, symplectic homology, intersection cohomology on surfaces and homologies in more advanced settings such as braided tensor categories. While the mathematical objects under consideration are different, the general pattern is the same as for the examples in this section. The variety of contexts in which they can be applied makes (co)homologies a very useful tool in many areas of mathematics.

While the examples treated so far illustrate the versatility and usefulness of (co)homologies, the treatment of this examples was too pedestrian and has serious limitations. In particular, the examples considered so far raise the following questions that call for a more systematic and abstract investigation of (co)homologies:

- Although we assigned homologies to objects in certain categories (topological spaces, bimodules over algebras, modules over group rings and representations of Lie algebras)
we did not consider the *morphisms* in these categories so far. Do morphisms in these categories (continuous maps between topological spaces, morphisms of bimodules, morphisms of modules or morphisms of representations) induce maps between the homologies associated to their objects? Is there a systematic way of including morphisms in the picture?

- What is the origin of the modules of (co)chains and the (co)boundary operators in the concrete examples? Is there a general construction or formalism that allows one to formulate (co)homology theories for objects in any category that satisfies certain assumptions? Are (co)chains necessarily realised as modules over certain rings and (co)boundary operators as module morphisms, or is there a more general framework?

- In all examples considered so far, the (co)boundary operators were obtained as an alternating sum over certain module homomorphisms that were largely combinatorial in nature such as the face maps in singular (co)homology and the maps that multiply two adjacent factors in a tensor product of algebras or a product of groups. Is this a general pattern oder a coincidence? What is the appropriate mathematical framework to formulate this question more precisely?

- How much arbitrariness is there in the definition of the (co)chains and (co)boundary operators? Are the (co)boundary operators introduced so far essentially the only way of defining these structures, or are there many other formulations that lead to equivalent definitions of (co)homologies? How much does the concrete choice of chain complex matter? Is there a way to define (co)homologies that relies less on the choice of modules of \(n\)-(co)chains and (co)boundary operators and more on the objects under consideration?

- What is the appropriate algebraic framework to analyse and discuss the similarities between different (co)homology theories in precise mathematical terms?

We will derive answers to these questions in the next sections. This requires a more abstract approach and relies heavily on the language of categories and functors.
3 Chain complexes, chain maps and chain homotopies

3.1 Abelian categories

In this section, we determine the most general mathematical setting for (co)chains, (co)boundary operators and (co)homologies. The starting point is the following observation. Although the examples in Section 2 were concerned with very different mathematical data, we always associated to this data a family of modules over a ring $R$ and a family of $R$-linear maps between them such that the composite of two subsequent maps vanishes. This allowed us to define (co)homologies as quotients of kernels and images of these $R$-linear maps.

This suggests that the appropriate setting for a general (co)homology theory could be categories of modules over rings. However, it turns out that this is neither the most general possibility nor an efficient viewpoint. Instead, we determine the most general setting for (co)homology theories abstractly, in terms of categories. We start with a category $\mathcal{C}$ and investigate which additional structure is needed in $\mathcal{C}$ in order to formulate (co)homology theories as in Section 2.

- All of the examples in Section 2 made use of direct sums of $R$-modules and the fact that there is a trivial $R$-module $\{0\}$, which can be viewed as a direct sum of $R$-modules over an empty index set. As the direct sum of $R$-modules is an example of a categorical coproduct, one should at least impose that coproducts in $\mathcal{C}$ exist for all finite families $(C_i)_{i \in I}$ of objects in $\mathcal{C}$. For symmetry and because this will follow automatically from this and another condition, we will also impose that products in $\mathcal{C}$ exist for all finite families of objects in $\mathcal{C}$.

- The (co)boundary operators in the examples from in Section 2 were defined as an alternating sum of certain $R$-module morphisms. To generalise this construction to a category $\mathcal{C}$, we need to be able to take sums of morphisms in $\mathcal{C}$ and to ensure that this is compatible with the composition of morphisms. Hence, we have to impose that all morphism sets $\text{Hom}_\mathcal{C}(X,Y)$ have the structure of abelian groups and that the group addition is compatible with the composition of morphisms in $\mathcal{C}$.

- To define (co)homologies in the examples from in Section 2, we considered kernels of $R$-module morphisms and took quotients by their images. Hence, the category $\mathcal{C}$ needs to be equipped with a concept of kernels and images that mimics the kernels and images of $R$-linear maps and gives rise to a sensible notion of homology.

The first two conditions lead to the concept of an additive category. Functors between additive categories that respect these conditions are called additive functors.

**Definition 3.1.1:** A category $\mathcal{C}$ is called additive if

(Add1) For all objects $C, C'$ of $\mathcal{C}$ the set of morphisms $\text{Hom}_\mathcal{C}(C,C')$ has the structure of an abelian group, and the composition of morphisms is $\mathbb{Z}$-bilinear: $g \circ (f + f') = g \circ f + g \circ f'$ and $(g + g') \circ f = g \circ f + g' \circ f$ for all morphisms $f, f' : X \to Y$ and $g, g' : Y \to Z$.

(Add2) Products and coproducts exist for all finite families of objects in $\mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories $\mathcal{C}, \mathcal{D}$ is called additive if for all objects $C, C'$ in $\mathcal{C}$ the map $F : \text{Hom}_\mathcal{C}(C,C') \to \text{Hom}_\mathcal{D}(F(C), F(C'))$ is a group homomorphism.

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Remark 3.1.2:

1. In particular, Definition 3.1.1 requires the existence of an empty product and an empty coproduct, a terminal object \( T = \Pi_\emptyset \) and an initial object \( I = \Pi_\emptyset \) (see Definition 1.1.15). In an additive category \( C \), these objects are isomorphic and hence zero objects: \( I \cong T \cong 0 \).

This follows because one has \( \text{Hom}_C(I, I) = \{1_I\} = \{0\} \) by definition of an initial object, where 0 denotes the neutral element of the abelian group \( \text{Hom}_C(I, I) \). If \( C \) is additive, this implies \( f = 1_I \circ f = 0 \circ f = 0 : C \to I \) for any morphism \( f : C \to I \), since the composition of morphisms is \( \mathbb{Z} \)-bilinear. It follows that \( \text{Hom}_C(C, I) = \{0\} \) and hence \( I \) is terminal.

2. It follows that for any two objects \( C, C' \) in an additive category \( C \), the neutral element of the abelian group \( 0 = \text{Hom}_C(C, C') \) is given by \( 0 = i_{C'} \circ t_C : C \to C' \).

3. Finite products and coproducts in additive categories are canonically isomorphic:
\[
\Pi_{i \in I} C_i \cong \Pi_{i \in I} C_i \quad \text{for all finite index sets } i \in I \text{ and objects } C_i \text{ in } C.
\]

The isomorphism is induced by the family \((f_j)_{i,j \in I}\) of morphisms \( f_{ij} = \delta_{ij} 1_{C_i} : C_i \to C_j \) with \( f_{ij} = 0 \) for \( i \neq j \) and \( f_{ii} = 1_{C_i} \). By the universal property of the product and the coproduct, there is a unique morphism \( f : \Pi_{k \in I} C_k \to \Pi_{k \in I} C_k \) with \( \pi_j \circ f \circ t_i = \delta_{ij} 1_{C_i} \).

The inverse of this morphism is given by \( f^{-1} = \Sigma_{i \in I} t_i \circ \pi_i : \Pi_{k \in I} C_k \to \Pi_{k \in I} C_k \), since
\[
\pi_k \circ f \circ f^{-1} = \Sigma_{i \in I} \pi_k \circ f \circ t_i \circ \pi_i = \Sigma_{i \in I} \delta_{ik} 1_{C_i} \circ \pi_i = \pi_k
\]
\[
f^{-1} \circ f \circ t_k = \Sigma_{i \in I} t_i \circ \pi_i \circ f \circ t_k = \Sigma_{i \in I} t_i \circ \delta_{ik} 1_{C_i} = t_k \quad \forall k \in I,
\]
and the universal property of the (co)product implies \( f \circ f^{-1} = 1_{\Pi_{i \in I} C_i} \), \( f^{-1} \circ f = 1_{\Pi_{i \in I} C_i} \).

4. The abelian group structure on the morphism sets \( \text{Hom}_C(C, C') \) in an additive category \( C \) is determined uniquely by its products and coproducts.

For a finite index set \( I \) and an object \( C \) in \( C \) we denote by \( \phi_C : \Pi_{i \in I} C \to \Pi_{i \in I} C \) the unique morphism with \( \pi_i \circ \phi_C \circ t_j = \delta_{ij} 1_C \) from 3. with inverse \( \phi_C^{-1} = \Sigma_{i \in I} t_i \circ \pi_i : \Pi_{i \in I} C \to \Pi_{i \in I} C \).

We also consider the unique morphism \( \Delta_C : C \to \Pi_{i \in I} C \) with \( \pi_i \circ \Delta_C = 1_C \) for all \( i \in I \), the unique morphism \( \nabla_C : \Pi_{i \in I} C \to C \) with \( \nabla \circ t_i = 1_C \) for all \( i \in I \) and for a finite family \((f_i)_{i \in I}\) of morphisms \( f_i : C \to D \), the unique morphism \( f : \Pi_{i \in I} C \to \Pi_{i \in I} D \) with \( \pi_j \circ f \circ t_i = \delta_{ij} f_i \) from 3. Then we have
\[
\nabla_D \circ \phi_D^{-1} \circ f \circ \phi_C^{-1} \circ \Delta_C = \Sigma_{i,j \in I} \nabla_D \circ t_i \circ \pi_i \circ f \circ t_j \circ \pi_j \circ \Delta_C
\]
\[
= \Sigma_{i,j \in I} \delta_{ij} 1_D \circ f_i \circ 1_C = \Sigma_{i \in I} \delta_{ij} 1_D \circ f_i \circ 1_C = \Sigma_{i \in I} f_i.
\]

Hence, we expressed the sum of the morphisms \( f_i \) in terms of quantities that are defined in terms of the product and coproduct in an additive category. (This includes the morphism \( f \), since the zero object that enters its definition is the empty coproduct.) As products and coproducts are unique up to unique isomorphisms, a given category \( C \) has at most one additive structure. Additivity is a property, not a choice of structure.

5. An object \( X \) in an additive category \( C \) is a product or coproduct of a finite family of objects \((C_i)_{i \in I}\) if and only if there are families \((i_j)_{j \in I}\) and \((p_j)_{j \in I}\) of morphisms \( i_j : C_j \to X \) and \( p_j : X \to C_j \) with \( p_j \circ i_k = \delta_{jk} 1_{C_j} \) and \( 1_X = \Sigma_{i \in I} i_j \circ p_j \) (Exercise 21).

6. A functor \( F : C \to D \) between additive categories \( C, D \) is additive if and only if it preserves finite products or finite coproducts (Exercise 21):
\[
F(\Pi_{i \in I} C_i) \cong \Pi_{i \in I} F(C_i), \quad F(\Pi_{i \in I} C_i) \cong \Pi_{i \in I} F(C_i) \quad \text{for all finite families of objects } (C_i)_{i \in I}.
\]
Example 3.1.3:

1. For any ring $R$ the category $R$-Mod of $R$-modules and $R$-linear maps is additive.
   Products and coproducts are products and direct sums of modules and exist for all families of modules. The set $\text{Hom}_R(M,N)$ of $R$-linear maps $f : M \to N$ is an abelian group with the pointwise addition, and this is compatible with their composition.

2. For any ring homomorphism $\phi : R \to S$, the functor $F_\phi : S$-Mod $\to R$-Mod that sends an $S$-module $(M,\triangleright)$ to the $R$-module $(M,\triangleright_R)$ with $r \triangleright_R m = \phi(r) \triangleright m$ and an $S$-linear map $f : (M,\triangleright) \to (M',\triangleright')$ to the associated $R$-linear map $f : (M,\triangleright_R) \to (M',\triangleright'_R)$ is additive.

3. Every full subcategory of an additive category $C$ in which finite products and coproducts exist, is an additive category as well.

4. For every small category $C$ and additive category $A$, the category $\text{Fun}(C,A)$ of functors $F : C \to A$ and natural transformations between them is an additive category.
   - The product of a family of functors $(F_i)_{i \in I}$ is the functor $\Pi_{i \in I} F_i : C \to A$ that assigns to an object $C$ the product $\Pi_{i \in I} F_i(C)$ and to a morphism $\alpha : C \to C'$ the unique morphism $\Pi_{i \in I} F_i(\alpha) : \Pi_{i \in I} F_i(C) \to \Pi_{i \in I} F_i(C')$ with $\pi_{iC} \circ \Pi_{i \in I} F_i(\alpha) = F_i(\alpha) \circ \pi_{iC}$, where $\pi_{iC} : \Pi_{i \in I} F_i(C) \to F_i(C)$ are the projection morphisms for the product in $A$.
   - The projection morphisms for $\Pi_{i \in I} F_i$ are the natural transformations $\pi_i : \Pi_{i \in I} F_i \to F_i$ with component morphisms $\pi_{iC} : \Pi_{i \in I} F_i(C) \to F_i(C)$.
   - Coproducts of functors are defined analogously, and the sum of two natural transformations $\eta,\kappa : F \to G$ is the natural transformation $\eta + \kappa : F \to G$ with component morphisms $(\eta + \kappa)_C = \eta_C + \kappa_C : F(C) \to G(C)$.

In any additive category $A$ we can consider a generalisation of chains, namely families $(C_n)_{n \in \mathbb{Z}}$ of objects in $C$, and boundary operators between them, namely families $(d_n)_{n \in \mathbb{Z}}$ of morphisms $d_n : C_n \to C_{n-1}$ with $d_{n-1} \circ d_n = 0 : C_n \to C_{n-2}$ for all $n \in \mathbb{Z}$. An analogous definition is possible for cochains and coboundary operators. However, to define homologies we also require a notion of kernels, images and quotients of kernels by images. In contrast to the standard definition of a kernel and image of an $R$-module morphism $f : M \to N$, as subsets of the modules $M$ and $N$, a sensible categorical notion of a kernel and image must be formulated purely in terms of morphisms and universal properties. It does not require additivity, but relies on the existence of a zero object $0$ in $C$ and the associated zero morphisms $0 = i_{C'} \circ t_C : C \to 0 \to C'$.

Definition 3.1.4: Let $C$ be a category with a zero object and $f : X \to Y$ a morphism in $C$.

1. A kernel of $f$ is a morphism $\iota : \text{ker}(f) \to X$ with the following universal property: $f \circ \iota = 0 : \text{ker}(f) \to Y$, and for every morphism $g : W \to X$ with $f \circ g = 0 : W \to Y$ there is a unique morphism $g' : W \to \text{ker}(f)$ with $\iota \circ g' = g$. 

\[
\begin{array}{cccc}
\text{ker}(f) & \xrightarrow{\iota} & X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow & & \\
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
W & & W & & \\
\end{array}
\]
2. A cokernel of \( f \) is a morphism \( \pi : Y \to \text{coker}(f) \) with the following universal property:\n\[ \pi \circ f = 0 : X \to \text{coker}(f), \text{and for every morphism } g : Y \to W \text{ with } g \circ f = 0 : X \to W \text{ there is a unique morphism } g' : \text{coker}(f) \to W \text{ with } g' \circ \pi = g. \]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \xrightarrow{\pi} \text{coker}(f) \\
\downarrow{0} & & \downarrow{g} \\
W & \xrightarrow{\exists g'} & \end{array}
\]

3. A kernel of a cokernel of \( f \) is called an image of \( f \) and denoted \( \iota' : \text{im}(f) \to Y \). A cokernel of a kernel of \( f \) is called a coimage of \( f \) and denoted \( \pi' : X \to \text{coim}(f) \).

Remark 3.1.5: As (co)kernels and (co)images are defined by a universal property, they are unique up to unique isomorphism: If \( \iota \) \( \phi \) \( \eta \) \( \chi \) \( \rho \) \( \xi \)

\[ \text{as } (\text{co})\text{kernels and (co)images are defined by a universal property, they are unique up to unique isomorphism: If } \iota : \ker(f) \to X, \eta : \ker(f) \to X \text{ are two kernels for } f : X \to Y, \text{ then there is a unique morphism } \phi : \ker(f) \to \ker(f)' \text{ with } \eta \circ \phi = \iota, \text{ and this morphism is an isomorphism. Analogous statements hold for cokernels, images and coimages.} \]

Example 3.1.6: Let \( R \) be a ring and \( f : M \to N \) an \( R \)-linear map.

- The inclusion map \( \iota : \ker(f) \to M \) is a kernel of \( f \) in \( R \text{-Mod.} \)
- The canonical surjection \( \pi : N \to N/\text{im}(f) \) is a cokernel of \( f \) in \( R \text{-Mod.} \)
- The canonical inclusion \( \iota' : \text{im}(f) \to N \) is an image of \( f \) in \( R \text{-Mod.} \)
- The canonical surjection \( \pi' : M \to M/\ker(f) \) is a coimage of \( f \) in \( R \text{-Mod.} \)

That \( \iota : \ker(f) \to M \) is a kernel of \( f \) follows, because \( f \circ \iota = 0 \) and for any \( R \)-linear map \( \phi : L \to M \) with \( f \circ \phi = 0 \), one has \( \text{im}(\phi) \subset \ker(f) \). The corestriction \( \phi' : L \to \ker(f), l \mapsto \phi(l) \) is an \( R \)-linear map with \( \iota \circ \phi' = \phi \). As \( \iota \) is injective, it is the only one.

That \( \pi : N \to N/\text{im}(f) \) is a cokernel of \( f \) follows, because \( \pi \circ f = 0 \) and for any \( R \)-linear map \( \psi : N \to P \) with \( \psi \circ f = 0 \) one has \( \text{im}(f) \subset \ker(\psi) \). By the universal property of the quotient there is a unique \( R \)-linear map \( \psi' : N/\text{im}(f) \to P, [n] \mapsto \psi(n) \) with \( \psi' \circ \pi = \psi \).

That the inclusion map \( \iota' : \text{im}(f) \to N \) is a kernel of \( \pi : N \to N/\text{im}(f) \) follows, because \( \pi \circ \iota' = 0 \) and for any \( R \)-linear map \( \chi : L \to N \) with \( \pi \circ \chi = 0 \), one has \( \text{im}(\chi) \subset \ker(\pi) = \text{im}(f) \). The corestriction \( \chi' : L \to \text{im}(f), l \mapsto \chi(l) \) satisfies \( \chi' \circ \iota' = \chi \) and is the only \( R \)-linear map with this property, since \( \iota' \) is injective.

That the canonical surjection \( \pi' : M \to M/\ker(f) \) is a cokernel of \( \iota : \ker(f) \to M \) follows because \( \pi' \circ \iota = 0 \) and any \( R \)-linear map \( \xi : M \to P \) with \( \xi \circ \iota = 0 \) satisfies \( \text{im}(\iota) = \ker(f) \subset \ker(\xi) \). The corestriction \( \xi' : L \to \text{im}(f), l \mapsto \xi(l) \) satisfies \( \xi' \circ \pi' = \xi \) and is the only \( R \)-linear map with this property.

In addition to kernels and cokernels, we also require an appropriate concept of injectivity and surjectivity and need to relate it to kernels and cokernels. Just as for kernels and cokernels, the appropriate notion of injectivity and surjectivity in a category needs to be formulated purely in terms of morphisms and universal properties. It is obtained from the observation that a map \( \iota : X \to Y \) is injective (a map \( \pi : X \to Y \) is surjective) if and only if \( \iota \circ f = \iota \circ g (f \circ \pi = g \circ \pi) \) implies \( f = g \) for all maps \( f, g : W \to X \) (for all maps \( f, g : Y \to Z \)). This notion of injectivity and surjectivity in Set generalises to any category.
Definition 3.1.7: Let $C$ be a category.

1. A morphism $\iota : X \to Y$ in $C$ is called a **monomorphism**, if $\iota \circ f = \iota \circ g$ for morphisms $f,g : W \to X$ implies $f = g$.

2. A morphism $\pi : X \to Y$ in $C$ is called an **epimorphism**, if $f \circ \pi = g \circ \pi$ for morphisms $f,g : Y \to Z$ implies $f = g$.

In diagrams, monomorphisms $\iota : M \to N$ are denoted $\begin{array}{c} M \xrightarrow{\iota} \end{array}$ $N$ and epimorphisms $\pi : N \to P$ are denoted $\begin{array}{c} N \xrightarrow{\pi} \end{array} P$.

Remark 3.1.8: Clearly, every isomorphism is a monomorphism and an epimorphism. However, a morphism that is a monomorphism and an epimorphism need not be an isomorphism. A counterexample is the inclusion morphism $\iota : \mathbb{Z} \to \mathbb{Q}$ in the category of unital rings.

We now relate epimorphisms and monomorphisms to (co)kernels and (co)images. Example 3.1.6 shows that in the category $R$-Mod the kernel $\iota : \ker(f) \to M$ of an $R$-linear map $f : M \to N$ is injective and its cokernel $\pi : N \to N/\im(f)$ is surjective. Moreover, the module morphism $0 \to M$ is a kernel of $f$ if and only if $f$ is injective and the module morphism $N \to 0$ is a cokernel of $f$ if and only if $f$ is surjective. Analogues of this hold in all additive categories.

Lemma 3.1.9: Let $C$ be an additive category.

1. All kernels of morphisms in $C$ are monomorphisms. A morphism $f : X \to Y$ is a monomorphism if and only if the morphism $i_X : 0 \to X$ is a kernel of $f$.

2. All cokernels of morphisms in $C$ are epimorphisms. A morphism $f : X \to Y$ is an epimorphism if and only if the morphism $t_Y : Y \to 0$ is a cokernel of $f$.

Proof:
We prove the first statement. The proof of the second one is analogous. Let $\iota : \ker(f) \to X$ be a kernel of $f : X \to Y$ and $g_1,g_2 : W \to \ker(f)$ morphisms with $\iota \circ g_1 = \iota \circ g_2$. Then we have $f \circ (\iota \circ g_1) = (f \circ \iota) \circ g_1 = 0 \circ g_1 = 0 : W \to Y$, and by the universal property of the kernel, there is a unique morphism $g' : W \to \ker(f)$ with $\iota \circ g' = \iota \circ g_1 = \iota \circ g_2$. The uniqueness implies $g' = g_1 = g_2$, and hence $\iota : W \to \ker(f)$ is a monomorphism.

Let now $f : X \to Y$ be a monomorphism. Then $f \circ i_X = i_Y = 0 : 0 \to Y$. If $g : W \to X$ is a morphism with $f \circ g = 0 : W \to X$ then $f \circ i_X \circ t_W = 0 : W \to X$ as well, and because $f$ is a monomorphism, it follows that $g = i_X \circ t_W$. Hence, $i_X : 0 \to X$ is a kernel of $f$. Conversely, if $i_X : 0 \to X$ is a kernel of $f$ and $g_1,g_2 : W \to X$ are morphisms with $f \circ g_1 = f \circ g_2$, then $f \circ (g_1 - g_2) = 0$ and by the universal property of the kernel, there is a unique morphism $g' : W \to 0$ with $i_X \circ g' = g_1 - g_2 = 0$. Since $g' = t_W : W \to 0$ is the only morphism from $W$ to $0$, we have $g_1 - g_2 = i_X \circ t_W = 0 : W \to X$ and $g_1 = g_2$. This shows that $f$ is a monomorphism.

This lemma shows that in any additive category, kernels are monomorphisms and cokernels epimorphisms, as expected from the corresponding statement for $R$-Mod. However, in the category $R$-Mod, the converse also holds. Every injective $R$-linear map $f : M \to N$ is a kernel, namely of its cokernel $\pi : N \to N/\im(f)$. This follows because $\pi \circ f = 0$, and for every $R$-linear map $g : L \to N$ with $\pi \circ g = 0$ one has $\im(g) \subset \ker(\pi) = \im(f)$. Hence, by injectivity of $f$ there is a unique $R$-linear map $g' : L \to M$ with $f \circ g' = g$. 

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Similarly, every surjective \( R \)-linear map \( f : M \to N \) is a cokernel of its kernel \( \iota : \ker(f) \to M \). One has \( f \circ \iota = 0 \) and \( \ker(f) = \im(\iota) \subseteq \ker(g) \) for every \( R \)-linear map \( g : M \to L \) with \( g \circ \iota = 0 \). As \( f \) is surjective, there is a unique \( R \)-linear map \( g' : N \to L, f(m) \to g(m) \) with \( g' \circ f = g \).

In contrast to the claims in Lemma \( \text{(3.1.9)} \) these statements do not hold automatically in an additive category and need to be imposed. If we also require that every morphism has a kernel and a cokernel, we obtain the notion of an abelian category, which we will use later as the framework for (co)homology.

We also consider functors between abelian categories that are compatible with the required structures. Clearly, such functors need to be additive and map kernels to kernels and cokernels to cokernels. We will see in the following that there are many additive functors that satisfy only one of the last two conditions and that these functors play an important role in (co)homology. For this reason, they also receive a name.

**Definition 3.1.10:**

1. An additive category is called **abelian** if it satisfies the following additional conditions:
   
   (Ab1) Every morphism has a kernel and a cokernel.
   
   (Ab2) Every monomorphism is a kernel of its cokernel or, equivalently, an image of itself.
   
   (Ab3) Every epimorphism is a cokernel of its kernel or, equivalently, a coimage of itself.

2. A functor \( F : \mathcal{A} \to \mathcal{B} \) between abelian categories \( \mathcal{A}, \mathcal{B} \) is called
   
   - **left exact** if it is additive and sends kernels in \( \mathcal{A} \) to kernels in \( \mathcal{B} \):
     
     if \( \iota : \ker(f) \to X \) is a kernel of \( f : X \to Y \), then \( F(\iota) : F(\ker(f)) \to F(X) \) is a kernel of \( F(f) : F(X) \to F(Y) \).
   
   - **right exact** if it is additive and sends cokernels in \( \mathcal{A} \) to cokernels in \( \mathcal{B} \):
     
     if \( \pi : Y \to \coker(f) \) is a cokernel of \( f : X \to Y \), then \( F(\pi) : F(Y) \to F(\coker(f)) \) is a kernel of \( F(f) : F(X) \to F(Y) \).
   
   - **exact** if it is left exact and right exact.

**Example 3.1.11:**

1. For any ring \( R \), the category \( R\text{-Mod} \) is abelian.

   By Example \( \text{(3.1.3)} \), it is additive, and by Example \( \text{(3.1.6)} \), every \( R \)-linear map \( f : M \to N \) has a kernel \( \iota : \ker(f) \to M \) and a cokernel \( \pi : N \to N/\im(f) \). We also showed before Definition \( \text{(3.1.10)} \) that every monomorphism in \( R\text{-Mod} \) is a kernel of its cokernel and every epimorphism a cokernel of its kernel.

2. For any abelian category \( \mathcal{A} \), the category \( \mathcal{A}^{\text{op}} \) is abelian. Kernels and cokernels in \( \mathcal{A}^{\text{op}} \) correspond to cokernels and kernels in \( \mathcal{A} \), respectively (Exercise \( \text{(26)} \)).

3. For any small category \( \mathcal{C} \) and any abelian category \( \mathcal{A} \) the category \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) of functors \( F : \mathcal{C} \to \mathcal{A} \) and natural transformations between them is abelian (Exercise \( \text{(27)} \)).

**Remark 3.1.12:**

1. One can show that in an abelian category \( \mathcal{A} \) a morphism that is both a monomorphism and an epimorphism is an isomorphism (Exercise \( \text{(25)} \)).
2. Like additivity, being abelian is a property of a category and not a choice of structure. If all objects in an additive category have kernels and cokernels that satisfy the conditions in Definition 3.1.10, they are unique up to unique isomorphism and determined by the additive structure.

3. Mitchell’s embedding theorem states that any small abelian category \( \mathcal{A} \) is equivalent to a full subcategory of the abelian category \( R\text{-Mod} \) for some ring \( R \), with an exact equivalence of categories. For a proof, see [Mi, p 151].

Although Mitchell’s embedding theorem allows one to interpret any small abelian category as a subcategory of the abelian category \( R\text{-Mod} \) for a suitable ring \( R \), it is still advantageous to work with general abelian categories. Firstly, there are also non-small abelian categories. Secondly, the construction of the associated ring \( R \) and the subcategory of \( R\text{-Mod} \) for an abelian category \( \mathcal{A} \) in Mitchell’s embedding theorem is implicit. It does not give rise to a description that is useful in concrete computations. It is often simpler to use the general formalism for abelian categories. However, we will sometimes use Mitchell’s embedding theorem to conduct proofs in \( R\text{-Mod} \) that become too cumbersome and technical in general abelian categories. This does not restrict generality of the proofs if the claims involve only a small full subcategory of \( \mathcal{A} \) that is again an abelian category.

By definition of an abelian category, kernels and cokernels exist for all morphisms and generalise the inclusion maps \( \iota : \ker(f) \to X \) and the canonical surjections \( \pi : Y \to Y/\text{im}(f) \) for \( R\)-linear maps \( f : X \to Y \). However, in order to define homologies we require one additional ingredient. If \( d_{n+1} : X_{n+1} \to X_n \) and \( d_n : X_n \to X_{n-1} \) are \( R\)-linear maps with \( d_n \circ d_{n+1} = 0 \), then \( \text{im}(d_{n+1}) \subseteq \ker(d_n) \), and there is an inclusion map \( \iota : \text{im}(d_{n+1}) \to \ker(d_n) \), which is a monomorphism in \( R\text{-Mod} \). This allows one to define the homologies as the quotients \( \ker(d_n)/\text{im}(d_{n+1}) \) or, equivalently, as the cokernels of the inclusion \( \iota : \text{im}(d_{n+1}) \to \ker(d_n) \). To generalise this construction to abelian categories, we require a monomorphism \( \phi : \text{im}(f) \to \ker(g) \) for all morphisms \( f : X \to Y \) and \( g : Y \to Z \) with \( g \circ f = 0 \). This is provided by the following lemma.

**Lemma 3.1.13:** Let \( \mathcal{A} \) be an abelian category.

1. Every morphism \( f : X \to Y \) in \( \mathcal{A} \) factorises as \( f = \iota'_f \circ \pi'_f \) where \( \iota'_f : \text{im}(f) \to Y \) is an image of \( f \) and \( \pi'_f : X \to \text{im}(f) \) a coimage of \( f \). This is called the **canonical factorisation** of \( f \).

2. If \( f : X \to Y \), \( g : Y \to Z \) are morphisms in \( \mathcal{A} \) with \( g \circ f = 0 \), there is a unique monomorphism \( \phi : \text{im}(f) \to \ker(g) \) such that the following diagram commutes

\[
\begin{array}{ccc}
\text{im}(f) & \xrightarrow{\exists! \phi} & \ker(g) \\
\pi'_f \downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
\end{array}
\]

\[ (16) \]

**Proof:**

1. For any morphism \( f : X \to Y \) in \( \mathcal{A} \), we have \( \pi_f \circ f = 0 \) for the cokernel \( \pi_f : Y \to \text{coker}(f) \). The universal property of the image \( \iota'_f : \text{im}(f) \to Y \) implies that there is a unique morphism \( \pi'_f : X \to \text{im}(f) \) with \( \iota'_f \circ \pi'_f = f \). We show that \( \pi'_f : X \to \text{im}(f) \) is an epimorphism. The first claim then follows because every epimorphism is its own coimage, or, equivalently, a cokernel of its kernel. By Exercise 22, the morphisms \( \pi'_f \) and \( f = \iota'_f \circ \pi'_f \) have the same kernel and hence the same coimage \( \pi'_f : X \to \text{im}(f) \).
To show that $\pi'_f : X \to \text{im}(f)$ is an epimorphism, let $\phi : \text{im}(f) \to U$ be a morphism with $\phi \circ \pi'_f = 0$. By the universal property of the kernel $\iota_\phi : \ker(\phi) \to \text{im}(f)$, there is a unique morphism $f'$ with $\iota_\phi \circ f' = \pi'_f$:

$$
\begin{array}{ccc}
\ker(\phi) & \xrightarrow{\iota_\phi} & \text{im}(f) \\
\downarrow{\exists f'_{\phi}} & & \downarrow{\phi} \\
X & \xrightarrow{f'} & Y.
\end{array}
$$

The morphism $\iota'_f \circ \iota_\phi : \ker(\phi) \to Y$ is a monomorphism as a composite of two monomorphisms. Hence, it is a kernel of its cokernel $\pi' : Y \to \text{coker}(\iota'_f \circ \iota_\phi)$. This implies

$$
\pi' \circ f = \pi' \circ (\iota'_f \circ \iota_\phi \circ f') = (\pi' \circ \iota'_f \circ \iota_\phi) \circ f' = 0 \circ f' = 0,
$$

and by the universal property of the cokernel $\pi_f : f \to \text{coker}(f)$ there is a unique morphism $\pi'' : \text{coker}(f) \to \text{coker}(\iota'_f \circ \iota_\phi)$ with $\pi'' \circ \pi_f = \pi'$

$$
\begin{array}{ccc}
\ker(\phi) & \xrightarrow{\iota_\phi} & \text{im}(f) \\
\downarrow{f} & & \downarrow{\phi} \\
X & \xrightarrow{f} & Y \\
\downarrow{\pi'_f} & & \downarrow{\pi'} \\
\text{coker}(f) & \xrightarrow{\text{coker}(\iota'_f \circ \iota_\phi)} & \text{coker}(f).
\end{array}
$$

This implies $\pi' \circ \iota'_f = (\pi'' \circ \pi_f) \circ \iota'_f = \pi'' \circ (\pi_f \circ \iota'_f) = \pi'' \circ 0 = 0$, since $\iota'_f : \text{im}(f) \to Y$ is a kernel of $\pi_f : Y \to \text{coker}(f)$. As $\iota'_f \circ \iota_\phi$ is a kernel of $\pi'$ and $\pi' \circ \iota'_f = 0$, the universal property of the kernel $\iota'_f \circ \iota_\phi$ implies that there is a unique morphism $\iota'' : \text{im}(f) \to \ker(\phi)$ with $\iota'_f \circ \iota_\phi \circ \iota'' = \iota'_f$. Because $\iota'_f$ is a monomorphism, it follows that $\iota_\phi \circ \iota'' = 1_{\text{im}(f)}$. As $\iota_\phi$ is a kernel of $\phi$, this implies $\phi = \phi \circ 1_{\text{im}(f)} = \phi \circ (\iota_\phi \circ \iota'') = (\phi \circ \iota_\phi) \circ \iota'' = 0 \circ \iota'' = 0$, and $\pi'_f : X \to \text{im}(f)$ is an epimorphism.

2. We consider the commuting diagram

$$
\begin{array}{ccc}
\text{im}(f) & \xrightarrow{\iota'_f} & \ker(g) \\
\downarrow{\pi_f} & & \downarrow{\iota_g} \\
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{\iota_g} \\
& Z.
\end{array}
$$

As $g \circ f = g \circ \iota'_f \circ \pi'_f = 0$ and $\pi'_f$ is an epimorphism, we have $g \circ \iota'_f = 0$. By the universal property of the kernel $\iota_g : \ker(g) \to Y$ there is a unique morphism $\phi : \text{im}(f) \to \ker(g)$ with $\iota_g \circ \phi = \iota'_f$. If $\phi \circ h = 0$ for some morphism $h : U \to \text{im}(f)$, then $0 = \iota_g \circ \phi \circ h = \iota'_f \circ h = 0 = \iota'_f \circ 0$, and because $\iota'_f$ is a monomorphism, we obtain $h = 0$. Hence, $\phi$ is a monomorphism. □

After clarifying the properties of abelian categories, we now focus on functors that are compatible with abelian categories - exact functors - and on functors that are partially compatible - left or right exact functors. It turns out that there are few exact functors, and most of them arise from certain canonical constructions such as products or coproducts or evaluation of a functor on objects and morphisms. Important examples are the following.
Example 3.1.14:

1. For any abelian category $\mathcal{A}$, the cartesian product $\mathcal{A} \times \mathcal{A}$ is abelian and the functors $\Pi: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\Pi: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are exact. (Exercise 28).

2. For any abelian category $\mathcal{A}$, small category $\mathcal{C}$ and object $C$ in $\mathcal{C}$, the evaluation functor $ev_C: \text{Fun}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$ that sends a functor $F: \mathcal{C} \to \mathcal{A}$ to the object $F(C)$ and a natural transformation $\eta: F \to G$ to the component morphism $\eta_C: F(C) \to G(C)$ is exact (Exercise 27).

Most functors that are relevant for homology are not exact, only left exact or right exact. In fact, we will see in Section 4 that we can view homologies as a measure of the non-exactness of left or right exact functors. The most important example of a right exact functor in homology is the functor $A \otimes_R- : \text{R-Mod} \to \text{Ab}$ for an $R$-right module $A$. It sends a $R$-module $B$ to the abelian group $A \otimes_R B$ and an $R$-linear map $f : B \to C$ to the group homomorphism $\text{id}_A \otimes f : A \otimes_R B \to A \otimes_R C$, $a \otimes b \mapsto a \otimes f(b)$.

Lemma 3.1.15:
For any right module $A$ over a ring $R$ the functor $A \otimes_R - : \text{R-Mod} \to \text{Ab}$ is right exact.

Proof:
Let $\pi : C \to D$ be a cokernel of an $R$-linear map $f : B \to C$. Then $\ker(\pi) = \text{im}(f)$ and $\pi$ is surjective. The latter implies that the group homomorphism $\text{id}_A \otimes \pi : A \otimes_R Y \to A \otimes_R Z$, $a \otimes y \mapsto a \otimes \pi(y)$ is surjective as well and the former implies $\text{im}(\text{id}_A \otimes f) \subseteq \ker(\text{id}_A \otimes \pi)$. To prove that $\text{id} \otimes \pi$ is a cokernel of $\text{id} \otimes f$, it is then sufficient to show that $\text{im}(\text{id}_A \otimes f) \supset \ker(\text{id}_A \otimes \pi)$.

For this, we consider the canonical surjection $p : A \otimes_R C \to A \otimes_R C / \text{im}(\text{id}_A \otimes f)$ and construct an $R$-linear map $q' : A \otimes_R D \to A \otimes_R C / \text{im}(\text{id}_A \otimes f)$ with $q' \circ (\text{id}_A \otimes \pi) = p$. The last equation then implies $\text{im}(\text{id}_A \otimes f) = \ker(p) \supset \ker(\text{id}_A \otimes \pi)$.

As $\pi$ is surjective, we can choose for every element $d \in D$ an element $i(d) \in \pi^{-1}(d)$ and obtain a map $i : C \to D$ with $\pi \circ i = \text{id}_D$. The map $q : A \times D \to A \otimes_R C / \text{im}(\text{id}_A \otimes f)$, $(a, d) \mapsto p(a \otimes i(d))$ satisfies

$q(a + a', d) = p((a + a') \otimes i(d)) = p(a \otimes i(d) + a' \otimes i(d)) = p(a \otimes i(d)) + p(a' \otimes i(d))$

$q(a, d + d') = p(a \otimes i(d + d')) = p(a \otimes i(d) + i(d')) + p(a \otimes (i(d + d') - i(d) - i(d')))$

$q(a \otimes r, d) = p(a \otimes (r \triangleright i(d))) = p(a \otimes i(r \triangleright d)) + p(a \otimes (r \triangleright i(d) - i(r \triangleright d)))$

The identity $\pi \circ i = \text{id}_D$ implies

$\pi(i(d + d') - i(d) - i(d')) = \pi \circ i(d + d') - \pi \circ i(d) - \pi \circ i(d') = d + d' - d - d' = 0$

$\pi(r \triangleright i(d) - i(r \triangleright d)) = r \triangleright \pi \circ i(d) = \pi \circ i(r \triangleright d) = r \triangleright d - r \triangleright d = 0$,

$\Rightarrow i(d + d') - i(d) - i(d'), r \triangleright i(d) - i(r \triangleright d) \in \ker(\pi) = \text{im}(f)$

$\Rightarrow a \otimes i(d + d') - i(d) - i(d'), a \otimes (r \triangleright i(d) - i(r \triangleright d)) \in \text{im}(\text{id}_A \otimes f) = \ker(p)$
for all $a \in A$, $d, d' \in D$ and $r \in R$. This shows that the map $q$ is $R$-bilinear, and by the universal property of the tensor product, it induces a unique group homomorphism

$$q' : A \otimes_R D \to A \otimes_R C / \text{im}(\text{id} \otimes f), \ a \otimes d \mapsto q(a, d) = p(a \otimes i(d)).$$

This group homomorphism satisfies

$$q' \circ (\text{id}_A \otimes \pi)(a \otimes c) = p(a \otimes i(\pi(c))) = p(a \otimes c) = p(a \otimes (i(\pi(c)) - c)) = p(a \otimes c) \quad \forall a \in A, c \in C,$$

since we have $\pi(i(c)) - c = \pi \circ (\pi(c) - c) = \pi(c) - \pi(c) = 0$ for all $c \in C$, which implies $\pi(i(c)) = \pi(c) = 0$. As $\pi$ is injective with $\ker(\pi) = \text{im}(f)$ and $a \otimes \pi(c) - c \in \ker(\pi) = \text{im}(f)$ for all $a \in A, c \in C$. \hfill \Box

**Remark 3.1.16:**
In general the functor $A \otimes_R - : R\text{-Mod} \to \text{Ab}$ is not left exact.

If one takes $R = \mathbb{Z}$, $A = \mathbb{Z}/n\mathbb{Z}$ and the morphisms $\iota : \mathbb{Z} \to \mathbb{Z}$, $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $z \mapsto nz$, $\bar{\pi} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $z \mapsto \bar{z}$ then $\iota$ is injective with $\ker(\pi) = \text{im}(\iota)$ and hence a kernel of $\pi$. But the group homomorphism $\text{id} \otimes \iota : \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}$ is given by $(\text{id} \otimes \iota)(k \otimes z) = k \otimes (nz) = \bar{n}k \otimes z = 0 \otimes z = 0$ for all $\bar{k} \in \mathbb{Z}/n\mathbb{Z}$ and $z \in \mathbb{Z}$. Hence, $\text{id} \otimes \iota = 0$ is not injective and cannot be a kernel of $\text{id} \otimes \pi$.

**Definition 3.1.17:**
A right module $A$ over a ring $R$ is called flat if the functor $A \otimes_R - : R\text{-Mod} \to \text{Ab}$ is exact.

The main examples of left exact functors are the functors $\text{Hom}(A, -) : \mathcal{A} \to \text{Ab}$ and $\text{Hom}(-, A) : \mathcal{A}^{\text{op}} \to \text{Ab}$ for an abelian category $\mathcal{A}$ and an object $A$ in $\mathcal{A}$. The functor $\text{Hom}(A, -)$ sends an object $B$ to the abelian group $\text{Hom}_A(A, B)$ and a morphism $f : B \to C$ to the group homomorphism $\text{Hom}(A, f) : \text{Hom}_A(A, B) \to \text{Hom}_A(A, C)$, $g \mapsto f \circ g$. The functor $\text{Hom}(-, A)$ sends an object $B$ in $\mathcal{A}$ to the abelian group $\text{Hom}_A(B, A)$ and a morphism $f : B \to C$ to the group homomorphism $\text{Hom}(f, A) : \text{Hom}_A(A, C) \to \text{Hom}_A(A, B)$, $g \mapsto g \circ f$.

**Lemma 3.1.18:** Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ a morphism in $\mathcal{A}$.

1. For any object $A$ in $\mathcal{A}$ the functor $\text{Hom}(A, -) : \mathcal{A} \to \text{Ab}$ is left exact:
   - A morphism $\iota : W \to X$ is a kernel of $f : X \to Y$ in $\mathcal{A}$ if and only if for all objects $A$ in $\mathcal{A}$ the morphism $\iota_* = \text{Hom}(A, \iota)$ in $\text{Ab}$ is a kernel of $f_* = \text{Hom}(A, f)$.

2. For any object $A$ in $\mathcal{A}$ the functor $\text{Hom}(-, A) : \mathcal{A}^{\text{op}} \to \text{Ab}$ is left exact:
   - A morphism $\pi : Y \to Z$ is a cokernel of $f : X \to Y$ in $\mathcal{A}$ if and only if for all objects $A$ in $\mathcal{A}$ the morphism $\pi^* = \text{Hom}(\pi, A)$ in $\text{Ab}$ is a kernel of $f^* = \text{Hom}(f, A)$.

**Proof:**
We prove the first claim. The proof of the second claim is analogous if one takes into account that kernels and cokernels in $\mathcal{A}$ are cokernels and kernels in $\mathcal{A}^{\text{op}}$, respectively.

As we work in $\text{Ab} = \mathbb{Z}\text{-Mod}$, Example 3.1.6 implies that the group homomorphism $\iota_*$ is a kernel of the group homomorphism $f_*$ if and only if (i) $\iota_*$ is injective and (ii) $\text{im}(\iota_*) = \ker(f_*)$. Condition (i) is satisfied if and only if $\iota \circ g = \iota \circ g'$ implies $g = g'$ for all $g, g' \in \text{Hom}_A(A, W)$, and this is equivalent to the statement that $\iota$ is a monomorphism. Condition (ii) is satisfied if and only if (iia) $f \circ \iota \circ h = 0$ for all objects $A$ in $\mathcal{A}$, $h \in \text{Hom}_A(A, W)$ and (iib) for every morphism $g : A \to X$ with $f \circ g = 0$ there is a morphism $g' : W \to X$ with $\iota \circ g' = g$. Condition (iia) is satisfied if $f \circ \iota = 0$, and by setting $A = W$ and $h = 1_W$ we see that (iia) implies $f \circ \iota = 0$. As $\iota$ is injective, condition (iib) then states that $\iota$ is a kernel of $f$. \hfill \Box
Remark 3.1.19: The functors $\text{Hom}(A, -)$ and $\text{Hom}(-, A)$ are in general not right exact. An example is the object $A = \mathbb{Z}/n\mathbb{Z}$ in the abelian category $\mathcal{A} = \text{Ab}$, for which the functors $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, -)$ and $\text{Hom}(-, \mathbb{Z}/n\mathbb{Z})$ are not exact.

The injective group homomorphism $\iota : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ and the surjective group homomorphism $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $z \mapsto \bar{z}$ satisfy $\ker(\pi) = \text{im}(\iota)$. Hence, $\pi$ is a cokernel of $\iota$ and $\iota$ a kernel of $\pi$.

However, because $\text{Hom}_{\text{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ and $\text{Hom}_{\text{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$, the morphism $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \pi) : \text{Hom}_{\text{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \text{Hom}_{\text{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ has a trivial image, is not surjective and cannot be a cokernel of $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \iota)$. Hence, $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, -)$ is not right exact.

Similarly, $\text{Hom}(\iota, \mathbb{Z}/n\mathbb{Z}) : \text{Hom}_{\text{Ab}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \text{Hom}_{\text{Ab}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, $g \mapsto g \circ \iota$ is trivial since $g \circ \iota(z) = g(nz) = ng(z) = 0$ for all $z \in \mathbb{Z}$ and group homomorphisms $g : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Hence, $\text{Hom}(\iota, \mathbb{Z}/n\mathbb{Z}) = 0$ is not surjective and cannot be a cokernel of $\text{Hom}(\pi, \mathbb{Z}/n\mathbb{Z})$. This shows that $\text{Hom}(-, \mathbb{Z}/n\mathbb{Z})$ is not right exact.

Objects $A$ in an abelian category $\mathcal{A}$ for which $\text{Hom}(A, -) : \mathcal{A} \to \text{Ab}$ or $\text{Hom}(-, A) : \mathcal{A}^{\text{op}} \to \text{Ab}$ are exact will play an important role in the following. For this reason, they receive a name, and we derive an alternative characterisation of such objects that is easier to handle.

Definition 3.1.20: An object $A$ in an abelian category $\mathcal{A}$ is called

- **projective** if the functor $\text{Hom}(A, -) : \mathcal{A} \to \text{Ab}$ is exact,
- **injective** if the functor $\text{Hom}(-, A) : \mathcal{A}^{\text{op}} \to \text{Ab}$ is exact.

Lemma 3.1.21: Let $\mathcal{A}$ be an abelian category.

1. An object $A$ in $\mathcal{A}$ is projective if and only if for every epimorphism $\pi : X \to Y$ and every morphism $f : A \to Y$ there is a morphism $f' : A \to X$ with $\pi \circ f' = f$.

$$
\begin{array}{ccc}
A & \xrightarrow{f'} & X \\
\downarrow{f} & & \downarrow{\pi} \\
0 & \rightarrow & Y
\end{array}
$$

2. An object $A$ in $\mathcal{A}$ is injective if and only if for every monomorphism $\iota : X \to Y$ and every morphism $f : X \to A$ there is a morphism $f' : Y \to A$ with $f' \circ \iota = f$.

$$
\begin{array}{ccc}
A & \xleftarrow{f} & 0 \\
\downarrow{f'} & & \downarrow{\iota} \\
X & \rightarrow & Y
\end{array}
$$

Proof:
We prove the first statement. The proof of the second one is analogous.

Let $A$ be projective. Then $\text{Hom}(A, -)$ is exact and maps kernels to kernels and cokernels to cokernels. As every epimorphism $\pi : X \to Y$ in an abelian category is a cokernel of its kernel $\iota : \ker(\pi) \to X$, the morphism $\text{Hom}(A, \pi) : \text{Hom}_{\mathcal{A}}(A, X) \to \text{Hom}_{\mathcal{A}}(A, Y)$ is a cokernel of $\text{Hom}(A, \iota)$ and hence an epimorphism by Lemma 3.1.9. This means that for every morphism $f : A \to Y$, there is a morphism $f' : A \to X$ with $\text{Hom}(A, \pi)(f') = \pi \circ f' = f$. 

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Conversely, if for every morphism \( f : A \rightarrow Y \) and epimorphism \( \pi : X \rightarrow Y \) there is a morphism \( f' : A \rightarrow X \) with \( \pi \circ f' = f \), then \( \text{Hom}(A, \pi) : \text{Hom}_A(A, X) \rightarrow \text{Hom}_A(A, Y) \) is an epimorphism for every epimorphism \( \pi : X \rightarrow Y \).

Let \( f : A \rightarrow X \) be a morphism with cokernel \( \pi : X \rightarrow Y \). By Lemma \[3.1.13\] we have \( f = \iota' \circ \pi' \) with a monomorphism \( \iota' \) and an epimorphism \( \pi' \) and by Exercise \[22\] the morphism \( \pi : X \rightarrow Y \) is a cokernel of the monomorphism \( \iota' \). As \( \iota' \) is a monomorphism, it is a kernel of its cokernel \( \pi : X \rightarrow Y \), and by left-exactness of \( \text{Hom}(A, -) \), this implies that \( \text{Hom}(A, \iota) \) is a kernel of the epimorphism \( \text{Hom}(A, \pi) \). As every epimorphism is a cokernel of its kernel, it follows that \( \text{Hom}(A, \iota) \) is a cokernel of \( \text{Hom}(A, \iota) \). As \( \text{Hom}(A, f) = \text{Hom}(A, \iota' \circ \pi') = \text{Hom}(A, \iota') \circ \text{Hom}(A, \pi') \) with an epimorphism \( \text{Hom}(A, \pi') \), Exercise \[22\] implies that \( \text{Hom}(A, \iota) \) is also a cokernel of \( \text{Hom}(A, f) \) and hence \( \text{Hom}(A, -) \) is right exact. \( \square \)

**Example 3.1.22:**

1. By Remark \[3.1.19\] the objects \( \mathbb{Z}/n\mathbb{Z} \) in the category \( \text{Ab} \) are neither projective nor injective.

2. For every ring \( R \), any free \( R \)-module is projective.
   
   If \( A \) is a free \( R \)-module with basis \( B \), \( \pi : X \rightarrow Y \) \( R \)-linear and surjective and \( f : A \rightarrow Y \) \( R \)-linear, then we can choose for every element \( b \in B \) an element \( f'(b) \in \pi^{-1}(f(b)) \) and obtain a map \( f'' : B \rightarrow X \) with \( \pi \circ f'' = f|_B \). As \( B \) is a basis of \( A \), there is a unique \( R \)-linear map \( f' : A \rightarrow X \) with \( f'|_B = f'' \). This implies \( \pi \circ f'|_B = f|_B \) and hence \( \pi \circ f' = f \).

3. The object \( \mathbb{Z} \) in \( \text{Ab} \) is projective, but not injective.
   
   The projectivity of \( \mathbb{Z} \) follows from 2. However, \( \mathbb{Z} \) is not injective, because for the monomorphism \( \iota : \mathbb{Z} \rightarrow \mathbb{Z}, \ z \mapsto nz \) with \( n > 1 \) and the group homomorphism \( f = \text{id}_\mathbb{Z} : \mathbb{Z} \rightarrow \mathbb{Z} \), there is no morphism \( f' : \mathbb{Z} \rightarrow \mathbb{Z} \) with \( f' \circ \iota = f = \text{id}_\mathbb{Z} \).

Lemma \[3.1.21\] not only gives simple criteria for projectivity and injectivity but also allows one to extend the notions of projectivity and injectivity from abelian categories to general categories. As the concepts in Lemma \[3.1.21\] are defined in any category, we can take the conditions in Lemma \[3.1.21\] as the definition of projectivity and injectivity in non-abelian categories.

### 3.2 Chain complexes and homology

We are now ready to define and investigate homology theories in general abelian categories \( \mathcal{A} \). The fundamental concept is that of a chain complex, which generalises the (co)chains and (co)boundary operators in the examples from Section \[2\]. It is obtained by replacing the modules of \( n \)-(co)chains by objects in an abelian category \( \mathcal{A} \) and the (co)boundary operators by morphisms such that subsequent morphisms compose to zero morphisms.

**Definition 3.2.1:** Let \( \mathcal{A} \) be an abelian category.

1. A chain complex \((X_\bullet, d_\bullet)\) in \( \mathcal{A} \) is a family \( X_\bullet = (X_n)_{n \in \mathbb{Z}} \) of objects and a family \( d_\bullet = (d_n)_{n \in \mathbb{Z}} \) of morphisms \( d_n : X_n \rightarrow X_{n-1} \) in \( \mathcal{A} \) with \( d_{n-1} \circ d_n = 0 \) for all \( n \in \mathbb{Z} \).

\[
\cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots
\]
2. A chain map \( f_* : (X_\bullet, d_\bullet) \rightarrow (X'_\bullet, d'_\bullet) \) is a family \( (f_n)_{n \in \mathbb{Z}} \) of morphisms \( f_n : X_n \rightarrow X'_n \) such that \( d'_n \circ f_n = f_{n-1} \circ d_n \) for all \( n \in \mathbb{Z} \).

\[
\begin{array}{cccccccc}
\cdots & d_{n+2} & X_{n+1} & d_{n+1} & X_n & d_n & X_{n-1} & d_{n-1} & \cdots \\
\downarrow f_{n+1} \quad & \quad \downarrow f_n \quad & \quad \downarrow f_{n-1} \quad & & & & & \\
\cdots & d'_{n+2} & X'_{n+1} & d'_{n+1} & X'_n & d'_n & X'_{n-1} & d'_{n-1} & \cdots 
\end{array}
\]

**Notation 3.2.2:** It is standard to omit subsequences of zero objects and morphisms between them from chain complexes:

- \( 0 \rightarrow X_m \xrightarrow{d_m} X_{m-1} \xrightarrow{d_{m-1}} \cdots \) stands for a chain complex with \( X_k = 0 \) for all \( k > m \). Such a chain complex is called **bounded above**. It is called **negative** if \( m = 0 \).
- \( \cdots \xrightarrow{d_{m+1}} X_m \xrightarrow{d_m} X_{m-1} \rightarrow 0 \) stands for a chain complex with \( X_k = 0 \) for all \( k < m - 1 \). Such a chain complex is called **bounded below**. It is called **positive** if \( m = 0 \).
- If \( X_k = 0 \) for all \( k < m \) and \( k > l > m \) the chain complex is called **finite** or **bounded** and denoted \( 0 \rightarrow X_l \xrightarrow{d_l} \xrightarrow{d_{l-1}} X_{l-1} \xrightarrow{d_{l-2}} \cdots X_0 \xrightarrow{d_0} X_1 \rightarrow 0 \).
- A chain complex \( \cdots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} 0 \rightarrow 0 \rightarrow \cdots \xrightarrow{d_{n-m}} X_{n-m-1} \xrightarrow{f_{n-m-1}} \cdots \) is viewed as two chain complexes, one with \( X_k = 0 \) for \( k < n \) and one with \( X_k = 0 \) for \( k \geq n - m \).

We also denote a chain complex \( (X_\bullet, d_\bullet) \) simply by \( X_\bullet \) when this causes no ambiguity.

**Remark 3.2.3:** Analogously, one defines a cochain complex \( (X^\bullet, d^\bullet) \) in an abelian category \( \mathcal{A} \) as a family \( X^\bullet = (X^n)_{n \in \mathbb{Z}} \) of objects \( X^n \) in \( \mathcal{A} \) together with a family \( d^\bullet = (d^n)_{n \in \mathbb{Z}} \) of morphisms \( d^n : X^n \rightarrow X^{n+1} \) with \( d^{n+1} \circ d^n = 0 \) for all \( n \in \mathbb{Z} \). However, we can restrict attention to chain complexes since a cochain complex \( (X^\bullet, d^\bullet) \) can be transformed into a chain complex \( (X_\bullet, d_\bullet) \) by setting \( X_n = X^{-n} \) and \( d_n = (-1)^n : X_n \rightarrow X_{n-1} \) for all \( n \in \mathbb{Z} \).

**Remark 3.2.4:**

1. Chain complexes and chain maps in an abelian category \( \mathcal{A} \) form a category \( \text{Ch}_\mathcal{A} \) with the composition of morphisms \( g_\bullet \circ f_\bullet = (g_n \circ f_n)_{n \in \mathbb{Z}} \) and the identity chain maps \( 1_{X_\bullet} = (1_{X_n})_{n \in \mathbb{Z}} \) as identity morphisms.

\[
\begin{array}{cccccccc}
\cdots & d_{n+2} & X_{n+1} & d_{n+1} & X_n & d_n & X_{n-1} & d_{n-1} & \cdots \\
\downarrow f_{n+1} \quad & \quad \downarrow f_n \quad & \quad \downarrow f_{n-1} \quad & & & & & \\
\cdots & d'_{n+2} & X'_{n+1} & d'_{n+1} & X'_n & d'_n & X'_{n-1} & d'_{n-1} & \cdots 
\end{array}
\]

2. The category \( \text{Ch}_\mathcal{A} \) is abelian:

- Coproducts of chain complexes given by \( \Pi_{i \in I} X^i_{\bullet} = (\Pi_{i \in I} X^i_n)_{n \in \mathbb{Z}} \) and the chain maps \( \Pi_{i \in I} d^i_{\bullet} = (\Pi_{i \in I} d^i_n)_{n \in \mathbb{Z}} \), where \( \Pi_{i \in I} d^i_n : \Pi_{i \in I} X^i_n \rightarrow \Pi_{i \in I} X^i_{n-1} \) is induced by the morphisms \( d^i_n : X^i_n \rightarrow X^i_{n-1} \) and the universal property of the coproduct.
• Products are defined analogously.

• The addition of chain maps is given by \( f_* + g_* = (f_n + g_n)_{n \in \mathbb{Z}}. \)

• Kernels and cokernels of a chain map \( f_* = (f_n)_{n \in \mathbb{Z}} : X_* \to X'_* \) are given by

\[
\ker(f_*) = (\ker(f_n) : X_n \to X_n')_{n \in \mathbb{Z}} \quad \text{and} \quad \operatorname{coker}(f_*) = (\operatorname{coker}(f_n) : X_n' \to X_n)_{n \in \mathbb{Z}}.
\]

3. Bounded chain complexes, chain complexes that are bounded below, positive chain complexes, chain complexes that are bounded above and negative chains complexes in \( \mathcal{A} \) form full abelian subcategories \( \text{Ch}_{\mathcal{A}^\text{fin}}, \text{Ch}_{\mathcal{A}^+}, \text{Ch}_{\mathcal{A}^\geq 0}, \text{Ch}_{\mathcal{A}^-} \) and \( \text{Ch}_{\mathcal{A}^\leq 0} \) of \( \text{Ch}_{\mathcal{A}}. \)

The last remark has important implications. It allows one to consider chain complexes in the abelian category \( \text{Ch}_{\mathcal{A}} \) of chain complexes in \( \mathcal{A} \) and to relate their homologies to the homologies of certain chain complexes in \( \mathcal{A} \). This leads to techniques that are useful for the computation of homologies. We will see basic examples of this phenomenon in Section 4.5. The general formalism that extends these basic examples are the so-called spectral sequences.

All the examples of (co)homologies from Section 2 define (co)chain complexes in abelian categories \( \mathcal{A} = k\text{-Mod} \) for some commutative ring \( k \). The objects \( X_n \) of the (co)chain complexes are the modules of \( n \)-(co)chains and the morphisms \( d_n \) the (co)boundary operators. The data that defines the (co)chain complexes from Section 2 are objects in a certain category \( \mathcal{C} \), such as \( \mathcal{C} = \text{Top}, \mathcal{C} = \mathcal{A}\text{-Mod}-A \) for a \( k \)-algebra \( A \), \( \mathcal{C} = k[G]\text{-Mod} \) for a group \( G \) or \( \mathcal{C} = \text{Rep}(\mathfrak{g}) \) for a Lie algebra \( \mathfrak{g} \). It turns out that morphisms in \( \mathcal{C} \) define (co)chain maps between these (co)chain complexes. As the assignment of (co)chain maps to morphisms in \( \mathcal{C} \) is compatible with the composition of morphisms and the identity morphisms, we can view them as functors \( F : \mathcal{C} \to \text{Ch}_{\mathcal{A}} \) \( (F : \mathcal{C}^\text{op} \to \text{Ch}_{\mathcal{A}}) \) from the category \( \mathcal{C} \) \( (\mathcal{C}^\text{op}) \) under investigation into the category \( \text{Ch}_{\mathcal{A}} \) of chain complexes in the abelian category \( \mathcal{A} = k\text{-Mod}. \)

**Example 3.2.5:**

1. The chain complex \( (C_\bullet(X,k), d_\bullet) \) in \( \mathcal{A} = k\text{-Mod} \) from Definition 2.1.2 is called the **singular chain complex** of \( X \) with coefficients in \( k \) and given by

\[
C_n(X,k) = \langle \sigma : \Delta^n \to X \text{ continuous} \rangle_k
\]

\[
d_n(\sigma)(t_1, ..., t_{n-1}) = \sigma(1-t_1 - ... - t_{n-1}, t_1, ..., t_{n-1}) - \sigma(0, t_1, ..., t_{n-1}) + \sigma(t_1, 0, t_2, ..., t_{n-1})
\]

\[
\pm \ldots + (-1)^{n-1} \sigma(t_1, ..., t_{n-2}, 0, t_{n-1}) + (-1)^n \sigma(t_1, ..., t_{n-1}, 0)
\]

A continuous map \( f : X \to Y \) induces a chain map \( C_\bullet(f,k) : C_\bullet(X,k) \to C_\bullet(Y,k) \) with \( C_n(f,k)(\sigma) = f \circ \sigma : \Delta^n \to X \) for all singular \( n \)-simplices \( \sigma : \Delta^n \to X \). This defines a functor \( C_\bullet(-,k) : \text{Top} \to \text{Ch}_{k\text{-Mod}}. \)

2. The cochain complex \( (C^\bullet(X,M), d^\bullet) \) in \( \mathcal{A} = k\text{-Mod} \) from Definition 2.1.12 is called the **singular cochain complex** of \( X \) with coefficients in \( M \) and given by

\[
C^n(X,M) = \text{Hom}_k(C_n(X,k), M)
\]

\[
d^n(\phi)(\sigma) = \phi(d_{n+1}\sigma) \quad \text{for all continuous} \quad \sigma : \Delta^{n+1} \to X.
\]

A continuous map \( f : X \to Y \) induces a cochain map \( C^\bullet(f,M) : C^\bullet(Y,M) \to C^\bullet(X,M) \) with \( C^n(f,k)(\phi) = \phi \circ f : C_n(X,k) \to M \) for all \( k \)-linear maps \( \phi : C_n(Y,k) \to M \). This defines a functor \( C^\bullet(-,M) : \text{Top}^\text{op} \to \text{Ch}_{k\text{-Mod}}. \)
3. Analogous statements hold for the **simplicial chain complex** \((C_\bullet(\Delta, k), d_\bullet)\) from Definition \[2.1.9\] and the **simplicial cochain complex** \((C^\bullet(\Delta), d^\bullet)\) from Definition \[2.1.13\] and for simplicial maps \(f : \Delta \to \Delta'\).

4. The chain complex \((C_\bullet(M, A), d_\bullet)\) in \(\mathcal{A} = \text{k-Mod}\) from Definition \[2.2.3\] is called the **Hochschild complex** of \(A\) with coefficients in \(M\) and given by

\[
C_n(A, M) = M \otimes_k A^n
\]

\[
d_n(m \otimes a_1 \otimes \ldots \otimes a_n) = m \cdot (a_1 a_2 \otimes \ldots \otimes a_n) - m \otimes a_1 a_2 \otimes \ldots \otimes a_n + (-1)^n m \otimes a_1 \otimes \ldots \otimes a_{n-1} a_n
\]

The cochain complex \((C^\bullet(M, A), d^\bullet)\) in \(\mathcal{A} = \text{k-Mod}\) from Definition \[2.2.4\] is called the **Hochschild cocomplex** of \(A\) with coefficients in \(M\) and given by

\[
C^n(A, M) = \text{Hom}_k(A^n, M)
\]

\[
d^n(\phi)(a_0 \otimes \ldots \otimes a_n) = a_0 \cdot \phi(a_1 \otimes \ldots \otimes a_n) - \phi(a_0 a_1 a_2 \otimes \ldots \otimes a_n) + \phi(a_0 \otimes a_1 a_2 \otimes a_3 \otimes \ldots \otimes a_n)
\]

\[
\quad \quad \quad \quad \quad \pm \ldots + (-1)^{n-1} \phi(a_0 \otimes \ldots \otimes a_{n-2} a_{n-1} a_n) - (-1)^n \phi(a_0 \otimes \ldots \otimes a_{n-1}) a_n
\]

Every \((A, A)\)-bimodule morphism \(f : M \to N\) defines (co)chain maps

\[
C_\bullet(A, f) : C_\bullet(A, M) \to C_\bullet(A, N) \quad \text{with} \quad C_n(A, f) = f \otimes \text{id}_A^n : C_n(A, M) \to C_n(A, N)
\]

\[
C^\bullet(A, f) : C^\bullet(A, M) \to C^\bullet(A, N) \quad \text{with} \quad C^n(A, f) : C^n(A, M) \to C^n(A, N), \phi \mapsto f \circ \phi.
\]

This defines functors \(C_\bullet(A, -), C^\bullet(A, -) : \text{A-Mod-A} \to \text{Ch}_{k\text{-mod}}\).

5. The cochain complex \((C^\bullet(G, M), d^\bullet)\) in \(\mathcal{A} = \text{k-Mod}\) from Definition \[2.3.1\] is called the **cochain complex of group cohomology** and given by

\[
C^n(G, M) = \text{Map}(G^n, M)
\]

\[
d^n(f)(g_0, \ldots, g_n) = g_0 \cdot f(g_1, \ldots, g_n) - f(g_0 g_1, g_2, \ldots, g_n) + f(g_0, g_1, g_2, g_3, \ldots, g_n)
\]

\[
\quad \quad \quad \quad \quad \pm \ldots + (-1)^{n-1} f(g_0, \ldots, g_{n-2}, g_{n-1} g_n) - (-1)^n f(g_0, \ldots, g_{n-1})
\]

Every morphism \(f : M \to N\) of \(k[G]\)-modules induces a cochain map

\[
C^\bullet(G, f) : C^\bullet(G, M) \to C^\bullet(G, N) \quad \text{with} \quad C^n(G, f) : C^n(G, M) \to C^n(G, N), \phi \mapsto f \circ \phi,
\]

and this defines a functor \(C^\bullet(G, -) : k[G]\text{-Mod} \to \text{Ch}_{k\text{-Mod}}\).

If we choose for the \(k\)-module \(M\) a \(k\)-module \(M\) with the trivial \(k[G]\)-module structure, then every group homomorphism \(f : G \to H\) defines a cochain map

\[
C^\bullet(f, M) : C^\bullet(H, M) \to C^\bullet(G, M)
\]

\[
\text{with} \quad C^n(f, M) : C^n(H, M) \to C^n(G, M), \phi \mapsto \phi \circ f^x^n.
\]

This defines a functor \(C^\bullet(-, M) : \text{Grp}^{op} \to \text{Ch}_{k\text{-Mod}}\).

6. The cochain complex \((C^\bullet(g, M), d^\bullet)\) in \(\mathcal{A} = \text{Vect}_F\) from Definition \[2.4.8\] is called the **Chevalley-Eilenberg complex** and given by

\[
C^n(g, M) = \text{Hom}_F(\Lambda^n g^*, M)
\]

\[
(d^n f)(x_0, \ldots, x_n) = \sum_{i=0}^n (-1)^i \rho(x_i) f(x_0, \ldots, \hat{x}_i, \ldots, x_n)
\]

\[
+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n).
\]
Every morphism $f : M \rightarrow N$ of $\mathfrak{g}$-representations induces a cochain map

$$C^\bullet(\mathfrak{g}, f) : C^\bullet(\mathfrak{g}, M) \rightarrow C^\bullet(\mathfrak{g}, N)$$

with $C^n(f) : C^n(\mathfrak{g}, M) \rightarrow C^n(\mathfrak{g}, N)$, $\phi \mapsto f \circ \phi$,

and this defines a functor $C^\bullet(\mathfrak{g}, -) : \text{Rep}(\mathfrak{g}) \rightarrow \text{Ch}_{\text{Vect}}$. In particular, we can consider for any Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ the trivial representation on a given vector space $M$ over $\mathbb{F}$. Then every Lie algebra morphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$ defines a cochain map

$$C^\bullet(f, M) : C^\bullet(\mathfrak{h}, M) \rightarrow C^\bullet(\mathfrak{g}, M)$$

with $C^n(f, M) : C^n(\mathfrak{h}, M) \rightarrow C^n(\mathfrak{g}, M)$, $\phi \mapsto \phi \circ f^\times n$.

This defines a functor $C^\bullet(-, M) : \text{Liealg}_{\mathfrak{g}}^{\text{op}} \rightarrow \text{Ch}_{\text{Vect}}$.

The (co)chain maps in these examples are structural or canonical in the sense that they are images of morphisms in the category $\mathcal{C}$ under investigation under a functor $F : \mathcal{C} \rightarrow \text{Ch}_\mathcal{A}$. We will see later that this is a general pattern and not specific to these examples. However, there are also less obvious chain maps that do not arise from such morphisms.

**Example 3.2.6:**

Let $k$ be a commutative ring, $G$ a group and $M$ a $k[G]$-module. Denote by $G^{\times n}_k$ the free $k$-module $(G^{\times n})_k$ with the $k[G]$-module structure $g \triangleright (g_1, \ldots, g_n) = (gg_1, \ldots, gg_n)$ and by $\text{Hom}_{k[G]}(G^{\times n}_k, M)$ the $k$-module of $k[G]$-linear maps $f : G^{\times n}_k \rightarrow M$.

Let $X^\bullet$ be the cochain complex in $k\text{-Mod}$ with $X^n = \text{Hom}_{k[G]}(G^{\times (n+1)}_k, M)$ for $n \in \mathbb{N}_0$ and coboundary operators $d^n = \Sigma_{i=0}^{n+1} (-1)^i d^n_i : X^n \rightarrow X^{n+1}$ with

$$d^n_i(\phi)(g_0, \ldots, g_{n+1}) = \phi(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}).$$

Then the $k$-linear maps $f^n : X^n \rightarrow C^n(M, G)$, $\phi \mapsto f^n(\phi)$ with

$$f^n(\phi)(g_1, \ldots, g_n) = \phi(1, g_1, g_1g_2, g_1g_2g_3, \ldots, g_1g_2 \cdots g_n).$$

define an invertible cochain map $f^\bullet : X^\bullet \rightarrow C^\bullet(G, M)$ (Exercise 33).

Example 3.2.6 exhibits the analogy between group cohomology and singular and simplicial cohomology of topological spaces. The latter are based on the face maps $f^n_i : \Delta^{n-1} \rightarrow \Delta^n$. These are the affine linear maps that send the standard simplex $\Delta^{n-1}$ to the $(n-1)$-face of $\Delta^n$ opposite the vertex $e_i$, that is, they omit the $(i+1)$th vertex $e_i$. Similarly, the coboundary operators of the cochain complex in Example 3.2.6 are defined in terms of the maps $f^n_i : G^{\times(n+1)} \rightarrow G^{\times n}$, $(g_0, \ldots, g_n) \mapsto (g_0, \ldots, \hat{g}_i, \ldots, g_n)$ that omit the $(i+1)$th copy of the group $G$. We will see later that this is not a coincidence but part of a general pattern.

After defining chain complexes and chain maps in general abelian categories $\mathcal{A}$, we can now construct their homologies. This is achieved by generalising the corresponding definitions in Section 2 as follows:

- the inclusion maps $t_n : \ker(d_n) \rightarrow X_n$ for the morphisms $d_n : X_n \rightarrow X_{n-1}$ in $R\text{-Mod}$ are replaced by their kernels $t_n : \ker(d_n) \rightarrow X_n$,
- the images of the morphisms $d_n : X_n \rightarrow X_{n-1}$ in $R\text{-Mod}$ are described by the images $t'_n : \text{im}(d_{n+1}) \rightarrow X_{n-1}$ and coinage $\pi'_n : X_n \rightarrow \text{im}(d_{n+1})$ from Lemma 3.1.13.
Definition 3.2.7: Let $X_\bullet$ be a chain complex in an abelian category $\mathcal{A}$

1. The $n$-cycle object of $X_\bullet$ is the kernel object $Z_n(X_\bullet) := \ker(d_n)$ of the morphism $d_n : X_n \to X_{n-1}$.
2. The $n$-boundary object of $X_\bullet$ is the image object $B_n(X_\bullet) := \text{im}(d_{n+1})$ of the morphism $d_{n+1} : X_{n+1} \to X_n$.
3. The $n$th homology of $X_\bullet$ is the cokernel object $H_n(X_\bullet) = \text{coker}(\phi_n)$ of the morphism $\phi_n : \text{im}(d_{n+1}) \to \ker(d_n)$ from (17).
4. The chain complex $X_\bullet$ is called exact in $X_n$ if $H_n(X_\bullet) = 0$ or, equivalently, if the monomorphism $\phi_n : \text{im}(d_{n+1}) \to \ker(d_n)$ is an isomorphism. It is called exact, an exact sequence or acyclic if it is exact in $X_n$ for all $n \in \mathbb{Z}$.

That $H_n(X_\bullet) = 0$ if and only if the monomorphism $\phi_n : \text{im}(d_{n+1}) \to \ker(d_n)$ is an isomorphism follows, because because any isomorphism $\phi : X \to Y$ in $\mathcal{A}$ is an epimorphism and hence has cokernel $\pi : Y \to 0$ by Lemma 3.1.13. Conversely, if a monomorphism $\phi : X \to Y$ in $\mathcal{A}$ has cokernel $0 : Y \to 0$, then it is an epimorphism and an isomorphism by Exercise 25. Hence, the homologies of a chain complex measure its failure to be exact.

It remains to investigate how chain maps between chain complexes affect their homologies. As the $n$th homology assigns to every chain complex $X_\bullet$ in $\mathcal{A}$ an object $H_n(X_\bullet)$ in $\mathcal{A}$, it is plausible that a chain map $f_\bullet : X_\bullet \to Y_\bullet$ should induce morphisms $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$ in $\mathcal{A}$ and that this should be compatible with the composition of chain maps and with identity chain maps. In other words, homologies should define functors from the category $\text{Ch}_\mathcal{A}$ of chain complexes in $\mathcal{A}$ to the underlying abelian category $\mathcal{A}$.

Proposition 3.2.8: Let $\mathcal{A}$ be an abelian category. Then the $n$th homology defines an additive functor $H_n : \text{Ch}_\mathcal{A} \to \mathcal{A}$ that assigns to a chain complex $X_\bullet$ its $n$th homology $H_n(X_\bullet)$ and to a chain map $f_\bullet : X_\bullet \to Y_\bullet$ the morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$ defined as the unique morphism for which the following diagram commutes.
im(d_{n+1}) \xrightarrow{\phi_n} ker(d_n) \xrightarrow{p_n} H_n(X_\bullet) = coker(\phi_n) \quad (18)

Proof:
1. We show that $H_n(f_\bullet)$ is well-defined:
As $f_\bullet$ is a chain map, $d_n' \circ f_n \circ \tau_n = f_{n-1} \circ d_n \circ \tau_n = 0$. By the universal property of the kernel $\iota_n' : ker(d_n') \to X_n$ there is a unique morphism $f_n : ker(d_n) \to ker(d_n')$ with $f_n \circ d_n' = f_n \circ d_n = 0$. From the diagram we have

$$\iota_n' \circ f_n \circ \phi_n \circ \pi_{n+1} = f_n \circ \tau_n \circ \phi_n \circ \pi_{n+1} = f_n \circ d_n+1 = d_n'+1 \circ f_{n+1} = \iota_n' \circ \phi_n' \circ \pi_{n+1}' \circ f_{n+1}.$$

As $\iota_n'$ is a monomorphism, it follows that $f_n \circ \phi_n \circ \pi_{n+1} = \phi_n' \circ \pi_{n+1}' \circ f_{n+1}$. The morphism $p_n' : ker(d_n') \to coker(\phi_n')$ is a cokernel of $\phi_n$, and this implies

$$p_n' \circ f_n \circ \phi_n \circ \pi_{n+1} = p_n' \circ \phi_n' \circ \pi_{n+1}' \circ f_{n+1} = 0 \circ \pi_{n+1}' \circ f_{n+1} = 0.$$

Because $\pi_{n+1}$ is an epimorphism, this implies $p_n' \circ f_n \circ \phi_n = 0$, and by the universal property of the cokernel $p_n : ker(d_n) \to coker(\phi_n)$, there is a unique morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(X'_\bullet)$ with $H_n(f_\bullet) \circ p_n = p_n' \circ f_n$.

2. We prove compatibility with identity and unit morphisms:
To show that $H_n(1_{X_\bullet}) = 1_{H_n(X_\bullet)}$ it is sufficient to note that diagram $\text{[18]}$ commutes if we set $f_k = 1_{X_k}, d_k' = d_k, \pi_k' = \pi_k, \iota_k' = \iota_k, p_k' = p_k, f_n = 1_{ker(d_n)}$ and $H_n(f_\bullet) = 1_{H_n(X_\bullet)}$. To show that $H_n(g_\bullet \circ f_\bullet) = H_n(g_\bullet) \circ H_n(f_\bullet)$ for all chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : X'_\bullet \to X''_\bullet$ we consider the commuting diagram obtained by composing diagrams $\text{[18]}$ for $f_\bullet$ and $g_\bullet$.

$$\psi_n \circ d_n \xrightarrow{\psi_n} H_n(X_\bullet) \quad (18)$$

with $\psi_n = \phi_n \circ p_{n+1}, \psi_n' = \phi_n' \circ p_n'$ and $\psi_n'' = \phi_n \circ p_n''$. As the morphism $H_n(f_\bullet) \circ H_n(g_\bullet)$ is defined uniquely by the commutativity of $\text{[18]}$, this shows that $H_n(g_\bullet \circ f_\bullet) = H_n(g_\bullet) \circ H_n(f_\bullet)$.

3. That $H_n : \text{Ch}_A \to A$ is additive follows because the diagram $\text{[18]}$ commutes if we set $f_n = f_n' + f_n''$, $f_n = f_n' + f_n''$ and $H_n(f_\bullet) = H_n(f_\bullet') + H_n(f_\bullet'')$. As the diagram defines $H_n(f_\bullet)$ uniquely, the claim follows.

\[\square\]
Remark 3.2.9: If $\mathcal{A} = R$-Mod for a ring $R$, the morphisms in diagram (18) are the following:

- $\pi_n : X_n \to \text{im}(d_n)$, $x \mapsto d_n(x)$ is the corestriction of $d_n : X_n \to X_{n-1}$,
- $\phi_n : \text{im}(d_{n+1}) \to \ker(d_n)$, $x \mapsto x$ is the inclusion map,
- $\iota_n : \ker(d_n) \to X_n$, $x \mapsto x$ is the inclusion map,
- $p_n : \ker(d_n) \to \ker(d_n)/\text{im}(d_{n+1})$, $x \mapsto [x]$ is the canonical surjection,
- $f_n : \ker(d_n) \to \ker(d_n')$, $x \mapsto f_n(x)$ is the restriction and corestriction $f_n : X_n \to X_n'$,
- $H_n(f \cdot) : \ker(d_n)/\text{im}(d_{n+1}) \to \ker(d_n')/\text{im}(d_{n+1}')$, $[x] \mapsto [f_n(x)]$ is the induced map between the quotient modules.

3.3 Chain homotopies

One reason why homologies are powerful is that there is another layer of structure beyond chain complexes and chain maps, namely chain homotopies between chain maps. The role of chain homotopies in homology theories is similar to the role of homotopies between continuous maps in homotopy theory. They define an equivalence relation on the set of chain maps between given chain complexes that is compatible with the composition of chain maps.

This allows one to form a new category whose objects are chain complexes and whose morphisms chain homotopy classes of chain maps between them. The isomorphisms in this category are chain homotopy classes of chain homotopy equivalences. They play a similar role as homotopy equivalences for topological spaces. In fact, the first examples of chain homotopies and chain homotopy equivalences were constructed from homotopies between continuous maps and homotopy equivalences between topological spaces.

Definition 3.3.1: Let $\mathcal{A}$ be an abelian category.

1. A chain homotopy $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ from a chain map $f_\bullet : X_\bullet \to X'_\bullet$ to $f'_\bullet : X_\bullet \to X'_\bullet$ in $\mathcal{A}$ is a family $(h_n)_{n \in \mathbb{Z}}$ of morphisms $h_n : X_n \to X'_{n+1}$ with

$$f_n - f'_n = h_{n-1} \circ d_n + d'_{n+1} \circ h_n \quad \forall n \in \mathbb{Z}.$$  

If there is a chain homotopy $h_\bullet : f_\bullet \Rightarrow f'_\bullet$, then $f_\bullet$ und $f'_\bullet$ are called chain homotopic, and one writes $f_\bullet \sim f'_\bullet$.

2. A chain map $f_\bullet : X_\bullet \to X'_\bullet$ is called a chain homotopy equivalence if there is a chain map $g_\bullet : X'_\bullet \to X_\bullet$ with $g_\bullet \circ f_\bullet \sim 1_{X_\bullet}$ and $f_\bullet \circ g_\bullet \sim 1_{X'_\bullet}$. In this case the chain complexes $X_\bullet$ und $X'_\bullet$ are called chain homotopy equivalent and one writes $X_\bullet \simeq X'_\bullet$.

Remark 3.3.2:

1. For given chain complexes $X_\bullet$, $X'_\bullet$ in an abelian category $\mathcal{A}$, the chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and chain homotopies between them form an abelian groupoid.

The composite of two chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : f'_\bullet \Rightarrow f''_\bullet$ is the chain homotopy $h_\bullet \circ h'_\bullet = (h_n + h'_n)_{n \in \mathbb{Z}} : f_\bullet \Rightarrow f''_\bullet$, the identity morphisms are trivial chain homotopies $1_{f_\bullet} = (0)_{n \in \mathbb{Z}}$ and the inverse of $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ is $h^{-1}_\bullet = (-h_n)_{n \in \mathbb{Z}} : f'_\bullet \Rightarrow f_\bullet$.

2. Chain homotopy defines an equivalence relation on the abelian groups $\text{Hom}_{\text{ch}\mathcal{A}}(X_\bullet, X'_\bullet)$ that is compatible with the composition of morphisms.
Remark 3.3.3: Let $(\cdot)$ and $g_\bullet, f_\bullet : X \to X'$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : g_\bullet \Rightarrow g'_\bullet$, the family of morphisms $k_n = (g'_{n+1} \circ h_n + h'_n \circ f_n)_{n \in \mathbb{Z}}$ is a chain homotopy $k_\bullet : g_\bullet \circ f_\bullet \Rightarrow g'_\bullet \circ f'_\bullet$ since
\[
g_n \circ f_n - g_n \circ f'_n = (g_n - g'_n) \circ f_n + g'_n \circ (f_n - f'_n)
= (h'_{n-1} \circ d'_n + d''_{n+1} \circ h'_n) \circ f_n + g'_n \circ (h_{n-1} \circ d_n + d'_{n+1} \circ h_n)
= (g'_n \circ h_{n-1} + h'_{n-1} \circ f_{n-1}) \circ d_n + d''_{n+1} \circ (g'_{n+1} \circ h_n + h'_n \circ f_n) = k_{n-1} \circ d_n + d''_{n+1} \circ k_n.
\]

3. We obtain a category $K(A)$, called the homotopy category of chain complexes in $A$, whose objects are chain complexes in $A$ and whose morphisms are chain homotopy classes of chain maps in $A$. The isomorphisms in $K(A)$ are chain homotopy classes of chain homotopy equivalences.

Although the definition of a chain homotopy looks very different from that of a chain map, chain homotopies are in fact chain maps. Just as homotopies between continuous maps $f, g : X \to X'$ are continuous maps $h : [0, 1] \times X \to X'$, chain homotopies between chain maps $f_\bullet, g_\bullet : X \to X'$ are chain maps from a certain chain complex constructed from $X_n$ to the chain complex $X'_n$. We illustrate this for the abelian category $R$-Mod, although it holds more generally.

**Remark 3.3.3:** Let $(X_\bullet, d_\bullet), (X'_\bullet, d'_\bullet)$ be chain complexes in $R$-Mod and $(Y_\bullet, d^Y_\bullet)$ given by
\[
Y_n = X_n \amalg X_n \amalg X_{n-1}, \quad d^Y_n : Y_n \to Y_{n-1}, \quad (x, x', x'') \mapsto (d_n(x) + x'', d_n(x') - x'', -d_{n-1}(x'')).
\]
Then chain maps $k_\bullet : (Y_\bullet, d^Y_\bullet) \to (X'_\bullet, d'_\bullet)$ are in bijection with triples $(f_\bullet, g_\bullet, h_\bullet)$ of chain maps $f_\bullet, g_\bullet : (X_\bullet, d_\bullet) \to (X'_\bullet, d'_\bullet)$ and a chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$.

**Proof:**
By the universal property of direct sums, a chain map $k_\bullet : (Y_\bullet, d^Y_\bullet) \to (X'_\bullet, d'_\bullet)$ is given by triples $(f_n, g_n, h_{n-1})$ of $R$-module morphisms $f_n, g_n : X_n \to X'_n$ and $h_{n-1} : X_{n-1} \to X'_n$ for all $n \in \mathbb{Z}$. We write $k_n(x, x', x'') = f_n(x) + g_n(x') + h_{n-1}(x'')$ for all $x, x' \in X_n, x'' \in X_{n-1}$. The condition that $k_\bullet$ is a chain map then reads
\[
d'_n \circ k_n(x, x', x'') = d'_n \circ f_n(x) + d'_n \circ g_n(x') + d'_n \circ h_{n-1}(x'')
= k_{n-1} \circ d'_n(x, x', x'') = f_{n-1} \circ d_n(x) + f_{n-1} \circ (x'') + g_{n-1} \circ d'_n(x') - g_{n-1}(x'') - h_{n-2} \circ d_{n-1}(x'').
\]
By setting $x' = x'' = 0$, $x = x'' = 0$ or $x = x' = 0$, one finds that this condition is equivalent to the statement that $f_\bullet$ and $g_\bullet$ are chain maps and $h_\bullet : f_\bullet \Rightarrow g_\bullet$ is a chain homotopy.

The analogy between chain complexes, chain maps and chain homotopies and topological spaces, continuous maps and homotopies also manifests itself in their homologies. Just as homotopic maps between topological spaces induce the same group homomorphisms between the homotopy groups, chain homotopic chain maps induce the same morphisms between homologies. As a consequence, chain homotopy equivalent chain complexes have isomorphic homologies. This allows one to view the homologies as functors $H_n : K(A) \to A$, where $K(A)$ is the homotopy category of chain complexes from Remark 3.3.2 with the same objects as $A$ and chain homotopy classes of morphisms in $A$ as morphisms.
**Proposition 3.3.4:** Let $\mathcal{A}$ be an abelian category.

1. Chain homotopic chain maps in $\mathcal{A}$ induce the same morphisms on the homologies: if $f_\bullet \sim g_\bullet$ then $H_n(f_\bullet) = H_n(g_\bullet)$ for all $n \in \mathbb{Z}$.

2. The $n$th homology induces a functor $H_n : K(\mathcal{A}) \to \mathcal{A}$ for all $n \in \mathbb{Z}$.

3. Chain homotopy equivalences induce isomorphisms on the homologies: if $X_\bullet \simeq X'_\bullet$, then $H_n(X_\bullet) \cong H_n(X'_\bullet)$ for all $n \in \mathbb{Z}$.

**Proof:**
To prove the first claim, let $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ be chain maps and $h_\bullet : f_\bullet \Rightarrow g_\bullet$ a chain homotopy. The morphism $H_n(X_\bullet) \to H_n(X'_\bullet)$ is defined by diagram (18) as the unique morphism with $H_n(f_\bullet) \circ p_n = p'_n \circ \bar{f}_n$.

As $p_n$ is an epimorphism, it is sufficient to show that $H_n(f_\bullet - g_\bullet) \circ p_n = p'_n \circ (\bar{f}_n - \bar{g}_n) = 0$. As $h_\bullet : f_\bullet \Rightarrow g_\bullet$ is a chain homotopy, we have

$$l'_n \circ (\bar{f}_n - \bar{g}_n) = (f_n - g_n) \circ \iota_n = h_{n-1} \circ d_n \circ \iota_n + d'_{n+1} \circ h_n \circ \iota_n = h_{n-1} \circ 0 + d'_{n+1} \circ h_n \circ \iota_n = l'_n \circ \phi'_n \circ \pi'_{n+1} \circ h_n \circ \iota_n.$$

As $l'_n$ is a monomorphism, this implies $\bar{f}_n - \bar{g}_n = (f_n - g_n) \circ \iota_n = h_{n-1} \circ d_n \circ \iota_n + d'_{n+1} \circ h_n \circ \iota_n = h_{n-1} \circ 0 + d'_{n+1} \circ h_n \circ \iota_n = l'_n \circ \phi'_n \circ \pi'_{n+1} \circ h_n \circ \iota_n.$

The second claim follow from the first, and so does the third: if $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : X'_\bullet \to X_\bullet$ are chain maps with $f_\bullet \circ g_\bullet \sim 1_{X_\bullet}$ and $g_\bullet \circ f_\bullet \sim 1_{X_\bullet}$, then one has

$$H_n(f_\bullet) \circ H_n(g_\bullet) = H_n(f_\bullet \circ g_\bullet) = H_n(1_{X'_\bullet}) = 1_{H_n(X'_\bullet)}$$

$$H_n(g_\bullet) \circ H_n(f_\bullet) = H_n(g_\bullet \circ f_\bullet) = H_n(1_{X_\bullet}) = 1_{H_n(X_\bullet)}$$

and hence $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(X'_\bullet)$ is an isomorphism with inverse $H_n(f_\bullet)^{-1} = H_n(g_\bullet)$.

A important application of Proposition 3.3.4 is to show that a given chain complex $X_\bullet$ has trivial homologies without computing them explicitly. For this, it is sufficient to construct a chain homotopy between the chain maps $1_{X_\bullet} : X_\bullet \to X_\bullet$ and $0_\bullet : X_\bullet \to X_\bullet$. The chain maps $f : X_\bullet \to 0_\bullet$ and $g : 0_\bullet \to X_\bullet$ then form a chain homotopy equivalence between $X_\bullet$ and the trivial chain complex $0_\bullet$, since they satisfy $f_\bullet \circ g_\bullet = 1_{X_\bullet}$ and $g_\bullet \circ f_\bullet = 0_{X_\bullet} \sim 1_{X_\bullet}$. This implies $H_n(X_\bullet) \cong H_n(0_\bullet) = 0$ for all $n \in \mathbb{Z}$ by Proposition 3.3.4.
Example 3.3.5: Let \( k \) be a commutative ring, \( G \) a group and \( A \) an algebra over \( k \).

1. The chain complex \((X_\bullet, d_\bullet)\) in \( \mathcal{A} = k\text{-Mod} \) given by

\[
X_n = \begin{cases} 
  A^\otimes(n+2) & n \geq -1 \\
  0 & n < -1 
\end{cases}
\]

\(d_n : X_n \to X_{n-1}, \ (a_0 \otimes \ldots \otimes a_{n+1}) \mapsto \sum_{i=0}^n (-1)^i a_i \otimes \ldots \otimes (a_{i+1}) \otimes \ldots \otimes a_{n+1}
\)

is exact, because the \( k \)-linear maps \( h_n : X_n \to X_{n+1}, \ (a_0 \otimes \ldots \otimes a_{n+1}) \mapsto (1 \otimes a_0 \otimes \ldots \otimes a_{n+1}) \) define a chain homotopy \( h_\bullet : 1_{X_\bullet} \Rightarrow 0_{X_\bullet} \):

\[
(h_{n-1} \circ d_n + d_{n+1} \circ h_n)(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n+1} (-1)^i h_{n-1}(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) + d_{n+1}(1 \otimes a_0 \otimes \ldots \otimes a_{n+1})
\]

This chain complex becomes an exact chain complex in \( k[G] \)-\text{-Mod} with the \( k[G] \)-module structures \( \triangleright : k[G] \times X_n \to X_n, \ g \triangleright (a_0 \otimes \ldots \otimes a_{n+1}) = (ga_0 \otimes \ldots \otimes ga_{n+1}) \) for \( n > -1 \) and the trivial \( k[G] \)-module structure on \( k \).

2. The chain complex \((X_\bullet, d_\bullet)\) in \( \mathcal{A} = k\text{-Mod} \) given by

\[
X_n = \begin{cases} 
  A^\otimes(n+2) & n \geq -1 \\
  0 & n < -1 
\end{cases}
\]

\(d_n : X_n \to X_{n-1}, \ (a_0 \otimes \ldots \otimes a_{n+1}) \mapsto \sum_{i=0}^n (-1)^i a_i \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_{n+1}
\)

is exact, because the \( k \)-linear maps \( h_n : X_n \to X_{n+1}, \ (a_0 \otimes \ldots \otimes a_{n+1}) \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_{n+1} \) define a chain homotopy \( h_\bullet : 1_{X_\bullet} \Rightarrow 0_{X_\bullet} \):

\[
(h_{n-1} \circ d_n + d_{n+1} \circ h_n)(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n+1} (-1)^i h_{n-1}(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) + d_{n+1}(1 \otimes a_0 \otimes \ldots \otimes a_{n+1})
\]

\[
= \sum_{i=0}^{n+1} (-1)^i a_0 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_{n+1} + a_0 \otimes \ldots \otimes a_{n+1}
\]

This chain complex becomes an exact chain complex in \( A\text{-Mod}-A \) with the \( (A, A) \)-bimodule structure \( b \triangleright (a_0 \otimes \ldots \otimes a_{n+1}) \triangleleft c = (ba_0) \otimes a_1 \otimes \ldots \otimes a_{n+1}c \) on \( A^\otimes(n+2) \).

Although these examples of chain complexes look similar to the cochain complex of group cohomology from Example 3.2.5 and to the Hochschild complex from Example 3.2.5, there is a fundamental difference, namely the absence of the \( k[G] \)-module or \( (A, A) \)-bimodule \( M \). In fact, the chain complexes of group cohomology and Hochschild (co)homology are obtained by applying the functor \( \text{Hom}(\_, \ M) \) to this chain complex or tensoring with the module \( M \). This will allow us to view group cohomology and Hochschild (co)homologies as the homologies of certain functors rather than homologies of chain complexes.

Important examples of chain homotopies originate in topology, and this motivates the name chain homotopy. Every homotopy between continuous maps induces a chain homotopy between the associated chain maps in singular homology.
Proposition 3.3.6: Let $k$ be a commutative ring and $C_\bullet(-, k) : \text{Top} \to \text{Ch}_k\text{-Mod}$ the functor from Example 3.2.5 that assigns to a topological space $X$ the singular chain complex $C_\bullet(X, k)$

$$C_n(X, k) = \langle \sigma : \Delta^n \to X \text{ continuous} \rangle_k \quad d_n(\sigma) = \Sigma_{i=0}^{n-1} (-1)^i \sigma \circ f_i^n$$

and to a continuous map $f : X \to Y$ the chain map

$$C_\bullet(f, k) : C_\bullet(X, k) \to C_\bullet(Y, k), \quad C_n(f, k)(\sigma) = f \circ \sigma : \Delta^n \to Y.$$  

1. Every homotopy $h : f \Rightarrow g$ between continuous maps $f, g : X \to Y$ induces a chain homotopy $C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k)$.

2. Homotopic maps $f, g : X \to Y$ induce the same morphisms between the singular homologies: $f \sim g \Rightarrow H_n(f, k) = H_n(g, k) : H_n(X, k) = H_n(Y, k)$ for all $n \in \mathbb{Z}$.

3. Homotopy equivalent topological spaces have isomorphic singular homologies: $X \simeq Y \Rightarrow H_n(X, k) \cong H_n(Y, k)$ for all $n \in \mathbb{Z}$.

**Proof:**

The second and the third claim follow from the first together with Proposition 3.3.4. The chain homotopy $C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k)$ induced by a homotopy $h : [0, 1] \times X \to Y$ from $f : X \to Y$ to $g : X \to Y$ is given by the **prism maps**, the affine linear maps

$$T_n^j : \Delta^{n+1} \to [0, 1] \times \Delta^n, \quad T_n^j(e_k) = \begin{cases} (0, e_k) & 0 \leq k \leq j \leq n \\ (1, e_{k-1}) & 0 \leq j < k \leq n + 1. \end{cases}$$

A direct computation shows that the the prism maps satisfy the relations

$$T_n^j \circ f_i^{n+1} = (\text{id}_{[0, 1]} \times f_i^n) \circ T_{n-1}^{j-1}, \quad \forall j > i \quad T_n^j \circ f_i^{n+1} = (\text{id}_{[0, 1]} \times f_i^{n-1}) \circ T_n^{j-1}, \quad \forall j < i - 1$$

$$T_n^i \circ f_{i+1}^{n+1} = T_n^{i-1} \circ f_{i+1}^n, \quad \forall i \in \{1, \ldots, n\} \quad T_0^i \circ f_0^{n+1} = i_1, \quad T_m^0 \circ f_m^{n+1} = i_0,$$  

(19)

where $i_t : \Delta^n \to [0, 1] \times \Delta^n, x \mapsto (t, x)$ is the inclusion map and $f_j^{n+1} : \Delta^n \to \Delta^{n+1}$ the face map from Definition 2.1.1. By composing the prism maps with the homotopy $h : [0, 1] \times X \to Y$ one obtains $k$-linear maps

$$C_n(h, k) : C_n(X, k) \to C_{n+1}(Y, k), \quad \sigma \mapsto \Sigma_{j=0}^n (-1)^j h \circ (\text{id}_{[0, 1]} \times \sigma) \circ T_n^j.$$  

(20)

A direct computation using (19) and the identities $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$ shows that the maps $C_n(h, k)$ define a chain homotopy $C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k)$ (Exercise 41).
Proposition 3.3.6 relates homotopy classes of continuous maps to chain homotopy classes of chain maps. Hence, we can view singular homology as a functor $hC_\bullet(-, k): h\text{Top} \to K(k\text{-Mod})$ from the category $h\text{Top}$ with topological spaces as objects and homotopy classes of continuous maps as morphisms to the homotopy category $K(k\text{-Mod})$ from Remark 3.3.2. With chain complexes in $k\text{-Mod}$ as objects and chain homotopy classes of chain maps as morphisms. Denoting by $C_\bullet(-, k): \text{Top} \to \text{Ch}_k\text{-Mod}$ the singular homology functor and by $P_{\text{Ch}}: k\text{-Mod} \to K(k\text{-Mod})$ and $P_{\text{Top}}: \text{Top} \to h\text{Top}$ the projection functors that send each object to itself and each morphism to its homotopy class, we obtain a commuting diagram

$$
\begin{array}{ccc}
\text{Top} & \xrightarrow{C_\bullet(-, k)} & \text{Ch}_k\text{-Mod} \\
\downarrow{P_{\text{Top}}} & & \downarrow{P_{\text{Ch}}} \\
\text{hTop} & \xrightarrow{hC_\bullet(-, k)} & K(k\text{-Mod}).
\end{array}
$$

### 3.4 Short exact sequences of chain complexes and the long exact homology sequence

As chain complexes and chain maps in an abelian category $\mathcal{A}$ form an abelian category $\text{Ch}_\mathcal{A}$ and the homologies are functors $H_n: \text{Ch}_\mathcal{A} \to \mathcal{A}$, it is natural to ask if these functors are exact or, more generally, how they behave with respect to exact sequences in $\text{Ch}_\mathcal{A}$. We will see in the following that the homology functors $H_n: \text{Ch}_\mathcal{A} \to \mathcal{A}$ are in general not exact. Instead, there is an exact sequence, the long exact homology sequence, that relates the homologies $H_n$ for different $n \in \mathbb{Z}$ of an exact sequence of chain complexes. This long exact homology sequence appears in many applications in algebra and topology and is one of the most useful tools for computing homologies. To derive this result, we require the concept of a short exact sequence in an abelian category $\mathcal{A}$, which can be viewed as an alternative description of kernels and cokernels and the relation between them.

**Definition 3.4.1:** Let $\mathcal{A}$ be an abelian category. A short exact sequence in $\mathcal{A}$ is an exact chain complex of the form $0 \to X \xrightarrow{j} Y \xrightarrow{\pi} Z \to 0$.

Short exact sequences are the shortest exact sequences that carry information that cannot be stated in a much simpler way. A chain complex of the form $0 \to X \to 0$ is exact if and only if the object $X$ is isomorphic to the zero object in $\mathcal{A}$, and a chain complex of the form $0 \to X \to Y \to 0$ is exact if and only if the morphism in the middle is an isomorphism. The information contained in short exact sequences is less trivial. The following example shows that they are generalisations of quotient modules.

**Example 3.4.2:** A short exact sequence $0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0$ in the abelian category $\mathcal{A} = R\text{-Mod}$ corresponds to a triple $(L, M, N)$ such that $L \subseteq M$ is a submodule and $N = M/L$ is the associated quotient module.

This follows because for any submodule $L \subseteq M$, the inclusion map $i: L \to M$ is a monomorphism in $R\text{-Mod}$ and the canonical surjection $\pi: M \to M/L$ is an epimorphism in $R\text{-Mod}$ with $\text{ker}(\pi) = \text{im}(i)$. Conversely, given a monomorphism $i: L \to M$ and an epimorphism $\pi: M \to N$ with $\text{ker}(\pi) = \text{im}(i)$, one has $L \cong \text{im}(i) \subseteq M$ and $N \cong M/\text{ker}(\pi) \cong M/\text{im}(i) \cong M/L$. 

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Short exact sequences are important because they are the fundamental building blocks of abelian categories. Instead of kernels and cokernels, we can take short exact sequences as the fundamental structures that characterise abelian categories and exact functors. A functor between abelian categories is exact if and only if it sends short exact sequences to short exact sequences. Left and right exact functors satisfy similar but weaker conditions.

**Lemma 3.4.3:**

1. A chain complex $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in an abelian category $\mathcal{A}$ is a short exact sequence if and only if one of the following equivalent conditions holds:
   
   (i) $f$ is a monomorphism and $g$ a cokernel of $f$.
   
   (ii) $g$ is an epimorphism and $f$ a kernel of $g$.

2. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$, $\mathcal{B}$ is
   
   - left exact if and only if $0 \to F(X) \xrightarrow{F(i)} F(Y) \xrightarrow{F(\pi)} F(Z)$ is an exact sequence in $\mathcal{B}$ for all short exact sequences $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ in $\mathcal{A}$,
   
   - right exact if and only if $F(X) \xrightarrow{F(i)} F(Y) \xrightarrow{F(\pi)} F(Z) \to 0$ is an exact sequence in $\mathcal{B}$ for all short exact sequences $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ in $\mathcal{A}$,
   
   - exact if and only if $0 \to F(X) \xrightarrow{F(i)} F(Y) \xrightarrow{F(\pi)} F(Z) \to 0$ is an exact sequence in $\mathcal{B}$ for all short exact sequences $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ in $\mathcal{A}$.

**Proof:**

1. As the morphism $i_X : 0 \to X$ is a monomorphism and hence an image of itself by Lemma 3.1.9 and the morphism $t_Z : Z \to 0$ has kernel $1_Z : Z \to Z$, we obtain the commuting diagram

   ![Diagram](image)

   - The commuting triangle on the left shows that $\phi_0$ is an isomorphism if and only if $0 \to X$ is a kernel of $f$, which by Lemma 3.1.9 is the case if and only if $f$ is a monomorphism.

   - The commuting triangle on the right shows that $\phi_2$ is an isomorphism if and only if $1_Z : Z \to Z$ is an image of $g$, which is equivalent to the statement that $Z \to 0$ is a cokernel of $g$. By Lemma 3.1.9 this is the case if and only if $g : Y \to Z$ is an epimorphism.

   - The commuting triangle in the middle states that $\phi_1$ is an isomorphism if and only if the image $\iota'_f : \text{im}(f) \to Y$ is a kernel of $g$. If $f$ is a monomorphism, it is an image of itself and hence the last statement is equivalent to the condition that $f : X \to Y$ is a kernel of $g : Y \to Z$.

   - If $g : Y \to Z$ is an epimorphism, then it is a cokernel of its kernel. Hence, by uniqueness of the cokernel and kernel, $\phi_1$ is an isomorphism if and only if $g$ is a cokernel of $\iota'_f : \text{im}(f) \to Y$.

As $\pi'_f : X \to \text{im}(f)$ is an epimorphism by Lemma 3.1.13 the morphisms $\iota'_f$ and $f = \iota'_f \circ \pi'_f$ have the same cokernel by Exercise 22. Hence, if $g$ is an epimorphism, then $\phi_1$ is an isomorphism if and only if $g$ is a cokernel of $f$.

2. By 1. the exactness of the sequence $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ is equivalent to the statements that $\iota$ is a monomorphism and $\pi$ a cokernel of $\iota$ or, equivalently, $\pi$ an epimorphism and $\iota$ a kernel of $\pi$. The proof of 1. also shows that the exactness of the first sequence in 2. is equivalent to the statement that $F(\iota)$ is a kernel of $F(\pi)$ and the exactness of the second sequence in 2. to the statement that $F(\pi)$ is a cokernel of $F(\iota)$. The claim then follows from Definition 3.1.10. 

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The characterisation of left and right exactness in Lemma 3.4.3 motivates the names left exact functor, right exact functor and exact functor. A left exact functor preserves exactness of a short exact sequence only in the first two objects on the left, whereas a right exact functor preserves it only if the first two objects on the right. The first two terms on the left belong to kernels and the first two terms on the right to cokernels.

The alternative definition of left exact, right exact and exact functors in terms of short exact sequences has many advantages. One of them is that it is easier to combine with natural transformations than the criteria in Definition 3.1.10. In particular, it shows directly that left exactness, right exactness and exactness are preserved under natural isomorphisms.

**Corollary 3.4.4:** Let \( F, G : \mathcal{A} \to \mathcal{B} \) be additive functors between abelian categories. If \( F \) and \( G \) are naturally isomorphic, then \( F \) is left exact, right exact or exact if and only if only if \( G \) is.

**Proof:**
Let \( \eta : F \to G \) be a natural isomorphism. For every short exact sequence \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) in \( \mathcal{A} \) the following diagram commutes by naturality of \( \eta \)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g)} & F(Z) & \longrightarrow & 0 \\
\downarrow{\eta_X} & & \downarrow{\eta_Y} & & \downarrow{\eta_Z} & & & & \\
0 & \longrightarrow & G(X) & \xrightarrow{G(f)} & G(Y) & \xrightarrow{G(g)} & G(Z) & \longrightarrow & 0.
\end{array}
\]

As the vertical arrows are isomorphisms, the first row is exact in \( F(X), F(Y) \) or \( F(Z) \), respectively, if and only if the second row is exact in \( G(X), G(Y) \) or \( G(Z) \). Hence, \( F \) is left exact, right exact or exact if and only if \( G \) is by Lemma 3.4.3. \( \square \)

Lemma 3.4.3 is a strong motivation to investigate how the homology functors \( H_n : \text{Ch}_\mathcal{A} \to \mathcal{A} \) interact with short exact sequences in the category \( \text{Ch}_\mathcal{A} \). Another motivation is that short exact sequences of chain complexes arise in many applications in group and Lie algebra (co)homology, Hochschild (co)homology and singular (co)homology. Examples are group (co)homologies with coefficients in submodules and singular homologies of subspaces.

**Example 3.4.5:** Let \( k \) be a commutative ring,

1. Let \( G \) be a group and \( 0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0 \) a short exact sequence of \( k[G] \)-modules. Then we obtain a short exact sequence of chain complexes

\[
0 \to C^\bullet(G, L) \xrightarrow{C^\bullet(G, L)} C^\bullet(G, M) \xrightarrow{C^\bullet(G, \pi)} C^\bullet(G, N) \to 0
\]

where \( C^\bullet(G, L) \) and \( C^\bullet(G, M) \) are the cochain complexes for group cohomology from Definition 2.3.1 and \( C^\bullet(G, \iota) \) and \( C^\bullet(G, \pi) \) the chain maps from Example 3.2.5.

2. Let \( X \) be a topological space and \( A \subset X \) a subspace. Then the inclusion map \( \iota : A \to X \) defines a short exact sequence of chain complexes

\[
0 \to C_\bullet(A, k) \xrightarrow{C_\bullet(\iota, k)} C_\bullet(X, k) \xrightarrow{\pi} C_\bullet(X, A, k) \to 0
\]

where \( C_\bullet(A, k) \) and \( C_\bullet(X, k) \) are the singular chain complexes from Definition 2.1.4 and \( C_\bullet(\iota, k) \) the chain map from Example 3.2.5. and \( C_\bullet(X, A, k) \) the chain complex with \( C_n(X, A, k) = C_n(X, k)/C_n(A, k) \) and \( d_n : C_n(X, A, k) \to C_{n-1}(X, A, k), [\sigma] \mapsto [d_n(\sigma)] \) for all singular \( n \)-simplexes \( \sigma : \Delta^n \to X \).
The first example can be used to compute group (co)homologies with coefficients in submodules or quotient modules, once one is able to relate the homologies for a short exact sequence of chain complexes. This is useful since group cohomologies with coefficients in free \(k[G]\)-modules are often particularly simple to compute and every \(k[G]\)-module can be written as quotient of a free \(k[G]\)-module by an appropriate submodule.

The second example is relevant for the computation of singular homologies of quotient spaces. One can show that if \(\emptyset \neq A \subset X\) is closed and a deformation retract of a neighbourhood of a point \(x \in X\), then \(H_n C_\bullet(X, A, k) \cong H_n(X/A, k)\), see for instance [H, Theorem 2.13].

We will now derive an exact sequence, the long exact homology sequence, that relates the homologies of a short exact sequence of chain complexes. To do so, we need the following technical lemma known as the snake lemma. It is called snake lemma, because it involves a morphism that is represented by a snakelike arrow in a commuting diagram.

**Lemma 3.4.6 (snake lemma):** Let \(\mathcal{A}\) be an abelian category and

\[
\begin{array}{ccc}
  L & \xrightarrow{f} & M & \xrightarrow{g} & N & \xrightarrow{0} \\
  \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & \\
  0 & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \\
\end{array}
\]

a commuting diagram in \(\mathcal{A}\) with exact rows. Then there are unique morphisms \(\bar{f}, \bar{g}, \bar{f}', \bar{g}'\) that make the following diagram commute

\[
\begin{array}{ccc}
  \ker(\alpha) & \xrightarrow{\exists! \bar{f}} & \ker(\beta) & \xrightarrow{\exists! \bar{g}} & \ker(\gamma) & \xrightarrow{\partial} & \\
  \downarrow{\iota_\alpha} & & \downarrow{\iota_\beta} & & \downarrow{\iota_\gamma} & & \\
  L & \xrightarrow{f} & M & \xrightarrow{g} & N & \xrightarrow{0} & \\
  \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
  0 & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & & \\
\end{array}
\]

and a unique morphism \(\partial : \ker(\gamma) \to \coker(\alpha)\), called the connecting morphism, that yields an exact sequence

\[
\ker(\alpha) \xrightarrow{\bar{f}} \ker(\beta) \xrightarrow{\bar{g}} \ker(\gamma) \xrightarrow{\partial} \coker(\alpha) \xrightarrow{\bar{f}'} \coker(\beta) \xrightarrow{\bar{g}'} \coker(\gamma).
\]

If \(f : L \to M\) is a monomorphism, then \(\bar{f} : \ker(\alpha) \to \ker(\beta)\) is a monomorphism as well. If \(g' : M' \to N'\) is an epimorphism, then \(\bar{g}' : \coker(\beta) \to \coker(\gamma)\) is an epimorphism as well.

**Proof:**

We prove the lemma for \(\mathcal{A} = R\text{-Mod}\). The general claim then follows from Mitchell’s embedding theorem, since we can restrict attention to a small full abelian subcategory of \(\mathcal{A}\). A general proof for abelian categories that does not use Mitchell’s embedding theorem is given in [McH, p202ff].
1. As the commutativity of the diagram (22) implies $f' \circ \alpha = \beta \circ f$, we have

$$\beta \circ f \circ \iota_\alpha = f' \circ \alpha \circ \iota_\alpha = f' \circ 0 = 0 \quad \pi_\beta \circ f' \circ \alpha = \pi_\beta \circ \beta \circ f = 0 \circ f = 0,$$  

and by the universal properties of $\iota_\beta : \ker(\beta) \to M$ and $\pi_\alpha : L' \to \coker(\alpha)$, there are unique morphisms $\bar{f} : \ker(\alpha) \to \ker(\beta)$ and $\bar{f}' : \coker(\alpha) \to \coker(\beta)$ with $\iota_\beta \circ \bar{f} = f \circ \iota_\alpha$ and $\bar{f}' \circ \pi_\alpha = \pi_\beta \circ f'$. Similarly, the commutativity of the diagram implies $g' \circ \beta = \gamma \circ g$, which yields

$$\gamma \circ g \circ \iota_\beta = g' \circ \beta \circ \iota_\beta = g' \circ 0 = 0 \quad \pi_\gamma \circ g' \circ \beta = \pi_\gamma \circ \gamma \circ g = 0 \circ g = 0,$$  

and by the universal property of $\iota_\gamma : \ker(\gamma) \to N$ and $\pi_\beta : M' \to \coker(\beta)$, there are unique morphisms $\bar{g} : \ker(\beta) \to \ker(\gamma)$ and $\bar{g}' : \coker(\beta) \to \coker(\gamma)$ satisfying $\iota_\gamma \circ \bar{g} = g \circ \iota_\beta$ and $\bar{g}' \circ \pi_\beta = \pi_\gamma \circ g$. If $f : L \to M$ is a monomorphism, then for all morphisms $k : X \to \ker(\alpha)$ with $\bar{f} \circ k = 0$ we have $\iota_\beta \circ \bar{f} \circ k = f \circ \iota_\alpha \circ k = 0$. Because $f \circ \iota_\alpha$ is a monomorphism, this implies $k = 0$ and $\bar{f}$ is a monomorphism as well. If $g' : M' \to N'$ is an epimorphism, then for all morphisms $k : \coker(\gamma) \to X$ with $k \circ g' = 0$ we have $0 = k \circ \bar{g} \circ \pi_\beta = k \circ \pi_\alpha \circ g$. Because $\pi_\alpha \circ g$ is an epimorphism, this implies $k = 0$ and $\bar{g}'$ is an epimorphism.

2. We show that the first and fourth row of the diagram (22) are exact:

The commutativity of the diagram (22) implies $\iota_\gamma \circ \bar{g} \circ \bar{f} = g \circ \iota_\beta \circ \bar{f} = g \circ f \circ \iota_\alpha = 0 \circ \iota_\alpha = 0$, and because $\iota_\gamma$ is injective, this implies $\bar{g} \circ \bar{f} = 0$ and $\im(\bar{f}) \subset \ker(\gamma)$. Conversely, for any $m \in \ker(\bar{g}) = \ker(\beta) \cap \ker(\gamma)$, by the exactness of the second row there is an $l \in L$ with $m = f(l)$. The commutativity of (22) implies $f' \circ \alpha(l) = \beta \circ f(l) = \beta(m) = 0$. As $f' : L' \to M'$ is a monomorphism, it follows that $\alpha(l) = 0$ and $l \in \ker(\alpha)$. Then we have $f'(l) = f(l) = m$ and $m \in \im(\bar{f})$, which shows that $\ker(\bar{g}) = \im(\bar{f})$.

Similarly, $\bar{g}' \circ \bar{f}' \circ \pi_\alpha = \bar{g}' \circ \pi_\beta \circ f' = \pi_\gamma \circ g' \circ f' = 0$ implies $\bar{g}' \circ \bar{f}' = 0$ and $\im(\bar{f}') \subset \ker(\bar{g}')$ because $\pi_\alpha$ is an epimorphism. If $x \in \ker(\bar{g}') \subset \coker(\beta)$, there is an element $m' \in M'$ with $\pi_\beta(m') = x$, since $\pi_\beta$ is an epimorphism, and it follows that $\pi_\gamma \circ \bar{g}'(m') = \bar{g}' \circ \pi_\beta(m') = \bar{g}'(x) = 0$. This implies $\bar{g}'(m') \in \ker(\pi_\gamma) = \im(\gamma)$. Hence, there is an element $n \in N$ with $\gamma(n) = \bar{g}'(m')$, and because $g$ is an epimorphism, an element $m \in M$ with $g(m) = n$. This implies $\gamma \circ g(m) = \bar{g}' \circ \beta(m) = \bar{g} \circ \beta(m) = g \circ (m')$, and $m' - \beta(m) \in \ker(\bar{g}') = \im(f')$. Hence, there is an element $l' \in L'$ with $f'(l') = m' - \beta(m)$, and $f'(\pi_\alpha(l')) = \pi_\beta(f(l')) = \pi_\beta(m' - \beta(m)) = \pi_\beta(m') = x$. This shows that $\ker(\bar{g}') = \im(f')$.

3. We construct the morphism $\ker(\gamma) \to L'/\im(\alpha)$:

Consider an element $n \in \ker(\gamma) \subset N$. Then by surjectivity of $g$ there is an $m \in M$ with $g(m) = n$. For any other element $m' \in M$ with $g(m') = n$, the exactness of the second row implies $m - m' \in \ker(g) = \im(f)$, and there is an $l \in L$ with $m' = f(l)$. By commutativity of (22) we have $\gamma \circ \beta(m) = \gamma \circ g(m) = \gamma(n) = 0$, that is $\beta(m) \in \ker(g')$, and analogously $\beta(m') \in \ker(g')$. By exactness of the third row one has $\ker(g') = \im(f')$, and there is an $l' \in L'$ with $f'(l') = \beta(m)$. As $f'$ is injective, this element $l' \in L$ is unique. Similarly, we obtain a unique element $l'' \in L'$ with $f(l'') = \beta(m') = \beta(m) + \beta \circ f(l) = f'(l') + f' \circ \alpha(l) = f'(l' + \alpha(l))$. As $f'$ is injective, this implies $l' = l' + \alpha(l)$, and $\pi_\alpha(l) = \pi_\alpha(l')$. We obtain a well-defined map

$$\partial : \ker(\gamma) \to L'/\im(\alpha), \quad n \mapsto \pi_\alpha(l') \quad \text{where} \quad n = g(m), \quad f'(l') = \beta(m), \quad \text{(23)}$$

which is an $R$-module morphism by construction, since $n = g(m)$, $f(l') = \beta(m)$ and $n' = g(m')$, $f(l'') = \beta(m')$ imply $n + n' = g(m + m')$ and $f(l' + l'') = \beta(m + m')$ and hence $\partial(n + n') = \pi_\alpha(l') + \pi_\alpha(l'') = \partial(n) + \partial(n')$.

4. The connecting homomorphism in (23) yields a sequence

$$\ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{\bar{g}} \ker(\gamma) \xrightarrow{\partial} L'/\im(\alpha) \xrightarrow{f} M'/\im(\beta) \xrightarrow{\bar{g}} N'/\im(\gamma).$$

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which is already exact in all entries except \( \ker(\gamma) \) and \( L'/\im(\alpha) \). To show the exactness in \( \ker(\gamma) \), consider an element \( n \in \im(\bar{g}) \). Then there is an element \( m \in \ker(\beta) \) with \( n = g(m) \), and \((23)\) yields an element \( l' \in L \) with \( f'(l') = \beta(m) = 0 \). The injectivity of \( f' \) implies \( l' = 0 \), and by definition of the connecting homomorphism in \((23)\) we have \( \partial(n) = 0 \). Hence, \( \im(\bar{g}) \subset \ker(\partial) \).

Conversely, if \( n \in \ker(\partial) \), then by \((23)\) there are elements \( m \in M \), \( l' \in L' \) with \( n = g(m) \), \( \beta(m) = f'(l') \) and \( \pi_\alpha(l') = 0 \). This implies \( l' \in \ker(\pi_\alpha) = \im(\alpha) \), and there is an \( l \in L \) with \( l' = \alpha(l) \). This yields \( \beta(m) = f' \circ \alpha(l) = \beta \circ f(l) \), and hence \( m = f(l) \in \ker(\beta) \). By exactness of the second row of \((22)\), this yields \( g(m - f(l)) = g(m) - g \circ f(l) = g(m) = n \), and we have \( n \in g(\ker(\beta)) = \im(\bar{g}) \). This shows that \( \ker(\partial) \subset \im(\bar{g}) \) and proves the exactness of the sequence in \( \ker(\gamma) \).

The proof of the exactness in \( \coker(\alpha) = L'/\im(\alpha) \) is analogous. If \( x \in \im(\bar{g}) \), then by definition of \( \partial \) in \((23)\), there are elements \( m \in M \) and \( l' \in L' \) with \( x = \pi_\alpha(l') \) and \( f'(l') = \beta(m) \). This implies \( f'(x) = f' \circ \pi_\alpha(l') = \pi_\beta \circ f'(l') = \pi_\beta \circ \beta(m) = 0 \) and \( x \in \ker(f') \). Conversely, if \( x \in \ker(f') \), then there is an \( l' \in L' \) with \( x = \pi_\alpha(l') \) and \( \pi_\beta \circ f'(l') = f' \circ \pi_\alpha(l') = f'(x) = 0 \). Hence, we have \( f'(l') \in \ker(\pi_\beta) = \im(\beta) \), and there is an \( m \in M \) with \( f'(l') = \beta(m) \). This implies \( \partial(g(m)) = \pi_\alpha(l') = x \) by definition of the connecting morphism in \((23)\), and hence \( x \in \im(\bar{g}) \). This shows that \( \im(\partial) = \ker(f') \) and proves the exactness in \( \coker(\alpha) = L'/\im(\alpha) \).

\( \square \)

By applying the snake lemma, we can determine the image of a short exact sequence of chain complexes in \( \mathcal{A} \) under the homology functors \( H_n : \text{Ch}_\mathcal{A} \to \mathcal{A} \). Although the individual homology functors \( H_n \) are not exact, it turns out that the homologies for different \( n \in \mathbb{Z} \) can be combined into an exact sequence in \( \mathcal{A} \).

**Theorem 3.4.7:** Every short exact sequence of chain complexes \( 0 \to L_0 \xrightarrow{f_0} M_0 \xrightarrow{g_0} N_0 \to 0 \) in an abelian category \( \mathcal{A} \) induces an exact sequence

\[
\cdots \xrightarrow{H_{n+1}(g_*)} H_{n+1}(N_0) \xrightarrow{\partial_{n+1}} H_n(L_0) \xrightarrow{H_n(f_*)} H_n(M_0) \xrightarrow{H_n(g_*)} H_n(N_0) \xrightarrow{\partial_n} H_{n-1}(L_0) \xrightarrow{H_{n-1}(f_*)} \cdots,
\]

the **long exact homology sequence**. The morphisms \( \partial_n : H_n(N_0) \to H_{n-1}(L_0) \) are called connecting morphisms.

**Proof:**

We prove the theorem for \( \mathcal{A} = \text{R-Mod} \). The general case then follows with Mitchell’s embedding theorem. For all \( n \in \mathbb{Z} \) the short exact sequence defines a commuting diagram with exact rows:

\[
\begin{array}{cccccccc}
0 & \to & L_{n+1} & \xrightarrow{f_{n+1}} & M_{n+1} & \xrightarrow{g_{n+1}} & N_{n+1} & \to & 0 \\
0 & \to & L_n & \xrightarrow{f_n} & M_n & \xrightarrow{g_n} & N_n & \to & 0.
\end{array}
\]

From the snake lemma, we obtain unique morphisms

\[
\bar{f}_{n+1} : \ker(d_{n+1}^L) \to \ker(d_{n+1}^M), \quad l \mapsto f_{n+1}(l), \\
\bar{g}_{n+1} : \ker(d_{n+1}^M) \to \ker(d_{n+1}^N), \quad m \mapsto g_{n+1}(m), \\
\bar{f}_n : \coker(d_{n+1}^L) \to \coker(d_{n+1}^M), \quad [l] \mapsto [f_n(l)], \\
\bar{g}_n : \coker(d_{n+1}^M) \to \coker(d_{n+1}^N), \quad [m] \mapsto [g_n(m)]
\]
for which the diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L_{n-1} & \xrightarrow{f_{n-1}} & M_{n} & \xrightarrow{g_{n}} & N_{n} & \longrightarrow & 0 \\
0 & \longrightarrow & L_{n} & \xrightarrow{f_{n}} & M_{n} & \xrightarrow{g_{n}} & N_{n} & \longrightarrow & 0 \\
& & \downarrow{d_{n+1}} & & \downarrow{d_{n+1}} & & \downarrow{d_{n+1}} & & \\
& & \ker(d_{n+1}^{L}) \xrightarrow{\exists!f_{n+1}} \ker(d_{n+1}^{M}) \xrightarrow{\exists!g_{n+1}} \ker(d_{n+1}^{N}) & & \check{f}_{n+1} & \check{g}_{n+1} & \check{f}_{n+1} & \check{g}_{n+1} & \check{f}_{n+1} \\
\end{array}
\]
commutes and a unique morphism \( \partial_{n+1} : \ker(d_{n+1}^{N}) \to \ker(d_{n+1}^{L}) \) that makes the sequence
\[
\ker(d_{n+1}^{L}) \xrightarrow{\check{f}_{n+1}} \ker(d_{n+1}^{M}) \xrightarrow{\check{g}_{n+1}} \ker(d_{n+1}^{N}) \xrightarrow{\partial_{n+1}} \ker(d_{n+1}^{L}) \xrightarrow{\check{f}_{n+1}} \ker(d_{n+1}^{M}) \xrightarrow{\check{g}_{n+1}} \ker(d_{n+1}^{N})
\]
exact. As \( f_{n+1} \) is a monomorphism and \( g_{n} \) an epimorphism, the morphism \( \check{f}_{n+1} \) is a monomorphism and the morphism \( \check{g}_{n} \) an epimorphism for all \( n \in \mathbb{Z} \).

The morphisms \( \check{d}_{n}^{X} : \ker(d_{n}^{X}) \to \ker(d_{n}^{X}) \), \([x] \mapsto [d_{n}(x)]\) induced by \( d_{n}^{X} : X_{n} \to X_{n-1} \) and the universal property of the quotient \( \ker(d_{n+1}^{X}) = X_{n}/\text{im}(d_{n+1}^{X}) \) satisfy the identities
\[
\check{f}_{n-1} \circ d_{n}^{X}([l]) = \check{f}_{n-1} \circ d_{n}^{X}(l) = d_{n}^{M} \circ f_{n}(l) = d_{n}^{M}([f_{n}(l)]) = \check{d}_{n}^{M} \circ \check{f}_{n}([l])
\]
\[
\check{g}_{n-1} \circ d_{n}^{X}(m) = \check{g}_{n-1} \circ d_{n}^{M}(m) = d_{n}^{N} \circ g_{n}(m) = d_{n}^{N}([g_{n}(m)]) = \check{d}_{n}^{N} \circ \check{g}_{n}([m])
\]
for all \( l \in L_{n}, m \in M_{n} \), and hence we obtain the following commuting diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(d_{n}^{L}) & \xrightarrow{\check{f}_{n}} & \ker(d_{n}^{M}) & \xrightarrow{\check{g}_{n}} & \ker(d_{n}^{N}) & \longrightarrow & 0 \\
0 & \longrightarrow & \ker(d_{n-1}^{L}) & \xrightarrow{\check{f}_{n-1}} & \ker(d_{n-1}^{M}) & \xrightarrow{\check{g}_{n-1}} & \ker(d_{n-1}^{N}) & \longrightarrow & 0 \\
\end{array}
\]
We construct the long exact homology sequence by applying the snake lemma to this diagram for all \( n \in \mathbb{Z} \). For this, note that the homologies of the three chain complexes are given by
\[
H_{n}(X_{n}) = \ker(d_{n}^{X})/\text{im}(d_{n+1}^{X}) = \ker(\check{d}_{n}^{X}) = \ker(d_{n+1}^{X})
\]
and satisfy the conditions
\[
\begin{align*}
\check{d}_{n}^{M} \circ H_{n}(f_{n}) &= \check{f}_{n} \circ \check{d}_{n}^{L} \\
\check{d}_{n}^{N} \circ H_{n}(g_{n}) &= \check{g}_{n} \circ \check{d}_{n}^{M} \\
H_{n-1}(f_{n}) \circ \pi_{n-1}^{L} &= \pi_{n-1}^{M} \circ \check{f}_{n} \\
H_{n-1}(g_{n}) \circ \pi_{n-1}^{M} &= \pi_{n-1}^{N} \circ \check{g}_{n-1}
\end{align*}
\]
where \( \check{d}_{n}^{X} : \ker(d_{n}^{X})/\text{im}(d_{n+1}^{X}) \to X_{n}/\text{im}(d_{n+1}^{X}) \) and \( \pi_{n-1}^{X} : \ker(d_{n-1}^{X}) \to \ker(d_{n-1}^{X})/\text{im}(d_{n}^{X}) \) are the inclusions and canonical surjections. The snake lemma then a yields for all \( n \in \mathbb{Z} \) commuting diagrams (25) and unique morphisms \( \partial_{n} : H_{n}(N_{\bullet}) \to H_{n-1}(L_{\bullet}) \) such that the sequences
\[
\ldots \xrightarrow{H_{n+1}(g_{n})} H_{n+1}(N_{\bullet}) \xrightarrow{\partial_{n+1}} H_{n}(L_{\bullet}) \xrightarrow{H_{n}(f_{n})} H_{n}(M_{\bullet}) \xrightarrow{H_{n}(g_{n})} H_{n}(N_{\bullet}) \xrightarrow{\partial_{n}} H_{n-1}(L_{\bullet}) \xrightarrow{H_{n-1}(f_{n})} \ldots
\]
are exact. Combining these exact sequences for different \( n \in \mathbb{Z} \) yields the long exact homology sequence. Explicitly, the connecting morphism \( \partial_{k} : H_{k}(N_{\bullet}) \to H_{k-1}(L_{\bullet}) \) is given by
\[
\partial_{k}([n]) = [l] \text{ where } n = g_{k}(m), f_{k-1}(l) = d_{k}^{M}(m) \text{ for some } m \in M_{k}, l \in Z_{k-1}(L_{\bullet}).
\]
As the connecting morphism is constructed implicitly by a diagram chase, it does not appear very intuitive at first. Nevertheless, it has nice properties and a conceptual interpretation. In particular, it is compatible with chain maps between short exact sequences of chain complexes. As a consequence, every chain map between short exact sequences in $\text{Ch}_A$ defines a chain map between the associated long exact homology sequences in $A$.

**Theorem 3.4.8:** Let $0 \to L_\bullet \overset{f_\bullet}{\to} M_\bullet \overset{g_\bullet}{\to} N_\bullet \to 0$ and $0 \to L'_\bullet \overset{f'_\bullet}{\to} M'_\bullet \overset{g'_\bullet}{\to} N'_\bullet \to 0$ short exact sequences of chain complexes in an abelian category $A$ and $\alpha_\bullet : L_\bullet \to L'_\bullet$, $\beta_\bullet : M_\bullet \to M'_\bullet$ and $\gamma_\bullet : N_\bullet \to N'_\bullet$ chain maps such that the following diagram in $\text{Ch}_A$ commutes

$$0 \to L_\bullet \overset{f_\bullet}{\to} M_\bullet \overset{g_\bullet}{\to} N_\bullet \to 0 \quad (26)$$

Then we obtain the following commuting diagram with exact rows

$$... \to H_{n+1}(L_\bullet) \overset{H_{n+1}(f_\bullet)}{\to} H_{n+1}(M_\bullet) \overset{H_{n+1}(g_\bullet)}{\to} H_{n+1}(N_\bullet) \overset{\partial_{n+1}}{\to} H_n(L_\bullet) \overset{H_n(f_\bullet)}{\to} H_n(M_\bullet) \overset{H_n(g_\bullet)}{\to} H_n(N_\bullet) \overset{\partial_n}{\to} H_{n-1}(L_\bullet) \overset{H_{n-1}(f_\bullet)}{\to} H_{n-1}(M_\bullet) \overset{H_{n-1}(g_\bullet)}{\to} H_{n-1}(N_\bullet) \overset{\partial_{n-1}}{\to} ...$$

where $\partial_n : H_n(N_\bullet) \to H_{n-1}(L_\bullet)$ and $\partial'_n : H_n(N'_\bullet) \to H_{n-1}(L'_\bullet)$ are the connecting morphisms.

**Proof:**

The squares in the diagram that do not involve the connecting morphisms commute by the commutativity of $(26)$ and because the homologies are functors. It is sufficient to show that

$$H_k(N_\bullet) \overset{\partial_k}{\to} H_{k-1}(L_\bullet)$$

and

$$H_k(N'_\bullet) \overset{\partial'_k}{\to} H_{k-1}(L'_\bullet)$$
commutes for all $k \in \mathbb{Z}$. We prove this for $\mathcal{A} = R$-mod. The general proof follows from Mitchell’s embedding theorem. By [24] the connecting morphisms $\partial_k : H_k(N_\bullet) \to H_{k-1}(L_\bullet)$ are characterised by the condition

$$\partial_k([n]) = [l] \quad \text{where} \quad n = g_k(m), \quad f_{k-1}(l) = d_k^M(m) \quad \text{for some} \quad m \in M_k, l \in Z_{k-1}(L_\bullet).$$

This implies $H_{k-1}(\alpha_\bullet) \circ \partial_k([n]) = H_{k-1}(\alpha_\bullet)([l]) = [\alpha_{k-1}(l)]$. The commutativity of the diagram [26] implies $\gamma_k \circ g_k = g'_k \circ \beta_k$ for all $k \in \mathbb{Z}$. This yields $\gamma_k(n) = \gamma_k \circ g_k(m) = g'_k(\beta_k(m))$ and

$$d_k^M(\beta_k(m)) = \beta_{k-1} \circ d_k^M(m) = \beta_{k-1} \circ f_{k-1}(l) = f'_{k-1}(\alpha_{k-1}(l))$$

where we used that $\beta_\bullet$ is a chain map and that $f'_\bullet \circ \alpha_\bullet = \beta_\bullet \circ f_\bullet$. By definition of the connecting morphism $\partial'_k$, this implies

$$\partial'_k \circ H_k(\gamma_\bullet)([n]) = \partial'_k([\gamma_k(n)]) = \partial'_k([g'_k(\beta_k(m))]) = [\alpha_{k-1}(l)] = H_{k-1}(\alpha_\bullet) \circ \partial_k([n])$$

for all $[n] \in H_k(N_\bullet)$ and $k \in \mathbb{Z}$. \hfill $\square$

**Remark 3.4.9:** The result in Theorem 3.4.8 is sometimes called the **naturality of the connecting morphism** and allows one to view the connecting morphisms as natural transformations between certain functors.

Denote by $\text{Sh}_{\text{Ch}_\mathcal{A}}$ is the full subcategory of $\text{Ch}_{\text{Ch}_\mathcal{A}}$ whose objects are short exact sequences in $\text{Ch}_\mathcal{A}$. Then Theorem 3.4.8 states that the connecting morphisms define natural transformations $\partial_n : H^3_n \to H^1_{n-1}$, where $H^3_n : \text{Sh}_{\text{Ch}_\mathcal{A}} \to \mathcal{A}$ and $H^1_{n-1} : \text{Sh}_{\text{Ch}_\mathcal{A}} \to \mathcal{A}$ are the functors that assign to a short exact sequence $0 \to L_\bullet \to M_\bullet \to N_\bullet \to 0$ in $\text{Ch}_\mathcal{A}$ the homologies $H_n(N_\bullet)$ and $H_{n-1}(L_\bullet)$, respectively, and to the chain map in [26] the morphisms $H_n(\gamma_\bullet) : H_n(N_\bullet) \to H_n(N'_\bullet)$ and $H_{n-1}(\alpha_\bullet) : H_{n-1}(L_\bullet) \to H_{n-1}(L'_\bullet)$.
4 Derived functors and (co)homologies

By comparing the results from Section 3 with the examples of (co)homologies in Section 2, we see that the concrete approach from Section 2 has certain drawbacks and difficulties. Although it was shown in Proposition 3.3.4 that the homologies of a chain complex depend only on its chain homotopy equivalence class, all definitions of (co)homology theories in Section 2 involved concrete choices of chain complexes. As it is difficult to see if two given chain complexes are chain homotopy equivalent, the consequences of these choices are hard to control.

There could be other chain complexes that are chain homotopy equivalent to the singular chain complex, the Hochschild (co)complex or the chain complexes for group and Lie algebra cohomology in Section 2 and would allow one to compute the (co)homologies in a much simpler way. In fact, it is nearly impossible to compute Hochschild (co)homologies from their definition in Section 2 for any non-trivial examples. With a better understanding of the possible choices of chain complexes one could compute (co)homologies more efficiently and treat more examples.

The concrete definitions in Section 2 are also conceptually unpleasant because they involve arbitrary choices. The (co)homologies in Section 2 encoded properties of an object in a certain category, namely a topological space, a bimodule over an algebra, a module over a group ring or a Lie algebra representation. However, these (co)homologies were defined as (co)homologies of certain (co)chain complexes constructed from this object. It is not clear if there are alternative chain complexes that could be assigned to the objects under consideration and would define non-equivalent (co)homology theories of topological spaces, algebras, groups and Lie algebras.

Moreover, the definitions of (co)homology theories in Section 2 did not systematically incorporate morphisms into the picture. We showed in Example 3.2.5 that morphisms such as continuous maps between topological spaces, morphisms of bimodules over an algebra or morphisms of modules over group rings or morphisms of Lie algebra representations give rise to chain maps between the associated chain complexes. It was also shown in Example 3.2.5 that these assignments define functors. However, it is not clear if this is a general pattern. We do not have a general formalism that systematically assigns chain complexes to objects in a category $\mathcal{C}$ and chain maps to morphisms between them, not even if $\mathcal{C}$ is abelian.

In this section, we develop a more systematic approach that associates homologies not to chain complexes in abelian categories but to certain additive functors between abelian categories. The idea is the following:

- Associate to each object $A$ in an abelian category $\mathcal{A}$ an exact chain complex $A_\bullet$, a so-called resolution of $A$. Suppose that the resolutions $A_\bullet$ are unique up to chain homotopy equivalence and constructed in such a way that any morphism $f : A \to A'$ in $\mathcal{A}$ extends to a chain map $f_\bullet : A_\bullet \to A'_\bullet$ that is unique up to chain homotopy.

- Apply an additive functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ to the resolutions of objects in $\mathcal{A}$ and chain maps between them. This associates a chain complex $F(A_\bullet)$ to each object $A$ in $\mathcal{A}$ that is again unique up to chain homotopy equivalence and a chain map $F(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ to each morphism $f : A \to A'$ that is unique up to chain homotopy.

- The homologies $H_n F(A_\bullet)$ depend only on the chain homotopy equivalence class of $F(A_\bullet)$ by Proposition 3.3.4 and Exercise 46. Hence, they are independent of the choice of the resolution $A_\bullet$ and depend only on the object $A$. 

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• The chain maps $F(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ that extend morphisms $f : A \to A'$ in $\mathcal{A}$ induce morphisms $H_n F(f_\bullet) : H_n F(A_\bullet) \to H_n F(A'_\bullet)$ in $\mathcal{B}$ that depend only on the chain homotopy class of $F(f_\bullet)$ by Proposition 3.3.4. Hence, they depend only on the morphism $f : A \to A'$ and not on the choice of the chain map $f_\bullet : A_\bullet \to A'_\bullet$.

• If this construction is compatible with the composition of morphisms in $\mathcal{A}$, it defines a collection of functors $H_n F : \mathcal{A} \to \mathcal{B}$ that send an object $A$ to the homology $H_n F(A_\bullet)$ and a morphism $f : A \to A'$ to the morphism $H_n F(f_\bullet) : H_n F(A_\bullet) \to H_n F(A'_\bullet)$.

We will see in the following that most examples from Section 2, namely Hochschild (co)homologies, group cohomologies and cohomologies of Lie algebras can be realised as homologies of functors in this way. This viewpoint has several advantages:

• It does not depend on non-canonical choices of chain complexes to define (co)homologies but defines them intrinsically as homologies of functors.

• It integrates different notions of homology and cohomology into a common conceptual framework and hence allows one to investigate them more systematically.

• It allows one to compute (co)homologies more efficiently through the choice of appropriate resolutions. This is a major advantage since the definitions in Section 2 can lead to very cumbersome computations.

4.1 Resolutions

To determine under which assumptions this idea can be implemented, we investigate the existence and uniqueness of resolutions and the extension of morphisms to chain maps between them. In this, we restrict attention to (co)chain complexes that are bounded below, like the (co)chain complexes from Section 2. More specifically, in agreement with the standard conventions in the literature, we require that all objects with index $< -1$ are zero objects and that the objects under consideration occur in the (co)chain complex with index -1.

Definition 4.1.1: Let $\mathcal{A}$ be an abelian category and $A$ an object in $\mathcal{A}$.

1. A left resolution of $A$ is an exact sequence in $\mathcal{A}$ of the form

   $$\ldots \xrightarrow{d_2} A_2 \xrightarrow{d_1} A_1 \xrightarrow{d_0} A_0 \xrightarrow{d_0} A \to 0$$

2. A right resolution of $A$ is an exact sequence in $\mathcal{A}$ of the form

   $$0 \to A \xrightarrow{d^{-1}} A^{-1} \xrightarrow{d^0} A^0 \xrightarrow{d^1} A^1 \xrightarrow{d^2} A^2 \xrightarrow{d^2} \ldots$$

A left or right resolution is called projective or injective if $A_n$ or $A^n$ is projective or injective for all $n \in \mathbb{N}_0$. If $\mathcal{A} = R$-Mod, then a left or right resolution is called free if $A_n$ or $A^n$ is a free $R$-module for all $n \in \mathbb{N}_0$ and flat if $A_n$ or $A^n$ is a flat $R$-module for all $n \in \mathbb{N}_0$. 

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Example 4.1.2: (Bar resolution for group cohomology)

Let $k$ be a commutative ring and $G$ a group. Consider the chain complex $X_\bullet$ in $k[G]$-Mod with

$$X_n = \begin{cases} (G^n)_{k[G]} & n \geq 0 \\
k & n = -1 \end{cases}$$

$$d_n(g_1, \ldots, g_n) = g_1 \triangleright (g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1})$$

$$d_0(g_1) = 1$$

and $X_n = 0$ for $n < -1$. It is exact, since the $k$-linear (but not $k[G]$-linear) maps

$$h_{-1} : k \to k[G], \quad 1 \mapsto 1, \quad h_n : X_n \to X_{n+1}, \quad g_0 \triangleright (g_1, \ldots, g_n) \mapsto (g_0, g_1, \ldots, g_n) \quad \text{for} \quad n \in \mathbb{N}_0$$

define a chain homotopy $h_\bullet : 1_{X_\bullet} \Rightarrow 0_{X_\bullet}$ in $k$-Mod. As the homologies of $X_\bullet$ as a chain complex in $k$-Mod and $k[G]$-Mod are the same, $X_\bullet$ is also exact in $k[G]$-Mod. This shows that $X_\bullet$ is a free left resolution of the trivial $k[G]$-module $k$ in $k[G]$-Mod, the \textbf{bar-resolution} of $k$.

Example 4.1.3: (Hochschild resolution)

Let $k$ be a commutative ring and $A$ an algebra over $k$. Then the chain complex $X_\bullet$ in $A$-Mod-$A$ from Example 3.3.5 2. with

$$X_n = A^\otimes(n+2) \quad n \geq -1$$

$$d_n : X_n \to X_{n-1}, \quad a_0 \otimes \ldots \otimes a_{n+1} \mapsto \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_{n+1}$$

$X_n = 0$ for $n < -1$ and the $(A, A)$-bimodule structure

$$\triangleright : A \otimes A^\text{op} \times X_n \to X_n, \quad (b \otimes c) \triangleright a_0 \otimes \ldots \otimes a_{n+1} = (ba_0) \otimes a_1 \otimes \ldots \otimes a_n \otimes (a_{n+1} c)$$

$$\triangleright : A \otimes A^\text{op} \times X_{-1} \to X_{-1}, \quad (b \otimes c) \triangleright a_0 = ba_0 c.$$

is exact since the $k$-linear maps

$$h_n : X_n \to X_{n+1}, \quad a_0 \otimes \ldots \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_{n+1}$$

define a chain homotopy $1_{X_\bullet} \Rightarrow 0_{X_\bullet}$ in $k$-Mod by Example 3.3.5. As the homologies of $X_\bullet$ in $k$-Mod and in $A$-Mod-$A$ are the same, $X_\bullet$ is also exact as a chain complex in $A$-Mod-$A$. This shows that $X_\bullet$ is a resolution of $A$ in $A$-Mod-$A$, the \textbf{Hochschild resolution} of $A$. As $(A, A)$-bimodule structures on a $k$-module $M$ are in bijection with $A \otimes A^\text{op}$-module structures via $a \triangleright m \leq b = (a \otimes b) \triangleright m$, we can also view this as a projective resolution in $A \otimes A^\text{op}$-Mod.

Example 4.1.4: (Chevalley-Eilenberg resolution for Lie algebra cohomology)

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\Lambda^n \mathfrak{g}$ is a free $U(\mathfrak{g})$-module with the module structure induced by the adjoint representation $\text{ad} : \mathfrak{g} \to \text{End}_\mathbb{F}(\Lambda^n \mathfrak{g})$, $\text{ad}(x)y = [x, y]$. The chain complex $X_\bullet$ in $U(\mathfrak{g})$-Mod with

$$X_n = U(\mathfrak{g}) \otimes \Lambda^n \mathfrak{g}, \quad X_{-1} = \mathbb{F}$$

$$d_n(y \otimes x_1 \wedge \ldots \wedge x_n) = \sum_{i=1}^{n} (-1)^i y x_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_n$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} y \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n$$

and $U(\mathfrak{g})$-module structure $\triangleright : U(\mathfrak{g}) \times X_n \to X_n, \quad z \triangleright (y \otimes x_1 \wedge \ldots \wedge x_n) = (zy) \otimes x_1 \wedge \ldots \wedge x_n$ is a projective left resolution of the trivial $U(\mathfrak{g})$-module $\mathbb{F}$ in $U(\mathfrak{g})$-Mod. The proof of this statement is long and can be found in [HS, VII.4].
**Example 4.1.5:** If \( R \) is a principal ideal domain, every finitely generated \( R \)-module \( N \) has a free left resolution that is a short exact sequence \( 0 \to L \xrightarrow{\pi} M \xrightarrow{\varepsilon} N \to 0 \).

This follows because every finitely generated \( R \)-module is of the form \( R^n \times R/q_1 R \times \ldots \times R/q_k R \) with \( n \in \mathbb{N}_0 \) and prime powers \( q_i \in R \) by the classification theorem. We can choose the free \( R \)-modules \( L = R^k \) and \( M = R^{n+k} \) and the \( R \)-linear maps \( \iota : L \to M \), \( (r_1, \ldots, r_k) \mapsto (q_1 r_1, \ldots, q_k r_k) \) and \( \pi : R^{n+k} \to N \), \( (r_1, \ldots, r_{n+k}) \mapsto (r_1, \ldots, r_n, \bar{r}_{n+1}, \ldots, \bar{r}_{n+k}) \).

For the idea at the beginning of this section we need resolutions that allow us to extend every morphism between objects to a chain map between their resolutions. We determine under which conditions on the resolutions this is possible.

For this we consider two objects \( A, A' \) in \( A \) with left resolutions \( A_*, A'_* \) and a morphism \( f : A \to A' \). The aim is to extend \( f : A \to A' \) to a chain map \( f_* : A_* \to A'_* \) with \( f_{-1} = f \)

\[
\begin{array}{ccccccccc}
\cdots & d_3 & A_2 & d_2 & A_1 & d_1 & A_0 & d_0 & A \\
& \uparrow \exists f_2 & \uparrow \exists f_1 & \uparrow \exists f_0 & & & & f \\
\cdots & d'_3 & A'_2 & d'_2 & A'_1 & d'_1 & A'_0 & d'_0 & A' \\
\end{array}
\]

This should be done step by step from the right. In the first step, we require that for any morphism \( f : A \to A' \) and epimorphisms \( d_0 : A_0 \to A \) and \( d'_0 : A'_0 \to A' \) there is a morphism \( f_0 : A_0 \to A'_0 \) with \( d'_0 \circ f_0 = f \circ d_0 \).

If we replace the morphism \( f \circ d_0 : A_0 \to A' \) in this condition by a general morphism \( g : A_0 \to A' \) in \( A \), this is equivalent to the projectivity of \( A_0 \) by Lemma [3.1.21]. Hence, we should consider left resolutions with *projective* objects \( A_0 \). We then attempt to extend \( f : A \to A' \) to a chain map \( f_* : A_* \to A'_* \) by iterating the construction of \( f_0 : A_0 \to A'_0 \). For this, we should impose that *each* object \( A_n \) in the left resolution \( A_* \) except \( A_{-1} = A \) is *projective*.

Indeed, we find that these conditions are sufficient to extend \( f : A \to A' \) to a chain map \( f_* : A_* \to A'_* \) and to ensure that \( f_* \) is unique up to chain homotopy. The corresponding conditions and results for *right resolutions* are obtained by identifying right resolutions and injective objects in \( A \) with left resolutions and projective objects in \( A^{op} \).

**Theorem 4.1.6:** (fundamental lemma of homological algebra)

Let \( A \) be an abelian category.

1. If \( A'_* = \ldots \xrightarrow{d'_1} A_0 \xrightarrow{d'_0} A' \to 0 \) is exact and \( A_* = \ldots \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \to 0 \) a chain complex in \( A \) with \( A_n \) projective for all \( n \in \mathbb{N}_0 \), then for every morphism \( f : A \to A' \) there is a chain map \( f_* : A_* \to A'_* \) with \( f_{-1} = f \), and this chain map is unique up to chain homotopy.

2. If \( A^{op}_* = 0 \to A' \xrightarrow{d'^{-1}} A^0 \xrightarrow{d^0} \ldots \) is exact and \( A^{op}_* = 0 \to A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} \ldots \) a chain complex in \( A \) with \( A^n \) injective for all \( n \in \mathbb{N}_0 \), then for every morphism \( f : A \to A' \) there is a chain map \( f^{op}_* : A^{op}_* \to A^{op}_* \) with \( f^{op}_{-1} = f \), and this chain map is unique up to chain homotopy.

**Proof:**
We prove the first statement, since the second statement is the first one for \( A^{op} \).

1. Construction of \( f_* \):
As \( A_0 \) is projective and \( d'_0 : A'_0 \to A' \) an epimorphism, by Lemma [3.1.21] there is a morphism \( f_0 : A_0 \to A'_0 \) for which the following diagram commutes.
As $d_0 \circ d_1 = 0$ and $d'_0 \circ d'_1 = 0$, by the universal property of the kernels $\iota_0 : \ker(d_0) \to A_0$ and $\iota'_0 : \ker(d'_0) \to A'_0$ there are unique morphisms $d_1 : A_1 \to \ker(d_0)$ and $d'_1 : A'_1 \to \ker(d'_0)$ with $\iota_0 \circ d_1 = d_1$ and $\iota'_0 \circ d'_1 = d'_1$. As $A'_* \to A_*$ is exact, we have $\ker(d'_0) \cong \text{im}(d'_1)$ and $d'_1 = \iota'_0 \circ d'_1$ is the canonical factorisation of $d'_1$ from Lemma 3.1.13. It follows that $d'_1$ is an epimorphism. As we have $d'_0 \circ f_0 \circ \iota_0 = f \circ d_0 \circ \iota_0 = 0$, by the universal property of the kernel $\iota'_0 : \ker(d'_0) \to A'_0$, there is a unique morphism $f'_0 : \ker(d_0) \to \ker(d'_0)$ with $\iota'_0 \circ f'_0 = f_0 \circ \iota_0$. As $A_1$ is projective and $d'_1 : A'_1 \to \ker(d'_0)$ an epimorphism, there is a morphism $f_1 : A_1 \to A'_1$ with $d'_1 \circ f_1 = f'_0 \circ d'_1$.

Iterating this procedure yields a chain map $f_* : A_* \to A'_*$ with $f_{-1} = f$.

2. Uniqueness of $f_*$:  
It is sufficient to show that any chain map $f_* : A_* \to A'_*$ with $f_{-1} = 0 : A \to A'$ is chain homotopic to $0_* : A_* \to A'_*$. We iteratively construct morphisms $h_n : A_n \to A'_n + 1$ for $n \geq -1$ with $f_n = h_n \circ d_n + d'_{n+1} \circ h_n$. For this, we set $h_{-1} = 0 : A \to A'_0$ and consider the commuting diagram from 1:

As $d'_0 \circ f_0 = 0 \circ d_0 = 0$, by the universal property of the kernel $\iota'_0 : A_0 \to \ker(d'_0)$ there is a unique morphism $f''_0 : A_0 \to \ker(d'_0)$ with $\iota'_0 \circ f''_0 = f_0$. As $A_0$ is projective and $d'_1 : A'_1 \to \ker(d'_0)$ an epimorphism, by Lemma 3.1.21 there is a morphism $h_0 : A_0 \to A'_1$ with $h_0 \circ d'_1 = f''_0$.
This implies \( d_1' \circ h_0 + h_{-1} \circ d_1 = d_1' \circ h_1 + 0 \circ d_1 = d_1' \circ h_0 = \iota_0' \circ d_1' \circ h_0 = \iota_0' \circ f''_0 = f_0 \) and 
\( d_1' \circ (f_1 - h_0 \circ d_1) = d_1' \circ f_1 - f_0 \circ d_1 = 0 = 0 \circ d_1 \). We can apply the same argument again and obtain the commuting diagram

\[
\begin{array}{ccc}
A_2 & \xrightarrow{d_2} & A_1 \\
\downarrow \exists h_1 & & \downarrow \exists f''_1 \\
\ker(d_1') & \xrightarrow{\iota_1} & 0 \\
A_2' \xrightarrow{d_1'} & A_1' \xrightarrow{d_0} & A_0 \\
\end{array}
\]

It follows that \( d_2' \circ h_1 = \iota_1' \circ d_2 \circ h_1 = \iota_1' \circ f''_1 = f_1 - h_0 \circ d_1 \) and \( f_1 = d_2' \circ h_1 + h_0 \circ d_1 \). Iterating this procedure yields a chain homotopy \( h : f_* \Rightarrow 0_{A_*} \).

Theorem 4.1.6 shows that projective resolutions of objects in an abelian category \( \mathcal{A} \) have the required extension properties for morphisms. They allow one to extend every morphism between objects in \( \mathcal{A} \) to a chain map between their resolutions that is unique up to chain homotopy. To implement the idea at the beginning of this section, we must ensure that each object \( A \) in \( \mathcal{A} \) has a projective left or injective right resolution

\[
A_* = \ldots \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \xrightarrow{0} \quad \text{or} \quad A^* = 0 \rightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \ldots
\]

The exactness of these chain complexes in \( A \) is equivalent to the statements that \( d_0 \) is an epimorphism and \( d^{-1} \) a monomorphism. We therefore must require at least that for each object \( A \) in \( \mathcal{A} \), there is a projective object \( A_0 \) and an epimorphism \( d_0 : A_0 \rightarrow A \) or an injective object \( A^0 \) and a monomorphism \( d^{-1} : A \rightarrow A^0 \).

**Definition 4.1.7:**

1. An abelian category \( \mathcal{A} \) has **enough projectives** if for every object \( A \) in \( \mathcal{A} \) there is a projective object \( P \) in \( \mathcal{A} \) and an epimorphism \( \pi : P \rightarrow A \).

2. An abelian category \( \mathcal{A} \) has **enough injectives** if for every object \( A \) in \( \mathcal{A} \) there is an injective object \( I \) in \( \mathcal{A} \) and a monomorphism \( \iota : A \rightarrow I \).

It turns out that the conditions in Definition 4.1.7 are not only necessary for the existence of projective and injective resolutions but also sufficient. They guarantee the existence of projective or injective resolutions for all objects in \( \mathcal{A} \). Their uniqueness up to chain homotopy equivalence then follows directly from Theorem 4.1.6.

**Theorem 4.1.8:** If an abelian category \( \mathcal{A} \) has enough projectives (injectives), then every object in \( \mathcal{A} \) has a projective left (injective right) resolution, and projective left (injective right) resolutions are unique up to chain homotopy equivalence.

**Proof:**

We prove the claim for projective left resolutions. The claim for injective right resolutions follows since injective right resolutions in \( \mathcal{A} \) are projective left resolutions in \( \mathcal{A}^{\text{op}} \).
Let $A$ be an object in $\mathcal{A}$. As $\mathcal{A}$ has enough projectives, there is a projective object $A_0$ and an epimorphism $d_0 : A_0 \to A$. For the kernel $\iota_0 : \ker(d_0) \to A_0$ there is a projective object $A_1$ and an epimorphism $d_1 : A_1 \to \ker(d_0)$. The morphism $d_1 = \iota_0 \circ d_1' : A_1 \to A_0$ satisfies $\im(d_1) = \ker(\coker(\iota_0 \circ d_1')) \cong \ker(\coker(\iota_0)) \cong \im(\iota_0) \cong \ker(d_0)$ since $d_1'$ is an epimorphism and $\iota_0$ a monomorphism and hence its own image.

This shows that the sequence $A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \to 0$ is exact. Iterating this procedure yields a projective resolution of $A$

$$\cdots \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \to 0$$

Let now $A_\bullet, A_\bullet'$ be two projective resolutions of $A$. Then by Theorem 4.1.6 there are chain maps $f_\bullet : A_\bullet \to A_\bullet'$ and $f'_\bullet : A_\bullet' \to A_\bullet$ with $f_{-1} = 1_A = f'_{-1}$. Their composites $g_\bullet = f'_\bullet \circ f_\bullet : A_\bullet \to A_\bullet$ and $g'_\bullet = f_\bullet \circ f'_\bullet : A_\bullet' \to A_\bullet'$ are chain maps with $g_{-1} = 1_A = g'_{-1}$. As the identity chain maps $1_{A_\bullet} : A_\bullet \to A_\bullet$ and $1_{A_\bullet'} : A_\bullet' \to A_\bullet'$ also satisfy this condition, we have $f'_{\bullet} \circ f_{\bullet} \sim 1_{A_\bullet}$ and $f_{\bullet} \circ f'_\bullet \sim 1_{A_\bullet'}$ by Theorem 4.1.6 and hence $f_\bullet : A_\bullet \to A_\bullet'$ is a chain homotopy equivalence.

Theorems 4.1.6 and 4.1.8 guarantee that in an abelian category $\mathcal{A}$ with enough projectives (injectives) every object has a projective (injective) resolution, unique up to chain homotopy equivalence, and every morphism lifts to a chain map between resolutions, unique up to chain homotopy. We can thus implement the idea at the beginning of this section in any abelian category with enough projectives (injectives).

Our main examples of abelian categories are the categories $R$-Mod of modules over a ring $R$. To apply the formalism to $R$-Mod, we therefore need to ensure that $R$-Mod has enough projectives or injectives. We also need to ensure that our standard resolutions, the bar resolution for group cohomology, the Hochschild resolution and the Chevalley-Eilenberg resolution for Lie algebra cohomology are indeed projective resolutions and derive derive general criteria for projectivity and injectivity in abelian categories.

### 4.2 Projectivity and injectivity criteria

Recall first the results on projectivity and injectivity from Section 3.1.

- By Definition 3.1.20 an object $A$ in an abelian category $\mathcal{A}$ is projective and injective, respectively, if the functors $\Hom(A, -) : \mathcal{A} \to \text{Ab}$ and $\Hom(-, A) : \mathcal{A}^{\text{op}} \to \text{Ab}$ are exact.
- By Lemma 3.1.21 projectivity of $A$ is equivalent to the existence of a morphism $f' : A \to X$ with $\pi \circ f' = f$ for every morphism $f : A \to Y$ and epimorphism $\pi : X \to Y$. Injectivity of $A$ is equivalent to the existence of a morphism $f' : Y \to A$ with $f' \circ \iota = f$ for every morphism $f : X \to A$ and monomorphism $\iota : X \to Y$.
- By Example 3.1.22 every free module over a ring $R$ is projective. The prototype of a non-projective and non-injective object is the abelian group $\mathbb{Z}/n\mathbb{Z}$, and the prototype of an object that is projective, but not injective is the abelian group $\mathbb{Z}$. 

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The fact that every free module in $R$-Mod is projective implies immediately that the category $R$-Mod has enough projectives. By Remark 1.2.1, every $R$-module $M$ is a quotient $M = F/L$ of a free and hence projective $R$-module $F$ by a submodule $L \subset F$, and the canonical surjection $\pi : F \to M$ is an epimorphism. The proof that $R$-Mod has enough injectives is more involved, see for instance [JS, Satz E.9] or [W, pp 39–41].

**Example 4.2.1:**

1. For any ring $R$, the category $R$-Mod has enough projectives.
2. For any ring $R$, the category $R$-Mod has enough injectives.

The fact that free $R$-modules are projective also guarantees directly that the bar resolution of group cohomology from Example 4.1.2 is a projective resolution, since it is a resolution by free $k[G]$-modules. However, for the Hochschild resolution and the Chevalley-Eilenberg resolution from Examples 4.1.3 and 4.1.4, the situation is more complicated. It is a priori not guaranteed that the $A \otimes A^\text{op}$-modules and $U(g)$-modules in these resolutions are free. In fact, finite tensor products $A \otimes_n A = A \otimes_k \cdots \otimes_k A$ of a $k$-algebra $A$ with itself need not even be projective $k$-modules. In order to derive good criteria for the projectivity of these standard resolutions, we need criteria for the projectivity of products, direct sums and tensor products in $R$-Mod.

Criteria for the projectivity of coproducts and the injectivity of products can be derived in more generality from the original definition of projective and injective objects in terms of exactness of the functors $\text{Hom}(A, -) : A \to \text{Ab}$ and $\text{Hom}(-, A) : A^\text{op} \to \text{Ab}$. The key idea is to notice that the functors $\text{Hom}(\Pi_{i \in I} X_i, -), \Pi_{i \in I} \text{Hom}(X_i, -) : A \to \text{Ab}$ are naturally isomorphic, and so are the functors $\text{Hom}(-, \Pi_{i \in I} X_i), \Pi_{i \in I} \text{Hom}(-, X_i) : A^\text{op} \to \text{Ab}$. The exactness of $\text{Hom}(\Pi_{i \in I} X_i, -)$ and $\text{Hom}(-, \Pi_{i \in I} X_i)$ amounts to the projectivity of the coproduct and the injectivity of the product, whereas the exactness of the functors $\Pi_{i \in I} \text{Hom}(X_i, -)$ and $\Pi_{i \in I} \text{Hom}(-, X_i)$ amounts to the projectivity and injectivity of all objects in the family.

**Lemma 4.2.2:** Let $(X_i)_{i \in I}$ be a family of objects in an abelian category $\mathcal{A}$ with a (co)product.

1. The coproduct $\Pi_{i \in I} X_i$ is projective if and only if $X_i$ is projective for all $i \in I$.
2. The product $\Pi_{i \in I} X_i$ is injective if and only if $X_i$ is injective for all $i \in I$.

**Proof:**

We prove the second claim by considering the functor $F = \text{Hom}(-, \Pi_{i \in I} X_i) : A^\text{op} \to \text{Ab}$ that assigns to an object $A$ in $\mathcal{A}$ the abelian group $\text{Hom}_A(A, \Pi_{i \in I} X_i)$, and to a morphism $f : A \to B$ in $A$ the group homomorphism

$$\text{Hom}(f, \Pi_{i \in I} X_i) : \text{Hom}_A(B, \Pi_{i \in I} X_i) \to \text{Hom}_A(A, \Pi_{i \in I} X_i), \quad g \mapsto g \circ f$$

and the functor $G = \Pi_{i \in I} \text{Hom}(X_i, -) : A^\text{op} \to \text{Ab}$ that assigns to an object $A$ the abelian group $\Pi_{i \in I} \text{Hom}_A(A, X_i)$ and to a morphism $f : A \to B$ the unique group homomorphism

$$\Pi_{i \in I} \text{Hom}(f, X_i) : \Pi_{i \in I} \text{Hom}_A(B, X_i) \to \Pi_{i \in I} \text{Hom}_A(A, X_i)$$

with $\pi_j \circ \Pi_{i \in I} \text{Hom}(f, X_i) = \text{Hom}(f, X_j) \circ \pi_j$ for all $j \in I$. By the universal property of the product we have isomorphisms of abelian groups

$$\eta_A : \text{Hom}_A(A, \Pi_{i \in I} X_i) \to \Pi_{i \in I} \text{Hom}_A(A, X_i), \quad g \mapsto (\pi_i \circ g)_{i \in I}$$
for each object $A$ in $\mathcal{A}$. The isomorphisms $\eta_A$ define a natural isomorphism $\eta: F \rightarrow G$, since for any morphism $f: A \rightarrow B$ the following diagram commutes

$$
\begin{array}{ccc}
\text{Hom}_A(B, \Pi_{i \in I} X_i) & \xrightarrow{\eta_B} & \Pi_{i \in I} \text{Hom}_A(B, X_i) \\
\downarrow \text{Hom}(f, \Pi_{i \in I} X_i) : g \mapsto \eta_B \circ g & & \downarrow \Pi_{i \in I} \text{Hom}(f, X_i) : (g_i)_{i \in I} \mapsto (g_i)_{i \in I}
\end{array}
$$

As $F$ and $G$ are naturally isomorphic, $F$ is exact if and only if $G$ is by Corollary 3.4.4. By Definition 3.1.20 the exactness of $F$ is equivalent to the injectivity of $\Pi_{i \in I} X_i$. By the universal property of the product, the exactness of $G$ is equivalent to the exactness of $\text{Hom}(\_ , X_i)$ for all $i \in I$ and hence to the injectivity of $X_i$ for all $i \in I$. □

By applying Lemma 4.2.2 to finite coproducts in the abelian category $\mathcal{A} = R\text{-Mod}$, we obtain a useful projectivity criterion in terms of direct sums. The compatibility between tensor products and direct sums then yields a criterion for the projectivity of tensor products over commutative rings that can be applied to the Hochschild resolution and Chevalley-Eilenberg resolution.

**Corollary 4.2.3:** Let $R$ be a ring.

1. An $R$-module $A$ is projective if and only if there is an $R$-module $B$ with $A \oplus B$ free.
2. If $R$ is commutative and $A_1$ and $A_2$ are projective, then $A_1 \otimes_R A_2$ is projective.

**Proof:**
1. By Example 3.1.22, free $R$-modules are projective and by Lemma 4.2.2, a direct sum $A \oplus B$ is projective if and only if $A$ and $B$ are projective. Hence, if $A \oplus B$ is free, it is projective and $A$ and $B$ are projective by Lemma 4.2.2.

Conversely, every $R$-module $A$ is a quotient $A = F/L$ of a free module $F$ by a submodule $L \subset F$, and the canonical surjection $\pi: F \rightarrow A$ is an epimorphism. If $A$ is projective, there is an $R$-linear map $f: A \rightarrow F$ with $\pi \circ f = \text{id}_A$ by Lemma 3.1.21. As $\text{id}_A$ is injective, $f$ is injective as well and hence an isomorphism onto its image $A \cong \text{im}(f) \subset F$. Hence $A \oplus \ker(\pi) \cong \text{im}(f) \oplus \ker(\pi) \cong F$.

2. If $A_1$ and $A_2$ are projective, then by 1. there are $R$-modules $B_i$ and free $R$-modules $F_i = \oplus I_i R$ with $A_i \oplus B_i = F_i$. The $R$-module $A_1 \otimes_R A_2$ is then projective by 1., since the compatibility of tensor products and direct sums implies

$$
\begin{align*}
(A_1 \otimes_R A_2) & \oplus (A_1 \otimes_R B_2 \oplus A_1 \otimes_R A_2 \oplus B_1 \otimes_R B_2) \\
\cong (A_1 \oplus B_1) \otimes_R (A_2 \oplus B_2) & = F_1 \otimes_R F_2 \cong (\oplus I_1 R) \otimes_R (\oplus I_2 R) \cong \oplus I_1 \times I_2 R \otimes_R R \cong \oplus I_1 \times I_2 R. & \quad \Box
\end{align*}
$$

It is worth mentioning that a similar reasoning can be used for flat modules. With an argument analogous to the proof of Lemma 4.2.2, one can show that a direct sum of $R$-modules is flat if and only if each summand is flat and by combining this with Corollary 4.2.3, one finds that every projective $R$-module is flat (Exercise 45).

**Corollary 4.2.4:** Let $R$ be a ring and $(M_i)_{i \in I}$ a family of $R$-modules.

1. The direct sum $\oplus_{i \in I} M_i$ is flat if and only if $M_i$ is flat for all $i \in I$.
2. Every projective $R$-module is flat.
Inductively, the second claim in Corollary 4.2.3 implies that all finite tensor products of projective modules over a commutative ring are projective. In particular, if \(A\) is an algebra over a commutative ring that is projective as a \(k\)-module, then all finite tensor products \(A^\otimes n = A \otimes_k A \otimes_k \ldots \otimes_k A\) are projective \(k\)-modules. This ensures that under the assumption that \(A\) is projective as a \(k\)-module, the Hochschild resolution is indeed a projective resolution. A similar reasoning shows that the Chevalley-Eilenberg resolution is a projective resolution.

**Corollary 4.2.5:** Let \(k\) be a commutative ring and \(F\) a field.

1. For any \(k\)-algebra \(A\) that is a projective \(k\)-module, the Hochschild resolution from Example 4.1.3 is a projective resolution of \(A\) in \(A \otimes A^{op}\text{-Mod}\).

2. For any Lie algebra \(\mathfrak{g}\) over \(F\), the Chevalley-Eilenberg resolution from Example 4.1.4 is a projective resolution of \(F\) in \(U(\mathfrak{g})\text{-Mod}\).

**Proof:**

1. We show that for any projective \(k\)-module \(M\) the module \(A \otimes_k M \otimes_k A\) is a projective \(A \otimes_k A^{op}\)-module with the canonical \(A \otimes_k A^{op}\)-module structure \((b \otimes c) \cdot (a \otimes m \otimes a') = (ba) \otimes m \otimes (a'b)\). The claim then follows because the \(A \otimes_k A^{op}\)-modules in the Hochschild resolution for \(n \in \mathbb{N}_0\) take this form with \(M = A^\otimes n\), and \(A^\otimes n\) is a projective \(k\)-module by Corollary 4.2.3.

For each projective \(k\)-module \(M\) and \(A \otimes_k A^{op}\)-module \(N\) the map

\[
\eta_N : \text{Hom}_{A \otimes_k A^{op}}(A \otimes_k M \otimes_k A, N) \to \text{Hom}_k(M, F(N)), \quad f \mapsto f' \quad \text{with} \quad f'(m) = f(1 \otimes m \otimes 1)
\]

is an isomorphism of abelian groups with inverse

\[
\eta_N^{-1} : \text{Hom}_k(M, F(N)) \to \text{Hom}_{A \otimes_k A^{op}}(A \otimes_k M \otimes_k A, N), \quad g \mapsto g' \quad \text{with} \quad g'(a \otimes m \otimes a') = (a \otimes a') \cdot g(m),
\]

and for every \(A \otimes A^{op}\)-linear map \(f : N \to N'\), the following diagram commutes

\[
\begin{array}{ccc}
\text{Hom}_{A \otimes_k A^{op}}(A \otimes_k M \otimes_k A, N) & \xrightarrow{\eta_N \circ g \circ f} & \text{Hom}_k(M, N) \\
\downarrow{g \circ f \circ g} & & \downarrow{g \circ f \circ g} \\
\text{Hom}_{A \otimes_k A^{op}}(A \otimes_k M \otimes_k A, N') & \xrightarrow{\eta_N^{-1} \circ g \circ g'} & \text{Hom}_k(M, N')
\end{array}
\]

We have shown the group isomorphism morphisms \(\eta_N\) define a natural isomorphism

\[
\eta : \text{Hom}_{A \otimes_k A^{op}}(A \otimes_k M \otimes_k A, -) \to \text{Hom}_k(M, F(-)),
\]

where \(F : A \otimes_k A^{op}\text{-Mod} \to k\text{-Mod}\) is the forgetful functor. As \(M\) is a projective \(k\)-module, the functor \(\text{Hom}_k(M, -)\) is exact. By Corollary 3.4.4 the functor \(\text{Hom}_{A \otimes_k A^{op}}(A \otimes_k M \otimes_k A, -)\) is exact as well, and hence \(A \otimes_k M \otimes_k A\) is a projective \(A \otimes_k A^{op}\)-module.

2. The claim follows from 1. by specialising to \(k = F\), \(A = U(\mathfrak{g})\) for a Lie algebra \(\mathfrak{g}\) over \(F\) and to trivial \(U(\mathfrak{g})\)-right module structures. The results in 1. imply that the \(U(\mathfrak{g})\)-module \(U(\mathfrak{g}) \otimes \Lambda^n \mathfrak{g}\) is a projective \(U(\mathfrak{g})\)-module since \(\Lambda^n \mathfrak{g}\) a free \(F\)-module, since \(F\) is a field. 

We have thus established that projective and injective resolutions exist for all objects in the abelian category \(R\text{-Mod}\) and that two of our standard resolutions, the bar-resolution and Chevalley-Eilenberg resolution, are indeed projective resolutions. For the Hochschild resolution, this holds if the \(k\)-algebra \(A\) is a projective \(k\)-module, which is always the case if \(k\) is a field. We can thus return to the general formalism in abelian categories and fully implement the idea form the beginning of this section.

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4.3 Derived functors

Given an abelian category $\mathcal{A}$ with enough projectives we assign to each object $A$ a projective left resolution $A_\bullet$, unique up to chain homotopy equivalence, and to each morphism $f : A \to A'$ a chain map $f_\bullet : A_\bullet \to A'_\bullet$ between the resolutions, unique up to chain homotopy. An additive functor $F : \mathcal{A} \to \mathcal{B}$ sends chain complexes $A_\bullet$ in $\mathcal{A}$ to chain complexes $F(A_\bullet)$ in $\mathcal{B}$, chain maps $f_\bullet : A_\bullet \to A'_\bullet$ to chain maps $F(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow g_\bullet$ to chain homotopies $H(f_\bullet) : F(f_\bullet) \Rightarrow F(g_\bullet)$ (see Exercise 46). It follows that for each object $A$ in $\mathcal{A}$ and projective resolution $A_\bullet$ with $A_{-1} = A$, the homologies $H_nF(A_\bullet)$ do not depend on the choice of $A_\bullet$. Similarly, for a morphism $f : A \to A'$ the morphisms $H_nF(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ do not depend on the choice of the chain map $f_\bullet : A_\bullet \to A'_\bullet$ that extends $f$. Hence, we can view the homologies $H_n(A_\bullet)$ as quantities associated with objects and the morphisms $H_n(f_\bullet) : H_n(A_\bullet) \to H_n(A'_\bullet)$ as quantities associated with morphisms in $\mathcal{A}$.

The assignment of the morphisms $H_nF(f_\bullet) : H_nF(A_\bullet) \to H_nF(A'_\bullet)$ to morphisms $f : A \to A'$ is compatible with the composition of morphisms and the identity morphisms. If $f_\bullet : A_\bullet \to A'_\bullet$ and $f'_\bullet : A'_\bullet \to A''_\bullet$ are chain maps with $f_{-1} = f : A \to A'$ and $f'_{-1} = f' : A' \to A''$, then their composite $f'_\bullet \circ f_\bullet : A_\bullet \to A''_\bullet$ is a chain map with $(f'_\bullet \circ f_\bullet)_{-1} = f'_{-1} \circ f_{-1} : A \to A''$. As $F : \mathcal{A} \to \mathcal{B}$ is an additive functor and $H_n : \text{Ch}_\mathcal{B} \to \mathcal{B}$ a functor, one has $H_nF(f'_\bullet \circ f_\bullet) = H_nF(f'_\bullet) \circ H_n(f_\bullet)$.

Similarly, the identity chain map $1_{A_\bullet} : A_\bullet \to A_\bullet$ extends $1_A : A \to A$, and the functor $H_nF : A \to \mathcal{B}$ sends it to $H_nF(1_{A_\bullet}) = 1_{H_n(A_\bullet)}$.

To ensure that the 0th homology is the object $F(A)$, one modifies the chain complex $F(A_\bullet)$ by removing the object at index -1 and considers the homologies of the resulting chain complex $F(A_{\bullet})_{\geq 0}$. This can be viewed as a normalisation condition. The construction then defines functors $L^nF : \mathcal{A} \to \mathcal{B}$, the left derived functors of $F$, and an analogous construction for injective resolutions yields the right-derived functors $R^nF : \mathcal{A} \to \mathcal{B}$. As they are useful and of interest mainly for functors $F : \mathcal{A} \to \mathcal{B}$ that are left or right exact, we restrict attention to these cases.

**Definition 4.3.1:** Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories.

1. If $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor, the **left derived functors** $L_nF : \mathcal{A} \to \mathcal{B}$ for $n \in \mathbb{N}_0$ are defined by:

   - $L_nF(A) = H_nF(A_\bullet)_{\geq 0}$, for a projective left resolution $A_\bullet$ of $A \in \text{Ob} \mathcal{A}$, where $F(A_\bullet)_{\geq 0}$ is the chain complex

     $F(A_\bullet)_{\geq 0} = \ldots \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} \ldots \xrightarrow{F(d_2)} F(A_1) \xrightarrow{F(d_1)} F(A_0) \to 0$.

   - $L_nF(f) = H_nF(f_\bullet)_{\geq 0}$ for all morphisms $f : A \to A'$, where $f_\bullet : A_\bullet \to A'_\bullet$ is a chain map between projective resolutions $A_\bullet$ of $A$ and $A'_\bullet$ of $A'$ with $f_{-1} = f$.

2. If $\mathcal{A}$ has enough injectives and $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, the **right derived functors** $R^nF : \mathcal{A} \to \mathcal{B}$ for $n \in \mathbb{N}_0$ are defined by:

   - $R^nF(A) = H^nF(A_\bullet)_{\geq 0}$, for an injective right resolution $A_\bullet$ of $A \in \text{Ob} \mathcal{A}$, where $F(A_\bullet)_{\geq 0}$ is the chain complex

     $F(A_\bullet)_{\geq 0} = 0 \xrightarrow{F(d_0)} F(A^0) \xrightarrow{F(d_1)} \ldots \xrightarrow{F(d_{n-1})} F(A^n) \xrightarrow{F(d_n)} \ldots$.

   - $R^nF(f) = H^nF(f_\bullet)_{\geq 0}$ for all morphisms $f : A \to A'$, where $f_\bullet : A_\bullet \to A'_\bullet$ is a chain map between injective right resolutions $A_\bullet$ of $A$ and $A'_\bullet$ of $A'$ with $f_{-1} = f$. 

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Remark 4.3.2:

1. The left (right) derived functors of a right (left) exact functor $F$ are additive (Exercise [51]).

2. If $F : \mathcal{A} \to \mathcal{B}$ is exact, then $R^n F = 0$ and $L_n F = 0$ for all $n > 0$. In this case, $F(P_\bullet)$ is exact for all projective resolutions $P_\bullet$ of $A$ and hence $L_n F(A) = H_n(F(P_\bullet)) = 0$ for all $n > 0$ and $A \in \mathcal{A}$. The reasoning for the right derived functors is similar.

3. More generally, for any exact functor $G : \mathcal{B} \to \mathcal{C}$ one has $L_n(GF') = G(L_n F')$ for all $n \in \mathbb{N}_0$ and right exact functors $F : \mathcal{A} \to \mathcal{B}$ and $R^n(GH) = G(R^n H)$ for all $n \in \mathbb{N}_0$ and left exact functors $H : \mathcal{A} \to \mathcal{B}$ (Exercise [52]).

4. If $A$ is a projective (injective) object in $\mathcal{A}$, then one has $L_n F(A) = 0$ for all $n > 0$ ($R^n F(A) = 0$ for all $n > 0$) for all right (left) exact functors $F : \mathcal{A} \to \mathcal{B}$.

This follows because $A_\bullet = 0 \to A \xrightarrow{1_A} A \to 0$ is a projective (injective) resolution of $A$. As $F(A_\bullet)$ is exact for any functor $F : \mathcal{A} \to \mathcal{B}$ all (co)homologies of the chain complex $F(A_\bullet)_{\geq 0}$ except the 0th (co)homology vanish.

5. Any natural transformation $\eta : F \to F'$ between right (left) exact functors $F : \mathcal{A} \to \mathcal{B}$ induces a family $(L_n \eta)_{n \in \mathbb{N}_0}$ of natural transformations $L_n \eta : L_n F \to L_n F'$ (a family $(R^n \eta)_{n \in \mathbb{N}_0}$ of natural transformations $R^n \eta : R^n F \to R^n F'$) (Exercise [50]).

As the 0th (co)homologies are easiest to compute and often directly related to the objects under investigation, it is natural to investigate them in the context of left or right derived functors. From the examples in Section 2 it seems plausible that they should coincide with the right or left exact functors under consideration. The following lemma shows that this is indeed the case. This is the main motivation for defining the left and right derived functors with the chain complexes $F(A_\bullet)_{\geq 0}$ and $F(A^\bullet)_{\geq 0}$ instead of $F(A_\bullet)$ and $F(A^\bullet)$.

Lemma 4.3.3: Let $\mathcal{A}, \mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives (injectives) and $F : \mathcal{A} \to \mathcal{B}$ right (left) exact. Then there is a natural isomorphism $L_0 F = F' (R^0 F \to F)$.

Proof:

We prove the claim for right exact functors. Let $A_\bullet$ be a projective resolution of $A$. Then the homology $H_0(F(A_\bullet)_{\geq 0}) = L_0 F(A)$ is defined by the diagram

$$\begin{array}{cccc}
\text{im}(F(d_1)) & - & - & - & - & - & - & - & \text{ker}(0) \\
\pi_1' \downarrow & & & & \downarrow & & & & \\
F(A_1) & \xrightarrow{\iota_1} & F(A_0) & \xrightarrow{\iota_0} & 0.
\end{array}$$

As $\iota_0$ is an isomorphism and $\pi_1'$ an epimorphism, we have

$$L_0 F(A) = \text{coker}(\phi_0) \cong \text{coker}(\iota_0 \circ \phi_0) = \text{coker}(\iota_1') \cong \text{coker}(\iota_1' \circ \pi_1') = \text{coker}(F(d_1)).$$

As $F : \mathcal{C} \to \mathcal{D}$ is right exact, it preserves cokernels, which implies $\text{coker}(F(d_1)) \cong F(\text{coker}(d_1))$.

As $A_\bullet$ is a projective resolution of $A$, we have $\text{coker}(d_1) \cong A$ and $\text{coker}(F(d_1)) \cong F(A)$. The naturality of the isomorphism $L_0 F(A) \to F(A)$ is a consequence of the fact that the following diagram commutes for any morphism $f : A \to A'$, projective resolutions $A_\bullet$ and $A'_\bullet$ of $A$ and $A'$ and chain map $f_\bullet : A_\bullet \to A'_\bullet$ with $f_{-1} = f$.

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Short exact sequences of chain complexes and the associated long exact homology sequences are one of the most important tools to compute homologies. As the left and right derived functors are defined as the homologies of certain chain complexes, it is natural to relate their values on short exact sequences in $\mathcal{A}$. The first step is then to extend short exact sequences of objects in $\mathcal{A}$ to short exact sequences of projective (injective) resolutions. As injective resolutions in $\mathcal{A}$ correspond to projective resolutions in $\mathcal{A}^{\text{op}}$, it is sufficient to consider projective resolutions.

**Lemma 4.3.4:** Let $\mathcal{A}$ be an abelian category with enough projectives, $0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0$ a short exact sequence in $\mathcal{A}$ and $L_\bullet$ and $N_\bullet$ projective resolutions of $L$ and $N$. Then there is a projective resolution $M_\bullet$ of $M$ and chain maps $\iota_\bullet : L_\bullet \to M_\bullet$ and $\pi_\bullet : M_\bullet \to N_\bullet$ with $i_{-1} = i$ and $\pi_{-1} = \pi$ such that $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0$ is a short exact sequence in $\text{Ch}_\mathcal{A}$.

**Proof:**

We define $M_n = L_n \amalg N_n$ for $n \geq 0$ and $M_{-1} = M$. As coproducts of projective objects are projective by Lemma 4.2.2, the object $M_n$ is projective for $n \in \mathbb{N}_0$. Let $i_n^1 : L_n \to L_n \amalg N_n$, $i_n^2 : N_n \to L_n \amalg N_n$ and $\pi_n^1 : L_n \amalg N_n \to L_n$, $\pi_n^2 : L_n \amalg N_n \to N_n$ be the canonical inclusion and projection morphisms for the coproduct. Then we have $\ker(\pi_n^1) \cong \ker(i_n^1)$ and $\ker(\pi_n^2) \cong \ker(i_n^2)$.

For $n > 0$ let $d_n^M : M_n \to M_{n-1}$ be the morphism with $d_n^M \circ i_n^1 = i_{n-1}^1 \circ d_n^L$ and $d_n^M \circ i_n^2 = i_{n-1}^2 \circ d_n^N$ induced by the universal property of the coproduct and define $d_0^M : L_0 \amalg N_0 \to M$ as follows.

As $N_0$ is projective and $\pi : M \to N$ an epimorphism, there is a morphism $d_0^N : N_0 \to M$ with $\pi \circ d_0^N = d_0^N$. By the universal property of the coproduct there is a unique morphism $d_0^M : L_0 \amalg N_0 \to M$ with $d_0^M \circ i_0^1 = \iota \circ d_0^L$ and $d_0^M \circ i_0^2 = d_0^N$. This implies

\[
\pi \circ d_0^M \circ i_0^1 = \pi \circ \iota \circ d_0^L = 0 = d_0^N \circ \pi_0^2 \circ i_0^1 \quad \pi \circ d_0^M \circ i_0^2 = \pi \circ d_0^N = d_0^N \circ \pi_0^2 \circ i_0^2
\]

and with the universal property of the coproduct $\pi \circ d_0^M = d_0^N \circ \pi_0^2$. The diagram
commutes. By definition, $M_\bullet$ is exact in $M_n$ for $n > 0$, since the chain complexes $L_\bullet$ and $N_\bullet$ are exact and the morphisms $d_0^M$ are induced by $d_0^L$ and $d_0^N$ via the universal property of the coproduct. This implies $\ker(d_0^M) \cong \ker(d_0^L) \amalg \ker(d_0^N) \cong \im(d_0^L_{n+1}) \amalg \im(d_0^N_{n+1}) \cong \im(d_0^M_n)$ for $n \in N_0$. The exactness in $M_0 = L_0 \amalg N_0$ and the surjectivity of $d_0^M$ follow by applying the snake lemma to the last two rows in the diagram. This yields an exact sequence

$$0 \to \ker(d_0^L) \overset{\phi}{\to} \ker(d_0^M) \overset{\psi}{\to} \ker(d_0^N) \overset{\partial}{\to} \coker(d_0^L) \to \coker(d_0^M) \to \coker(d_0^N) \to 0,$$  \hfill (27)

As $L_\bullet$ and $N_\bullet$ are exact, we have $\coker(d_0^L) = 0$ and $\coker(d_0^N) = 0$, which implies $\coker(d_0^M) = 0$ by exactness of (27), and hence $d_0^M : M_0 \to M$ is an epimorphism. It also follows from the exactness of (27) that $\phi$ is a monomorphism and $\psi$ an epimorphism with $\im(\phi) \cong \ker(\psi)$. With the exactness of $L_\bullet$ and $N_\bullet$ and the definition of $d_1^M$ this implies $\ker(d_0^M) \cong \ker(d_0^L) \amalg \ker(d_0^N) \cong \im(d_1^M)$. This shows that $M_\bullet$ is an exact sequence with an epimorphism $d_0^M : M_0 \to M$ and hence a projective resolution of $M$.

The inclusion morphisms $\iota_1^A : L_n \to L_n \amalg N_n$ and $\iota : L \to M$ define a chain map $\iota_\bullet : L_\bullet \to M_\bullet$ and the projection morphisms $\pi_n^2 : L_n \amalg N_n \to N_n$ and $\pi : M \to N$ a chain map $\pi_\bullet : M_\bullet \to N_\bullet$. The former are monomorphisms, the latter epimorphisms, $\iota_n$ is a kernel of $\pi_n$ and $\iota$ a kernel of $\pi$. This shows that we have a short exact sequence of chain complexes $0 \to L_\bullet \overset{\iota}{\to} M_\bullet \overset{\pi}{\to} N_\bullet \to 0$. □

Using this lemma and the long exact homology sequence from Theorem 3.4.7 we can relate the values of left and right derived functors on a short exact sequence of objects in $\mathcal{A}$. This yields a long exact sequence of derived functors, and morphisms of short exact sequences in $\mathcal{A}$ induce chain maps between these long exact sequences.

**Theorem 4.3.5:** Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, $F : \mathcal{A} \to \mathcal{B}$ an additive functor.

1. If $\mathcal{A}$ has enough projectives and $F$ is right exact, then for every short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ in $\mathcal{A}$ one has a **long exact sequence of left derived functors**

$$\ldots \overset{L_1F(\iota)}{\longrightarrow} L_1F(B) \xrightarrow{L_1F(\pi)} L_1F(C) \xrightarrow{\partial_0} L_0F(A) \xrightarrow{L_0F(\iota)} L_0F(B) \xrightarrow{L_0F(\pi)} L_0F(C) \to 0,$$

and for every chain map between short exact sequences

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\pi} C \longrightarrow 0 \hfill (28)$$

one has a commuting diagram

$$\ldots \overset{\partial_{n+1}^{\alpha}}{\longrightarrow} L_nF(A) \xrightarrow{L_nF(\iota)} L_nF(B) \xrightarrow{L_nF(\pi)} L_nF(C) \xrightarrow{\partial_n} L_{n-1}F(A) \xrightarrow{L_{n-1}F(\iota)} \ldots \hfill (29)$$

$$\ldots \overset{\partial_{n+1}^{\beta}}{\longrightarrow} L_nF(A') \xrightarrow{L_nF(\iota')} L_nF(B') \xrightarrow{L_nF(\pi')} L_nF(C') \xrightarrow{\partial_n} L_{n-1}F(A') \xrightarrow{L_{n-1}F(\iota')} \ldots$$

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2. If \( A \) has enough injectives and \( F \) is left exact, then for every short exact sequence \( 0 \to A \to B \xrightarrow{\pi} C \to 0 \) one has a \textbf{long exact sequence of right derived functors}

\[
0 \to R^0F(A) \xrightarrow{R^0F(\iota)} R^0F(B) \xrightarrow{R^0F(\pi)} R^0F(C) \xrightarrow{\varrho^0} R^1F(A) \xrightarrow{R^1F(\iota)} R^1F(B) \xrightarrow{R^1F(\pi)} \ldots
\]

and for every chain map between short exact sequences

\[
\begin{array}{c}
0 \\ A \xrightarrow{\iota} B \xrightarrow{\pi} C \xrightarrow{\gamma} 0 \\
A' \xrightarrow{\iota'} B' \xrightarrow{\pi'} C' \xrightarrow{\gamma'} 0
\end{array}
\]

one has a commuting diagram

\[
\begin{array}{c}
\ldots \xrightarrow{\varrho^{n-1}} R^nF(A) \xrightarrow{R^nF(\iota)} R^nF(B) \xrightarrow{R^nF(\pi)} R^nF(C) \xrightarrow{\varrho^n} R^{n+1}F(A) \xrightarrow{R^{n+1}F(\iota)} \ldots
\end{array}
\]

\[
\begin{array}{c}
\ldots \xrightarrow{\varrho^{n-1}} R^nF(A') \xrightarrow{R^nF(\iota')} R^nF(B') \xrightarrow{R^nF(\pi')} R^nF(C') \xrightarrow{\varrho^n} R^{n+1}F(A') \xrightarrow{R^{n+1}F(\iota')} \ldots
\end{array}
\]

\textbf{Proof:}

By Remark 4.3.2, left and right derived functors are additive. We prove the remaining claims for right exact functors then follow by considering \( A^\text{op} \).

By Lemma 4.3.4 there are projective resolutions \( A_\bullet, B_\bullet \) and \( C_\bullet \) of \( A, B, C \) that form a short exact sequence of chain complexes \( 0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0 \). By applying the functor \( F \) and removing the lowest terms of the sequence, we obtain an exact sequence

\[
0 \to F(A_\bullet)_{\geq 0} \xrightarrow{F(\iota_\bullet)} F(B_\bullet)_{\geq 0} \xrightarrow{F(\pi_\bullet)} F(C_\bullet)_{\geq 0} \to 0.
\]

The exactness of this sequence follows from the construction of \( 0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0 \) in Lemma 4.3.4. For \( n \geq 0 \), we have \( B_n = A_n \amalg C_n \) with the inclusion and the projection morphisms \( \iota_n = \iota_n^1 : A_n \to A_n \amalg C_n \) and \( \pi_n = \pi_n^2 : A_n \amalg C_n \to C_n \) of the (co)product. As \( F \) is additive, preserves finite (co)products and hence the sequence

\[
0 \to F(A_n)_{\geq 0} \xrightarrow{F(\iota_n^1)=1} F(A_n \amalg C_n) \cong F(A_n) \amalg F(C_n) \xrightarrow{F(\pi_n^2)=1} F(C_n) \to 0
\]

is exact for all \( n \in \mathbb{N}_0 \). The first statement then follows from Theorem 3.4.7 about the long exact homology sequence, since the left derived functors are given by \( L_nF(X) = H_n(F(X_\bullet)_{n \geq 0}) \) for \( X = A, B, C \).

Given two short exact sequences \( 0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0 \) and \( 0 \to A' \xrightarrow{\iota'} B' \xrightarrow{\pi'} C' \to 0 \) and morphisms \( \alpha : A \to A', \beta : B \to B' \) and \( \gamma : C \to C' \) as in (28), we can choose projective resolutions \( A_\bullet, B_\bullet, C_\bullet \) and \( A'_\bullet, B'_\bullet, C'_\bullet \) that form two short exact sequences of chain complexes by Lemma 4.3.4. Theorem 4.1.6 then yields chain maps \( \alpha_\bullet : A_\bullet \to A'_\bullet, \beta_\bullet : B_\bullet \to B'_\bullet \) and \( \gamma_\bullet : C_\bullet \to C'_\bullet \) with \( \alpha_n = \alpha, \beta_n = \beta \) and \( \gamma_n = \gamma \). Applying \( F \) and omitting the lowest terms gives the following commuting diagram with exact rows

\[
\begin{array}{c}
0 \xrightarrow{F(\iota_\bullet)} F(A_\bullet)_{\geq 0} \xrightarrow{F(\pi_\bullet)} F(B_\bullet)_{\geq 0} \xrightarrow{F(\gamma_\bullet)} 0
\end{array}
\]

and Theorem 3.4.8 yields the commuting diagram (29). \qed
Remark 4.3.6: Theorem 4.3.5 implies that $L_n F = 0$ for all $n > 0$ ($R^n F = 0$ for all $n > 0$) of a right (left) exact functor $F : \mathcal{A} \to \mathcal{B}$ if and only if $F$ is exact.

The only if-statement follows directly from the definition. Theorem 4.3.5 implies that if all left derived functors $L_n F$ vanish, we have short exact sequence

$$0 \to L_0 F(A) \xrightarrow{L_0 F(\iota)} L_0 F(B) \xrightarrow{L_0 F(\pi)} L_0 F(C) \to 0.$$ 

As $L_0 F \cong F$, this implies the exactness of $0 \to F(A) \xrightarrow{F(\iota)} F(B) \xrightarrow{F(\pi)} F(C) \to 0$ for each short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ in $\mathcal{A}$ and hence the exactness of $F$ by Lemma 3.4.3.

4.4 The functors Tor and Ext

The results from the last subsection allow us to consider the left (right) derived functors of right (left) exact functors $F : \mathcal{A} \to \mathcal{B}$ whenever the category $\mathcal{A}$ has enough projectives (injectives). In particular, these conditions are satisfied for the category $R$-Mod for any ring $R$. The most important examples of right and left exact functors in this setting are the functors $M \otimes_R - : R$-Mod $\to$ Ab and $\text{Hom}_R(-,N) : R$-Mod$^\text{op} \to$ Ab for a fixed $R$-right module $M$ and $R$-left module $N$, which are right and left exact, respectively, by Lemma 3.1.15 and Lemma 3.1.18.

Definition 4.4.1: Let $R$ be a ring, $M$ an $R$-right module and $N$ an $R$-left module.

1. The left derived functors of the right exact functor $M \otimes_R - : R$-Mod $\to$ Ab are denoted $L_n (M \otimes_R -) = \text{Tor}^R_n (M, -)$.
2. The right derived functors of the left exact functor $\text{Hom}_R(-,N) : R$-Mod$^\text{op} \to$ Ab are denoted $R^n \text{Hom}_R(-,N) = \text{Ext}^R_n (-, N)$.

Remark 4.4.2:

1. To compute the value of the functors Tor and Ext on an $R$-module $A$, one uses projective resolutions of $A$. For Tor, this holds by definition of the left derived functors. For Ext this holds because because an injective resolution of $A$ in $R$-Mod$^\text{op}$ is the same as a projective resolution of $A$ in $R$-Mod.
2. One has $\text{Ext}^n_R (M,N) = 0$ for all $n > 0$ and $R$-modules $M$ if and only if $N$ is injective and $\text{Tor}^R_n (M,N) = 0$ for all $n > 0$ and $R$-modules $N$ if and only if $M$ is flat.
3. All $R$-linear maps $f : M \to M'$ and $g : N \to N'$ define natural transformations $f \otimes_R - : M \otimes_R - \to M' \otimes_R -$ and $\text{Hom}(-,g) : \text{Hom}(-,N) \to \text{Hom}(-,N')$. By Remark 4.3.2, they induce natural transformations $\text{Tor}^R_n (f,-) : \text{Tor}^R_n (M,-) \to \text{Tor}^R_n (M',-) \text{ and natural transformations } \text{Ext}^n_R (-,g) : \text{Ext}^n_R (-,N) \to \text{Ext}^n_R (-,N')$.

It turns out that most of the (co)homology theories in Section 2 are nothing but the functors Tor and Ext for specific choices of rings. This gives a simpler description of these cohomology theories and places them in a common framework.
Example 4.4.3: (Group cohomology)

Let $k$ be a commutative ring and $G$ a group. The bar resolution from Example 4.1.2

$$X_n = \begin{cases} (G^n)_{k[G]} & n \geq 0 \\ k & n = -1 \end{cases}$$

$$d_n(g_1, \ldots, g_n) = g_1 \triangleright (g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1})$$

$$d_0(g_1) = 1.$$

is a free and hence projective resolution of the trivial $k[G]$-module $k$.

Applying the functor $\text{Hom}(\_, M) : k[G]-\text{Mod}^{op} \to k-\text{Mod}$ for a $k[G]$-module $M$ to the associated chain complex $(X_\bullet)_{\geq 0}$ with $X_{-1}$ replaced by 0 yields a cochain complex $Z^\bullet$ in $k$-Mod

$$Z^n = \text{Hom}_{k[G]}((G^n)_{k[G]}, M) \cong \text{Map}(G^n, M)$$

$$d^n(\phi)(g_0, \ldots, g_n) = g_0 \triangleright \phi(g_1, \ldots, g_n) + \sum_{i=1}^{n} (-1)^i \phi(g_0, \ldots, g_{i-1} g_i, \ldots, g_n) + (-1)^{n+1} \phi(g_0, \ldots, g_{n-1})$$

that is isomorphic to the cochain complex $C^\bullet(G, M)$ of group cohomology. Hence, we have

$$H^n(G, M) = \text{Ext}^n_{k[G]}(k, M).$$

Example 4.4.4: (Hochschild (co)homology)

Let $k$ be a commutative ring, $A$ an algebra over $k$ that is projective as a $k$-module. Then by Example 4.1.3 and Corollary 4.2.5 the chain complex $X_\bullet$ with

$$X_n = A^{\otimes (n+2)} \quad n \geq -1$$

$$d_n : X_n \to X_{n-1}, \quad a_0 \otimes \ldots \otimes a_{n+1} \mapsto \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_{n+1}.$$ 

is a projective resolution of $A$ in $A \otimes A^{op}$-Mod.

Applying the functor $M \otimes_{A \otimes A^{op}} \_ : A \otimes A^{op}$-Mod $\to$ $k$-Mod for an $(A, A)$-bimodule $M$ to the associated chain complex $(X_\bullet)_{\geq 0}$ with $X_{-1}$ removed yields a chain complex $Y_\bullet$ in $k$-Mod with

$$Y_n = M \otimes_{A \otimes A^{op}} A^{\otimes (n+2)} \quad n \in \mathbb{N}_0$$

$$d_n(m \otimes a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i m \otimes a_0 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_{n+1},$$

and the $k$-linear maps

$$f_n : M \otimes_{A \otimes A^{op}} A^{\otimes (n+2)} \to M \otimes_{k} A^{\otimes n}, \quad m \otimes a_0 \otimes \ldots \otimes a_{n+1} \mapsto (a_{n+1} \triangleright m \otimes a_0) \otimes a_1 \otimes \ldots \otimes a_n$$

define an invertible chain map in $k$-Mod from $Y_\bullet$ to the Hochschild complex $C_\bullet(A, M)$ from Definition 2.2.3. Hence, we have

$$H_n(A, M) = \text{Tor}^n_{A \otimes A^{op}}(M, A).$$

Applying the functor $\text{Hom}_{A \otimes A^{op}}(\_, M) : A \otimes A^{op}$-Mod$^{op} \to k$-Mod to $(X_\bullet)_{\geq 0}$ yields a cochain complex $Z^\bullet$ in $k$-Mod with

$$Z^n = \text{Hom}_{A \otimes A^{op}}(A^{\otimes (n+2)}, M) \quad n \in \mathbb{N}_0$$

$$d^n(\phi)(a_0 \otimes \ldots \otimes a_{n+1} \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i \phi(a_0 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_{n+1}),$$

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and the \( k \)-linear maps

\[
f_n : \text{Hom}_{A \otimes A^{op}}(A^{\otimes (n+2)}, M) \to \text{Hom}_k(A^{\otimes n}, M), \quad f_n(\phi)(a_1 \otimes \ldots \otimes a_n) = \phi(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)
\]
define an invertible cochain map in \( k \text{-Mod} \) from \( Z^n \) to the Hochschild cocomplex \( C^\bullet(A, M) \) from Definition 2.2.4. This implies

\[
H^n(A, M) = \text{Ext}^n_{A \otimes A^{op}\text{-Mod}}(A, M).
\]

**Example 4.4.5:** (Lie algebra cohomology)

Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{F} \). Then the Chevalley-Eilenberg complex \( X_\bullet \) from Example 4.1.4 is

\[
X_n = U(\mathfrak{g}) \otimes \Lambda^n \mathfrak{g}, \quad X_{-1} = \mathbb{F}
\]

\[
d_n(y \otimes x_1 \wedge \ldots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} yx_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_n
\]

\[
+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} y \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n
\]

with \( U(\mathfrak{g}) \)-module structure \( \triangleright : U(\mathfrak{g}) \times X_n \to X_n, \ z \triangleright (y \otimes x_1 \wedge \ldots \wedge x_n) = (zy) \otimes x_1 \wedge \ldots \wedge x_n \) is a projective resolution of the trivial \( U(\mathfrak{g}) \)-module \( \mathbb{F} \) in \( U(\mathfrak{g})\text{-Mod} \).

Applying the functor \( \text{Hom}(\cdot, M) : U(\mathfrak{g})\text{-Mod} \to \text{Vect}_{\mathbb{F}} \) to \( X_\bullet \) for an \( U(\mathfrak{g}) \)-module \( M \) and omitting the term \( X_{-1} \) yields the chain complex \( W_\bullet \) in \( \text{Vect}_{\mathbb{F}} \) with

\[
W_n = \text{Hom}_{U(\mathfrak{g})\text{-Mod}}(U(\mathfrak{g}) \otimes \Lambda^n \mathfrak{g}, M) \cong \text{Hom}_{\mathbb{F}}(\Lambda^n \mathfrak{g}, M)
\]

\[
d_n(f)(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i x_i \triangleright f(x_1, \ldots, \hat{x}_i, \ldots, x_n)
\]

\[
+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots x_n).
\]

This is the cochain complex of Lie algebra cohomology, and we have

\[
H^n(\mathfrak{g}, M) = \text{Ext}^n_{U(\mathfrak{g})\text{-Mod}}(\mathbb{F}, M).
\]

This allows us to give an alternative definition of the (co)homology theories from Section 2 in terms of the functor \( \text{Tor} \) and \( \text{Ext} \). This definition is conceptually clearer since it does not rely on concrete choices of chain complexes. It is also much better for computations, because it allows one to compute (co)homologies from any projective resolution of the objects under consideration. In many cases, there are much simpler projective resolutions than the standard resolutions. A good choice of resolution simplifies the computations considerably.

**Definition 4.4.6:** Let \( k \) be a commutative ring and \( \mathbb{F} \) a field.

1. Group homology of a group \( G \) with coefficients in a \( k[G] \)-right module \( M \) and group cohomology of \( G \) with coefficients in a \( k[G] \)-left module \( M \) are given by

\[
H_n(G, M) = \text{Tor}^k_{n[G]}(M, k) \quad H^n(G, M) = \text{Ext}^n_{k[G]}(k, M).
\]

2. Hochschild homology and cohomology of an algebra \( A \) that is a projective \( k \)-module with coefficients in an \((A, A)\)-bimodule \( M \) are given by

\[
H_n(A, M) = \text{Tor}^n_{A \otimes A^{op}}(M, A) \quad H^n(A, M) = \text{Ext}^n_{A \otimes A^{op}}(A, M).
\]
3. Lie algebra homology of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with coefficients in a right $U(\mathfrak{g})$-module $M$ and Lie algebra cohomology of $\mathfrak{g}$ with coefficients in a $U(\mathfrak{g})$-module $M$ are given by

$$H_n(\mathfrak{g}, M) = \text{Tor}^U_{n}(\mathfrak{g}) \text{-Mod}(M, \mathbb{F})$$

$$H^n(\mathfrak{g}, M) = \text{Ext}^n_{U(\mathfrak{g}) \text{-Mod}}(\mathbb{F}, M).$$

Example 4.4.7: (Group (co)homologies of cyclic groups)

We compute $H_n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ and $H^n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ for the trivial $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$-module $\mathbb{Z}$.

To distinguish the group multiplication in $\mathbb{Z}/m\mathbb{Z}$ from the addition in $\mathbb{R} := \mathbb{Z}/m\mathbb{Z}$ we identify $\mathbb{Z}/m\mathbb{Z}$ with the subgroup $\mathbb{Z}/m\mathbb{Z} = \{e^{2\pi ik/m} \mid k = 0, 1, \ldots, m - 1\} \subset \mathbb{C}^\times$.

- We consider the following chain complex in $\text{R-Mod}$

$$\cdots \xrightarrow{d_4} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_3} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_2} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_1} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_0} \mathbb{Z} \to 0$$

$$d_n(e^{2\pi ik/m}) = \begin{cases} 1 & n = 0 \\ e^{2\pi ik/m} - e^{2\pi i(k+1)/m} & n \text{ odd} \\ 1 + e^{2\pi i/m} + \ldots + e^{2\pi i(m-1)/m} & n > 0 \text{ even} \end{cases} \quad (30)$$

where $+$ stands for the addition in $R = \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$. This is a free (and hence projective) resolution of $\mathbb{Z}$ in $\text{R-Mod}$ since $\text{im}(d_0) = \mathbb{Z}$ and

$$\text{ker}(d_{2n+1}) = \{\lambda(1 + e^{2\pi i/m} + \ldots + e^{2\pi i(m-1)/m}) \mid \lambda \in \mathbb{Z}\} = \text{im}(d_{2n+2})$$

$$\text{ker}(d_{2n}) = \{\lambda_0 1 + \lambda_1 e^{2\pi i/m} + \ldots + \lambda_{m-1} e^{2\pi i(m-1)/m} \mid \lambda_i \in \mathbb{Z}, \lambda_0 + \ldots + \lambda_m = 0\} = \text{im}(d_{2n+1}).$$

- To compute $H_n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$, we apply the functor $\mathbb{Z} \otimes_R - : \text{R-Mod} \to \text{Ab}$ to the free resolution $(30)$ and omit the last entry on the right. As $\phi : \mathbb{Z} \otimes_R R \to \mathbb{Z}$, $z \otimes r \mapsto z \cdot r$ is an isomorphism in $\text{Ab}$ with inverse $\phi^{-1} : \mathbb{Z} \to \mathbb{Z} \otimes_R R$, $z \mapsto z \otimes 1$ and

$$\begin{align*}
(\text{id}_\mathbb{Z} \otimes d_{2n+1})(z \otimes e^{2\pi ik/m}) &= z \otimes (e^{2\pi ik/m} - e^{2\pi i(k+1)/m}) = z \otimes 1 - z \otimes 1 = 0 \\
(\text{id}_\mathbb{Z} \otimes d_{2n})(z \otimes e^{2\pi ik/m}) &= z \otimes (1 + e^{2\pi i/m} + \ldots + e^{2\pi i(m-1)/m}) = mz \otimes 1 = \phi^{-1}(m \cdot \phi(z \otimes e^{2\pi ik/m})),
\end{align*}$$

we have an isomorphism of chain complexes

$$\begin{align*}
\cdots &\xrightarrow{\phi} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_4} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_3} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_2} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_1} \mathbb{Z} \otimes_R R \to 0 \\
\xrightarrow{0} &\xrightarrow{\phi} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_4} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_3} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_2} \mathbb{Z} \otimes_R R \xrightarrow{\text{id} \otimes d_1} \mathbb{Z} \otimes_R R \to 0
\end{align*}$$

and the group homologies are given by

$$H_n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = \text{Tor}^\mathbb{Z}_{n}[\mathbb{Z}/m\mathbb{Z}](\mathbb{Z}, \mathbb{Z}) = \begin{cases} \ker(0)/\text{im}(0) = \mathbb{Z} & n = 0 \\
\ker(z \mapsto mz)/\text{im}(0) = 0 & n > 0 \text{ even} \\
\ker(0)/\text{im}(z \mapsto mz) = \mathbb{Z}/m\mathbb{Z} & n \text{ odd}.
\end{cases}$$

- For the cohomologies $H^n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$, we apply $\text{Hom}_R(-, \mathbb{Z}) : \text{R-Mod}^{op} \to \text{Ab}$ to the free resolution $(30)$ and omit the first term on the left. As the $R$-module $R$ is cyclic with generator 1 and $\mathbb{Z}$ is equipped with the trivial $R$-module structure, any $R$-linear map $f : R \to \mathbb{Z}$ satisfies
and we obtain an isomorphism of cochain complexes $H^\phi f = f(1)$ for all $k \in \{0, \ldots, m - 1\}$. Hence, the map 

\[ \phi : \text{Hom}_R(R, \mathbb{Z}) \to \mathbb{Z}, f \mapsto f(1) \]

is an isomorphism in $\text{Ab}$ with inverse $\phi^{-1} : \mathbb{Z} \to \text{Hom}_R(R, \mathbb{Z})$, $z \mapsto f(z)$ with $f_z(e^{2\pi i/m}) = z$. The coboundary operators satisfy 

\[ \phi \circ \text{Hom}_R(d_{2n+1}, \mathbb{Z})(f) = f(d_{2n+1}(1)) = f(1) - f(e^{2\pi i/m}) = 0 \]

and we obtain an isomorphism of cochain complexes

\[ 0 \to \text{Hom}_R(R, \mathbb{Z}) \xrightarrow{\text{Hom}(d_1, \mathbb{Z})} \text{Hom}_R(R, \mathbb{Z}) \xrightarrow{\text{Hom}(d_2, \mathbb{Z})} \text{Hom}_R(R, \mathbb{Z}) \xrightarrow{\text{Hom}(d_3, \mathbb{Z})} \text{Hom}_R(R, \mathbb{Z}) \xrightarrow{\text{Hom}(d_4, \mathbb{Z})} \cdots \]

\[ 0 \to \mathbb{Z} \xrightarrow{z \mapsto mz} \mathbb{Z} \xrightarrow{z \mapsto mz} \mathbb{Z} \xrightarrow{z \mapsto mz} \cdots \]

The group cohomologies $H^n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ are the cohomologies of this cochain complex:

\[ H^n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = \text{Ext}^n_{\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]}(\mathbb{Z}, \mathbb{Z}) = \begin{cases} 
\ker(0)/\text{im}(0) = \mathbb{Z} & n = 0 \\
\ker(z \mapsto mz)/\text{im}(0) = 0 & n \text{ odd} \\
\ker(0)/\text{im}(z \mapsto mz) = \mathbb{Z}/m\mathbb{Z} & n \text{ even}.
\end{cases} \]

**Example 4.4.8:** (Hochschild homologies of the tensor algebra)

Let $\mathbb{F}$ be a field and $V$ a vector space over $\mathbb{F}$. The tensor algebra $T(V)$ is the $\mathbb{F}$-vector space $T(V) = \bigoplus_{n=0}^{\infty} V^\otimes n$ with $V^\otimes 0 := \mathbb{F}$ and the multiplication given by concatenation 

\[ (v_1 \otimes \cdots \otimes v_n) \cdot (v_{n+1} \otimes \cdots \otimes v_{n+k}) = v_1 \otimes \cdots \otimes v_{n+k} \quad \forall n, k \in \mathbb{N}, v_i \in V. \]

- We consider the chain complex $X_\bullet$ given by 

\[ 0 \to T(V) \otimes V \otimes T(V) \xrightarrow{d_1} T(V) \otimes T(V) \xrightarrow{d_0} T(V) \to 0 \]

\[ d_1(x \otimes v \otimes y) = (x \otimes v) \otimes y - x \otimes (v \otimes y), \quad d_0(x \otimes y) = x \cdot y. \]

It is a free resolution of the $T(V) \otimes T(V)^{op}$-module $T(V)$ in $T(V) \otimes T(V)^{op}$-$\text{Mod}$, since we have $d_0 \circ d_1 = 0$ and the $\mathbb{F}$-linear maps

\[ h_{-1} : T(V) \to T(V) \otimes T(V), \quad x \mapsto x \otimes 1 \]

\[ h_0 : T(V) \otimes T(V) \to T(V) \otimes V \otimes T(V), \quad x \otimes (v_1 \cdots v_n) \mapsto -\sum_{i=1}^n xv_1 \cdots v_{i-1} \otimes \otimes v_{i+1} \cdots v_n \]

define a chain homotopy from $1_{X_\bullet} : X_\bullet \to X_\bullet$ to $0_{X_\bullet} : X_\bullet \to X_\bullet$.

- To compute the Hochschild homologies, we omit the first term on the right and apply the functor $T(V) \otimes_R - : \text{R-Mod} \to \text{Vect}_\mathbb{F}$ for $R := T(V) \otimes T(V)^{op}$. As the $\mathbb{F}$-linear maps

\[ \phi : T(V) \otimes_R T(V) \otimes V \otimes T(V) \to T(V) \otimes V, \quad w \otimes (x \otimes v \otimes y) \mapsto ywx \otimes v \]

\[ \psi : T(V) \otimes_R T(V) \otimes T(V) \to T(V), \quad w \otimes (x \otimes y) \mapsto ywx \]

are linear isomorphisms with inverses

\[ \phi^{-1} : T(V) \otimes V \to T(V) \otimes_R (T(V) \otimes V \otimes T(V)), \quad w \otimes v \mapsto w \otimes (1 \otimes v \otimes 1) \]

\[ \psi^{-1} : T(V) \to T(V) \otimes_R (T(V) \otimes T(V)), \quad w \mapsto w \otimes (1 \otimes 1) \]
and we have the identity
\[ \psi \circ (\text{id} \otimes d_1)(w \otimes x \otimes v \otimes y) = \psi(w \otimes (xv) \otimes y) - w \otimes x \otimes (vy)) = \phi(w \otimes x \otimes y) \cdot v - v \cdot \phi(w \otimes x \otimes y), \]
we obtain an isomorphism of chain complexes
\[
\begin{array}{c}
0 \longrightarrow T(V) \otimes_R (T(V) \otimes V \otimes T(V)) \xrightarrow{id \otimes d_1} T(V) \otimes_R (T(V) \otimes T(V)) \longrightarrow 0 \\
0 \longrightarrow T(V) \otimes V \xrightarrow{\phi} T(V) \otimes T(V) \xrightarrow{\psi} T(V) \longrightarrow 0
\end{array}
\]

The Hochschild homologies of \( T(V) \) are the homologies of this chain complex. Denoting by \( \tau_n : V^\otimes n \to V^\otimes n \), \( v_1 \ldots v_n \mapsto v_n \otimes v_1 \otimes \ldots \otimes v_{n-1} \) the linear map that cyclically permutes the factors in the tensor product \( V^\otimes n \) we obtain
\[
H_n(T(V), T(V)) = \begin{cases} 
T(V)/\text{im}(d'_1) & n = 0 \\
\ker(d'_1) & n = 1 \cong \begin{cases} 
\mathbb{F} \oplus V \oplus \oplus_{n \geq 2} V^\otimes n/(\text{id}_{V^\otimes n} - \tau_n)V^\otimes n & n = 0 \\
V \oplus \oplus_{n \geq 2} \{x \in V^\otimes n \mid \tau_n(x) = x\} & n = 1 \\
0 & n > 1.
\end{cases}
\end{cases}
\]

For \( V = \mathbb{F} \) we obtain the polynomial algebra \( \mathbb{F}[x] = T(\mathbb{F}) \) and its Hochschild homologies
\[
H_n(\mathbb{F}[x], \mathbb{F}[x]) = \begin{cases} 
\mathbb{F}[x] & n = 0 \\
x\mathbb{F}[x] & n = 1 \\
0 & n > 1.
\end{cases}
\]

While we clarified the interpretation of Tor and Ext in the context of Hochschild (co)homology, group (co)homology and Lie algebra (co)homology in Section 2, we do not know what properties of the \( R \)-modules are encoded in Tor and Ext for a general rings \( R \).

By considering the functors \( \text{Tor}_n^R : R\text{-Mod} \to \text{Ab} \) for finitely generated modules over a principal ideal domain, we find that they are related to the torsion of the modules, which motivates the name Tor. For this, recall that every finitely generated module over a principal ideal domain \( R \) is isomorphic to a module \( M = R^n \times R/p_1^{n_1} \times \ldots \times R/p_k^{n_k} \) with \( n \in \mathbb{N}_0 \), \( n_i \in \mathbb{N} \) and prime elements \( p_i \in R \). By Lemma 1.2.21 we have \( M \cong R^n \oplus \text{Tor}_R(M) \) with \( \text{Tor}_R(M) = R/p_1^{n_1} R \times \ldots \times R/p_k^{n_k} R \), and by Lemma 1.2.17 every submodule of a free \( R \)-module is free.

**Example 4.4.9:** (\( \text{Tor}_n^R \) for finitely generated modules over principal ideal domains)

1. Let \( R \) be a principal ideal domain. We compute \( \text{Tor}_n^R(R/qR, R/pR) \) for \( p, q \in R \).

   By Lemma 4.3.3 we have
   \[
   \text{Tor}_0^R(R/qR, R/pR) = L_0(R/qR \otimes_R -)(R/pR) = R/qR \otimes_R R/pR = R/\text{gcd}(p, q)R.
   \]

   To compute the higher homologies, we use the free (and therefore projective) resolution
   \[
   0 \to R \xrightarrow{d_1 : r \mapsto pr} R \xrightarrow{d_0 = \pi} R/pR \to 0.
   \]

   By applying the functor \( R/qR \otimes_R - : \text{Ab} \to \text{Ab} \), omitting the last term on the right and using the isomorphism \( R/qR \otimes_R R \cong R/qR \), we obtain the chain complex
   \[
   0 \to R/qR \xrightarrow{d_1 : r \mapsto pr} R/qR \to 0.
   \]

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Its first homology is given by
\[ \text{Tor}_1^R(R/qR, R/pR) = \ker(d_1) = \{ x \in R/qR \mid q \cdot x \} \cong R/\gcd(p, q)R, \]
while all higher homologies vanish. This yields
\[ \text{Tor}_k^R(R/qR, R/pR) \cong \begin{cases} R/\gcd(p, q)R & k = 0, 1 \\ 0 & k \geq 2. \end{cases} \]

2. We compute \( \text{Tor}_k^R(R, R/pR) \). By Lemma 4.3.3 \( \text{Tor}_0^R(R, R/pR) \cong R \otimes_R R/pR \cong R/pR \). Applying the functor \( R \otimes_R - \) to the free resolution \( (31) \), omitting the first term on the right and using the isomorphism \( R \otimes_R R \cong R \) we obtain a chain complex isomorphic to
\[ 0 \to R \xrightarrow{d_1: r \mapsto pr} R \to 0. \]
whose first homology is given by \( \text{Tor}_1^R(R, R/pR) = \ker(d_1) = 0 \), and whose higher homologies vanish as well. This yields
\[ \text{Tor}_k^R(R, R/pR) \cong \begin{cases} R/pR & k = 0 \\ 0 & k \geq 1. \end{cases} \]

3. We compute \( \text{Tor}_k^R(R/qR, R) \) and \( \text{Tor}_k^R(R, R) \). By Lemma 4.3.3 we have
\[ \text{Tor}_0^R(R/qR, R) \cong R/qR \otimes_R R \cong R/qR \]
\[ \text{Tor}_0^R(R, R) \cong R \otimes_R R \cong R. \]
To compute the higher homologies, we use the free resolution \( 0 \to R \xrightarrow{id} R \to 0 \). In this case, applying \( R/qR \otimes_R - \) and \( R \otimes_R - \) and omitting the first term on the right yields chain complexes isomorphic to \( 0 \to R/qR \to 0 \) and \( 0 \to R \to 0 \) with trivial homologies and
\[ \text{Tor}_k^R(R/qR, R) \cong \begin{cases} R/qR & k = 0 \\ 0 & k \geq 1, \end{cases} \]
\[ \text{Tor}_k^R(R, R) \cong \begin{cases} R & k = 0 \\ 0 & k \geq 1. \end{cases} \]
As the left derived functors are additive and every finitely generated \( R \)-module is of the form \( M = R^n \times \text{Tor}_R(M) \) with the torsion submodule given by \( \text{Tor}_R(M) = \sum_{i=1}^n R/\gcd(q_i)R \) for prime powers \( q_i \in R \), Example 4.4.9 shows that \( \text{Tor}_n^R(M, N) = \{0\} \) for all \( n > 0 \) if \( M \) or \( N \) are torsion free. Hence, \( \text{Tor}_n^R \) is related to the torsion of the modules \( M, N \).

This example also shows a more general pattern. For any ring \( R \) and \( R \)-module \( M \) that has a short exact sequence as a projective resolution, all torsion functors \( \text{Tor}_n^R(N, M) \) for \( n \geq 2 \) and arbitrary \( R \)-modules \( N \) vanish. If \( M \) is projective, then \( \text{Tor}_n^R(N, M) = 0 \) as well.

By a similar computation, we can determine the functors \( \text{Ext}_n^R : R\text{-Mod} \to R\text{-Mod} \) for principal ideal domains \( R \) (Exercise 55). To motivate its name, we show that it classifies extensions of modules. This can be achieved in more generality than for the functor \( \text{Tor} \), since \( \text{Tor}_n^R(M, -) \) is defined only for modules over rings, whereas the functors \( \text{Hom}(-, A) : A \to \text{Ab} \) and \( \text{Ext}_n^A = R^n\text{Hom}(-, A) : A \to \text{Ab} \) are defined for any abelian category \( A \). For our purposes, it is sufficient to consider the case \( A = R\text{-Mod} \) for some ring \( R \) and the functor \( \text{Ext}_1^{R\text{-Mod}} \).
**Definition 4.4.10:** Let $R$ be a ring and $N$ an $R$-module.

1. An **extension** of $N$ by an $R$-module $L$ is a short exact sequence $0 \rightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \rightarrow 0$.

2. Two extensions $0 \rightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \rightarrow 0$ and $0 \rightarrow L' \xrightarrow{i'} M' \xrightarrow{\pi'} N \rightarrow 0$ are called **equivalent** if there is an isomorphism $f : M \rightarrow M'$ for which the following diagram commutes

\[
\begin{array}{ccc}
0 & \xrightarrow{} & L \xrightarrow{i} M \xrightarrow{\pi} N \xrightarrow{} 0 \\
\downarrow{\text{id}_L} & \equiv & \downarrow{f} \\
0 & \xrightarrow{} & L \xrightarrow{i'} M' \xrightarrow{\pi'} N \xrightarrow{} 0.
\end{array}
\]

3. One says an extension **splits**, if it is equivalent to an extension of the form

\[
0 \rightarrow L \xrightarrow{i} L \oplus N \xrightarrow{\pi_2} N \rightarrow 0,
\]

where $i_1 : L \rightarrow L \oplus N$ and $\pi_2 : L \oplus N \rightarrow N$ denote the inclusion map for the first and the projection map for the second factor in the direct sum.

**Proposition 4.4.11:** Let $L, N$ be modules over a ring $R$. Then equivalence classes of extensions of $N$ by $L$ are in bijection with elements of $\operatorname{Ext}^1_R(N, L)$. Extensions that split correspond to the element $0 \in \operatorname{Ext}^1_R(N, L)$.

**Proof:**

1. We define a map $\phi : \operatorname{Ex}(N, L)/\sim \rightarrow \operatorname{Ext}^1_R(N, L)$ from the set $\operatorname{Ex}(N, L)/\sim$ of equivalence classes of extensions of $N$ by $L$ to $\operatorname{Ext}^1_R(N, L)$.

For every extension $0 \rightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \rightarrow 0$ the long exact sequence of derived functors from Theorem 4.3.5 yields an exact sequence

\[
\cdots \rightarrow \operatorname{Ext}^0_R(M, L) \xrightarrow{i_*} \operatorname{Ext}^1_R(L, L) = \operatorname{Hom}_R(L, L) \xrightarrow{\partial^1} \operatorname{Ext}^1_R(N, L) \rightarrow \cdots \tag{32}
\]

By assigning to the extension $0 \rightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \rightarrow 0$ the element $\partial^1(\text{id}_L) \in \operatorname{Ext}^1_R(N, L)$ we obtain a map $\phi : \operatorname{Ex}(N, L) \rightarrow \operatorname{Ext}^1_R(N, L)$ from the set of extensions of $N$ by $L$ to $\operatorname{Ext}^1_R(N, L)$.

If $0 \rightarrow L \xrightarrow{i'} M' \xrightarrow{\pi'} N \rightarrow 0$ is another extension equivalent to $0 \rightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \rightarrow 0$, there is an $R$-linear isomorphism $f : M \rightarrow M'$ with $f \circ i = i'$ and $\pi' \circ f = \pi$. The naturality of the connecting morphism then yields the following commuting diagram with exact rows

\[
\begin{array}{ccc}
\cdots & \xrightarrow{} & \operatorname{Hom}_R(M, L) \xrightarrow{g \circ g_0} \operatorname{Hom}_R(L, L) \xrightarrow{\partial^1} \operatorname{Ext}^1_R(N, L) \xrightarrow{} \cdots \\
\downarrow{g \circ g_0} & \equiv & \downarrow{\text{id}} \\
\cdots & \xrightarrow{} & \operatorname{Hom}_R(M', L) \xrightarrow{g \circ g_0'} \operatorname{Hom}_R(L, L) \xrightarrow{\partial^1} \operatorname{Ext}^1_R(N, L) \xrightarrow{} \cdots,
\end{array}
\]

which implies $\partial^1(\text{id}_L) = \partial^1(\text{id}_L)$. This shows that $\phi : \operatorname{Ex}(N, L) \rightarrow \operatorname{Ext}^1_R(N, L)$ is constant on equivalence classes of extensions and induces a map $\phi : \operatorname{Ex}(N, L)/\sim \rightarrow \operatorname{Ext}^1_R(N, L)$.

2. We show that $\phi$ is surjective by constructing for each element $m \in \operatorname{Ext}^1_R(N, L)$ an extension $m_\ast$ with $\phi(m_\ast) = m$.

As $R$-Mod has enough projectives, there is a projective $R$-module $X$ and an epimorphism $\pi : X \rightarrow N$. By choosing $Y = \ker(\pi)$ and the inclusion $i : \ker(\pi) \rightarrow X$, we obtain an exact
sequence $0 \to Y \xrightarrow{\iota} X \xrightarrow{\pi} N \to 0$. As $X$ is projective, we have $\text{Ext}^1_R(X, L) = 0$ for all $R$-modules $L$. As $\text{Ext}^0_R(X, L) = \text{Hom}_R(X, L)$ and $\text{Ext}^0_R(Y, L) = \text{Hom}_R(Y, L)$, Theorem 4.3.5 yields the following long exact sequence of right derived functors

$$
\text{Hom}_R(X, L) \xrightarrow{f \mapsto \phi f} \text{Hom}_R(Y, L) \xrightarrow{\partial^1} \text{Ext}^1_R(N, L) \to \text{Ext}^1_R(X, L) = 0 \to \ldots
$$

This implies that $\partial^1 : \text{Hom}_R(Y, L) \to \text{Ext}^1_R(M, L)$ is surjective: for every element $m \in \text{Ext}^1_R(N, L)$ there is an $R$-linear map $f : Y \to L$ with $m = \partial^1(f)$.

To construct an extension $m_\bullet = 0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ with $\phi(m_\bullet) = m$, we consider the $R$-linear map $g = i_1 \circ \iota - i_2 \circ f : Y \to X \oplus L$, where $i_1 : X \to X \oplus L$ and $i_2 : L \to X \oplus L$ are the inclusions for the direct sum. We set $M = (X \oplus L)/\text{im}(g)$ and denote by $p : X \oplus L \to M$ the canonical surjection. By the universal property of the direct sum, there is a unique morphism $\pi'' : X \oplus L \to N$ with $\pi'' \circ i_1 = \pi$ and $\pi'' \circ i_2 = 0$. As $\pi$ is an epimorphism, $\pi''$ is an epimorphism as well, and it satisfies $\pi'' \circ g(y) = \pi \circ \iota(y) = 0$. By the universal property of the kernel, there is a unique epimorphism $\pi' : M \to N$ with $\pi' \circ p = \pi''$.

$$
\begin{array}{ccccccc}
L & \xrightarrow{i_2} & X \oplus L & \xrightarrow{p} & M \\
\uparrow{\iota_1} & & \uparrow{\pi''} & & \uparrow{1 \oplus \pi'} \\
0 & \xrightarrow{\iota} & X & \xrightarrow{\pi} & N & \xrightarrow{0} \\
\end{array}
$$

By composing the inclusion map $i_2 : L \to X \oplus L$ with the canonical surjection $p$, we obtain an injection $\iota' = p \circ i_2 : L \to M$. As the monomorphism $p \circ i_1 : X \to M$ satisfies $\iota' \circ f = p \circ i_1 \circ \iota$ and $\pi' \circ p \circ i_1 = \pi'' \circ i_1 = \pi$, we have a commuting diagram

$$
\begin{array}{ccccccc}
0 & \xrightarrow{0} & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & N & \xrightarrow{0} \\
| & \downarrow{f} & \downarrow{id} & & \downarrow{p \circ i_1} & | id \\
0 & \xrightarrow{0} & L & \xrightarrow{\iota'} & M & \xrightarrow{\pi'} & N & \xrightarrow{0} \\
\end{array}
$$

Its first row is exact by assumption, and the second, because $\pi'$ is surjective, $\iota'$ is injective and

$$
\ker(\pi') = \{[[x, l]] \mid x \in X, l \in L, \pi(x) = 0\} = \{[[\iota(y), l]] : y \in Y, l \in L\} = \{[[0, l + f(y)]] : l \in L, y \in Y\} = \{[[0, l]] : l \in L\} = \text{im}(\iota').
$$

The naturality of the connecting morphism then yields the commuting diagram

$$
\begin{array}{ccc}
\text{Hom}_R(Y, L) & \xrightarrow{\partial^1} & \text{Ext}^1_R(N, L) \\
\downarrow{h \mapsto h \circ f} & & \downarrow{\text{id}} \\
\text{Hom}_R(L, L) & \xrightarrow{\partial^1} & \text{Ext}^1_R(N, L),
\end{array}
$$

which implies $m = \partial^1(f) = \partial^1(\text{id}_L \circ f) = \partial^1(\text{id}_L)$. Hence, $m_\bullet = 0 \to L \xrightarrow{\iota'} M \xrightarrow{\pi'} N \to 0$ is an extension of $N$ by $L$ with $\phi(m_\bullet) = m$ and $\phi$ is surjective.

3. For injectivity of $\phi$, it is sufficient to show that any extension $0 \to L \xrightarrow{\iota''} M' \xrightarrow{\pi''} N \to 0$ with $\partial''(\text{id}_L) = m$ is equivalent to the one constructed in 2.

As $X$ is projective and $\pi''$ is an epimorphism, there is an $R$-linear map $h' : X \to M'$ with $\pi'' \circ h' = \pi$. This implies $\pi'' \circ h' \circ \iota = \pi \circ \iota = 0$. By the universal property of the kernel $\iota''$ there
By the 5-Lemma diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & L \\
\downarrow{f'} & & \downarrow{h'} \\
0 & \rightarrow & X \\
\downarrow{f} & & \downarrow{h} \\
0 & \rightarrow & M \\
\end{array}
\]

\[
\begin{array}{ccc}
\pi'' & \rightarrow & N \\
\downarrow{id_N} & & \downarrow{id_N} \\
\pi & \rightarrow & N \\
\end{array}
\]

As $\partial^1(id_L) = \partial^1(f) = \partial^1(f') = m = \partial^m(id_L)$, we have $f - f' \in \ker(\partial^1)$ and by exactness of \([32]\), an $R$-linear map $k : X \rightarrow L$ with $f - f' = k \circ \iota$. The $R$-linear map $r = h' + \iota'' \circ k + \iota'' : X \oplus L \rightarrow M'$, $x + l \mapsto h'(x) + \iota''(k(x)) + \iota''(l)$ satisfies

\[
r \circ g = h' \circ \iota + \iota'' \circ k \circ \iota - \iota'' \circ f = \iota'' \circ (f' + k - f) = \iota'' \circ 0 = 0
\]

and hence induces a unique $R$-linear map $r' : M \rightarrow M'$ with $r' \circ p = r$. As we have

\[
r' \circ \iota' = r' \circ p \circ i_2 = r \circ i_2 = \iota''
\]

\[
\pi'' \circ r' \circ p = \pi'' \circ r = \pi'' \circ h' + \pi'' \circ \iota'' \circ (k + id_L) = \pi'' \circ h' = \pi
\]

we obtain another commuting diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & L \\
\downarrow{id_L} & & \downarrow{id_L} \\
0 & \rightarrow & M \\
\end{array}
\]

\[
\begin{array}{ccc}
\iota' & \rightarrow & M \\
\downarrow{id_N} & & \downarrow{id_N} \\
\iota'' & \rightarrow & N \\
\end{array}
\]

\[
\begin{array}{ccc}
\pi' & \rightarrow & N \\
\downarrow{id_N} & & \downarrow{id_N} \\
\pi'' & \rightarrow & N \\
\end{array}
\]

By the 5-Lemma $r' : M \rightarrow M'$ is an isomorphism. This shows that $0 \rightarrow L \xrightarrow{\iota''} M' \xrightarrow{\pi''} N \rightarrow 0$ and $0 \rightarrow L \xrightarrow{\iota'} M \xrightarrow{\pi'} N \rightarrow 0$ are equivalent and $\phi$ is injective.

4. To prove the last statement, it is sufficient to note that the exactness of \([32]\) implies that for every extension $X_\bullet = 0 \rightarrow L \xrightarrow{\iota} M \xrightarrow{\pi} N \rightarrow 0$ with $\phi(X_\bullet) = 0$ the induced morphism $\text{Hom}(\iota, L) : \text{Hom}_R(M, L) \rightarrow \text{Hom}_R(L, L)$, $f \mapsto f \circ \iota$ is surjective. Hence, there is an $R$-linear map $f : M \rightarrow L$ with $f \circ \iota = id_L$, and by Exercise \([38]\) the extension splits. \(\square\)

### 4.5 Tor and Ext as bifunctors

To get a full understanding of the functors Tor and Ext, it remains to clarify one issue. The functors Tor and Ext were introduced in Definition 4.4.1 as, respectively, the left derived functors $\text{Tor}^R_n(-, L) : \text{R-Mod} \rightarrow \text{Ab}$ for an $R$-right module $L$ and the right derived functors $\text{Ext}^R_n(-, M) = R^n\text{Hom}_R(-, M) : \text{R-Mod}^{\text{op}} \rightarrow \text{Ab}$ for an $R$-left module $M$.

This involved arbitrary choices, namely tensoring on the right and considering the $R$-linear map into the module $M$. Instead of the right exact functor $L \otimes_R - : \text{R-Mod} \rightarrow \text{Ab}$ and the left exact functor $\text{Hom}_R(-, M) : \text{R-Mod}^{\text{op}} \rightarrow \text{Ab}$, we could have considered the right exact functor $\text{Tor}^R_n(-, M) = L_n(- \otimes_R M)$ and the left exact functor $\text{Hom}_R(M, -) : \text{R-Mod} \rightarrow \text{Ab}$, with their left and right derived functors $\text{Tor}^R_n(-, M) = L_n(- \otimes_R M)$ and $\text{Ext}^R_n(-, M) = R^n\text{Hom}_R(M, -)$.

We have to determine how the resulting functors Tor and Ext are related to Tor and Ext. For this, we consider chain complexes in the abelian category $\text{Ch}_{\text{R-Mod}}$ of chain complexes in $\text{R-Mod}$ or, equivalently, double complexes in $\text{R-Mod}$. 

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Definition 4.5.1: Let $\mathcal{A}$ be an abelian category.

1. A **double complex** in $\mathcal{A}$ is a family $X_{**} = (X_{i,j})_{i,j \in \mathbb{Z}}$ of objects in $\mathcal{A}$ together with two families $(d^h_{i,j})_{i,j \in \mathbb{Z}}$ and $(d^v_{i,j})_{i,j \in \mathbb{Z}}$ of morphisms $d^h_{i,j} : X_{i,j} \to X_{i-1,j}$ and $d^v_{i,j} : X_{i,j} \to X_{i,j-1}$, the **horizontal** and **vertical differentials**.

\[
d^h_{i-1,j} \circ d^v_{i,j} = -d^h_{i,j-1} \circ d^v_{i,j} : X_{i,j} \to X_{i-1,j-1} \quad d^h_{i-1,j} \circ d^h_{i,j} = 0 \quad d^v_{i-1,j} \circ d^v_{i,j} = 0.
\]

2. A **morphism of double complexes** $f_{**} : X_{**} \to Y_{**}$ is a family $(f_{i,j})_{i,j \in \mathbb{Z}}$ of morphisms $f_{i,j} : X_{i,j} \to Y_{i,j}$ in $\mathcal{A}$ that satisfy for all $i, j \in \mathbb{Z}$

\[
d^h_{i,j} \circ f_{i,j} = f_{i-1,j} \circ d^h_{i,j} \quad d^v_{i,j} \circ f_{i,j} = f_{i,j-1} \circ d^v_{i,j}.
\]

A double complex $X_{**}$ is called **bounded on the left** (bounded below) if there is an $n \in \mathbb{Z}$ with $X_{i,j} = 0$ for all $i < n$ and $j \in \mathbb{Z}$ (with $X_{i,j} = 0$ for all $j < n$ and $i \in \mathbb{Z}$).

Remark 4.5.2:

1. Double complexes and morphisms of double complexes in $\mathcal{A}$ form a category $D\text{Ch}_A$.

2. We can regard double complexes and morphisms of double complexes in $\text{Ch}_A$ and vice versa.

Every double complex $X_{**}$ defines a family of chain complexes $X^h_{**} = (X^h_{i,j})_{i,j \in \mathbb{Z}}$ with differentials $d^h_{i,j} = (d^h_{i,j})_{i,j \in \mathbb{Z}}$, and the morphisms $d^v_{i,j} = (-1)^i d^h_{i,j} : X^h_{i,j} \to X^h_{i,j-1}$ define chain maps $d^v_{i,j} : X^h_{i,j} \to X^h_{i,j-1}$:

\[
d^v_{i-1,j} \circ d^v_{i,j} = (-1)^{i-1} d^h_{i,j-1} \circ d^h_{i,j} = (-1)^i d^h_{i,j} \circ d^v_{i,j} = d^h_{i,j-1} \circ d^v_{i,j} \quad \forall i, j \in \mathbb{Z}.
\]

Hence, we have a chain complex $(X^h_{**}, d^v_{**})$ in $\text{Ch}_A$ with $X^h_j = X^h_{i,j}$ and boundary operators $d^v_j = d^v_{i,j} : X^h_{i,j} \to X^h_{i,j-1}$. A morphism $f_{**} : X_{**} \to Y_{**}$ of double complexes yields chain maps $f^h_{i,j} = (f_{i,j})_{i,j \in \mathbb{Z}} : X^h_{i,j} \to Y^h_{i,j}$ that define a chain map $f_{**} : X_{**} \to Y_{**}$.

Conversely, every chain complex $(X^h_{**}, d^v_{**})$ in $\text{Ch}_A$ with $X^h_j = (X^h_{i,j})_{i \in \mathbb{Z}}$ defines a double complex $X_{**} = (X_{i,j})_{i,j \in \mathbb{Z}}$ in $\mathcal{A}$ with $X_{i,j} = X^h_{i,j}$, $d^h_{i,j} = d^h_{i,j}$, $d^v_{i,j} = (-1)^i d^v_{i,j}$.

The minus sign in the vertical differential is sometimes called the **Koszul sign trick**.

Double complexes or, equivalently, chain complexes in $\text{Ch}_A$ are relevant for our question, because tensoring two families $(L_i)_{i \in I}$ and $(M_j)_{j \in J}$ of, respectively, $R$-right and left modules yields a double complex in $R\text{-Mod}$, and so do the abelian groups $	ext{Hom}_R(N_i, M_j)$ for two families $(N_i)_{i \in I}$ and $(M_j)_{j \in J}$ of $R$-left modules.

Example 4.5.3: Let $R$ be a ring.

1. If $L_{**}$ is a chain complex in $R^{op}\text{-Mod}$ and $M_{**}$ a chain complex in $R\text{-Mod}$, then we obtain a double complex $X_{**}$ in $\text{Ab}$ given by

\[
X_{i,j} = L_i \otimes_R M_j, \quad d^h_{i,j} = d^L_i \otimes \text{id} M_j, \quad d^v_{i,j} = (-1)^i \text{id} L_i \otimes d^M_j.
\]

2. If $M_{**}, N_{**}$ are chain complexes in $R\text{-Mod}$, we obtain a double complex $Y_{**}$ in $\text{Ab}$ with

\[
Y_{i,j} = \text{Hom}_R(M_{-i}, N_j), \quad d^h_{i,j} : f \mapsto f \circ d^M_{-i+1}, \quad d^v_{i,j} : f \mapsto (-1)^i d^N_j \circ f.
\]
Double complexes in \( \mathcal{A} \) allow one to construct chain complexes in \( \mathcal{A} \) that are obtained by combining all objects \( X_{ij} \) with fixed \( i + j \) into a product or coproduct and combining their horizontal and vertical differentials into a chain map. This yields the so-called total complexes. In order to define them for general double complexes, one needs to require countable products and coproducts exist for all chain complexes in \( \mathcal{A} \). If one restricts attention to double complexes that are bounded on the left and below, this requirement is not necessary.

**Lemma 4.5.4:** Let \( \mathcal{A} \) be an abelian category in which products and coproducts exist for all countable families of objects in \( \mathcal{A} \).

1. Every double complex \( X_{**} \) in \( \mathcal{A} \) defines two chain complexes \( \text{Tot}_{**}^h(X_{**}) \) and \( \text{Tot}_{**}^v(X_{**}) \) in \( \mathcal{A} \), the total complexes of \( X_{**} \), given by

\[
\text{Tot}_{**}^h(X_{**}) = \Pi_{i+j=n} X_{i,j}, \quad d_{i,j}^h \circ t_{i,j}^h = t_{i-1,j}^h \circ d_{i,j}^h + t_{i,j-1}^h \circ d_{i,j}^v,
\]

\[
\text{Tot}_{**}^v(X_{**}) = \Pi_{i+j=n} X_{i,j}, \quad \pi_{i,j}^v \circ d_{i,j}^v = d_{i+1,j}^h \circ \pi_{i,j+1}^v + d_{i,j+1}^v \circ \pi_{i,j}^v,
\]

where \( t_{i,j}^h : X_{i,j} \to \text{Tot}_{**}^h(X_{**}) \) and \( \pi_{i,j}^v : \text{Tot}_{**}^v(X_{**}) \to X_{i,j} \) are the inclusion and projection morphisms for the coproduct and product.

2. Every morphism \( f_{**} : X_{**} \to Y_{**} \) induces chain maps

\[
f_{**}^h : \text{Tot}_{**}^h(X_{**}) \to \text{Tot}_{**}^h(Y_{**}) \quad \quad f_{**}^v : \text{Tot}_{**}^v(X_{**}) \to \text{Tot}_{**}^v(Y_{**})
\]

3. This defines functors \( \text{Tot}_{**}^h, \text{Tot}_{**}^v : \text{DCh}_\mathcal{A} \to \text{Ch}_\mathcal{A} \).

**Proof:**
We prove the claims for the chain complex \( \text{Tot}_{**}^h(X_{**}) \). The proof for \( \text{Tot}_{**}^v(X_{**}) \) is analogous.

1. From the definition of \( d_{i,j}^h : \text{Tot}_{**}^h \to \text{Tot}_{**}^h \) we obtain for all \( i, j \in \mathbb{Z} \) and \( n = i + j \)

\[
d_{i,j}^h \circ d_{i,j}^h \circ t_{i,j}^{n+2}_{i+1,j+1} = d_{i,j}^h \circ t_{i,j}^{n+1}_{i+1,j+1} \circ d_{i,j}^h + d_{i,j}^h \circ t_{i,j}^n_{i+1,j+1} + d_{i,j}^v \circ t_{i,j}^n_{i+1,j+1} + t_{i,j}^n_{i+1,j+1} = 0.
\]

By the universal property of the coproduct, this implies \( d_{i,j}^h \circ d_{i,j}^h = 0 \) for all \( n \in \mathbb{Z} \).

2. If \( f_{**} : X_{**} \to Y_{**} \) is a morphism of double complexes, then we have

\[
d_{i,j}^h \circ f_{i,j}^{n+1} = d_{i,j}^h \circ f_{i,j}^{n+1} = d_{i,j}^v \circ f_{i,j}^{n+1} + d_{i,j}^h \circ f_{i,j}^{n+1} + d_{i,j}^v \circ f_{i,j}^{n+1} + d_{i,j}^h \circ f_{i,j}^{n+1} = 0.
\]

for \( i, j \in \mathbb{Z} \) and \( n = i + j \). This implies \( d_{i,j}^v \circ f_{i,j}^{n+1} = d_{i,j}^h \circ f_{i,j}^{n+1} \) by the universal property of the coproduct and shows that \( f_{**}^h : \text{Tot}_{**}^h(X_{**}) \to \text{Tot}_{**}^h(Y_{**}) \) is a chain map. As this is compatible with the composition and the identity morphisms, we obtain a functor \( \text{Tot}_{**}^h : \text{DCh}_\mathcal{A} \to \text{Ch}_\mathcal{A} \).

The total complexes of a double complexes allow one to describe double complexes in an abelian category \( \mathcal{A} \) in terms of chain complexes in \( \mathcal{A} \). Moreover, one has sufficient conditions on the double complexes that imply the exactness of the associated total complexes in \( \mathcal{A} \). Whenever a double complex is bounded from below or from the left, the exactness of the row or column complexes guarantees that its total complexes are also exact.
Lemma 4.5.5: Let $X_{**} = (X_{i,j})_{i,j \in \mathbb{Z}}$ be a double complex in an abelian category $A$ in which products and coproducts exist for all countable families of objects. Then:

1. The total complex $\text{Tot}^H_0(X_{**})$ is exact if all row complexes $X^i_\bullet = (X_{i,j})_{j \in \mathbb{Z}}$ are exact and $X_{**}$ is bounded below or if all column complexes $X^j_\bullet = (X_{i,j})_{i \in \mathbb{Z}}$ are exact and $X_{**}$ is bounded on the left.

2. The total complex $\text{Tot}^H_0(X_{**})$ is exact if all row complexes $X^i_\bullet = (X_{i,j})_{j \in \mathbb{Z}}$ are exact and $X_{**}$ is bounded on the left or if all column complexes $X^j_\bullet = (X_{i,j})_{i \in \mathbb{Z}}$ are exact and $X_{**}$ is bounded below.

Proof:
We prove the claims for $A = R$-Mod and for exact row complexes. The proof for exact column complexes is analogous.

1. It is sufficient to prove exactness in $\text{Tot}^H_0(X_{**}) = \Pi_{n \in \mathbb{Z}}X_{-n,n} = \Pi_{n \in \mathbb{N}_0}X_{-n,n}$ for a double complex $X_{**}$ with $X_{i,j} = 0$ for all $j < 0$. All other cases can be obtained from this by renumbering the rows and columns. Let $x = (x_n)_{n \in \mathbb{N}_0}$ with $x_n \in X_{-n,n}$ be in $\ker(d^H_0) \subset \text{Tot}^H_0(X_{**})$. Then $d^H_0(x) = 0$. Hence we can choose $y_n = 0$ for $n > 0$ and inductively construct elements $y_{n-m} \in X_{-i+m+1,i-m}$ for $m \in \mathbb{N}_0$.

For $m = 0$, the exactness of the row complexes and the identity $d^b_{-i,i}(x_i) = 0$ implies that there is an element $y_i \in X_{-i+1,i}$ with $d^b_{-i+1,i}(y_i) = x_i$. Suppose we constructed for $k \in \{i, i-1, \ldots, j+1\}$ elements $y_k \in X_{-k+1,k}$ that satisfy $x_k = d^b_{k+1,k}(y_k)$. Then we have

$$d^b_{-j,j}(x_j - d^e_{-j+1,j}(y_{j+1})) = d^b_{-j,j}(x_j) + d^b_{-j+1,j}(y_{j+1})$$

and by exactness of the row complex there is a $y_j \in X_{-j+1,j}$ such that $d^e_{-j+1,j}(y_j) = x_j - d^b_{-j+1,j}(y_{j+1})$. For $j = -1$, we obtain $x_{-1} = 0$ and $d^b_{-1,0}(y_0) = 0$. Hence we can choose $y_{n-1} = 0$ for $n \leq -1$ and obtain an element $y = (y_n)_{n \in \mathbb{N}_0} \in \text{Tot}^H_0$ with $d^H_0(y) = x$.

2. It is sufficient to prove exactness in $\text{Tot}^H_0(X_{**}) = \Pi_{n \in \mathbb{Z}}X_{-n,n} = \Pi_{n \in \mathbb{N}_0}X_{-n,n}$ for double complexes $X_{**}$ with $X_{i,j} = 0$ for $i < 0$ and all other cases are obtained by renumbering the rows and columns. Let $x = (x_n)_{n \in \mathbb{N}_0}$ in $\ker(d^H_0) \subset \text{Tot}^H_0(X_{**})$ with $x_n \in X_{-n,n}$. Then we have

$$d^b_{n,n}(x_n) + d^e_{n-1,n+1}(x_{n-1}) = 0 \quad \text{for } n \geq 1 \quad d^b_{0,0}(x_0) = 0.$$

We construct an element $y = (y_n)_{n \in \mathbb{N}_0} \in \text{Tot}^H_0(X_{**})$ with $y_n \in X_{n+1,n}$ and $d^H_1(y) = x$ by induction over $n \in \mathbb{N}_0$. For $n = 0$ the exactness of the row complex and the identity $d^0_{0,0}(x) = 0$ imply that there is an element $y_0 \in X_{1,0}$ with $d^b_{1,0}(y_0) = x_0$. Suppose we constructed elements $y_k \in X_{k+1,k}$ with $x_k = d^b_{k+1,k}(y_k)$. Then

$$d^b_{n,n}(x_n) + d^e_{n-1,n+1}(x_{n-1}) = 0 \quad \text{for } n \geq 1 \quad d^b_{0,0}(x_0) = 0.$$

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and the exactness of the row complexes implies that there is an element \( y_n \in X_{n+1,-n} \) with 
\[
d_{n+1,-n}(y_n) = x_n - d_{n,-n+1}(y_{n-1}).
\]
Hence we constructed an element \( y = (y_n)_{n \in \mathbb{N}_0} \in \text{Tot}^\bullet_1(X_{\bullet \bullet}) \) with \( d^1_0(y) = x \).

We can now apply these results to the double complexes from Example 4.5.3. By tensoring a projective resolution \( L_\bullet \) of an \( R \)-right module \( L \) and a projective resolution \( M_\bullet \) of an \( R \)-left module \( M \) and omitting the terms with \( L_{-1} = L \) or with \( M_{-1} = M \), we obtain bounded double complexes with, respectively, exact rows or columns. By Lemma 4.5.5 the associated total complexes are exact. Tensoring the resolution \( L_\bullet \) with \( M \) or the resolution \( M_\bullet \) with \( L \) over \( R \) yields a short exact sequence of chain complexes. The associated long exact homology sequence then relates \( \text{Tor}^R_n(L, M) \) to \( \text{Tor}^R_n(L, M) \). An analogous procedure for \( R \)-left modules \( M, N \) relates \( \text{Ext}^R_n(M, N) = \text{Ext}^R_n(M, N) \) and \( \text{Ext}^R_n(M, N) = \text{Ext}^R_n(M, N) \).

**Theorem 4.5.6:** Let \( R \) be a ring. Then for all \( R \)-left modules \( M, N \) and \( R \)-right modules \( L \)
\[
\text{Tor}^R_n(L, M) \cong \text{Tor}^R_n(L, M) \quad \text{Ext}^R_n(M, N) \cong \text{Ext}^R_n(M, N) \quad \forall n \in \mathbb{N}_0.
\]
and for all \( R \)-linear maps \( f : L \to L' \), \( g : M \to M' \) and \( h : N \to N' \)
\[
\text{Tor}^R_n(f, g) = L_n(- \otimes g)(f) = L_n(f \otimes -)(g) = \text{Tor}^R_n(f, g) \\
\text{Ext}^R_n(g, h) = R^n\text{Hom}_R(-, h)(g) = \text{Ext}^R_n(g, h).
\]

**Proof:**
We prove \( \text{Tor}^R_n(L, M) = \text{Tor}^R_n(L, M) \). The proof of \( \text{Ext}^R_n(M, N) = \text{Ext}^R_n(M, N) \) is analogous.

We choose projective resolutions \( L_\bullet \) of \( L \) and \( M_\bullet \) of \( M \) in, respectively, \( R^{op}\)-Mod and \( R\)-Mod and denote by \( \overline{L}_\bullet = L_{\bullet \geq 0} \) and \( \overline{M}_\bullet = M_{\bullet \geq 0} \) the associated chain complexes with \( L_{-1} = L \) and \( M_{-1} = M \) replaced by 0. We consider the double complexes \( X_{\bullet \bullet} \) and \( Y_{\bullet \bullet} \) from Example 4.5.3 in \( \text{Ab} \) with 
\[
X_{i,j} = L_i \otimes R \overline{M}_j, \quad Y_{i,j} = \overline{L}_i \otimes R M_j, \quad d^1_{i,j} = d_i \otimes \text{id}_{M_j}, \quad d^0_{i,j} = (-1)^i \text{id}_{L_i} \otimes d^M_j. \tag{33}
\]
As \( \overline{L}_i \) and \( \overline{M}_i \) are projective for all \( i \geq -1 \), they are flat by Exercise 45. As \( L_\bullet \) and \( M_\bullet \) are exact, it follows that all column complexes \( Y_{\bullet \bullet} \) and row complexes \( X_{\bullet \bullet} \) are exact as well. As \( X_{\bullet \bullet} \) and \( Y_{\bullet \bullet} \) are bounded from the left and from below, it follows that the total complexes \( \text{Tot}^\bullet_1(X_{\bullet \bullet}) \) and \( \text{Tot}^\bullet_1(Y_{\bullet \bullet}) \) are exact by Lemma 4.5.5.

Denoting by \( L'_\bullet \) and \( M'_\bullet \) the chain complexes with \( L'_{-1} = L \), \( M'_{-1} = M \) and \( L'_i = M'_i = 0 \) for \( i \neq -1 \), we have short exact sequences of chain complexes
\[
0 \to L'_\bullet \xrightarrow{\delta_L} L_\bullet \xrightarrow{\pi_L} \overline{L}_\bullet \to 0 \quad 0 \to M'_\bullet \xrightarrow{\delta_M} M_\bullet \xrightarrow{\pi_M} \overline{M}_\bullet \to 0.
\]
As \( \delta_L \) and \( \delta_M \) are projective for all \( j \in \mathbb{N}_0 \), this defines short exact sequences of chain complexes
\[
0 \to L'_\bullet \otimes R M_j \xrightarrow{\delta_L \otimes \text{id}_{M_j}} L_\bullet \otimes R M_j \xrightarrow{\pi_L \otimes \text{id}_{M_j}} \overline{L}_\bullet \otimes R M_j \to 0 \\
0 \to L_j \otimes R M'_\bullet \xrightarrow{\text{id}_L \otimes \pi'_M} L_j \otimes R M_\bullet \xrightarrow{\text{id}_L \otimes \pi'_M} L_j \otimes R \overline{M}_\bullet \to 0
\]
in \( \text{Ab} \) for all \( j \in \mathbb{N}_0 \) and short exact sequences of double complexes
\[
0 \to L'_{\bullet \bullet} \xrightarrow{\delta_{L_{\bullet \bullet}}} X_{\bullet \bullet} \xrightarrow{\pi_{L_{\bullet \bullet}}} Z_{\bullet \bullet} \to 0 \quad 0 \to M'_{\bullet \bullet} \xrightarrow{\delta_{M_{\bullet \bullet}}} Y_{\bullet \bullet} \xrightarrow{\pi_{M_{\bullet \bullet}}} Z_{\bullet \bullet} \to 0, \tag{34}
\]

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where $L_{\bullet}'$, $M_{\bullet}'$ and $Z_{\bullet}$ are the double complexes with

$L_{-1,j}' = L \otimes_R \overline{M}_j$, $L'_{i,j} = 0$ for $i \neq -1$, $M_{i,-1}' = \overline{L}_i \otimes_R M$, $M_{i,j}' = 0$ for $i \neq -1$, $Z_{i,j} = \overline{L}_i \otimes_R \overline{M}_j$.

The differentials of these double complexes are given by the differentials of $L_{\bullet}'$, $M_{\bullet}'$, $\overline{L}_{\bullet}$ and $\overline{M}_{\bullet}$, as in (33). As all of these chain complexes are bounded below and on the left, Lemma 4.5.4 and 4.5.5 yields short exact sequences of total complexes

\[0 \to \Tot^H_\bullet(L_{\bullet}') \xrightarrow{\Tot^H_{\bullet}(\partial)} \Tot^H_\bullet(X_{\bullet}) \xrightarrow{\Tot^H_{\bullet}(\partial)} \Tot^H_\bullet(Z_{\bullet}) \to 0 \tag{35}\]

\[0 \to \Tot^H_\bullet(M_{\bullet}') \xrightarrow{\Tot^H_{\bullet}(\partial)} \Tot^H_\bullet(Y_{\bullet}) \xrightarrow{\Tot^H_{\bullet}(\partial)} \Tot^H_\bullet(Z_{\bullet}) \to 0.\]

As $\Tot^H_\bullet(X_{\bullet})$ and $\Tot^H_\bullet(Y_{\bullet})$ are exact by Lemma 4.5.5, all homologies $H_n(\Tot^H_\bullet(X_{\bullet}))$ and $H_n(\Tot^H_\bullet(Y_{\bullet}))$ vanish. Their long exact homology sequences from Theorem 3.4.7 take the form

\[\cdots \to H_{n+1}(\Tot^H_\bullet(Z_{\bullet})) \xrightarrow{\partial_{n+1}} H_n(\Tot^H_\bullet(L_{\bullet}')) \to 0 \to H_n(\Tot^H_\bullet(Z_{\bullet})) \xrightarrow{\partial_n} H_{n-1}(\Tot^H_\bullet(L_{\bullet}')) \to 0 \cdots\]

and $\partial_n : H_n(\Tot^H_\bullet(Z_{\bullet})) \to H_{n-1}(\Tot^H_\bullet(L_{\bullet}'))$, $\partial_n^M : H_n(\Tot^H_\bullet(Z_{\bullet})) \to H_{n-1}(\Tot^H_\bullet(M_{\bullet}'))$ are isomorphisms for all $n \in \mathbb{N}$. By definition, we have

$$\Tot^H_{n-1}(L_{\bullet}') = L_\otimes_R \overline{M}_n = (L_\otimes_R -)(\overline{M}_n) \quad \Tot^H_{n-1}(M_{\bullet}') = \overline{L}_n \otimes_R M = (- \otimes_R M)(\overline{L}_n)$$

and hence

$$\Tor^R_n(L, M) = L_n(- \otimes_R M)(L) = H_{n-1}(\Tot^H_\bullet(M_{\bullet}')) \cong H_n(\Tot^H_\bullet(Z))$$

$$\cong H_{n-1}(\Tot^H_\bullet(L_{\bullet}')) = L_n(L_\otimes_R -)(M) = \Tor^R_n(L, M).$$

All $R$-linear maps $f : L \to L'$ and $g : M \to M'$ extend to chain maps $f_\bullet : L_\bullet \to L_{\bullet}'$ and $g_\bullet : M_\bullet \to M_{\bullet}'$ between the projective resolutions, to chain maps between the associated short exact sequences of double complexes in (34) and to chain maps between the associated short exact sequences of total complexes in (35). By Proposition 3.4.8 they induce chain maps between the associated long exact homology sequences, and this implies

$$\Tor^R_n(f, g) = L_n(- \otimes g)(f) = L_n(f \otimes -)(g) = \Tor^R_n(f, g),$$

where $\Tor^R_n(f, -) = L_n(f \otimes -) : \Tor^R_n(L, -) \to \Tor^R_n(L', -)$ is the natural transformation from Remark 4.4.2 3. and $\Tor^R_n(-, g) = L_n(- \otimes g)$ its counterpart for $\Tor^R_n$.

Theorem 4.5.6 shows that the choice involved in the definition or Tor and Ext, namely the decision to tensor on the left and to take $R$-linear maps into a given module are of no consequence, since tensoring on the right and taking $R$-linear maps from a module lead to the same functors. Moreover, it shows that Tor and Ext are functors in both arguments and that applying morphisms in the first and in the second argument commutes. In other words, they define functors from the product categories $\Tor^R_n : \text{R-mod}^\text{op} \times \text{R-mod}$ and $\Ext^n_R : \text{R-mod}^\text{op} \times \text{R-mod}$. Such functors are also called bifunctors.

**Corollary 4.5.7:** For any ring $R$, the functors Tor and Ext define a family of functors

$$\Tor^R_n : \text{R-mod}^\text{op} \times \text{R-mod} \to \text{Ab} \quad \Ext^n_R : \text{R-mod}^\text{op} \times \text{R-mod} \to \text{Ab}.$$
5 Simplicial methods

In the last section, we derived a unified description of certain (co)homologies in terms of the functors Tor and Ext. These functors are obtained as the left derived functors of the functor \( L \otimes_R - : R\text{-Mod} \to \text{Ab} \) for an \( R \)-right module \( L \) and the right derived functors of the functor \( \text{Hom}( -, M) : R \text{-Mod}^{op} \to \text{Ab} \) for an \( R \)-left module \( M \). This description of (co)homologies in terms of Tor and Ext includes Hochschild (co)homologies, group (co)homologies and (co)homologies of Lie algebras and defines them without a specific choice of chain complex. The standard chain complexes that were used to define these (co)homologies in Section 2 arise from specific projective resolutions.

That the resulting description is independent of the choice of resolutions is conceptually nice and advantageous in computations. However, it does not explain the origin and mathematical structure of the standard resolutions such as the Hochschild resolution, the bar resolution for group cohomology and the Chevalley-Eilenberg resolution for Lie algebra cohomology. Although these standard resolutions are just specific choices of resolutions, they are distinguished by the fact that they work globally, for all possible algebras, groups or Lie algebras under consideration.

Another interesting feature of the standard chain complexes from Section 2 is that they have a very similar combinatorial structure. In all examples from Section 2, the boundary operators are given as alternating sums \( d_n = \sum_{i=0}^{n} (-1)^i d_i^n \) for certain \( R \)-linear maps \( d_i^n : C_n \to C_{n-1} \). In all cases, these \( R \)-linear maps \( d_i^n : C_n \to C_{n-1} \) exhibit similar commutation relations, derived in Lemma 2.1.3 and 2.2.5. These commutation relations guarantee that the boundary operators satisfy the identities \( d_n \circ d_{n+1} = 0 \) that characterise a chain complex.

In the case of singular and simplicial homology, the boundary operators have a geometrical interpretation. They are defined by the face maps, that send the standard \((n-1)\)-simplex \( \Delta^{n-1} \) to the face opposite the vertex \( e_i \) in the standard \( n \)-simplex \( \Delta^n \). However, in the end the description is purely combinatorial and relies only on the ordering of the \( n+1 \) vertices in \( \Delta^n \). The geometrical interpretation of the face maps in singular and simplicial homology also does not explain why similar combinatorial structures arise in the context of Hochschild (co)homology, group (co)homology and (co)homology of Lie algebras.

This suggests that the combinatorial structure and the associated commutation relations between the maps \( d_i^n \) could be a global pattern that characterises chain complexes. One could define chain complexes in any abelian category \( \mathcal{A} \) by identifying a collection of morphisms \( d_i^n : X_n \to X_{n-1} \) in \( \mathcal{A} \) with similar commutation relations and defining the boundary operators as alternating sums \( d_n = \sum_{i=0}^{n} (-1)^i d_i^n \). The question is how to find such collections of morphisms \( d_i^n \) and which chain complexes in \( \mathcal{A} \) can be obtained in this way.

In this section, we investigate this construction systematically and show that up to chain homotopy equivalence all positive chain complexes in an abelian category \( \mathcal{A} \) can be obtained from this construction. This is the famous Dold-Kan correspondence. It is not an isolated result but the foundation of a general combinatorial approach to homologies that also incorporates chain maps and chain homotopies into the picture. Among others, it leads to a systematic construction of resolutions from (co)monads and adjoint functors.
5.1 The simplex category

The first step is a precise formulation of the combinatorics in the examples in Section 2. The suitable framework is a category. This category must encode the combinatorics of the standard \( n \)-simplexes \( \Delta^n \) for all \( n \in \mathbb{N}_0 \), and of all maps between them that are obtained by composing face maps. Hence, we require a category with one object for each \( n \in \mathbb{N}_0 \). To account for the \( n+1 \) ordered vertices of \( \Delta^n \) we can choose for this object the finite ordinal \([n+1] = \{0, 1, \ldots, n\}\).

Face maps between standard \( n \)-simplexes are maps \( f_n : \Delta^{n-1} \to \Delta^n \) that are determined by their behaviour on the vertices and respect their ordering. Hence, we can describe them as strictly increasing maps \( f : [n] \to [n+1] \).

For reasons that will become apparent later, it makes sense to introduce an additional object \([0] = \emptyset\) and to consider all weakly monotonic maps between finite ordinals. This leads to the so-called augmented simplex category or algebraist’s simplex category. The simplex category or topologist’s simplex category is the full subcategory obtained by omitting the object \([0] = \emptyset\).

Definition 5.1.1:

1. The augmented simplex category \( \Delta \) has as objects the finite ordinal numbers \([n] = \{0, 1, \ldots, n-1\}\) for \( n \in \mathbb{N}_0 \) with \([0] = \emptyset\). The morphisms \( f : [n] \to [m] \) are monotonic maps \( f : \{0, \ldots, n-1\} \to \{0, \ldots, m-1\} \), and their composition is the composition of maps.

2. The simplex category \( \Delta^+ \) is the full subcategory of \( \Delta \) with objects \([n]\) for \( n \in \mathbb{N} \).

To understand the simplex category and apply it to homological algebra, we need a more detailed understanding of its morphisms, in particular the morphisms that generalise face maps between standard \( n \)-simplexes. Clearly, the face maps correspond to the injective morphisms \( \delta^i_n : [n] \to [n+1] \) that skip the element \( i \in [n+1] \). There are also surjective counterparts of the face maps, the degeneracies \( \sigma^i_n : [n+1] \to [n] \) that send \( j \) and \( j+1 \) to \( j \). It turns out that any morphism in \( \Delta \) can be expressed uniquely a product of these maps, with \( \delta^0_n : [0] \to [1] \) corresponding to the empty map.

Proposition 5.1.2: (factorisation in the simplex category)

1. Every morphism \( f : [m] \to [n] \) in \( \Delta \) can be expressed uniquely as a composite
   \[
   f = \delta^i_{n-1} \circ \cdots \circ \delta^i_{n-l} \circ \sigma^{j_1}_{m-1} \circ \cdots \circ \sigma^{j_k}_{m-1}
   \]
   for \( n = m - l + k \), \( 0 \leq i_k < \cdots < i_1 < n \), \( 0 \leq j_1 < \cdots < j_k < m - 1 \). Of the face maps \( \delta^i_n : [n] \to [n+1] \) and the degeneracies \( \sigma^i_n : [n+1] \to [n] \) for \( i \in \{0, \ldots, n\} \) and \( j \in \{0, \ldots, n-1\} \),
   \[
   \delta^i_n(k) = \begin{cases} 
   k & 0 \leq k < i \\
   k+1 & i \leq k < n 
   \end{cases}
   \]
   \[
   \sigma^i_n(k) = \begin{cases} 
   k & 0 \leq k \leq j \\
   k-1 & j < k \leq n. 
   \end{cases}
   \]

2. The morphisms \( \delta^i_n : [n] \to [n+1] \) and \( \sigma^i_n : [n+1] \to [n] \) satisfy the relations
   \[
   \delta^j_{n+1} \circ \delta^i_n = \delta^{i+1}_{n+1} \circ \delta^n_i \quad \text{for} \quad i \leq j
   \]
   \[
   \sigma^j_n \circ \sigma^i_{n+1} = \sigma^i_n \circ \sigma^{j+1}_n \quad \text{for} \quad i \leq j
   \]
   \[
   \sigma^i_n \circ \delta^i_n = \begin{cases} 
   \delta^i_{n-1} \circ \sigma^i_{n-1} & \text{for} \quad i < j \\
   \delta^i_{n-1} \circ \sigma^i_{n-1} & \text{for} \quad i > j + 1 
   \end{cases}
   \]
Proof:
1. Every monotonic map $f : [m] \rightarrow [n]$ is determined uniquely by the sets
\[ M_\delta = \{i_1, ..., i_k\} = [n] \setminus \text{im}(f) \quad M_\sigma = \{j_1, ..., j_l\} = \{x \in [m - 1] \mid f(x) = f(x + 1)\} \].

If $M_\delta$ and $M_\sigma$ are ordered such that $0 \leq i_k < ... < i_1 < n$ and $0 \leq j_1 < ... < j_l < m - 1$, then $f$ can be factorised uniquely as $f = g \circ h$ with an injective monotonic map $g : [m - l] \rightarrow [n]$ and a surjective monotonic map $h : [m] \rightarrow [m - l]$ given by
\[
g(r) = \begin{cases} r & 0 \leq r < i_k \\ r + s & i_{k-s+1} \leq r < i_{k-s} \\ r + k & i_1 \leq r \end{cases} \quad \quad h(r) = \begin{cases} r & r \leq j_1 \\ r - s & j_s < r \leq j_{s+1} \\ r - l & j_l < r. \end{cases}
\]
This implies $g = \delta^{i_1}_n \circ ... \circ \delta^{i_{m-l}}_{m-1}$ and $h = \sigma^{j_1}_{m-1} \circ ... \circ \sigma^{j_l}_{m-1}$.

2. The relations between the maps $\delta^i_n : [n] \rightarrow [n + 1]$ and $\sigma^i_n : [n + 1] \rightarrow [n]$ follow by a direct computation. For $0 \leq i \leq j \leq n - 1$, we have
\[
\delta^{i+1}_{n+1} \circ \delta^i_n(k) = \begin{cases} \delta^{i+1}_{n+1}(k) & 0 \leq k < j \\ \delta^{i+1}_{n+1}(k+1) & j \leq k \leq n - 1 \end{cases} = \begin{cases} 0 \leq k < i \\ k + 1 & i \leq k < j \\ k + 2 & j \leq k \leq n - 1 \end{cases}
\]
\[
\delta^{j+1}_{n+1} \circ \delta^i_n(k) = \begin{cases} \delta^{j+1}_{n+1}(k) & 0 \leq k < i \\ \delta^{j+1}_{n+1}(k+1) & i \leq k \leq n - 1 \end{cases} = \begin{cases} 0 \leq k < i \\ k + 1 & i \leq k < j \\ k + 2 & j \leq k \leq n - 1, \end{cases}
\]
and the computations for the other relations are similar. \qed

Remark 5.1.3: As the relations (37) allow one to transform any product of the morphisms $\delta^i_n$ and $\sigma^i_n$ into an ordered product of the form (36) and the factorisation in (36) is unique, there can be no further relations between the morphisms $\delta^i_n$ and $\sigma^i_n$. All relations between them are obtained by composing (37) with morphisms and using the relations for identity morphisms.

One says that $\Delta$ is generated as a category or presented as a category by the morphisms $\delta^i_n$ and $\sigma^i_n$ subject to the relations (37).

The relations between the face maps in (37) resemble the relations between the face maps $f^i_n$ from singular and simplicial (co)homology from Lemma 2.1.3 and between the maps $d^i_n$ and $l^n$ from Hochschild (co)homology in Lemma 2.2.5. More precisely, the relations for the cohomologies coincide with the relations (37) while the composition of the face maps is reversed for the homologies. As any functor $F : \Delta^+ \rightarrow C$ into a category $C$ preserves the relations between the face maps and functors $F : \Delta^{+\text{op}} \rightarrow C$ reverse them, this suggests to construct chain complexes in an abelian category $C$ from functors $F : \Delta^+ \rightarrow C$ and $F : \Delta^{+\text{op}} \rightarrow C$, respectively. Such functors are called cosimplicial and simplicial objects in $C$. It will be useful to consider such functors for general categories $C$, not just abelian ones.
**Definition 5.1.4:** Let $\mathcal{C}$ be a category.

1. A **simplicial object** in $\mathcal{C}$ is a functor $F : \Delta^{+ op} \to \mathcal{C}$. An **augmented simplicial object** in $\mathcal{C}$ is a functor $F : \Delta^{op} \to \mathcal{C}$.

2. A **morphism of simplicial objects** from $F : \Delta^{+ op} \to \mathcal{C}$ to $G : \Delta^{+ op} \to \mathcal{C}$ is a natural transformation $\eta : F \to G$. A **morphism of augmented simplicial objects** from $F : \Delta^{op} \to \mathcal{C}$ to $G : \Delta^{op} \to \mathcal{C}$ is a natural transformation $\eta : F \to G$.

3. A **cosimplicial object** in $\mathcal{C}$ is a functor $F : \Delta^+ \to \mathcal{C}$. An **augmented cosimplicial object** in $\mathcal{C}$ is a functor $F : \Delta \to \mathcal{C}$.

4. A **morphism of cosimplicial objects** from $F : \Delta^+ \to \mathcal{C}$ to $G : \Delta^+ \to \mathcal{C}$ is a natural transformation $\eta : F \to G$. A **morphism of augmented cosimplicial objects** from $F : \Delta \to \mathcal{C}$ to $G : \Delta \to \mathcal{C}$ is a natural transformation $\eta : F \to G$.

**Remark 5.1.5:**

1. As the morphisms $\delta^i_n : [n] \to [n+1]$ and $\sigma^i_n : [n+1] \to [n]$ from Proposition 5.1.2 generate the simplicial category $\Delta^+$ subject to the relations (37), a (co)simplicial object is determined uniquely by the images of the objects $[n]$ for $n \in N_0$ and the morphisms $\delta^i_n$ and $\sigma^i_n$, which must satisfy relations analogous to (37).

Hence, a simplicial object in $\mathcal{C}$ can be defined equivalently as a family $(C_n)_{n \in N_0}$ of objects in $\mathcal{C}$ together with morphisms $d^i_n : C_n \to C_{n-1}$, the **face operators**, and $s^i_n : C_n \to C_{n+1}$, the **degeneracies**, for $0 \leq i \leq n$ that satisfy the simplicial identities

$$
\begin{align*}
d^i_n \circ d^i_{n+1} &= d^i_n \circ d^{i+1}_{n+1} \quad \text{for } i \leq j \\
s^i_{n+1} \circ s^j_n &= s^{i+1}_{n+1} \circ s^i_n \quad \text{for } i \leq j \\
d^i_{n+1} \circ s^j_n &= \begin{cases} 
  s^{j-1}_{n-1} \circ d^i_n & i < j \\
  1_{C_n} & i \in \{j, j+1\} \\
  s^j_{n-1} \circ d^{i-1}_n & j+1 < i \leq n+1.
\end{cases}
\end{align*}
$$

They correspond to a functor $F : \Delta^{+ op} \to \mathcal{C}$ with $C_n = F([n+1])$, $d^i_n = F(\delta^i_n)$ and $s^i_n = F(\sigma^i_{n+1})$. The shift in indices is a standard convention.

2. Similarly, one can define a cosimplicial object in $\mathcal{C}$ as a family $(C^n)_{n \in N_0}$ of objects in $\mathcal{C}$ together with morphisms $d^n_i : C^{n-1} \to C^n$ and $s^n_i : C^{n+1} \to C^n$ for $0 \leq i \leq n$ that satisfy the cosimplicial identities

$$
\begin{align*}
d^{i+1}_n \circ d^n_j &= d^{i+1}_{j+1} \circ d^n_i \quad \text{for } i \leq j \\
s^{i-1}_j \circ s^n_i &= s^{i-1}_{j+1} \circ s^n_{j+1} \quad \text{for } i \leq j \\
s^n_j \circ d^n_{i+1} &= \begin{cases} 
  d^n_i \circ s^{i-1}_{j-1} & i < j \\
  1_{C_n} & i \in \{j, j+1\} \\
  d^n_{i-1} \circ s^{i-1}_j & i > j+1.
\end{cases}
\end{align*}
$$

They define a functor $F : \Delta^+ \to \mathcal{C}$ with $C_n = F([n+1])$, $d^n_i = F(\delta^i_n)$, $s^n_i = F(\sigma^i_{n+1})$. 

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functors are often very simple and obtained from a functor into the category Set. One has to compose the (co)simplicial objects with functors into an abelian category. These (co)chain complexes from (co)simplicial objects in non-abelian categories such as Top or Set, any (co)simplicial object in an abelian category gives rise to a (co)chain complex. To obtain (co)chain complexes from (co)simplicial objects in non-abelian categories such as Top or Set, one has to compose the (co)simplicial objects with functors into an abelian category. These functors are often very simple and obtained from a functor into the category Set.

Remark 5.1.6:

1. Simplicial objects in a category \(\mathcal{D}\) and morphisms of simplicial objects form a category, namely the category \(\text{Fun}(\Delta^{+\text{op}}, \mathcal{D})\). Cosimplicial objects and morphisms of cosimplicial objects in \(\mathcal{C}\) form the category \(\text{Fun}(\Delta^+, \mathcal{D})\). Analogous statements hold for augmented (co)simplicial objects and morphisms of augmented (co)simplicial objects.

2. Let \(G : \mathcal{C} \to \mathcal{D}\) be a functor. Then for any simplicial object \(F : \Delta^{+\text{op}} \to \mathcal{C}\) the functor \(GF : \Delta^{+\text{op}} \to \mathcal{D}\) is a simplicial object in \(\mathcal{D}\), and for any simplicial morphism \(\eta : F \to F'\) the natural transformation \(G\eta : GF \to GF'\) with component morphisms \(G\eta[n] = G(\eta[n]) : GF([n]) \to GF'([n])\) is a simplicial morphism in \(\mathcal{D}\). Analogous statements hold in the cosimplicial and in the augmented case.

As expected, the (co)chain complexes for singular and simplicial (co)homology, for Hochschild (co)homology, group cohomology and cohomology of Lie algebras from Section 2 all arise from (co)simplicial objects in \(k\text{-Mod}\), where \(k\) is a commutative ring. The additional information contained in an augmented (co)simplicial object defines the associated standard resolution.

Example 5.1.7: (Hochschild homology and cohomology)
Let \(A\) be an algebra over a commutative ring \(k\) and \(M\) an \((A,A)\)-bimodule.

1. The functor \(F : \Delta^{\text{op}} \to A \otimes A^{\text{op}}\text{-Mod}\) with
\[
C_n = F([n+1]) = A^\otimes(n+2), \quad n \geq -1
\]
\[
d^n_i = F(\delta^n_i) : A^\otimes(n+2) \to A^\otimes(n+1), \quad a_0 \otimes \ldots \otimes a_{n+1} \mapsto a_0 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_{n+1}
\]
\[
s^n_i = F(\sigma^{n+1}_i) : A^\otimes(n+2) \to A^\otimes(n+3), \quad a_0 \otimes \ldots \otimes a_{n+1} \mapsto a_0 \otimes \ldots \otimes a_{i-1} \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_{n+1}.
\]
is an augmented simplicial object in \(A \otimes A^{\text{op}}\text{-Mod}\). The \((A,A)\)-bimodules \(C_n\) and the morphisms \(d^n_i : C_n \to C_{n-1}\) are the ones of the Hochschild resolution in Example 4.1.3.

2. Composing its restriction \(F^+ : \Delta^{+\text{op}} \to A \otimes A^{\text{op}}\text{-Mod}\) to the full subcategory \(\Delta^{+\text{op}} \subset \Delta^{\text{op}}\) with \(M \otimes A \otimes A^{\text{op}}\text{-} -: A \otimes A^{\text{op}}\text{-Mod} \to k\text{-Mod}\) yields a simplicial object \((M \otimes A \otimes A^{\text{op}}\text{-} \cdot \cdot)F^+\) in \(k\text{-Mod}\) that defines the chain complex of Hochschild homology from Definition 2.2.3.

3. Composing \(F^+ : \Delta^{+\text{op}} \to A \otimes A^{\text{op}}\text{-Mod}\) with \(\text{Hom}_{A \otimes A^{\text{op}}\text{-} \cdot \cdot}(-, M) : A \otimes A^{\text{op}}\text{-Mod}^{\text{op}} \to k\text{-Mod}\) yields a cosimplicial object \((\text{Hom}_{A \otimes A^{\text{op}}\text{-} \cdot \cdot}(-, M) F^+)\) in \(k\text{-Mod}\) that defines the cochain complex of Hochschild cohomology from Definition 2.2.4.

4. By specialising to the case \(A = k[G]\) for a group \(G\) and bimodules with trivial \(k[G]\text{-}\cdot\cdot\cdot\text{-}\)module structures, we obtain the corresponding statements for group (co)homology. By considering the algebra \(A = U(\mathfrak{g})\) for a Lie algebra \(\mathfrak{g}\) and bimodules with the trivial \(U(\mathfrak{g})\text{-}\cdot\cdot\cdot\text{-}\)module structures, we obtain Lie algebra (co)homology.

The categories \(k\text{-Mod}\) and \(A \otimes A^{\text{op}}\text{-Mod}\) in Example 5.1.7 are abelian, and we will see later that any (co)simplicial object in an abelian category gives rise to a (co)chain complex. To obtain (co)chain complexes from (co)simplicial objects in non-abelian categories such as Top or Set, one has to compose the (co)simplicial objects with functors into an abelian category. These functors are often very simple and obtained from a functor into the category Set.
Example 5.1.8: (Singular homology and cohomology)

1. The family of standard $n$-simplexes $(\Delta^n)_{n \in \mathbb{N}_0}$ with the face maps and degeneracy maps

$$d^n_i = f^n_i : \Delta^{n-1} \to \Delta^n \quad \text{and} \quad s^n_i : \Delta^{n+1} \to \Delta^n$$

$$f^n_i(e_k) = \begin{cases} e_k & 0 \leq k < i \\ e_{k+1} & i \leq k \leq n \end{cases} \quad \text{and} \quad s^n_i(e_k) = \begin{cases} e_k & 0 \leq k < i \\ e_{k-1} & i \leq k \leq n + 1 \end{cases}$$

form a cosimplicial object $F_\Delta : \Delta^+ \to \text{Top}$ with

$$F_\Delta([n]) = \Delta^{n-1}, \quad F_\Delta(\delta^n_i) = f^n_i, \quad F_\Delta(\sigma^n_i) = s^{n-1}_i.$$

More generally, for a morphism $\alpha : [m+1] \to [n+1]$ in $\Delta^+$ the morphism $F_\Delta(\alpha)$ is the affine-linear map $F_\Delta(\alpha) : \Delta^m \to \Delta^n$ with $F_\Delta(\alpha)(k) = e_{\alpha(k)}$ for all $k \in \{0, ..., m\}$.

2. Let $X$ be a topological space and $\text{Hom}_{\text{Top}}(-, X) : \text{Top} \to \text{Set}^{\text{op}}$ the functor that assigns

- to a topological space $Y$ the set $\text{Hom}_{\text{Top}}(Y, X)$ of continuous maps $f : Y \to X$
- to a continuous map $\sigma : Y \to Z$ the map

$$\text{Hom}(\sigma, X) : \text{Hom}_{\text{Top}}(Z, X) \to \text{Hom}_{\text{Top}}(Y, X), \quad g \mapsto g \circ \sigma.$$ 

By composing $\text{Hom}_{\text{Top}}(-, X)$ with the functor $F_\Delta$ from 1., we obtain a simplicial object $C^X = \text{Hom}(-, X) \circ F_\Delta : \Delta^{\text{op}} \to \text{Set}$ given by

$$C^X([n+1]) = C^n_n = \text{Hom}_{\text{Top}}(\Delta^n, X),$$

$$C^X(\delta^n_i) = d^n_i : C^n_n \to C^n_{n-1}, \quad \sigma \mapsto \sigma \circ f^n_i, \quad C^X(\sigma^n_{n+1}) = s^n_i : C^n_n \to C^n_{n+1}, \quad \sigma \mapsto \sigma \circ s^n_i.$$ 

3. This defines a functor $\text{Sing} : \text{Top} \to \text{Fun}(\Delta^{\text{op}} \text{Set})$ that assigns

- to a topological space $X$ the simplicial object $\text{Sing}(X) = C^X : \Delta^{\text{op}} \to \text{Set}$
- to a continuous map $f : X \to Y$ the simplicial morphism $\text{Sing}(f) : C^n_n \to C^n_Y$ with component morphisms $\text{Sing}(f)_n : C^n_n \to C^n_Y, \sigma \mapsto f \circ \sigma$.

4. Let $k$ be a commutative ring and $\langle \rangle_k : \text{Set} \to \text{k-Mod}$ the functor that assigns

- to a set $X$ the free $k$-module $\langle X \rangle_k$
- to a map $f : X \to Y$ the induced $k$-module homomorphism $\langle f \rangle_k : \langle X \rangle_k \to \langle Y \rangle_k$.

By composing this functor with the functor $C^X$ from 2. we obtain a simplicial object $C^{X,k} = \langle \rangle_k \circ \text{Hom}(-, X) \circ F : \Delta^{\text{op}} \to \text{k-Mod}$. This simplicial object defines the chain complex of singular homology from Definition 2.1.2

$$C^{X,k}([n+1]) = C_n(X, k) = \langle \text{Hom}_{\text{Top}}(\Delta^n, X) \rangle_k,$$

$$C^{X,k}(\delta^n_i) = d^n_i : C_n(X, k) \to C_{n-1}(X, k), \quad \sigma \mapsto \sigma \circ f^n_i,$$

$$C^{X,k}(\sigma^n_{n+1}) = s^n_i : C_n(X, k) \to C_{n+1}(X, k), \quad \sigma \mapsto \sigma \circ s^n_i.$$ 

5. By combining the functors $\text{Sing} : \text{Top} \to \text{Fun}(\Delta^{\text{op}} \text{Set})$ from 3. and $\langle \rangle_k : \text{Set} \to \text{k-Mod}$, from 4. we obtain a functor $\text{Top} \to \text{Fun}(\Delta^{\text{op}}, \text{k-Mod})$ that assigns to a topological space $X$ the functor $C^{X,k} : \Delta^{\text{op}} \to \text{k-Mod}$ and to a continuous map $f : X \to Y$ the natural transformation $f^{X,k} : C^{X,k} \to C^{Y,k}, \sigma \mapsto f \circ \sigma$. This is the functor that defines the chain complex of singular homology from Definition 2.1.2

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6. By composing $C^{X,k} : \Delta^{+,op} \to k\text{-Mod}$ with $\text{Hom}_k(-, M) : k\text{-Mod} \to k\text{-Mod}$ for a $k$-module $M$, we obtain a functors $\text{Hom}_k(-, M) \circ C^{X,k} : \Delta^{+} \to k\text{-Mod}$ for each topological space $X$ and a functor $\text{Top} \to \text{Fun}(\Delta^{+}, k\text{-Mod})$ that defines the chain complexes of singular cohomology from Definition 2.1.12.

This example motivates the shift in indices in Remark 5.1.5 and the topologist’s version of the simplex category without the object $[0] = \emptyset$. There is no need to view the empty topological space as a standard (-1)-simplex. However, the algebraist’s version of the simplex category has other advantages that will become apparent in the next section.

Simplicial objects in the category Set are called simplicial sets and natural transformations between them are called simplicial maps. They play an important role in modern approaches to topology. In particular, they allow one to systematically construct semisimplicial complexes, for an accessible introduction see [F]. The information in a simplicial set $S : \Delta^{+,op} \to \text{Set}$ is precisely the data needed to construct a semisimplicial complex by gluing $n$-simplexes.

Example 5.1.9: (Geometric realisation)

- The geometric realisation of a simplicial set $S : \Delta^{+,op} \to \text{Set}$ is the topological space $\text{Geom}(S)$ obtained as follows. One equips all sets $S_n = S([n+1])$ with the discrete topology and forms the quotient space

$$\text{Geom}(S) = (\prod_{n \in \mathbb{N}_0} S_n \times \Delta^n) / \sim$$

with the equivalence relation $(S(\alpha)x,p) \sim (x, F_\Delta(\alpha)p)$ for $\alpha \in \text{Hom}_\Delta^{+,op}([m+1],[n+1])$, where $F_\Delta(\alpha) : \Delta^m \to \Delta^n$, $e_k \to e_{\alpha(k)}$ is the affine map from Example 5.1.8.

- The topological space $\text{Geom}(S)$ is a semisimplicial complex (Exercise 61).

The simplicial set $S : \Delta^{+,op} \to \text{Set}$ describes the construction of $\text{Geom}(S)$ by gluing standard simplexes. The elements of the sets $S_n$ label the $n$-simplexes in the semisimplicial complex, and the maps $S(\alpha) : S_n \to S_m$ for a morphism $\alpha : [m+1] \to [n+1]$ in $\Delta^{+}$ specify the gluing pattern, as shown in Figure 1.

- For any simplicial map $\eta : S \to S'$ with component morphisms $\eta_{[n+1]} : S_n \to S'_n$, one obtains a continuous map $\text{Geom}(\eta) : \text{Geom}(S) \to \text{Geom}(S')$ given by

$$\text{Geom}(\eta)[(x,p)] = [(\eta_{[n+1]}(x),p)] \quad \forall (x,p) \in S_n \times \Delta^n.$$

It is a simplicial map between the simplicial complexes $\text{Geom}(S)$ and $\text{Geom}(S')$.

- As these assignments are compatible with the composition of morphisms and unit morphisms in $\text{Fun}(\Delta^{+}, \text{Set})$, they define a functor $\text{Geom} : \text{Fun}(\Delta^{+,op}, \text{Set}) \to \text{Top}$.

It should be noted that (co)simplicial objects and (co)simplicial morphisms are not the only structures investigated in simplicial approaches to homological algebra. There is also a notion of a simplicial homotopy that defines an equivalence relation on the set of simplicial morphisms between fixed simplicial objects $S, S' : \Delta^{+,op} \to \mathcal{A}$ and generalises the notion of chain homotopy. Moreover, there is a concept of simplicial homotopy groups for simplicial sets that satisfy certain additional conditions. These simplicial homotopy groups behave like the homotopy groups of topological spaces and are related to them by the geometric realisation functor. Details on these constructions can be found in [W] Chapter 8.3.
5.2 Dold-Kan correspondence

Our main motivation to consider simplicial objects is that simplicial objects in abelian categories $\mathcal{A}$ define chain complexes in $\mathcal{A}$. As suggested by the simplicial relations (38), the chain complex for a simplicial object $S : \Delta^{+\text{op}} \to \mathcal{A}$ is obtained by taking an alternating sum over the face maps $d^n_i = S(\delta^n_i)$. The degeneracies do not enter the definition of this chain complex, but they also carry relevant information. Their images define a subcomplex with trivial homologies that can be removed to obtain a more efficient description.

To describe this construction in a general abelian category $\mathcal{A}$, we consider for each finite family $(f_i)_{i \in I}$ of morphisms $f_i : X \to Y$ the morphism $f : \coprod_{i \in I} X \to Y$ with $f \circ \iota_i = f_i : X \to Y$ induced by the universal property of the coproduct. We define the object $\bigcap_{i \in I} \ker(f_i) := \ker(f)$ as its kernel object and the object $\bigoplus_{i \in I} \im(f_i) := \im(f)$ as its image object. For $\mathcal{A} = R\text{-Mod}$ these objects are given as the intersection of the submodules $\bigcap_{i \in I} \ker(f_i) \subset X$ and the sum of submodules $\bigoplus_{i \in I} \im(f_i) \subset Y$, as suggested by the notation.

**Proposition 5.2.1:** Let $S : \Delta^{+\text{op}} \to \mathcal{A}$ be a simplicial object in an abelian category $\mathcal{A}$ and

\[
S_n := S([n+1]), \quad d^n_i := S(\delta^n_i) : S_n \to S_{n-1}, \quad s^n_i := S(\sigma^n_{n+1}) : S_n \to S_{n+1}
\]

for $n \in \mathbb{N}_0$ and $0 \leq i \leq n$.

1. The following are positive chain complexes in $\mathcal{A}$:
   - The **Moore complex** $MS_\bullet$ with
     \[
     MS_n = S_n, \quad d_n = \sum_{i=0}^{n} (-1)^i d^n_i : MS_n \to MS_{n-1}.
     \]
   - The **normalised chain complex** $NS_\bullet$ with
     \[
     NS_n = \bigcap_{i=0}^{n-1} \ker(d^n_i) \subset S_n, \quad d_n = (-1)^n d^n_i : NS_n \to NS_{n-1}.
     \]
   - The **degenerate chain complex** $DS_\bullet$ with
     \[
     DS_n = \bigoplus_{i=0}^{n-1} \im(s^n_{i+1}) \subset S_n, \quad d_n = \sum_{i=0}^{n} (-1)^i d^n_i : DS_n \to DS_{n-1}.
     \]

2. They are related by the identity $MS_\bullet = NS_\bullet \amalg DS_\bullet$.

3. The chain complexes $MS_\bullet$ and $NS_\bullet$ are chain homotopy equivalent, and the chain complex $DS_\bullet$ is chain homotopy equivalent to the trivial chain complex. This implies for all $n \in \mathbb{N}_0$

\[
H_n(MS_\bullet) = H_n(NS_\bullet) = H_n(DS_\bullet) = 0.
\]
Proof:
We prove the claims for $A = R$-Mod.

1. That $MS_\bullet$ is a chain complex in $R$-Mod follows directly from the simplicial relations (38). They imply for $n \in \mathbb{N}$

$$d_{n-1} \circ d_n = \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} d_{n-1}^i \circ d_n^i$$
$$= \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_{n-1}^i \circ d_n^i + \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{n-1}^i \circ d_n^i$$
$$= \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j+1} d_{n-1}^i \circ d_n^i + \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{n-1}^i \circ d_n^i = 0.$$  

2. We show that $NS_\bullet$ and $DS_\bullet$ are chain complexes in $R$-Mod. From the simplicial relations (38) we have $d_{n-1}^i \circ d_n^i(x) = d_{n-1}^i \circ d_n^i(x) = 0$ for all $0 \leq i < n$ and $x \in NS_n$, and this implies $d_n(\NS_n) \subset \NS_{n-1}$. This shows that $\NS_\bullet$ is a chain complex. Similarly, we obtain for all $0 \leq i \leq n-1$ and $x \in S_{n-1}$

$$d_n(s_{n-1}^i(x)) = \sum_{i=0}^{n} (-1)^{i} d_n^i \circ d_n^i s_{n-1}^i(x)$$
$$= \sum_{i=0}^{n} (-1)^{i} s_{n-2}^i \circ d_n^i s_{n-1}^i(x) + (-1)^i x + \sum_{n} (-1)^{i} s_{n-2}^i \circ d_n^i(x)$$
$$= \sum_{i=0}^{n-1} (-1)^{i} s_{n-2}^i \circ d_n^i s_{n-1}^i(x) + \sum_{n} (-1)^{i} s_{n-2}^i \circ d_n^i(x) = \sum_{i=0}^{n-2} \im(s_{n-2}^i).$$

This shows that $d_n(DS_n) \subset DS_{n-1}$ and $DS_\bullet$ is a chain complex in $R$-Mod.

3. We show that $MS_n = NS_n \oplus DS_n$ for all $n \in \mathbb{N}_0$. To see that $DS_n \cap NS_n = \{0\}$, let $0 \neq x \in DS_n \cap NS_n$ and set $j = \max\{k \in \{0,...,n-1\} \mid x \in d_{n}^{-1}(\im(s_{n-1}))\}$. Then we have $x = \sum_{i=j}^{n-1} s_{n-1}^i(x_i) \neq 0$, $0 \leq j \leq n-1$, and $x \in NS_n$ implies

$$0 = d_n^i(x) = \sum_{i=j}^{n-1} d_n^i \circ s_{n-1}^i(x_i) = x_j + \sum_{i=j+1}^{n-1} s_{n-2}^i \circ d_n^i(x_i).$$

If $j = n-1$, it follows that $x_{n-1} = 0$ and $x = s_{n-1}^{n-1}(x_{n-1}) = 0$. If $j < n-1$, we have

$$s_{n-1}^j(x_j) = -\sum_{i=j+1}^{n-1} s_{n-2}^i \circ d_n^i(x_i) = -\sum_{i=j+1}^{n-1} s_{n-2}^i \circ d_n^i(x_i) \in d_{n-1}^{-1}(\im(s_{n-2}^i))$$

and $x \in d_{n}^{-1}(\im(s_{n-1}^i))$, in contradiction to the maximality of $j$. Hence, $NS_n \cap DS_n = \{0\}$.

To show that $MS_n = NS_n + DS_n$, let $x \in MS_n$ and $j_x = \min\{k \in \{0,...,n\} \mid d_n^k(x) \neq 0\}$. If $j_x = n$, then $x \in NS_n$. If $j_x = j < n$, we have $x = x_1 + \sum_{i=0}^{j} y_i \in NS_n + DS_n$.

$$d_n^k(x_1) = d_n^k(x - s_{n-1}^j \circ d_n^j(x)) = d_n^k(x) - d_n^k(x) = 0$$

for $k = \{0,...,j-1\}$. We thus decomposed $x$ as $x = x_1 + y_1$ with $y_1 \in DS_n$ and an element $x_1 \in MS_n$ with $j_x = j_x + 1$. By iterating this procedure, we obtain elements $y_1,...,y_k \in DS_n$ and $x_1,...,x_k \in MS_n$ with $x_i = x_{i+1} + y_{i+1}$ and $x_k \in NS_n$. This shows that $x = x_k + \sum_{i=1}^{k} y_i \in NS_n + DS_n$ and hence $MS_n = NS_n + DS_n$.

4. It remains to show that the chain complex $NS_\bullet$ is chain homotopy equivalent to $MS_\bullet$. For this, we consider for $j \in \mathbb{N}_0$ the subcomplexes $NS_j = (\NS_n)_{n \in \mathbb{N}_0} \subset MS_\bullet$ in $R$-Mod with

$$NS_j = \begin{cases} \cap_{i=0}^{j} \ker(d_n^i) & n \geq j + 2 \\ \NS_n & 0 \leq n \leq j + 1 \end{cases}$$
and set $NS^1_\bullet = MS_\bullet$. That $NS^1_\bullet$ is indeed a chain complex follows because $d_n(NS^1_n) \subset NS^1_{n-1}$ for $n \leq j + 1$ by 2., and the simplicial relations imply for $j + 2 \leq n$, $x \in NS^1_n$ and $0 \leq i \leq j$
\v
$d_{n-1} \circ d_n(x) = \sum_{k=0}^n (-1)^k d_{n-1} \circ d^n_k(x) = \sum_{k=j+1}^n (-1)^k d_{n-1} \circ d^n_k(x) = \sum_{k=j+1}^n (-1)^k d^{k-1}_{n-1} \circ d^n_k(x) = 0$.

This shows that $d_n(NS^1_n) \subset NS^1_{n-1}$ for all $n, j \in \mathbb{N}_0$ and $NS^1_\bullet \subset MS_\bullet$ is a subcomplex.

As $NS^j_{n+1} \subset NS^j_n$ for all $j \geq -1$, the inclusion maps $\iota^j_n : NS^j_{n+1} \to NS^j_n$ define chain maps $\iota^j_\bullet : NS^j_\bullet \to NS^j_\bullet$. We show that $\iota^j_\bullet : NS^j_\bullet \to NS^j_\bullet$ is a chain homotopy equivalence by constructing chain maps $f^j_\bullet : NS^j_\bullet \to NS^j_{n+1}$ with $f^j_n \circ \iota^j_n = 1_{NS^j_{n+1}}$ and a chain homotopies $t^j_\bullet : 1_{NS^j_\bullet} \Rightarrow \iota^j_\bullet \circ f^j_\bullet$. For this, we consider for $j \geq -1$ and $n \in \mathbb{N}_0$ the $R$-linear maps
\v
$f^j_n : NS^j_n \to NS^j_{n+1}, \quad x \mapsto \begin{cases} x - s^{j+1}_n \circ d^{j+1}_n(x) & n \geq j + 2 \\
 x & n \leq j + 1 \end{cases}$

that take values in $NS^j_{n+1}$ by \([10]\) They define chain maps $f^j_\bullet : NS^j_\bullet \to NS^j_{n+1}$, since
\v
d_n \circ f^j_n(x) = (-1)^n d^n_n(x) = (-1)^n f^j_{n-1} \circ d^n_n(x) = f^j_{n-1} \circ d_n(x) \quad n \leq j + 1
\v
d_n \circ f^j_n(x) = (-1)^n d^n_n(x) - (-1)^n d^n_n \circ s^{j+1}_n \circ d^{j+1}_n(x) = d_n(x) = f^j_{n-1} \circ d_n(x) \quad n = j + 2
\v
d_n \circ f^j_n(x) = \sum_{k=j+2}^{n+1} (-1)^{k} d^k_n(x) = \sum_{k=j+2}^{n+1} (-1)^{k} d^k_n \circ s^{j+1}_n \circ d^{j+1}_n(x) \quad n > j + 2
\v
and satisfy $1_{NS^j_{n+1}} = f^j_n \circ \iota^j_n : NS^j_{n+1} \to NS^j_{n+1}$ by definition. For $j \geq -1$ the $R$-linear maps
\v
t^j_n : NS^j_n \to NS^j_{n+1}, \quad x \mapsto \begin{cases} (-1)^{j+1} s^{j+1}_n(x) & n \geq j + 1 \\
 0 & n < j + 1 \end{cases}$

define chain homotopies $t^j_\bullet : 1_{NS^j_\bullet} \Rightarrow \iota^j_\bullet \circ f^j_\bullet$, since one has
\v
d_{n+1} \circ t^j_{n-1} \circ d_n(x) = 0 = x - t^j_n \circ f^j_n(x) \quad n < j + 1
\v
d_{n+1} \circ t^j_{n-1} \circ d_n(x) = (-1)^{n+j+1} d^j_{n+1} \circ s^{j+1}_n \circ d^{j+1}_n(x) + (-1)^{n+j} d^{j+1}_{n+1} \circ s^{j+1}_n(x) \quad n = j + 1
\v
d_{n+1} \circ t^j_{n-1} \circ d_n(x)
\v\v
= \sum_{k=j+2}^{n+1} (-1)^{j+k+1} d^k_{n+1} \circ s^{j+1}_n(x) + \sum_{k=j+1}^{n+1} (-1)^{k+j+1} s^{j+1}_n \circ d^{j+1}_n(x) \quad n \geq j + 2
\v
We now show that the inclusion map $i^j_\bullet : NS^j_\bullet \to MS^j_\bullet$ is a chain homotopy equivalence. For this, we note that $i^j_n = i^{j-1}_n \circ \cdots \circ i^{j-2}_n : NS^1_n \to MS^1_n$ and consider for $n \in \mathbb{N}_0$ the $R$-linear maps
\v
g_n = f^{n-2}_n \circ f^{n-3}_n \circ \cdots \circ f^0_n \circ f^{-1}_n : S_n \to NS_n
\v
h_n = \sum_{k=-2}^{n-2} i^{n-1}_k \circ \cdots \circ i^{k+1}_n \circ f^k_n \circ \cdots \circ f^{-1}_n : S_n \to S_{n+1}.
\v
The $R$-linear maps $g_n$ define a chain map $g_\bullet : MS^1_\bullet \to NS^1_\bullet$ because the maps $f^j_\bullet : N^j_\bullet \to N^j_{n+1}$ are chain maps for all $j \geq -1$:
\v
d_n \circ g_n = d_n \circ f^{n-2}_n \circ f^{n-3}_n \circ \cdots \circ f^0_n \circ f^{-1}_n = f^{n-2}_n \circ d_n \circ f^{n-3}_n \circ \cdots \circ f^0_n \circ f^{-1}_n
\v= \cdots = f^{n-2}_n \circ f^{n-3}_n \circ \cdots \circ f^0_n \circ d_n \circ f^{-1}_n = f^{n-2}_n \circ f^{n-3}_n \circ \cdots \circ f^0_n \circ f^{-1}_n \circ d_n = g_{n-1} \circ d_n.
They also satisfy $1_{NS} = g_\bullet \circ t_\bullet : NS \to NS$ since $1_{NS}^j = f^j_0 \circ t^j_0$ for all $j \geq -1$:

$$g_n \circ t_n = f^{n-2}_0 \circ f^{n-3}_0 \circ \ldots \circ f^0_0 \circ f^{-1}_n \circ \ldots \circ f^{-1}_n = f^{n-2}_0 \circ f^{n-3}_0 \circ \ldots \circ f^0_0 \circ t^0_0 \circ \ldots \circ t^{-2}_n = \ldots = f^{n-2}_0 \circ t^{n-2}_n = \text{id}_{NS}.$$

The maps $h_n : S_n \to S_{n+1}$ define a chain homotopy $h_\bullet : 1_{MS} \Rightarrow t_\bullet \circ g_\bullet$ since $t^j_0 \circ f^j_0$ is a chain homotopy for all $j \geq -1$:

$$d_{n+1} \circ h_n + h_{n-1} \circ d_n = \sum_{k=-2}^{n-2} d_{n+1} \circ t_{n+1}^k \circ \ldots \circ t_{n+1}^k \circ f^{k+1}_n \circ \ldots \circ f^{-1}_n + \sum_{k=-2}^{n-3} t_{n+1}^k \circ \ldots \circ t_{n+1}^k \circ f^{k+1}_n \circ \ldots \circ f^{-1}_n \circ d_n,$$

and

$$= \ldots$$

This shows that $g_\bullet : MS \to NS$ and $t_\bullet : NS \to MS$ are chain homotopy equivalences. By Proposition 3.3.4 this implies $H_n(MS_n) = H_n(NS_n)$ for all $n \in \mathbb{N}_0$. As $MS_n = NS_n \amalg DS_n$, it follows that $H_n(MS_n) = H_n(NS_n) \amalg H_n(DS_n)$ and $H_n(DS_n) = 0$. □

The fact that the degenerate carrier chain complex $DS_n$ is chain homotopic to the trivial chain complex has a geometrical interpretation in simplicial and singular homology. It states that degenerate $n$-simplexes that are of the form $\sigma = \tau \circ s^{n-1}_i : \Delta^n \to X$ with an $(n-1)$-simplex $\tau : \Delta^{n-1} \to X$ do not contribute to the homologies. This motivates the restriction to $n$-simplexes $\sigma : \Delta^n \to X$ with $\sigma|_{\Delta^n} : \Delta^n \to X$ injective in the definition of a (semi)simplicial complex.

The assignments of chain complexes in an abelian category $A$ to simplicial objects $S : \Delta^{+op} \to A$ from Proposition 5.2.1 can also be extended to simplicial morphisms. As expected, one finds that every simplicial morphism $\eta : S \to T$ defines a chain map $M\eta_\bullet : MS_\bullet \to MT_\bullet$ between the associated Moore complexes. This chain map restricts to chain maps between the degenerate complex theorems and the normalised chain complexes. As these assignments are compatible with the composition of simplicial morphisms and the unit morphisms, they define functors from the category $\text{Fun}(\Delta^{+op}, A)$ of simplicial objects in $A$ to the category $\text{Ch}_{A \geq 0}$ of positive chain complexes in $A$.

**Proposition 5.2.2:** Let $S(A) = \text{Fun}(\Delta^{+op}, A)$ be the category of simplicial objects in an abelian category $A$. Then the following are functors:

1. The Moore complex functor $M : S(A) \to \text{Ch}_{A \geq 0}$ that sends
   - a simplicial object $S : \Delta^{+op} \to A$ to the chain complex $MS_\bullet$,
   - a simplicial morphism $\eta : S \to T$ to the chain map $M\eta_\bullet : MS_\bullet \to MT_\bullet$ with $M\eta_n = \eta_{[n+1]} : S_n \to T_n$.

2. The normalised chain complex functor $N : S(A) \to \text{Ch}_{A \geq 0}$ that sends
   - a simplicial object $S : \Delta^{+op} \to A$ to the chain complex $NS_\bullet$
   - a simplicial morphism $\eta : S \to T$ to the chain map $N\eta_\bullet : NS_\bullet \to NT_\bullet$ induced by $M\eta_\bullet : MS_\bullet \to MT_\bullet$. 

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Proof:
1. Let $S, T : \Delta^+ \to \mathcal{A}$ be simplicial objects in $\mathcal{A}$ with associated Moore complexes $MS_\bullet$ and $MT_\bullet$ in $\mathcal{A}$ and $\eta : S \to T$ a natural transformation. We show that its component morphisms $M\eta_n = \eta_{[n+1]} : S_n \to T_n$ define a chain map $M\eta_\bullet : MS_\bullet \to MT_\bullet$. This follows from the naturality of $\eta$, which implies that for all morphisms $\alpha : [m] \to [n]$ in $\Delta^+$ one has

$$M\eta_{m-1} \circ S(\alpha) = \eta_{[m]} \circ S(\alpha) \xrightarrow{\text{nat}} T(\alpha) \circ \eta_{[n]} = T(\alpha) \circ M\eta_{n-1}. \quad (41)$$

In particular, this holds for the morphisms $\delta^i_k : [n] \to [n+1]$, and we obtain

$$d^T_n \circ M\eta_n = \Sigma^i_{i=0}(-1)^iT(\delta^i_n) \circ \eta_{[n+1]} \xrightarrow{\text{nat}} \Sigma^i_{i=0}(-1)^i\eta_{[n]} \circ S(\delta^i_n) = M\eta_{n-1} \circ d^S_n.$$

2. We show that the assignments of chain complexes to simplicial objects and chain maps to natural transformations are compatible with the composition of morphisms and the identity morphisms. For $\eta = \text{id}_S : S \to S$ we have $(M\text{id}_S)_n = 1_{S([n+1])} = 1_{S_n} : MS_n \to MS_n$, and we obtain the identity chain map $\text{id}_{MS_\bullet} : MS_\bullet \to MS_\bullet$. For simplicial objects $R, S, T : \Delta^{+op} \to \mathcal{A}$ and natural transformations $\eta : R \to S$ and $\kappa : S \to T$, the composite natural transformation $\kappa\eta : R \to T$ has component morphisms $(\kappa\eta)_{[n]} = \kappa_{[n]} \circ \eta_{[n]}$. This shows that the morphisms $M\kappa\eta_n : R_n \to T_n$ are given by $M\kappa\eta_n = (\kappa_{[n+1]} \circ \eta_{[n+1]} = M\kappa_n \circ M\eta_n$, and the chain maps satisfy $M\kappa_\bullet \circ M\eta_\bullet$.

3. To prove the statement for the normalised chain complex functor, it is sufficient to show that for every natural transformation $\eta : S \to T$ the chain map $M\eta_\bullet : MS_\bullet \to MT_\bullet$ between the associated Moore complexes restricts to a chain map $N\eta_\bullet : NS_\bullet \to NT_\bullet$. By applying (41) to the morphisms $\delta^i_n : [n] \to [n+1]$, we obtain $d^T_n \circ M\eta_n(x) = M\eta_{n-1} \circ d^S_n(x)$ for all $i \in \{0, \ldots, n\}$. As the chain complex $NS_\bullet$ is given by $NS_n = \cap^{n+1}_{i=0}\ker(d^S_i)$, this implies $M\eta_n(NS_n) \subset NT_n$, and hence $M\eta_\bullet$ induces a chain map $N\eta_\bullet : NS_\bullet \to NT_\bullet$. $\square$

The Moore complex functor explains the similarities between the (co)chain complexes in Section 2. These examples are obtained either from the simplicial object for Hochschild homology in Example 5.1.7 or from the simplicial object for singular homology in Example 5.1.8 by applying the Moore complex functor. Although this is a nice explanation for the specific form of the chain complexes in Section 2, it is not sufficient in itself. One would like to know if all positive chain complexes in an abelian category $\mathcal{A}$ are obtained from simplicial objects in $\mathcal{A}$, possibly up to chain homotopy equivalence. Surprisingly, the answer to this question is yes.

Theorem 5.2.3: (Dold-Kan correspondence)
For any abelian category $\mathcal{A}$ the normalised chain complex functor $N : \text{Fun}(\Delta^{+op}, \mathcal{A}) \to \text{Ch}_{\mathcal{A} \geq 0}$ is an equivalence of categories between the category $\text{Fun}(\Delta^{+op}, \mathcal{A})$ of simplicial objects in $\mathcal{A}$ and the category $\text{Ch}_{\mathcal{A} \geq 0}$ of positive chain complexes in $\mathcal{A}$.

Proof:
Instead of showing that the normalised chain complex functor $N : \text{Fun}(\Delta^{+op}, \mathcal{A}) \to \text{Ch}_{\mathcal{A} \geq 0}$ from Propositions 5.2.1 and 5.2.2 is an equivalence of categories, we show this for the functor $N' : \text{Fun}(\Delta^{+op}, \mathcal{A}) \to \text{Ch}_{\mathcal{A} \geq 0}$ that associates to a simplicial object $S : \Delta^{+op} \to \mathcal{A}$ the chain complex $N'S_\bullet$ with $N'S_n = NS_n$ and $d'_n = (-1)^n d_n : NS_n \to NS_{n-1}$. As the morphisms $\tau^n_n = (-1)^n 1_{NS_n} : NS_n \to N'S_n$ induce isomorphisms of chain complexes $\tau^n_\bullet : NS_\bullet \to N'S_\bullet$, with $\tau^n_T \circ \eta_{[n+1]} = \eta_{[n+1]} \circ \tau^n_n$ for all natural transformations $\eta : S \to T$, the chain maps $\tau^n : N(S) \to N'(S)$ define a natural isomorphism $\tau : N \to N'$ and hence $N$ is an equivalence of categories if and only if $N'$ is.
To show that \( N' : \text{Fun}(\Delta^{+\text{op}}, A) \to \text{Ch}_{\geq 0} \) is an equivalence of categories, we construct a functor \( K : \text{Ch}_{\geq 0} \to \text{Fun}(\Delta^{+\text{op}}, A) \) such that \( N'K = \text{id}_{\text{Ch}_{\geq 0}} \) and \( KN' \) is naturally isomorphic to the identity functor \( \text{id}_{\text{Fun}(\Delta^{+\text{op}}, A)} \).

**Step 1:** We define \( K : \text{Ch}_{\geq 0} \to \text{Fun}(\Delta^{+\text{op}}, A) \) on the objects of \( \text{Ch}_{\geq 0} \):

To a positive chain complex \( C_\bullet \) in \( A \) we assign the functor \( K^C = K(C_\bullet) : \Delta^{+\text{op}} \to A \) given by

\[
K^C([n+1]) = K^C_n = \coprod_{0 \leq p \leq n, \sigma : [n+1] \to [p+1]} C_p,
\]

where the coproduct runs over all monotonic surjections \( \sigma : [n+1] \to [p+1] \) with \( 0 \leq p \leq n \).

For a morphism \( \alpha : [m+1] \to [n+1] \) in \( \Delta^+ \) we define \( K^C(\alpha) : K^C_n \to K^C_m \) by the following procedure. We denote by \( \iota_\sigma : C_p \to K^C_n \) the inclusion morphism for the factor associated with \( \sigma \) in the coproduct (42) and define \( K^C(\alpha) \) as the unique morphism with \( K^C(\alpha) \circ \iota_\sigma = \iota_\tau \circ K^C(\alpha)_\tau^\sigma \) for all \( 0 \leq p \leq n, 0 \leq q \leq m \), monotonic surjections \( \sigma : [n+1] \to [p+1] \), \( \tau : [m+1] \to [q+1] \), and \( \alpha \).

\[
\begin{array}{c}
K^C_n \xrightarrow{K^C(\alpha)_\sigma^\tau} K^C_m \\
\downarrow \iota_\sigma \quad \downarrow \iota_\tau \\
C_p \xrightarrow{K^C(\alpha)_\tau^\sigma} C_q.
\end{array}
\]

The morphisms \( K^C(\alpha)_\tau^\sigma : C_p \to C_q \) are defined as follows. The factorisation of morphisms in \( \Delta^+ \) from Proposition 5.1.2 implies that for every monotonic map \( \alpha : [m+1] \to [n+1] \) and every monotonic surjection \( \sigma : [n+1] \to [p+1] \) in \( \Delta^+ \), there is a unique \( q \leq \min(m, p) \), a unique monotonic surjection \( \sigma_\alpha : [m+1] \to [q+1] \) and monotonic injection \( \alpha_\sigma : [q+1] \to [p+1] \) with \( \sigma \circ \alpha = \alpha_\sigma \circ \sigma_\alpha \). We set

\[
K^C(\alpha)_\sigma^\tau = \begin{cases} 
\delta_{\sigma_\alpha}^\tau c_p : C_p \to C_q & q = p \\
\delta_{\alpha_\sigma}^\tau d_p : C_p \to C_{p-1} & q = p - 1 \\
0 : C_p \to C_q & q < p - 1.
\end{cases}
\]

If \( \alpha : [m+1] \to [n+1] \) is a monotonic surjection, we have \( \sigma_\alpha = \sigma \circ \alpha : [m+1] \to [p+1] = [q+1] \) and \( \alpha_\sigma = 1_{[p+1]} \), which implies \( K^C(\alpha)^\tau_\sigma = \delta_{\alpha_\sigma}^\tau 1_c_p \). In this case, \( K^C(\alpha) : K^C_n \to K^C_m \) is the morphism in \( A \) that sends the copy of \( C_p \) in \( K^C_m \) associated with \( \sigma : [n+1] \to [p+1] \) to the copy of \( C_p \) in \( K^C_n \) associated with \( \sigma_\alpha \).

In particular, \( K^C(1_{[n+1]}) = 1_{K^C_n} : K^C_n \to K^C_n \).

To show that this defines a functor \( K^C : \Delta^{+\text{op}} \to A \), it remains to show compatibility with the composition of morphisms in \( \Delta^+ \). For monotonic maps \( \beta : [l+1] \to [m+1] \) and \( \alpha : [m+1] \to [n+1] \), we obtain numbers \( 0 \leq q \leq p \), \( 0 \leq r \leq q \), monotonic surjections \( \sigma_\alpha : [m+1] \to [q+1] \) and \( (\sigma_\alpha)_\beta : [r+1] \to [q+1] \) and monotonic injections \( \alpha_\sigma : [q+1] \to [p+1] \), \( \beta_{\sigma_\alpha} : [r+1] \to [q+1] \) such that the following diagram commutes

\[
\begin{array}{c}
\xymatrix{
[l+1] \ar[r]_{\beta} & [m+1] \ar[d]_{\sigma_\alpha} & [n+1] \\
[r+1] \ar[r]_{\beta_{\sigma_\alpha}} & [q+1] \ar[u]^{\sigma_\alpha} & [p+1].
}\end{array}
\]
This proves (45) and shows that the top and bottom quadrilaterals commute by definition of $K$ because the right-hand side vanishes by (44), and the only non-trivial contribution to the left hand side is proportional to $d_{p-1} \circ d_p$, which vanishes as well. In the other cases, we obtain from (44)

$$
K^C(\alpha \circ \beta)_{\mu}^\rho = \delta_{\mu_{\alpha \circ \beta}}^\rho \cdot 1_{C_p} = \delta_{(\mu_{\alpha})_{\beta}}^\rho \cdot 1_{C_p} = \Sigma_{\nu} \delta_{\nu_{\beta}}^{\rho_{\nu}} \delta_{\mu_{\alpha}}^{\nu_{\mu}} \cdot 1_{p} = \Sigma_{\nu} K^C(\beta)_{\nu}^\rho \circ K^C(\alpha)_{\mu}^\nu \quad r = p
$$

$$
K^C(\alpha \circ \beta)_{\mu}^\rho = \delta_{\mu_{\alpha \circ \beta}}^{\rho_{\alpha \circ \beta}} \cdot d_p = \delta_{(\mu_{\alpha})_{\beta}}^{\rho_{\mu}} \cdot d_p = \Sigma_{\nu} \delta_{\nu_{\beta}}^{\rho_{\nu}} \delta_{\mu_{\alpha}}^{\nu_{\mu}} \delta_{\delta_{p_{-1}}}^{\nu_{\delta_{p_{-1}}}} \cdot d_p = \Sigma_{\nu} K^C(\beta)_{\nu}^\rho \circ K^C(\alpha)_{\mu}^\nu \quad r = p - 1.
$$

This proves (45) and shows that $K^C : \Delta^{+\text{op}} \to \mathcal{A}$ is a simplicial object.

**step 2:** We define $K : \text{Ch}_{\mathcal{A} \geq 0} \to \text{Fun}(\Delta^{+\text{op}}, \mathcal{A})$ on the morphisms of $\text{Ch}_{\mathcal{A} \geq 0}$:

Let $C_\bullet$ and $C'_\bullet$ be positive chain complexes in $\mathcal{A}$, let $K^C, K^{C'} : \Delta^{+\text{op}} \to \mathcal{A}$ be the associated simplicial objects from step 1, and let $f_\bullet : C_\bullet \to C'_\bullet$ be a chain map. We define a natural transformation $K^f = K(f_\bullet) : K^C \to K^{C'}$ by its component morphisms $K^f_{[n+1]} : K^C_n \to K^{C'}_n$.

We define them via the universal property of the coproduct in (42) as the unique morphism with $K^f_{[n+1]} \circ \iota_\sigma = \iota'_\sigma \circ f_p$ for all $0 \leq p \leq n$ and monotonic surjections $\sigma : [n+1] \to [p+1]$. This implies

$$
K^{C'}(\alpha) \circ K^f_{[n+1]} \circ \iota_\sigma = K^{C'}(\alpha) \circ \iota'_\sigma \circ f_p = \iota'_\tau \circ K^{C'}(\alpha)_{\sigma}^\tau \circ f_p = \iota'_\tau \circ f_q \circ K^C(\alpha)_{\sigma}^\tau = K^f_{[m+1]} \circ \iota_\tau \circ K^C(\alpha)_{\sigma}^\tau.
$$

As $\iota_\sigma$ is a monomorphism, it follows that the rectangle in the middle commutes as well.
• step 3: We show that $K : \text{Ch}_{A \geq 0} \to \text{Fun}(\Delta^{+\text{op}}, A)$ is a functor:

Setting $K^C = K^{C'}$, $C_p = C'_p$, $\iota_p = \iota'_p$, $f_p = 1_{C_p}$ and $K^{f}_{[n+1]} = 1_{K^C_p}$ in (46), we find that the diagram commutes for all $p \leq n \in \mathbb{N}_0$ and monotonic surjections $\sigma : [n+1] \to [p+1]$. This shows that $K(1_{C_p}) = \text{id}_{K^C} : K^C \to K^C$ is the identity natural transformation.

If $C_\bullet$, $C'_\bullet$, $C''_\bullet$ are positive chain complexes in $A$ and $f_\bullet : C_\bullet \to C'_\bullet$, $f'_\bullet : C'_\bullet \to C''_\bullet$ chain maps, then composing the commutative diagrams (46) for $f_\bullet$ and $f'_\bullet$ yields the commuting diagram

\[
\begin{array}{c}
K^C_n \\
\downarrow K^{f}_{[n+1]} \downarrow \quad \downarrow K^{f'}_{[n+1]} \downarrow \\
K'^C_n \\
\downarrow \sigma \downarrow \quad \downarrow \iota'_p \downarrow \\
K''_n \\
\end{array}
\]

\[
\begin{array}{c}
C_p \\
\downarrow f_p \\
C'_p \\
\downarrow f'_p \\
C''_p \\
\end{array}
\]

\[
\begin{array}{c}
f'_p f_p \\
\end{array}
\]

This shows that $K'^{f'}_{[n+1]} = K'^{f'}_{[n+1]} \circ K^{f}_{[n+1]}$ for all $n \in \mathbb{N}_0$ and $K((f'_\bullet) \circ f_\bullet) = K^{f'} \circ K^f = K(f'_\bullet) \circ K(f_\bullet)$.

• step 4: We show that $N'K = \text{id}_{\text{Ch}_{A \geq 0}}$:

The functor $N'K : \text{Ch}_{A \geq 0} \to \text{Ch}_{A \geq 0}$ sends a chain complex $C_\bullet$ in $A$ to the chain complex $N'(K^C)$ for the simplicial object $K^C : \Delta^{+\text{op}} \to A$ from step 1 and a chain map $f_\bullet : C_\bullet \to C'_\bullet$ to the chain map $N'(K^f)$ for the simplicial morphism $K^f : K^C \to K^{C'}$ from step 2, we show that $N'(K^C) = C_\bullet$ and $N'(K^f) = 1_{C_\bullet}$.

By the discussion after (44), the morphism $K^C(\alpha) : K^C_n \to K^C_m$ assigned to a monotonic surjection $\alpha : [m+1] \to [n+1]$ sends the component $C_p$ in $K^C_n$ in (42) associated with a monotonic surjection $\sigma : [n+1] \to [p+1]$ to the component of $C_p$ in $K^C_m$ associated with $\sigma \circ \alpha : [m+1] \to [p+1]$.

As every monotonic surjection $\alpha : [m+1] \to [n+1]$ is a unique composite of the monotonic surjections $\delta_k^j : [k+1] \to [k]$, all components of $C_p$ in (42) except the one for $n = p$ and $\sigma = 1_{[n+1]}$ are in the images of the maps $s_{n-1}^j = K^C(\delta_k^j)$. This shows that the $R$-modules of the degenerate chain complex and the chain complex $N'K$ are given by

\[DK_n^C = +\sum_{j=0}^{n-1} \text{im}(s_{n-1}^j) = +\sum_{j=0}^{n} K^C(\sigma_j^j) = \prod_{0 \leq p < n, \sigma : [n+1] \to [p+1]} C_p \quad N'K_n^C = C_n.\]

The morphisms $d_n = K^C(\delta_n^n) : C_n = N'K_n^C \to N'K_{n-1}^C = C_{n-1}$ can be computed from (44). In this case, we have the factorisation $1_{[n+1]} \circ \delta_n^n = \delta_{n+1}^n \circ 1_{[n]}$ with $p = q+1 = n+1$, and (44) yields $K^C(\delta_n^n) = d_n : C_n \to C_{n-1}$. Hence, we have $N'K(C_\bullet) = N'(K^C) = C_\bullet$ for all positive chain complexes $C_\bullet$ in $A$.

For every chain map $f_\bullet : C_\bullet \to C'_\bullet$, we obtain a natural transformation $K^f = K(f_\bullet) : K^C \to K^{C'}$ given by diagram (46). As the only summand for $\sigma = 1_{[n+1]}$ in (42) contributes to $N'K_n^C$, we have $N'(K(f_\bullet)) = N'(K^f) = f_n : C_n \to C'_n$ and $N'K = \text{id}_{\text{Ch}_{A \geq 0}}$. 

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• **step 5:** We construct a natural transformation $\eta : K N' \to \id_{\Fun(\Delta^{op}, A)}$:

A natural transformation $\eta : K N' \to \id_{\Fun(\Delta^{op}, A)}$ assigns to simplicial objects $S : \Delta^{op} \to A$ natural transformations $\eta^S : K N'(S) \to S$, such that $\eta^S \circ K N'(\tau) = \tau \circ \eta^T$ for every natural transformation $\tau : S \to T$. Each natural transformation $\eta^S : K N'(S) \to S$ is determined by its component morphisms $\eta^S_{[n+1]} : K^{N^S}_{n} S_n$. From (42) we have

$$K^{N^S}_{n} S_n = K N'(S)([n + 1]) = \prod_{0 \leq p \leq n, \sigma : [n+1] \to [p+1]} NS_p.$$  \hspace{1cm} (47)

We denote by $i_p : NS_p \to S_p$ the inclusion map and define the morphisms $\eta^S_{[n+1]} : K^{N^S}_{n} S_n$ as the unique morphism induced by the universal property of the coproduct and the morphisms $S(\sigma) \circ i_p : NS_p \to S_n$ for each monotonic surjection $\sigma : [n + 1] \to [p + 1]$. Each natural transformation $\eta^S : K N'(S) \to S$ assigned to each simplicial object $S : \Delta^{op} \to A$ is sufficient to prove that the expressions in (49) are equal for monotonic surjections $\sigma : [m + 1] \to [q + 1]$ and monotonic maps $\alpha : [m + 1] \to [n + 1]$. The two sides of this equation are given by

$$\eta^S_{[m+1]} \circ K^{N^S}_{n} (\alpha) \circ i_\sigma = S(\alpha) \circ \eta^S_{[n+1]} \circ i_\sigma, \hspace{1cm} \hspace{1cm} (49)$$

where $\sigma \circ \alpha = \alpha_\sigma \circ \sigma_\alpha$ with a monotonic injection $\alpha_\sigma : [q + 1] \to [p + 1]$, a monotonic surjection $\sigma_\alpha : [m + 1] \to [q + 1]$ and $i_p : NS_p \to S_p$ and $i_q : NS_q \to S_q$ denote the inclusions. As we have $NS_n = \cap_{j=0}^{n-1} ker(\partial_n^j) = \cap_{j=0}^{n-1} ker(S(\partial_n^j))$, $S_n = NS_n \Pi DS_n$ and $\alpha : [m + 1] \to [n + 1]$ can be factorised as in Proposition 5.1.2, it is sufficient to prove that the expressions in (49) are equal for monotonic surjections $\alpha$ and the maps $\delta_n^j : [n] \to [n + 1]$. In the first case, we have $p = q$, $\sigma_\alpha = \sigma \circ \alpha$ and $K^{N^S}(\alpha)_{\sigma_\alpha} = 1_{NS_p}$ and the two sides of the equation agree. In the second case we have $p = q + 1$, $\alpha_\sigma = \delta_n^p \circ \sigma_\alpha = \sigma_\alpha \circ \delta_n^p$ and $i_q \circ K^{N^S}(\delta_n^p)_{\sigma_\alpha} = d_n^{\sigma_\alpha} = S(\delta_n^p) \circ i_p$. Hence we assigned to each simplicial object $S : \Delta^{op} \to A$ a natural transformation $\eta^S : K N'(S) \to S$.

To prove that this defines a natural transformation $\eta : K N \to \id_{\Fun(\Delta^{op}, A)}$ we show that for each natural transformation $\tau : S \to T$ we have $\eta^T \circ K N(\tau) = \tau \circ \eta^S$ or, equivalently, $\eta^T_{[n+1]} \circ K N(\tau_{[n+1]}) = \tau_{[n+1]} \circ \eta^S_{[n+1]}$ for all $n \in N_0$. With the definition (48) of $\eta^S$ and (46) of $K N(\tau) = K N^T(\tau)$ we compute

$$\eta^T_{[n+1]} \circ K N(\tau_{[n+1]}) \circ i_\sigma = \eta^T_{[n+1]} \circ i_\sigma \circ \eta^S_{[n+1]} \circ N(\tau_{[n+1]}) = T(\sigma) \circ i_p^T \circ N(\tau_{[n+1]}) = T(\sigma) \circ \tau_{[n+1]} \circ i_p^S,$$

where $i_p^S : NS_p \to S_p = T_p$ and $i_p^T : NT_p \to T_p$ denote the inclusion morphisms for the subcomplexes $NS_\bullet \subset MS_\bullet$ and $NT_\bullet \subset MT_\bullet$. With the universal property of the coproduct, this implies $\eta^T_{[n+1]} \circ K N(\tau_{[n+1]}) = \tau_{[n+1]} \circ \eta^S_{[n+1]}$ for all $n \in N_0$ and $\eta^T \circ K N(\tau) = \tau \circ \eta^S$. This shows that the natural isomorphisms $\eta^S : K N'(S) \to S$ define a natural transformation $\eta : K N' \to S$.  

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We prove that the morphisms \( \eta^S_{n+1} : K_n^\mathrm{NS} \rightarrow S_n \) are isomorphisms for all simplicial objects \( S : \Delta^{op} \rightarrow \mathcal{A} \) and \( n \in \mathbb{N}_0 \) by induction over \( n \).

For \( n = 0 \) we have \( K_0^\mathrm{NS} = S_0 = S([1]) \) since \( \sigma = 1_{[1]} : [1] \rightarrow [1] \) is the only surjective morphism in the coproduct in (47) and \( \eta^S_{[1]} = \text{id}_{S_0} : S_0 \rightarrow S_0 \) by (48).

Suppose that it is shown that \( \eta^S_k : K_k^\mathrm{NS} \rightarrow S_k \) is an isomorphism for all \( k \leq n - 1 \). As \( \eta^S_k : KN^S \rightarrow S \) is a natural transformation, we obtain with the induction hypothesis

\[
\eta^S_{n-1}(\sigma^i_n) = \eta^S_n(\sigma^i_n) \circ (\eta^S_{n-1})^{-1} \quad \Rightarrow \quad \text{im}(s^j_{n-1}) \subseteq \text{im}(\eta^S_{n-1}) \forall j \in \{0, ..., n-1\}.
\]

By choosing \( \sigma = 1_{[n+1]} \) in (48) we obtain \( \eta^S_{[n+1]} \circ \iota_{1_{[n+1]}} = \iota_n \) and \( NS_n \subseteq \text{im}(\eta^S_{[n+1]}) \). As we have \( S_n = NS_n \amalg DS_n \) with \( DS_n = +_{j=0}^{n-1} \text{im}(s^j_{n-1}) \) for all \( n \in \mathbb{N}_0 \) by Proposition 5.2.1 this shows that \( \eta^S_{[n+1]} : K_n^\mathrm{NS} \rightarrow S_n \) is an epimorphism.

Suppose now that \( x \in \ker(\eta^S_{[n+1]}) \). Then by definition of \( K_n^\mathrm{NS} \) in (47) there is a monotonic surjection \( \sigma : [n+1] \rightarrow [p+1] \) and an element \( x' \in NS_p \) with \( x = \iota_{\sigma}(x') \) and \( x' \in \ker(\eta^S_{[n+1]} \circ \iota_{\sigma}) \).

With the relations (37) in \( \Delta^{op} \), we can construct a monotonic injection \( \delta : [p+1] \rightarrow [n+1] \) with \( \sigma \circ \delta = 1_{[p+1]} \) and obtain from (48)

\[
S(\delta) \circ \eta^S_{[n+1]} \circ \iota_{\sigma} = S(\delta) \circ S(\sigma) \circ \iota_p = S(\sigma \circ \delta) \circ \iota_p = S(1_{[p+1]}) \circ \iota_p = \iota_p.
\]

As \( \iota_p : NS_p \rightarrow S_p \) is a monomorphism, we have \( \ker(\eta^S_{[n+1]} \circ \iota_{\sigma}) = 0 \), which implies \( x' = 0 \) and \( x = \iota_{\sigma}(x') = 0 \). This shows that \( \eta^S_{[n+1]} : K_n^\mathrm{NS} \rightarrow S_n \) is an isomorphism.

The proof of Theorem 5.2.3 appears rather lengthy and technical, but this is largely due to the fact that all necessary computations were carried out explicitly and in great detail. The essential idea in the proof is the construction of the functor \( K : \mathbf{Ch}_{\mathcal{A} \geq 0} \rightarrow \mathbf{Fun}(\Delta^{op}, \mathcal{A}) \) in (42), (43) and (44). Once this is done, all other constructions in the proof are essentially determined by compatibility with this construction, and the claims can then be verified by routine computations.

The construction of \( K \) addresses the fundamental problem that the boundary operator of a positive chain complex in \( \mathcal{A} \) involves only the face maps, whereas a simplicial object also requires information about the degeneracies.

This problem can be stated more precisely by considering instead of simplicial objects in \( \mathcal{A} \) so-called semisimplicial objects, functors \( S : \Delta^+_{m_j} \rightarrow \mathcal{A} \), where \( \Delta^+_{m_j} \subseteq \Delta^+ \) is the subcategory with only injective monotonic maps as morphisms. Morphisms of semisimplicial objects are natural transformations between semisimplicial objects, and one obtains again a functor category \( \mathbf{Fun}(\Delta^+_{m_j}, \mathcal{A}) \). As all injective monotonic maps in the simplex category \( \Delta^+_{m_j} \) are composites of face maps, it seems plausible that a positive chain complex in \( \mathcal{A} \) contains enough information to define a semisimplicial object \( S : \Delta^+_{m_j} \rightarrow \mathcal{A} \). The non-trivial part is to extend this to a simplicial object.

We will see in Exercise 62 that the procedure used in the definition of \( K \) defines a functor \( L : \mathbf{Fun}(\Delta^+_{m_j}, \mathcal{A}) \rightarrow \mathbf{Fun}(\Delta^{op}, \mathcal{A}) \) and in Exercise 63 that this functor \( L \) is left adjoint to the restriction functor \( R : \mathbf{Fun}(\Delta^{op}, \mathcal{A}) \rightarrow \mathbf{Fun}(\Delta^+_{m_j}, \mathcal{A}) \) that restricts a simplicial object in \( \mathcal{A} \) to a semisimplicial one by forgetting the images of the degeneracies. As a consequence, one finds
that the functor $K : \text{Ch}_{\mathcal{A}_{\geq 0}} \to \text{Fun}(\Delta^{op}, \mathcal{A})$ is left adjoint to the normalised chain complex functor $N : \text{Fun}(\Delta^{+op}, \mathcal{A}) \to \text{Ch}_{\mathcal{A}_{\geq 0}}$, see [W Exercise 8.4.2]. Hence, the essential idea in Dold-Kan correspondence can be understood in terms of \textbf{adjoint functors}. Moreover, one can show that for every semisimplicial object $K : \Delta^{+op}_{inj} \to \mathcal{A}$ the functor $LK : \Delta^{+op} \to \mathcal{A}$ is a left \textbf{Kan extension} of $K$ along the inclusion functor $\iota : \Delta^{+op}_{inj} \to \Delta^{+op}$, see [W Exercise 8.1.5]. This shows that the construction of $K$, which seems surprising and unintuitive at first, is natural from the perspective of category theory.

It should also be noted that Dold-Kan correspondence extends to simplicial homotopies. As explained at the end of Section \ref{sec:kan-correspondence} there is a notion of simplicial homotopy $h : f \Rightarrow g$ that relates simplicial morphisms $f, g : S \to S'$ in an abelian category $\mathcal{A}$. By Proposition \ref{prop:simplicial-homotopy} the simplicial morphisms $f, g$ define chain maps $Nf, Ng : \text{NS}_* \to \text{NS}'_*$ between the normalised complexes associated with $S$ and $S'$. Similarly, one can show that every simplicial homotopy $h : f \Rightarrow g$ defines a chain homotopy $Nh : Nf \Rightarrow Ng$ [W Lemma 8.3.13]. Conversely, every chain homotopy $h_* : f_* \Rightarrow g_*$ between chain maps $f, g : C_* \to C'_*$ defines a simplicial homotopy $K(h) : K(f) \Rightarrow K(g)$, where $K : \text{Ch}_{\mathcal{A}_{\geq 0}} \to \text{Fun}(\Delta^{+op}, \mathcal{A})$ is the functor from the Dold-Kan correspondence [W Section 8.4, p273ff]. This allows one to formulate Dold-Kan correspondence as an equivalence of categories between the homotopy category of chain complexes from Remark \ref{rem:dold-kan-equiv} and the homotopy category of simplicial objects, whose morphisms are homotopy equivalence classes of simplicial morphisms.

\section{Simplicial objects from algebra objects in tensor categories}

The Dold-Kan correspondence shows that positive chain complexes in an abelian category $\mathcal{A}$ and chain maps between them are obtained from simplicial objects and simplicial morphisms in $\mathcal{A}$ via the normalised chain complex functor. Hence, we can investigate positive chain complexes in $\mathcal{A}$ by studying simplicial objects in $\mathcal{A}$ and vice versa.

In this section, we derive a general method that allows one to \textit{construct} simplicial objects and morphisms from much simpler data. This method works for categories that are equipped with additional structure, namely a categorical \textbf{tensor product} that generalises the concept of a tensor product over a commutative ring $k$.

To generalise the tensor product in $k$-\textit{Mod} from Definition \ref{def:tensor-product} to other categories, we have to formulate it in such a way that it involves only objects, morphisms, functors and natural transformations. We already established in Theorem \ref{thm:tensor-product-properties} that the tensor product of modules over a commutative ring $k$ defines a functor $\otimes : k\text{-Mod} \times k\text{-Mod} \to k\text{-Mod}$. This suggests that one should view a tensor product in a general category $\mathcal{C}$ as a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that satisfies certain additional conditions.

Of the properties of the tensor product in Lemma \ref{lem:tensor-product-properties} only the second and the fourth can be formulated in general categories $\mathcal{C}$ without additional structures. They state, respectively, that $k$ acts as a unit for the tensor product and that the tensor product over $k$ is associative. The fact that $k$ acts as a unit for the tensor product is encoded in the $k$-module isomorphisms

\[ l_M : k \otimes_k M \to M, \lambda \otimes m \mapsto \lambda m \quad r_M : M \otimes_k k \to M, m \otimes \lambda \mapsto \lambda m \]

from Lemma \ref{lem:tensor-product-properties}. If we denote by $k \times \id : k\text{-Mod} \to k\text{-Mod} \times k\text{-Mod}$ the functor that assigns to a $k$-module $M$ the pair $(k, M)$ and to a $k$-linear map $f : M \to M'$ the pair $(\id_k, f)$, then the $k$-module isomorphisms $l_M : k \otimes_k M \to M$ and $r_M : M \otimes_k k \to M$ relate the functors.
\(\otimes(k \times \text{id}) : \text{k-Mod} \to \text{k-Mod}\) and \(\otimes(\text{id} \times k) : \text{k-Mod} \to \text{k-Mod}\) to the identity functor \(\text{id}_{\text{k-Mod}}\). Similarly, the associativity of the tensor product is encoded in the \(k\)-module isomorphisms

\[
a_{M,N,P} : (M \otimes_k N) \otimes_k P \to M \otimes_k (N \otimes_k P), \quad (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)
\]

from Lemma 1.2.27 that relate the value of the functors \(\otimes(\otimes \times \text{id})\) and \(\otimes(\text{id} \times \otimes)\) on the triple \((M, N, P)\) of objects in \(\text{k-Mod}\).

The \(k\)-linear isomorphisms \(l_M, r_M\) and \(a_{M,N,P}\) commute with \(k\)-linear maps. For all \(k\)-linear maps \(f : M \to M',\ g : N \to N'\) and \(h : P \to P'\) we have

\[
a_{M',N',P'} \circ ((f \otimes g) \otimes h) = (f \otimes (g \otimes h)) \circ a_{M,N,P}, \quad l_M \circ (\text{id}_k \otimes f) = f \circ l_M, \quad r_M \circ (f \otimes \text{id}_k) = f \circ r_M.
\]

We can therefore interpret \(a_{M,N,P}, l_M\) and \(r_M\) as component morphisms of natural isomorphisms \(a : \otimes(\otimes \times \text{id}) \to \otimes(\text{id} \times \otimes),\ l : \otimes(\text{id} \times k) \to \text{id}\) and \(r : \otimes(\text{id} \times k) \to \text{id}\). If we also take into account the compatibility conditions between multiple composites of these isomorphisms, we obtain the following definition that generalises tensor products over commutative rings.

**Definition 5.3.1:**

A monoidal category or tensor category is a sextuple \((\mathcal{C}, \otimes, e, a, l, r)\) consisting of

- a category \(\mathcal{C}\),
- a functor \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\), the tensor product,
- an object \(e\) in \(\mathcal{C}\), the tensor unit,
- a natural isomorphism \(a : \otimes(\otimes \times \text{id}_\mathcal{C}) \to \otimes(\text{id}_\mathcal{C} \times \otimes)\), the associator,
- natural isomorphisms \(r : \otimes(\text{id}_\mathcal{C} \times e) \to \text{id}\) and \(l : \otimes(e \times \text{id}_\mathcal{C}) \to \text{id}\), the unit constraints,

subject to the following two conditions:

1. **pentagon axiom:** for all objects \(U, V, W, X\) of \(\mathcal{C}\) the following diagram commutes

\[
\begin{array}{ccc}
(U \otimes V) \otimes W \otimes X & \xrightarrow{\begin{array}{c} \text{\(a_{U,V,W,X}\)} \end{array}} & U \otimes (V \otimes (W \otimes X)) \\
\downarrow{\text{\(a_{U,V,W} \otimes 1_X\)}} & & \downarrow{\text{\(1_U \otimes a_{V,W,X}\)}} \\
(U \otimes (V \otimes W)) \otimes X & \xrightarrow{\begin{array}{c} \text{\(a_{U,V \otimes W,X}\)} \end{array}} & U \otimes ((V \otimes W) \otimes X).
\end{array}
\]

2. **triangle axiom:** for all objects \(V, W\) of \(\mathcal{C}\) the following diagram commutes

\[
\begin{array}{ccc}
(V \otimes e) \otimes W & \xrightarrow{\begin{array}{c} \text{\(a_{V,e,W}\)} \end{array}} & V \otimes (e \otimes W) \\
\downarrow{\text{\(r_V \otimes 1_W\)}} & & \downarrow{\text{\(1_V \otimes l_W\)}} \\
V \otimes W. & & V \otimes W.
\end{array}
\]

A monoidal category is called strict if \(a, r\) and \(l\) are identity natural transformations.

**Remark 5.3.2:**

1. The tensor unit and the unit constraints are determined by \(\otimes\) uniquely up to unique isomorphism: If \(e'\) is an object in \(\mathcal{C}\) with natural isomorphisms \(r' : \text{id}_\mathcal{C} \times e' \to \text{id}_\mathcal{C}\) and \(l' : e' \times \text{id}_\mathcal{C} \to \text{id}_\mathcal{C}\), then there is a unique isomorphism \(\phi : e \to e'\) with \(r'_X \circ (1_X \times \phi) = r_X\) and \(l'_X \circ (\phi \times 1_X) = l_X\). (Exercise).
2. One can show that if $C, D$ are objects of a monoidal category $(C, \otimes, e, a, l, r)$ and $f, g : C \to D$ morphisms in $C$ that are obtained by composing identity morphisms, component morphisms of the associator $a$ and component morphisms of the left and right unit constraints $l, r$ with the composition of morphisms and the tensor product, then $f$ and $g$ are equal. This is MacLane’s famous coherence theorem. A proof of this statement can be found in [McC, Chapter VI.2] and [K, Chapter XI.5].

3. The name monoidal category is motivated by the fact that for a monoidal category $(C, \otimes, e, a, l, r)$ the endomorphisms of the tensor unit form a commutative monoid $(\text{End}_C(e), \circ)$. This is a consequence of the coherence theorem.

Note that a given category can have several non-equivalent monoidal structures. Specifying the functor $\otimes : C \times C \to C$ is a choice of structure, while the associativity constraint, the tensor unit and unit constraints are essentially determined by the functor $\otimes$.

When considering functors between monoidal categories and natural transformations between them, it makes sense to ask that these functors and natural transformations respect the monoidal structures - up to isomorphisms. This leads to the concept of a monoidal functor or tensor functor and a monoidal natural transformation.

**Definition 5.3.3:** Let $(C, \otimes_C, e_C, a^C, l^C, r^C)$ and $(D, \otimes_D, e_D, a^D, l^D, r^D)$ be monoidal categories.

1. A **monoidal functor** or **tensor functor** from $C$ to $D$ is a triple $(F, \phi^e, \phi^\otimes)$ of
   - a functor $F : C \to D$,
   - an isomorphism $\phi^e : e_D \to F(e_C)$ in $D$,
   - a natural isomorphism $\phi^\otimes : \otimes_D(F \times F) \to F \otimes_C$,
   that satisfy the following axioms:
   
   (a) **compatibility with the associativity constraint:**
   
   for all objects $U, V, W$ of $C$ the following diagram commutes
   
   $$(F(U) \otimes (F(V)) \otimes F(W)) \xrightarrow{\phi^\otimes_U \otimes 1_{F(W)}} F(U) \otimes (F(V) \otimes F(W))$$
   
   $$(\phi^\otimes_U \otimes 1_{F(W)}) \downarrow \quad \downarrow 1_U \otimes \phi^\otimes_V \cdot W$$
   
   $$F(U \otimes V) \otimes F(W) \quad F(U) \otimes F(V) \otimes W$$
   
   $$(\phi^\otimes_U \otimes \cdot W) \downarrow \quad \downarrow \phi^\otimes_{U \otimes V} \cdot W$$
   
   $$F((U \otimes V) \otimes W) \xrightarrow{F(a^C_{U,V,W})} F(U \otimes (V \otimes W)).$$

   (b) **compatibility with the unit constraints:**

   for all objects $V$ of $C$ the following diagrams commute

   $$(F(V) \xrightarrow{\phi^\otimes_V} F(e_C) \otimes F(V))$$
   
   $$\downarrow \phi^\otimes_V \quad \downarrow \phi^\otimes_{F(V,e_C)}$$
   
   $$F(V) \quad F(e_C \otimes V)$$
   
   $$(F(V) \xrightarrow{F(a^C_{e_C,V})} F(e_C) \otimes F(V))$$
   
   $$\downarrow \phi^\otimes_{F(V,e_C)} \quad \downarrow \phi^\otimes_{F(V,e_C)}$$
   
   $$F(V) \quad F(V \otimes e_C).$$

   A monoidal functor $(F, \phi^e, \phi^\otimes)$ is called **strict** if $\phi^e = 1_{e_D}$ and $\phi^\otimes = \text{id}_{F \otimes_C}$ is the identity natural transformation. It is called a **monoidal equivalence** if $F : C \to D$ is an equivalence of categories.
2. Let \((F, \phi^e, \phi^{\otimes}), (F', \phi'^e, \phi'^{\otimes}) : C \to D\) be monoidal functors. A **monoidal natural transformation** is a natural transformation \(\eta : F \to F'\) for which the following diagrams commute:

(a) **compatibility with** \(\phi^e\) and \(\phi'^e\):

\[
\begin{array}{ccc}
F(e_C) & \xrightarrow{\eta_C} & F'(e_C) \\
\phi^e & \downarrow & \phi'^e \\
e_D & \xrightarrow{\eta_D} & e_D
\end{array}
\]

(b) **compatibility with** \(\phi^{\otimes}\) and \(\phi'^{\otimes}\): for all objects \(C, C'\) in \(C\)

\[
\begin{array}{ccc}
F(C) \otimes F(C') & \xrightarrow{\eta_C \otimes \eta'_C} & F'(C) \otimes F'(C') \\
\phi^{\otimes}_{C,C'} & \downarrow & \phi'^{\otimes}_{C,C'} \\
F(C \otimes C') & \xrightarrow{\eta_{C \otimes C'}} & F'(C \otimes C').
\end{array}
\]

A monoidal natural transformation \(\eta : F \to F'\) is called a **monoidal isomorphism** if \(\eta_C : F(C) \to F'(C)\) is an isomorphism for all objects \(C\) in \(C\).

There are many examples of tensor categories. Many of the categories from algebra or topology considered so far have the structure of a tensor category, some of them even several non-equivalent ones.

**Example 5.3.4:**

1. For any commutative ring \(k\), the category \(k\)-Mod is a tensor category with:
   - the functor \(\otimes : k\text{-Mod} \times k\text{-Mod} \to k\text{-Mod}\) that assigns to a pair \((M, N)\) of \(k\)-modules the \(k\)-module \(M \otimes_k N\) and to a pair \((f, g)\) of \(k\)-linear maps \(f : M \to M', g : N \to N'\) the linear map \(f \otimes g : M \otimes_k N \to M' \otimes_k N', m \otimes n \mapsto f(m) \otimes g(n)\),
   - the tensor unit \(e = k\),
   - the associator with component isomorphisms \(a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P), (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)\),
   - the unit constraints with component morphisms \(r_M : M \otimes_k k \to M, m \otimes \lambda \mapsto \lambda m\) and \(l_M : k \otimes_k M \to M, \lambda \otimes m \mapsto \lambda m\).

For any ring homomorphism \(\phi : k \to l\), the functor \(F_\phi : l\text{-Mod} \to k\text{-Mod}\) that sends an \(l\)-module \((M, \triangleright_k)\) to the \(k\)-Module \((M, \triangleright_k)\) with \(\triangleright_k m := \phi(\lambda) \triangleright_k m\) and every \(l\)-linear map to itself is a tensor functor.

2. For any small category \(C\), the category \(\text{End}(C)\) of endofunctors \(F : C \to C\) and natural transformations between them is a **strict** monoidal category with:
   - the functor \(\otimes : \text{End}(C) \times \text{End}(C) \to \text{End}(C)\) that assigns to a pair \((F, G)\) of functors \(F, G : C \to C\) the functor \(FG : C \to C\) and to a pair \((\mu, \eta)\) of natural transformations \(\mu : F \to F', \eta : G \to G'\) the natural transformation \(\mu \eta : FG \to F'G'\) with component morphisms \((\mu \eta)_C = \mu_{G(C)} \circ F(\eta_C) = F'(\eta_C) \circ \mu_{G(C)} : FG(C) \to F'G'(C)\),
   - the identity functor as the tensor unit: \(e = \text{id}_C\).
3. The categories Set and Top are monoidal categories with:
   - the functor $\otimes : \text{Set} \times \text{Set} \to \text{Set}$ that assigns to a pair of sets $(X, Y)$ their cartesian product $X \times Y$ and to a pair $(f, g)$ of maps $f : X \to X'$, $g : Y \to Y'$ the product map $f \times g : X \times Y \to X' \times Y'$,
   - the functor $\otimes : \text{Top} \times \text{Top} \to \text{Top}$ that sends a pair $(X, Y)$ of topological spaces the product space $X \times Y$ and a pair of continuous maps $f : X \to X'$, $g : Y \to Y'$ to the product map $f \times g$,
   - the one-point set $\{p\}$ and the one-point space $\{p\}$ as the tensor unit,
   - the associators with component morphisms $a_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$, $((x,y),z) \mapsto (x,(y,z))$,
   - the unit constraints with component morphisms $r_X : X \times \{p\} \to X$, $(x,p) \mapsto x$ and $l_X : \{p\} \times X \to X$, $(p,x) \mapsto x$.
   - The forgetful functor $F : \text{Top} \to \text{Set}$ is a strict monoidal functor.
   - The functor $F : \text{Set} \to k\text{-Mod}$ that assigns to a set $X$ the free $k$-module $F(X) = \langle X \rangle_k$ and to a map $f : X \to Y$ the unique $k$-linear map $F(f) : \langle X \rangle_k \to \langle Y \rangle_k$ with $F(f)|_X = f$ is a monoidal equivalence. Its coherence data is given by $\phi^c : k \to \langle p \rangle_k$, $\lambda \mapsto \lambda p$ and $\phi^\otimes_{X,Y} : \langle X \rangle_k \otimes_k \langle Y \rangle_k \to \langle X \times Y \rangle_k$, $x \otimes y \mapsto (x,y)$.

4. Any category $\mathcal{C}$ with finite (co)products is a tensor category with:
   - the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that sends a pair of objects to their (co)product and a pair of morphisms to the associated morphism between (co)products induced by the universal property,
   - the empty (co)product, i.e. the final (initial) object in $\mathcal{C}$ as the tensor unit,
   - the associators induced by the universal properties of the (co)products,
   - the unit constraints induced by the universal properties of the (co)products.

This includes:
   - any abelian category $\mathcal{A}$,
   - the category Set with the disjoint union of sets and the empty set, or with the Cartesian product of sets and the 1-point set,
   - the category Top with the sum of topological spaces and the empty space, or with the product of topological spaces and the 1-point space,
   - the category $\text{Top}^1$ of pointed topological spaces with wedge sums and the one-point space or with products of pointed spaces and the one-point space,
   - the category Grp with the direct product of groups and the trivial group or with the free product of groups and the trivial group.

5. For any commutative ring $k$, the category $\text{Ch}_{k\text{-Mod}}$ of chain complexes in $k\text{-Mod}$ is a monoidal category with the tensor product of chain complexes given by
   $$(A \bullet \otimes B \bullet)_n = \bigoplus_{j=0}^n A_j \otimes_k B_{n-j}, \quad d_n^{A \otimes B}(a \otimes b) = d_j^A(a) \otimes b + (-1)^k a \otimes d_n^B(b)$$
   for $a \in A_j$, $b \in B_{n-j}$ and with the tensor product of chain maps given by
   $$(f \bullet \otimes g \bullet)_n(a \otimes b) = f_j(a) \otimes g_{n-j}(b) \quad \text{for} \quad a \in A_j, b \in B_{n-j}.$$ 

6. For any monoidal category $\mathcal{C}$ and small category $\mathcal{X}$, the category $\text{Fun}(\mathcal{X}, \mathcal{C})$ is a monoidal category with $F \otimes G : \mathcal{X} \to \mathcal{C}$ given by $(F \otimes G)(X) = F(X) \otimes G(X)$ for all objects $X$ in $\mathcal{X}$ and $\eta \otimes \kappa : F \otimes G \to F' \otimes G'$ by $(\eta \otimes \kappa)_X = \eta_X \otimes \kappa_X : F(X) \otimes G(X) \to F'(X) \otimes G'(X)$ for
all natural transformations \( \eta : F \to F' \) and \( \kappa : G \to G' \). The tensor unit is the constant functor \( I : \mathcal{X} \to \mathcal{C} \) with \( I(X) = e \) and \( I(\alpha) = 1_e \) for all objects \( X \) and morphisms \( \alpha : X \to Y \) in \( \mathcal{X} \). The associator and the unit constraints are induced by the associators and unit constraints in \( \mathcal{C} \).

7. In particular, 4. and 6. imply that for any abelian category \( \mathcal{A} \), the category \( \text{Fun}(\Delta^{op}, \mathcal{A}) \) of simplicial objects and simplicial morphisms in \( \mathcal{A} \) is a monoidal category.

Example 5.3.4 unites the definitions of homotopies between continuous maps and chain homotopies. Although the latter were defined by a technical condition in Definition [3.3.1] it was shown in Remark [3.3.3] that a chain homotopy \( h : f \Rightarrow g \) between chain maps \( f, g : X_\bullet \to Y_\bullet \) in \( R\text{-Mod} \) can be viewed as a chain map \( h : Y_\bullet \to X'_\bullet \) for a certain chain complex \( X'_\bullet \) constructed from \( X_\bullet \). By comparing the definition of the chain complex \( Y_\bullet \) in Remark [3.3.3] with the tensor product in the category \( \text{Ch}_{R\text{-Mod}} \) in Example [5.3.4] one finds that the chain complex \( Y_\bullet \) is given as the tensor product \( Y_\bullet = \Delta_1^+ \otimes X_\bullet \), where \( \Delta_1^+ \) is the chain complex \( \Delta_1^+ = 0 \to R \to R \oplus R \to 0 \). This is precisely the chain complex for simplicial complex consisting of one 1-simplex and two 0-simplexes that describes the unit interval \([0,1]\). Hence, a chain homotopy is a chain map \( h : \Delta_1^+ \otimes X_\bullet \to X'_\bullet \), just as a homotopy between continuous maps \( f, g : X \to X' \) is a continuous map \( h : [0,1] \otimes X \to X' \).

The reason why tensor categories are relevant for simplicial methods in homological algebra is that the augmented simplex category is a tensor category with a particularly simple structure. This does not hold for the simplex category, and this is the reason why the augmented simplex category is preferable from the algebraic viewpoint.

Example 5.3.5:
The augmented simplex category \( \Delta \) from Definition [5.1.1] is a strict tensor category with:

- the functor \( \otimes : \Delta \times \Delta \to \Delta \) that assigns to a pair \(([m],[n])\) of ordinals the ordinal \([m] \otimes [n] = [m+n]\) and to a pair \((f,g)\) of monotonic maps \( f : [m] \to [m'] \) and \( g : [n] \to [n'] \) the monotonic map \( f \otimes g : [m+n] \to [m'+n'] \) given by concatenation of \( f \) and \( g \)

\[
(f \otimes g)(i) = \begin{cases} f(i) & 0 \leq i < m \\ m' + g(i - m) & m \leq i < n + m. \end{cases}
\]

- the ordinal \([0] = \emptyset\) as the tensor unit.

Lemma 5.3.6: The augmented simplex category \( \Delta \) is presented as a strict tensor category by the object \([1]\) and the morphisms \( \sigma_1^0 : [2] \to [1], \delta_0^0 : [0] \to [1] \), subject to the relations

\[
\sigma_1^0 \circ (\sigma_1^0 \otimes 1_{[1]}) = \sigma_1^0 \circ (1_{[1]} \otimes \sigma_1^0) \quad \sigma_1^0 \circ (1_{[1]} \otimes \delta_0^0) = 1_{[1]} = \sigma_1^0 \circ (\delta_0^0 \otimes 1_{[1]}).
\] (50)

In other words:

1. Every object \([n]\) is a multiple tensor product of the object \([1]\) with itself.
2. Every morphism in \( \Delta \) is given as a multiple composite and tensor product of the morphisms \( \sigma_1^0, \delta_0^0 \) and identity morphisms.
3. All relations between morphisms in \( \Delta \) arise either from the properties of a tensor category or from (50) via the tensor product and the composition of morphisms.
Proof:
By definition of the tensor product in $\Delta$ we have $[m] \otimes [n] = [m + n]$ for all $m, n \in \mathbb{N}_0$, and this implies inductively $[n] = [n - 1] \otimes [1] = [n - 2] \otimes [1] \otimes [1] = ... = [1] \otimes^n$ for all $n \in \mathbb{N}$. For $n = 0$ we have the empty tensor product $[0] = [1]^{\otimes 0}$. To show that every morphism in $\Delta$ is a composite of the morphisms $\sigma^0_0$ and $\delta^0_0$, it is sufficient to prove this for the morphisms $\delta^i_n : [n] \to [n + 1]$ and $\sigma^i_n : [n + 1] \to [n]$, since these morphisms generate $\Delta$ as a category by Proposition 5.1.2. These morphisms are given by

$$
\delta^i_n = 1_{[i]} \otimes \delta^0_0 \otimes 1_{[n - i]} : [n] \to [n + 1], \quad \sigma^i_n = 1_{[i]} \otimes \sigma^0_1 \otimes 1_{[n - i - 1]} : [n + 1] \to [n]. \tag{51}
$$

The defining relations between the morphisms $\delta^i_n$ and $\sigma^i_n$ in (37) follow from the definition of the tensor product in $\Delta$ and from the relations (50). For instance, we have

$$
\sigma^i_{n-1} \circ \sigma^i_n = (1_{[i]} \otimes \sigma^0_1 \otimes 1_{[n - i - 2]}) \circ (1_{[i]} \otimes \sigma^0_1 \otimes 1_{[n - i - 1]}) = 1_{[i]} \otimes (\sigma^0_1 \circ (\sigma^0_1 \circ 1_{[1]})) \otimes 1_{[n - i - 2]} = 1_{[i]} \otimes (\sigma^0_1 \circ (\sigma^0_1 \circ 1_{[1]})) \otimes 1_{[n - i - 2]} = (1_{[i]} \otimes \sigma^0_1 \otimes 1_{[n - i - 2]}) \circ (1_{[i+1]} \otimes \sigma^0_1 \otimes 1_{[n - i - 2]}) = \sigma^i_{n-1} \circ \sigma^{i+1}_n,
$$

and the proof of the other relations in (37) are similar (Exercise). As the relations (37) are the defining relations of the augmented simplicial category $\Delta$ (see Remark 5.1.3), the claim follows. □

Lemma 5.3.6 gives a more efficient description of the augmented simplicial category $\Delta$ with fewer relations than the description in terms of face maps and degeneracies in Proposition 5.1.2. It involves only two morphisms and the two relations in (50). The first relation resembles an associativity condition and the second resembles a unit law for a monoid or an algebra. This can be made precise by generalising the concept of an algebra over a commutative ring to an algebra in a tensor category.

An algebra over commutative ring $k$ is a $k$-module $A$ together with an associative $k$-bilinear map $\cdot : A \times A \to k$ and a unit $1_A \in A$ with $1_A \cdot a = a \cdot 1_A = a$ for all $a \in A$. By the universal property of the tensor product over $k$, we can also view the multiplication as a $k$-linear map $\mu : A \otimes_k A \to A$, $a \otimes a' \mapsto a \cdot a'$. The unit element of $A$ can be encoded in a $k$-linear map $\eta : k \to A$, $\lambda \mapsto \lambda 1_A$. The condition that $\mu$ is associative and that $1_A$ is a unit then read

$$
\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu) \circ a_{A,A,A} \quad \mu \circ (\text{id} \otimes \eta) \circ r_A^{-1} = \mu \circ (\eta \otimes \text{id}) \circ l_A^{-1} = \text{id}_A,
$$

where $a_{A,A,A} : (A \otimes_k A) \otimes_k A \to A \otimes_k (A \otimes_k A)$ is the associator, $r_A : A \otimes_k k \to A$, $l_A : k \otimes_k A \to A$ are its left and right unit constraints. This definition of an algebra generalises to any tensor category. Moreover, one obtains a dual concept, the so-called coalgebra, by reversing the directions of the multiplication and unit map.

**Definition 5.3.7:** Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a tensor category.

1. A **monoid** or **algebra object** in $\mathcal{C}$ is a triple $(A, \mu, \eta)$ of an object $A$ in $\mathcal{C}$ and morphisms $\mu : A \otimes A \to A$, $\eta : e \to A$, the **multiplication** and **unit morphism**, such that the following diagrams commute

\[
\begin{array}{ccc}
(A \otimes A) \otimes A & \xrightarrow{a_{A,A,A}} & A \otimes (A \otimes A) \\
\mu \otimes 1_A & \xrightarrow{1_A \otimes \mu} & 1_A \otimes \mu \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
\quad
\begin{array}{ccc}
A \otimes A & \xrightarrow{1_A \otimes \eta} & A \otimes e \\
\mu & \xrightarrow{\mu} & \mu \\
e \otimes A & \xrightarrow{\eta \otimes 1_A} & e \otimes A
\end{array}
\]
2. A **comonoid** or **coalgebra object** in \( C \) is a triple \((C, \Delta, \epsilon)\) of an object \( C \) in \( C \) and morphisms \( \Delta : C \to C \otimes C, \epsilon : C \to e \), the **comultiplication** and **counit morphism**, such that the following diagrams commute

\[
\begin{array}{ccc}
(C \otimes C) \otimes C & \overset{ac_{C,C}}{\iff} & C \otimes (C \otimes C) \\
C \otimes C & \overset{\Delta}{\iff} & C \\
\end{array}
\]

\[
\begin{array}{ccc}
C \otimes C & \overset{1_C \otimes \Delta}{\iff} & C \otimes C \\
C & \overset{\Delta}{\iff} & C \\
\end{array}
\]

\[
\begin{array}{ccc}
C \otimes e & \overset{1_C \otimes \epsilon}{\iff} & e \otimes C \\
C & \overset{\Delta}{\iff} & C \\
\end{array}
\]

Example 5.3.8:

1. A monoid in the tensor category \((\text{Set}, \times, \{p\})\) is simply a monoid \((M, \cdot, e)\). The multiplication morphism is the monoid multiplication \(\mu : M \times M \to M, (m, m') \mapsto m \cdot m'\) and the unit morphism is given by \(\eta : \{p\} \to M, p \mapsto e\).

2. A monoid in the tensor category \((k\text{-Mod}, \otimes, k)\) for a commutative ring \(k\) is precisely a \(k\)-algebra. In particular, monoids in \((\text{Vect}_F, \otimes, F)\) are algebras over \(F\).

3. Every set \(X\) is a comonoid in \((\text{Set}, \times, \{p\})\) and every topological space \(X\) is a comonoid in \((\text{Top}, \times, \{p\})\) with the diagonal map \(\Delta : X \to X \times X, x \mapsto (x, x)\) as the comultiplication and the counit \(\epsilon : X \to \{p\}, x \mapsto p\).

4. Every group \(G\) is a comonoid in the tensor category \((\text{Grp}, \times, \{e\})\) with the diagonal map \(\Delta : G \to G \times G, g \mapsto (g, g)\) and the map \(\epsilon : G \to \{e\}, g \mapsto e\). Every **abelian** group \((G, \cdot, e_G)\) is a monoid in the tensor category \((\text{Grp}, \times, \{e\})\) with the multiplication and unit morphisms \(\cdot : G \times G \to G\) and \(\eta : \{e\} \to G, e \mapsto e_G\).

5. Let \(C\) be a small category. A monoid in the strict monoidal category \(\text{End}(C) = \text{Fun}(C, C)\) from Example 5.3.4 2. is called a **monad**. It is a triple \((T, \mu, \eta)\) of a functor \(T : C \to C\) together with natural transformations \(\mu : T^2 \to T\) and \(\eta : \text{id}_C \to T\) such that the following diagrams commute

\[
\begin{array}{ccc}
T^3 & \overset{T \mu}{\iff} & T^2 \\
\mu T & \downarrow & \mu \\
T^2 & \overset{\mu}{\iff} & T \\
\end{array}
\]

\[
\begin{array}{ccc}
T & \overset{\eta T}{\iff} & T^2 \\
\mu & \downarrow & \mu \\
T^2 & \overset{\eta}{\iff} & T \\
\end{array}
\]

A comonoid in the strict tensor category \(\text{End}(C)\) is called a **comonad**. It is a triple \((C, \Delta, \epsilon)\) of a functor \(C : C \to C\) together with natural transformations \(\Delta : C \to C^2\) and \(\epsilon : C \to \text{id}_C\) such that the following diagrams commute

\[
\begin{array}{ccc}
C^3 & \overset{C \Delta}{\iff} & C^2 \\
\Delta C & \downarrow & \Delta \\
C^2 & \overset{\Delta}{\iff} & C \\
\end{array}
\]

\[
\begin{array}{ccc}
C & \overset{\epsilon C}{\iff} & C^2 \\
C^2 & \overset{C \epsilon}{\iff} & C \\
\text{id}_C & \downarrow \quad \Delta \quad & \text{id}_C \\
C & \overset{\Delta}{\iff} & C \\
\end{array}
\]

6. The triple \(([1], \sigma_1^0, \delta_0^0)\) of the ordinal \([1] = \{0\}\), the map \(\sigma_1^0 : [2] \to [1]\) and the empty map \(\delta_0^0 : \emptyset \to [1]\) from Lemma 5.3.6 is a monoid in the strict tensor category \((\Delta, \otimes, \emptyset)\). This follows from the definition of the tensor product in \(\Delta\) in Example 5.3.3 and the relations \([51]\) in the augmented simplex category \(\Delta\).
The monoid \([1], \sigma_0, \delta_0\) in the augmented simplex category \(\Delta\) plays a special role in homology. By Lemma 5.3.6 the augmented simplex category \(\Delta\) is generated as a tensor category by the object \([1]\) and the morphisms \(\sigma_0^i : [2] \to [1], x \mapsto 0\) and the empty map \(\delta_0^0 : [0] \to [1]\). Its defining relations (50) are precisely the defining relations for monoid in a tensor category. This allows one to characterise tensor functors from \(\Delta\) to a strict tensor category \(\mathcal{C}\) by monoids in \(\mathcal{C}\) and tensor functors from \(\Delta^{op}\) to \(\mathcal{C}\) by comonoids. Every comonoid in \(\mathcal{C}\) defines a simplicial object in \(\mathcal{C}\) and every monoid in \(\mathcal{C}\) a cosimplicial object in \(\mathcal{C}\). This is sometimes called the *universality* of the augmented simplex category.

**Proposition 5.3.9: (Universality of \(\Delta\))**

Let \((\mathcal{C}, \otimes, e, a, l, r)\) be a strict tensor category.

1. For every monoid \((A, \mu, \eta)\) in \(\mathcal{C}\), there is a unique strict tensor functor \(F : \Delta \to \mathcal{C}\) with \(F([1]) = A\), \(F(\sigma_0^i) = \mu\) and \(F(\delta_0^0) = \eta\).

2. For every comonoid \((C, \Delta, \epsilon)\) in \(\mathcal{C}\), there is a unique strict tensor functor \(F : \Delta^{op} \to \mathcal{C}\) with \(F([1]) = C\), \(F(\sigma_0^i) = \Delta\) and \(F(\delta_0^0) = \epsilon\).

**Proof:**

We prove 1., since 2. is obtained from 1. by reversing the direction of morphisms. By Lemma 5.3.6 the augmented simplex category \(\Delta\) is generated as a strict tensor category by the object \([1]\) and the morphisms \(\sigma_0^i : [2] \to [1]\) and \(\delta_0^0 : [0] \to [1]\) subject to the associativity and the unit relations (50). If \(F : \Delta \to \mathcal{C}\) is a strict tensor functor, then \(F\) is determined on the objects by \(F([1])\) since it satisfies \(F([0]) = e\) and \(F([n]) = F([1])^\otimes n = F([1])^\otimes n\) for all \(n \in \mathbb{N}_0\). Similarly, \(F\) is determined on the morphisms of \(\Delta\) by \(F(\sigma_0^i) : F([1])^\otimes F([1]) \to F([1])\) and \(F(\delta_0^0) : e \to F([1])\), since every morphism in \(\Delta\) is a composite of \(\sigma_0^0\) and \(\delta_0^0\) via the composition \(\circ\) and the tensor product \(\otimes\). The relations (50) then imply that \((A, \mu, \eta) = (F([1]), F(\sigma_0^0), F(\delta_0^0))\) is a monoid in \(\mathcal{C}\). Conversely, given a monoid \((A, \mu, \eta)\) in \(\mathcal{C}\), we can define a strict tensor functor \(F : \Delta \to \mathcal{C}\) by setting \(F([1]) = A, F(\sigma_0^i) = \mu\) and \(F(\delta_0^0) = \eta\), since the relations (50) are the defining relations of \(\Delta\).

**Corollary 5.3.10:** Let \((\mathcal{C}, \otimes, e, a, l, r)\) be a strict tensor category. Then:

1. Every monoid \((A, \mu, \eta)\) in \(\mathcal{C}\) defines an augmented cosimplicial object \(F : \Delta \to \mathcal{C}\) in \(\mathcal{C}\) with \(F([n]) = A^\otimes n\) and
   \[F(\sigma_0^i) = 1_A^\otimes \otimes \mu \otimes 1_A^\otimes(n-i-1) : A^\otimes(n+1) \to A^\otimes n,\]
   \[F(\delta_0^0) = 1_A^\otimes \otimes \eta \otimes 1_A^\otimes(n-i) : A^\otimes n \to A^\otimes(n+1).\]

2. Every comonoid \((C, \Delta, \epsilon)\) in \(\mathcal{C}\) defines an augmented simplicial object \(F : \Delta^{op} \to \mathcal{C}\) in \(\mathcal{C}\) with \(F([n]) = C^\otimes n\) and
   \[F(\sigma_0^i) = 1_C^\otimes \otimes \Delta_0 \otimes C^\otimes(n-i-1) : C^\otimes(n+1) \to C^\otimes n,\]
   \[F(\delta_0^0) = 1_C^\otimes \otimes \epsilon \otimes 1_C^\otimes(n-i) : C^\otimes(n+1) \to C^\otimes n.\]

In fact, the restriction to strict tensor categories in Proposition 5.3.9 and Corollary 5.3.10 is not necessary. It just simplifies the statement of the result. Mac Lane’s coherence theorem (see Remark 5.3.2) allows one to extend this result to non-strict tensor categories \(\mathcal{C}\) by replacing the strict tensor functor in Proposition 5.3.9 by a tensor functor and including associators and left and right unit constraints into the formulas. The resulting functor is then unique up to natural isomorphisms constructed from associators and unit constraints in \(\mathcal{C}\). We will make use of this in the following and also consider monoids and comonoids in non-strict tensor categories and the associated augmented cosimplicial and simplicial objects.
Example 5.3.11: Every group \( G \) is a comonoid in \((\text{Set}, \times, \{x\})\) with \( \Delta : G \to G \times G \), \( g \mapsto (g, g) \) and \( \epsilon : G \to \{x\} \), \( g \mapsto x \). The associated augmented simplicial object \( F : \Delta^{\text{op}} \to \text{Set} \) is given by \( F([n]) = G^{\times n} \) for all \( n \in \mathbb{N}_0 \) and
\[
F(\sigma_n') : G^{\times n} \to G^{\times (n+1)}, \quad (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, \ldots, g_n)
\]
\[
F(\delta_n') : G^{\times (n+1)} \to G^{\times n}, \quad (g_1, \ldots, g_{n+1}) \mapsto (g_1, \ldots, g_{i-1}, g_i+1, \ldots, g_{n+1}).
\]
By comparing this example to the chain complexes in Examples 3.2.6, 3.3.5 and 4.1.2, we see that it is related to the bar resolution of group cohomology.

Note, however, that this construction for group cohomology cannot be generalised to algebras over a commutative ring \( k \) since the diagonal map \( \Delta : A \to A \otimes A \) and the map \( \epsilon : A \to k \), \( a \mapsto 1 \) are not \( k \)-linear. Note also that this is not sufficiently general for our purposes. We do not want to associate augmented simplicial objects to specific groups or algebras, Lie algebras or topological spaces but to associate them systematically to all groups, algebras, Lie algebras and topological spaces at once. This suggests that the relevant comonoids should be given by functors. For this reason, we consider comonoids in a category \( \text{End}(\mathcal{D}) = \text{Fun}(\mathcal{D}, \mathcal{D}) \) of endofunctors, the comonads from Example 5.3.8.

Example 5.3.12: Let \((C, \Delta, \epsilon)\) be a comonad in a small category \( \mathcal{D} \).

- By Proposition 5.3.9 and Corollary 5.3.10 the comonad \((C, \Delta, \epsilon)\) determines a unique augmented simplicial object \( S_C : \Delta^{\text{op}} \to \text{End}(\mathcal{D}) \) given by
\[
S_C([n]) = C^n : \mathcal{D} \to \mathcal{D}, \quad S_C(\sigma_n') = C^n \Delta^{n-1} : C^n \to C^{n+1}, \quad S_C(\delta_n') = C^n \epsilon C^{n-1} : C^{n+1} \to C^n.
\]
- This defines a functor \( F_C : \mathcal{D} \to \text{Fun}(\Delta^{\text{op}}, \mathcal{D}) \) that assigns
  - to an object \( D \) in \( \mathcal{D} \) the functor \( F_C(D) : \Delta^{\text{op}} \to \mathcal{D} \) with \( F_C(D)([n]) = C^n(D) \) and \( F_C(D)(\alpha) = S_C(\alpha)D : C^n(D) \to C^m(D) \) for all monotonic maps \( \alpha : [m] \to [n] \),
  - to a morphism \( f : D \to D' \) in \( \mathcal{D} \) the natural transformation \( F_C(f) : F_C(D) \to F_C(D') \) with component morphisms \( F_C(f)_[n] = C^n(f) : C^n(D) \to C^n(D') \).
- Post-composition with a functor \( H : \mathcal{D} \to \mathcal{A} \) into an abelian category \( \mathcal{A} \) defines a functor \( HF_C : \mathcal{D} \to \text{Fun}(\Delta^{\text{op}}, \mathcal{A}) \) that assigns
  - to an object \( D \) the functor \( HF_C(D) : \Delta^{\text{op}} \to \mathcal{A} \) with \( HF_C(D)([n]) = HC^n(D) \) and \( HF_C(D)(\alpha) = HS_C(\alpha)D : HC^n(D) \to HC^m(D) \) for all monotonic maps \( \alpha : [m] \to [n] \),
  - to a morphism \( f : D \to D' \) in \( \mathcal{D} \) the natural transformation \( HF_C(f) : HF_C(D) \to HF_C(D') \) with component morphisms \( HF_C(f)_[n] = HC^n(f) : HC^n(D) \to HC^n(D') \).
- By restricting the functor \( HF_C(D) : \Delta^{\text{op}} \to \mathcal{A} \) to the full subcategory \( \Delta^{+\text{op}} \subset \Delta^{\text{op}} \) we obtain a simplicial object in the abelian category \( \mathcal{A} \).
- Applying the Moore complex functor \( M : \text{Fun}(\Delta^{+\text{op}}, \mathcal{A}) \to \text{Ch}_{n \geq 0, \mathcal{A}} \) from Proposition 5.2.2, we obtain a functor \( G : \mathcal{D} \to \text{Ch}_{n \geq 0, \mathcal{A}} \) that assigns
  - to an object \( D \) in \( \mathcal{D} \) the Moore complex \( MHF_C(D)_\bullet \) in \( \mathcal{A} \),
  - to a morphism \( f : D \to D' \) the chain map \( MHF_C(f)_\bullet : MHF_C(D)_\bullet \to MHF_C(D')_\bullet \).
- Applying the normalised chain complex functor \( N : \text{Fun}(\Delta^{+\text{op}}, \mathcal{A}) \to \text{Ch}_{A \geq 0} \) from Proposition 5.2.2, we obtain a functor \( G' : \mathcal{D} \to \text{Ch}_{A \geq 0} \) that assigns.
- to an object $X$ in $\mathcal{D}$ the normalised chain complex $\text{NHF}_C(X)_\bullet$ in $\mathcal{A}$,
- to a morphism $f : X \to Y$ in $\mathcal{C}$ the chain map $\text{NHF}_C(f)_\bullet : \text{NHF}_C(X)_\bullet \to \text{NHF}_C(Y)_\bullet$.

Example 5.3.12 gives a general formalism that allows one to construct chain complexes in an abelian category $\mathcal{A}$ from a comonad in a small category $\mathcal{C}$ and a functor $H : \mathcal{C} \to \mathcal{A}$. If $\mathcal{C}$ is already abelian, we can choose $H = \text{id}_C$ and apply the Moore complex functor directly to the functor $F_C$ in Example 5.3.12. We can also drop the requirement that $\mathcal{C}$ is small, since we can always restrict attention to a small full subcategory of $\mathcal{C}$.

We now focus on the case, where $\mathcal{C}$ is abelian and investigate the chain complexes defined by comonads in $\mathcal{C}$. It turns out that under a small additional assumption, the resulting chain complexes generalise the pattern observed in Example 5.3.11. Via the construction in Example 5.3.12 comonads in an abelian category $\mathcal{A}$ define resolutions of objects in $\mathcal{A}$.

**Definition 5.3.13:** Let $(C, \Delta, \epsilon)$ be a monad in an abelian category $\mathcal{A}$. An object $A$ in $\mathcal{A}$ is called $C$-projective if there is a morphism $f : A \to C(A)$ with $\epsilon_A \circ f = 1_A$.

**Proposition 5.3.14:** Let $(C, \Delta, \epsilon)$ be a comonad in an abelian category $\mathcal{A}$. Then for any $C$-projective object $A$ in $\mathcal{A}$ the chain complex

$$C(A)_\bullet = \ldots \xrightarrow{d_2} C^2(A) \xrightarrow{d_1} C(A) \xrightarrow{\epsilon_A} A \to 0$$

with $d_n = \sum_{i=0}^n (-1)^i C^i\epsilon C^{n-i} : C^{n+1}(A) \to C^n(A)$ is exact. It is called the canonical resolution of $A$ defined by the comonad $C$.

**Proof:**

Let $f : A \to C(A)$ a morphism with $\epsilon_A \circ f = 1_A$. We consider the morphisms

$$h_n = (-1)^n C^{n+1}(f) : C^{n+1}(A) \to C^{n+2}(A) \text{ for } n \in \mathbb{N}_0 \quad h_{-1} = f : A \to C(A)$$

and show that they define a chain homotopy $h_\bullet : 1_{C(A)}_\bullet \Rightarrow 0_{C(A)}_\bullet$. By Proposition 3.3.4 we then have $H_n(C(A)_\bullet) = 0$ for all $n \in \mathbb{N}_0$.

The boundary operator of $C(A)_\bullet$ is given by $d_n = \sum_{i=0}^n (-1)^i d_i : C^{n+1}(A) \to C^n(A)$, where $d_i^\ast = C^i\epsilon C^{n-i} : C^{n+1}(A) \to C^n(A)$ are the component morphisms of the natural transformation $C^\ast \epsilon C^{n-\ast} : C^{n+1} \to C^n$. This implies

$$d_{n+1}^\ast \circ h_n = (-1)^{n+1} C^{n+1}(\epsilon_A) \circ C^{n+1}(f) = (-1)^{n+1} C^{n+1}(\epsilon_A \circ f) = (-1)^{n+1} 1_{C^{n+1}(A)}$$

$$d_i^\ast \circ h_0 = -\epsilon_C(A) \circ C(f) = -f \circ \epsilon_A$$

$$d_{n+1}^\ast \circ h_n = (-1)^{n+1} C^n(\epsilon_C^{n+1-i}(A)) \circ C^{n+1}(f) = (-1)^{n+1} C^n(f) \circ C^n(\epsilon_C^{n+1-i}(A)) = -h_{n-1} \circ d_i^\ast$$

Combining these expressions and taking an alternating sum over the morphisms $d_i^\ast$ we obtain

$$d_{n+1}^\ast \circ h_n + h_{n-1} \circ d_n = \sum_{i=0}^n (-1)^i d_{n+1}^\ast \circ h_n + \sum_{i=0}^n (-1)^i h_{n-1} \circ d_i^\ast = 1_{C^{n+1}(A)} + \sum_{i=0}^n (-1)^{i+1} h_{n-1} \circ d_i^\ast = 1_{C^{n+1}(A)}$$

$$d_{1}^\ast \circ h_0 + h_{-1} \circ d_0 = d_1^\ast \circ h_0 - d_1^\ast \circ h_0 + f \circ d_0^\ast = -f \circ \epsilon_A + 1_{C(A)} + f \circ \epsilon_A = 1_{C(A)}$$

This shows that $h_\bullet : 1_{C(A)}_\bullet \Rightarrow 0_{C(A)}_\bullet$ is a chain homotopy and $H_n(C(A)_\bullet) = 0$ for all $n \in \mathbb{N}_0$. □
5.4 Resolutions from adjoint functors

The results of the last subsection give a systematic procedure to construct resolutions of objects in an abelian category \( \mathcal{A} \) from a comonad in \( \mathcal{A} \). This raises the question how to find comonads in \( \mathcal{A} \) and if there is a systematic way of doing so. It turns out that comonads are rather common, since they arise from adjoint functors. Pairs of adjoint functors occur in many situations and are directly related to universal properties of certain constructions.

**Definition 5.4.1:** Let \( \mathcal{C}, \mathcal{D} \) be categories and \( F : \mathcal{C} \to \mathcal{D}, \ G : \mathcal{D} \to \mathcal{C} \) functors.

One calls \( F \) left adjoint to \( G \) and \( G \) right adjoint to \( F \) and writes \( F \dashv G \), if the functors \( \text{Hom}(F(\cdot), \cdot), \text{Hom}(\cdot, G(\cdot)) : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \text{Set} \) are naturally isomorphic.

In other words, there is a family of bijections \( \phi_{C,D} : \text{Hom}_\mathcal{C}(C,G(D)) \to \text{Hom}_\mathcal{D}(F(C),D) \), indexed by objects \( C \) in \( \mathcal{C} \) and \( D \) in \( \mathcal{D} \), such that the diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(C,G(D)) & \xrightarrow{\phi_{C,D}} & \text{Hom}_\mathcal{D}(F(C),D) \\
\downarrow^{\text{Hom}(f,G(g))} & & \downarrow^{\text{Hom}(F(f),g)} \\
\text{Hom}_\mathcal{C}(C',G(C')) & \xrightarrow{\phi_{C',D'}} & \text{Hom}_\mathcal{D}(F(C'),D').
\end{array}
\]

(52)

commutes for all morphisms \( f : C' \to C \) in \( \mathcal{C} \) and \( g : D \to D' \) in \( \mathcal{D} \).

**Example 5.4.2:**

1. **Forgetful functor and freely generated modules:**  
   Let \( R \) be a ring. The functor \( \langle \cdot \rangle_R : \text{Set} \to \text{R-Mod} \) that sends a set \( A \) to the free \( R \)-module \( \langle A \rangle_R \) generated by \( A \) and a map \( f : A \to B \) to the \( R \)-linear map \( \langle f \rangle_R : \langle A \rangle_R \to \langle B \rangle_R \) with \( \langle f \rangle_R \circ \iota_A = \iota_B \circ f \) is left adjoint to the forgetful functor \( G : \text{R-Mod} \to \text{Set} \).

   The map \( \phi_{A,M} : \text{Hom}_\text{Set}(A,G(M)) \to \text{Hom}_\text{R-Mod}(\langle A \rangle_R,M) \), that sends a map \( f : A \to M \) to the unique \( R \)-linear map \( \phi_{A,M}(f) : \langle A \rangle_R \to M \) with \( \phi_{A,M}(f) \circ \iota_A = f \) is a bijection for all sets \( A \) and \( R \)-modules \( M \). It satisfies (52) since

   \[
   \phi_{A',M'}(G(g) \circ h \circ f) \circ \iota_{A'} = g \circ h \circ f = g \circ \phi_{A,M}(h) \circ \iota_A \circ f = g \circ \phi_{A,M}(h) \circ \langle f \rangle_R \circ \iota_{A'}
   \]

   for all \( R \)-linear maps \( g : M \to M' \) and maps \( h : A \to M, \ f : A' \to A \).

2. **Inclusion functor for abelian groups and abelisation:**
   The abelisation functor \( \text{Ab} : \text{Grp} \to \text{Ab} \) that assigns to a group \( G \) the abelian group \( \text{Ab}(G) = G/[G,G] \) and to a group homomorphism \( f : G \to H \) the induced group homomorphism \( \text{Ab}(f) : G/[G,G] \to H/[H,H] \) with \( \text{Ab}(f) \circ \pi_G = \pi_H \circ f \) is left adjoint to the inclusion functor \( I : \text{Ab} \to \text{Grp} \).

   The map \( \phi_{G,A} : \text{Hom}_\text{Grp}(G,I(A)) \to \text{Hom}_\text{Ab}(\text{Ab}(G),A) \), that sends a group homomorphism \( f : G \to A \) to the unique group homomorphism \( \phi_{G,A}(f) : \text{Ab}(G) \to A \) with \( \phi_{G,A}(f) \circ \pi_G = f \) is a bijection for all groups \( G \) and abelian groups \( A \) that satisfies (52).
3. Tensor products and Hom-functors:

- For any $R$-right module $M$, the functor $M \otimes_R - : R\text{-Mod} \to \text{Ab}$ is left adjoint to the functor $\text{Hom}(M, -) : \text{Ab} \to R\text{-Mod}$.

- For any $R$-left module $N$ the functor $- \otimes_R N : R^{\text{op}}\text{-Mod} \to \text{Ab}$ is left adjoint to the functor $\text{Hom}(-, N) : \text{Ab} \to R^{\text{op}}\text{-Mod}$.

We prove the claim for $R$-right modules $M$. For an abelian group $A$ and $R$-left module $L$ we equip $\text{Hom}_{\text{Ab}}(M, A)$ with the $R$-module structure $(r \triangleright \phi)(m) = \phi(m \triangleleft r)$ and define

$$\phi_{L,A} : \text{Hom}_{R\text{-Mod}}(L, \text{Hom}_{\text{Ab}}(M, A)) \to \text{Hom}_{\text{Ab}}(M \otimes_R L, A), \quad \psi : L \to \text{Hom}_{\text{Ab}}(M, A), \ l \mapsto \psi_l \mapsto \chi : M \otimes_R L \to A, \ m \otimes l \mapsto \psi_l(m).$$

The map $\chi : M \otimes_R L \to A, m \otimes l \mapsto \psi_l(m)$ is well defined, since the $R$-linearity of the map $\psi : L \to \text{Hom}_{\text{Ab}}(M, A)$ implies that $\chi' : M \times L \to A, (m, l) \mapsto \psi_l(m)$ is $R$-bilinear: $\chi'(m, r \triangleright l) = \psi_{r \triangleright l}(m) = (r \triangleright \psi_l)(m) = \psi_l(m \triangleleft r) = \chi'(m \triangleleft r, l)$ for all $r \in R, l \in L$ and $m \in M$. By the universal property of the tensor product, it induces a unique group homomorphism $\chi : M \otimes_R L \to A$ with $\chi(m \otimes l) = \chi(m, l)$.

The inverse of $\phi_{L,A}$ is given by

$$\phi_{L,A}^{-1} : \text{Hom}_{\text{Ab}}(M \otimes_R L, A) \to \text{Hom}_{R\text{-Mod}}(L, \text{Hom}_{\text{Ab}}(M, A))$$

$$\chi : M \otimes_R L \to A, \ l \mapsto \psi_l \quad \mapsto \quad \psi : L \to \text{Hom}_{\text{Ab}}(M, A), \ l \mapsto \psi_l \text{ with } \psi_l(m) = \chi(m \otimes l).$$

As we have $\psi_{r \triangleright l}(m) = \chi(m \otimes (r \triangleright l)) = \chi((m \triangleleft r) \otimes l) = \psi_l(m \triangleleft r)$, the map $\psi_l$ is indeed $R$-linear, and a short computation shows that the diagram (52) commutes for all $R$-linear maps $f : L' \to L$ and all group homomorphisms $g : A \to A'$.

4. Singular homology and geometric realisation:

The geometric realisation functor $\text{Geom} : \text{Fun}(\Delta^{\text{op}}, \text{Set}) \to \text{Top}$ from Example [5.1.9] is left adjoint to the singular functor $\text{Sing} : \text{Top} \to \text{Fun}(\Delta^{\text{op}}, \text{Set})$ from Example [5.1.8].

For every functor $S : \Delta^{\text{op}} \to \text{Set}$ and topological space $X$ the map

$$\phi_{S,X} : \text{Hom}_{\text{Fun}(\Delta^{\text{op}}, \text{Set})}(S, \text{Sing}(X)) \to \text{Hom}_{\text{Top}}(\text{Geom}(S), X)$$

$$\eta \quad \mapsto \quad \phi_{S,X}(\eta) : \text{Geom}(S) \to X, [(x, p)] \mapsto \eta_{[n+1]}(x)(p)$$

is a bijection with inverse

$$\phi_{S,X}^{-1} : \text{Hom}_{\text{Top}}(\text{Geom}(S), X) \to \text{Hom}_{\text{Fun}(\Delta^{\text{op}}, \text{Set})}(S, \text{Sing}(X))$$

$$f \quad \mapsto \quad \phi_{S,X}^{-1}(f) \text{ with } \phi_{S,X}^{-1}(f)_{[n+1]}(x)(p) = f([(x, p)]).$$

for which the diagram (52) commutes (Exercise).
Proposition 5.4.3: Let $\phi : R \to S$ be a ring homomorphism and $\text{Res} : S\text{-Mod} \to R\text{-Mod}$ the restriction functor that sends an $S$-module $(M, \triangleright_S)$ to the $R$-module $(M, \triangleright_R)$ with $r \triangleright_R m = \phi(r) \triangleright_S m$ and every $S$-linear map $f : M \to M'$ to itself.

1. The induction functor $\text{Ind} = S \otimes_R -$ : $R\text{-Mod} \to S\text{-Mod}$ that sends
   - an $R$-module $M$ to the $S$-module $\text{Ind}(M) = S \otimes_R M$ with $s \triangleright (s' \otimes m) = (ss') \otimes m$,
   - an $R$-linear map $f : M \to M'$ to the $S$-linear map $\text{Ind}(f) = \text{id}_S \otimes f$

   is left adjoint to $\text{Res}$.

2. The coinduction functor $\text{Coind} = \text{Hom}_R(S, -) : R\text{-Mod} \to S\text{-Mod}$ that sends
   - an $R$-module $M$ to the $S$-module $\text{Hom}_R(S, M)$ with $(s \triangleright f)(s') = f(s' \cdot s)$,
   - an $R$-linear map $f : M \to M'$ to $\text{Hom}_R(S, f) : g \mapsto f \circ g$

   is right adjoint to $\text{Res}$.

Proof:
1. By Lemma 1.2.27 the $(S, R)$-bimodule structure on $S$ given by $s \triangleright s' = s \cdot s'$ and $s \triangleleft r = s \cdot \phi(r)$ defines an $S$-left-module structure on the abelian group $S \otimes_R M$ given by $s \triangleright (s' \otimes m) = (s \cdot s') \otimes m$.
   To show that $\text{Ind}$ is left adjoint to $\text{Res}$, we consider the group homomorphisms

   $\phi_{M,N} : \text{Hom}_R(M, \text{Res}(N)) \to \text{Hom}_S(\text{Ind}(M), N)$, \hspace{1cm} $\phi_{M,N}(f)(s \otimes m) = s \triangleright f(m)$

   $\psi_{M,N} : \text{Hom}_S(\text{Ind}(M), N) \to \text{Hom}_R(M, \text{Res}(N))$, \hspace{1cm} $\psi_{M,N}(g)(m) = g(1 \otimes m)$.

   A direct computation shows that $\phi_{M,N}^{-1} = \psi_{M,N}$. To prove that the diagram (52) commutes, we compute for all $R$-linear maps $f : M' \to M$, $h : M \to N$ and $S$-linear maps $g : N \to N'$

   \[ g \circ \phi_{M,N}(h) \circ (\text{id}_S \otimes f)(s \otimes m') = g \circ \phi_{M,N}(h)(s \otimes f(m')) = g(s \triangleright h \circ f(m')) \]

   \[ = s \triangleright (g \circ h \circ f(m')) = \phi_{M',N'}(g \circ h \circ f)(s \otimes m'). \]

2. We consider the ring $S$ with the $R$-left module structure $r \triangleright s = \phi(r) \cdot s$ and the abelian group $\text{Hom}_R(S, M)$ with the $S$-left module structure $(s \triangleright f)(s') = f(s' \cdot s)$. To show that $\text{Coind}$ is right adjoint to $\text{Res}$ and $\text{Res}$ left adjoint to $\text{Coind}$, we consider the maps

   $\phi_{M,N} : \text{Hom}_R(\text{Res}(N), M) \to \text{Hom}_S(N, \text{Hom}_R(S, M))$, \hspace{1cm} $\phi_{M,N}(f)(s) = f(s \triangleright n)$

   $\psi_{M,N} : \text{Hom}_S(N, \text{Hom}_R(S, M)) \to \text{Hom}_R(\text{Res}(N), M)$, \hspace{1cm} $\psi_{M,N}(g)(n) = g(n)(1)$.

   As in 1., one can show that $\phi_{M,N}^{-1} = \psi_{M,N}$ and that $\phi_{M,N}$ makes the diagram (52) commute. \qed

Corollary 5.4.4: For every ring $S$, the functor $\text{Ind} = S \otimes_Z -$ : $Z\text{-Mod} \to S\text{-Mod}$ is left adjoint to the forgetful functor $\text{Res} : S\text{-Mod} \to Z\text{-Mod}$, and the coinduction functor $\text{Coind} = \text{Hom}_Z(S, -) : Z\text{-Mod} \to S\text{-Mod}$ is right adjoint to the forgetful functor $\text{Res} : S\text{-Mod} \to Z\text{-Mod}$.

Proof:
This is Proposition 5.4.3 for $R = Z$, where $\text{Res} : S\text{-Mod} \to Z\text{-Mod}$ is the forgetful functor. \qed

These examples show that adjoint functors arise in many contexts in algebra and topology and are often related to certain canonical constructions such as forgetful functors, freely generated modules or tensoring over a ring. To show that pairs of adjoint functors define comonads, we require an alternative characterisation of adjoint functors in terms of natural transformations.
Proposition 5.4.5: A functor \( F : \mathcal{C} \to \mathcal{D} \) is left adjoint to \( G : \mathcal{D} \to \mathcal{C} \) if and only if there are natural transformations \( \epsilon : FG \to \text{id}_\mathcal{D} \) and \( \eta : \text{id}_\mathcal{C} \to GF \) such that
\[
G(\epsilon_D) \circ \eta_{G(D)} = 1_{G(D)}, \quad \epsilon_{F(C)} \circ F(\eta_C) = 1_{F(C)} \quad \forall C \in \text{Ob} \mathcal{C}, \ D \in \text{Ob} \mathcal{D}. \quad (53)
\]

Proof:
1. Let \( F : \mathcal{C} \to \mathcal{D} \) be left adjoint to \( G : \mathcal{D} \to \mathcal{C} \). Then there are bijections
\[
\phi_{G(D),D} : \text{Hom}_\mathcal{C}(G(D), G(D)) \to \text{Hom}_\mathcal{D}(FG(D), D)
\]
\[
\phi_{C,F(C)}^{-1} : \text{Hom}_\mathcal{D}(F(C), F(C)) \to \text{Hom}_\mathcal{C}(C, GF(C))
\]
for all objects \( C \) in \( \mathcal{C} \) and \( D \) in \( \mathcal{D} \). We define the natural transformations \( \epsilon : FG \to \text{id}_\mathcal{D} \) and \( \eta : \text{id}_\mathcal{C} \to GF \) by specifying their component morphisms:
\[
\epsilon_D := \phi_{G(D),D}(1_{G(D)}): FG(D) \to D \quad \eta_C := \phi_{C,F(C)}^{-1}(1_{F(C)}): C \to GF(C).
\]
The commuting diagram (52) in Definition 5.4.1 implies for all morphisms \( f : D \to D' \) in \( \mathcal{D} \):
\[
\epsilon_D' \circ FG(f) = \phi_{G(D'),D'}(1_{G(D')}) \circ FG(f) \quad \phi_{G(D),D}(1_{G(D)} \circ G(f)) = \phi_{G(D),D}(G(f))
\]
\[
= \phi_{G(D),D}(G(f) \circ 1_{G(D)}) \quad f \circ \phi_{G(D),D}(1_{G(D)}) = f \circ \epsilon_D.
\]
This shows that the morphisms \( \epsilon_D : FG(D) \to D \) define a natural transformation \( \epsilon : FG \to \text{id}_\mathcal{D} \). Diagram (52) then implies for all objects \( C \) in \( \mathcal{C} \)
\[
\epsilon_{F(C)} \circ F(\eta_C) = \phi_{G(C),F(C)}(1_{G(C)}) \circ F(\phi_{C,F(C)}^{-1}(1_{F(C)}))
\]
\[
\phi_{C,F(C)}(GF(C) \circ \phi_{C,F(C)}^{-1}(1_{F(C)})) = \phi_{C,F(C)} \circ \phi_{C,F(C)}^{-1}(1_{F(C)}) = 1_{F(C)}.
\]
The proofs for \( \eta : \text{id}_\mathcal{C} \to GF \) and \( G(\epsilon_D) \circ \eta_{G(D)} = 1_{G(D)} \) are analogous.

2. Let \( \epsilon : FG \to \text{id}_\mathcal{D} \) and \( \eta : \text{id}_\mathcal{C} \to GF \) be natural transformations that satisfy (53). Consider for all objects \( C \) in \( \mathcal{C} \) and \( D \) in \( \mathcal{D} \) the maps
\[
\phi_{C,D} = \text{Hom}(1_{F(C)}, \epsilon_D) \circ F : \text{Hom}_\mathcal{C}(C, G(D)) \to \text{Hom}_\mathcal{D}(F(C), D), \quad f \mapsto \epsilon_D \circ F(f)
\]
\[
\psi_{C,D} = \text{Hom}(\eta_C, 1_{G(D)}) \circ G : \text{Hom}_\mathcal{D}(F(C), D) \to \text{Hom}_\mathcal{C}(C, G(D)), \quad g \mapsto G(g) \circ \eta_C.
\]
Then we have for all morphisms \( f : C \to G(D) \) in \( \mathcal{C} \) and \( g : F(C) \to D \) in \( \mathcal{D} \)
\[
\psi_{C,D} \circ \phi_{C,D}(f) = G(\epsilon_D) \circ GF(f) \circ \eta_C \quad \phi_{C,D} \circ \psi_{C,D}(g) = \epsilon_D \circ FG(g) \circ F(\eta_C)
\]
\[
\phi_{C,D} \circ \psi_{C,D}(g) = \epsilon_D \circ FG(g) \circ F(\eta_C) \quad \psi_{C,D} \circ \phi_{C,D}(f) = G(\epsilon_D) \circ GF(f) \circ \eta_C.
\]
This shows that \( \psi_{C,D} = \phi_{C,D}^{-1} \) and \( \phi_{C,D} : Hom_\mathcal{C}(C, G(D)) \to Hom_\mathcal{D}(F(C), D) \) is a bijection. To verify that the diagram (52) in Definition 5.4.1 commutes, consider morphisms \( f : C' \to C \), \( h : C \to G(D) \) in \( \mathcal{C} \) and \( g : D \to D' \) in \( \mathcal{D} \) and compute
\[
\phi_{C',D'}(G(g) \circ h \circ f) = \epsilon_{D'} \circ FG(g) \circ F(h) \circ F(f) \equiv g \circ \phi_{C,D}(h) \circ F(f).
\]
With Proposition 5.4.5 we can show that every pair of adjoint functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) defines a monad in \( \mathcal{C} \) and a comonad in \( \mathcal{D} \). One on hand, this motivates why monads and comonads play an important role in homological algebra. They arise from simple and global constructions that define the underlying adjoint functors. If we take the opposite viewpoint of a comonad as the fundamental structure that defines canonical resolutions, this motivates why the resolutions for the homology and cohomology theories from section 2 are all obtained by iterating constructions such as tensor products or direct products of groups.
Proposition 5.4.6: Let \( F : \mathcal{C} \to \mathcal{D} \) be left adjoint to \( G : \mathcal{D} \to \mathcal{C} \) with natural transformations \( \epsilon : FG \to \text{id}_\mathcal{D} \) and \( \eta : \text{id}_\mathcal{C} \to GF \) satisfying (53). Then \( (GF, G\epsilon F, \eta) \) is a monad in \( \mathcal{C} \) and \( (F, F\eta G, \epsilon) \) is a comonad in \( \mathcal{D} \).

Proof:
We prove the claim for the comonad by verifying that \( (GF, F\eta G, \epsilon) \) satisfies the conditions in Example 5.3.8, 6. The component morphisms of \( \Delta = F\eta G : FG \to FGFG \) are given by \( \Delta_D = (\eta_{\mathcal{G}(D)}) : FG(D) \to FGFG(D) \). From the naturality of \( \eta : \text{id}_\mathcal{C} \to GF \) we then obtain the first condition in Example 5.3.8, 6.

\[
FG(\Delta_D) \circ \Delta_D = FG(\eta_{\mathcal{G}(D)}) \circ F(\eta_{\mathcal{G}(D)}) = F(GF(\eta_{\mathcal{G}(D)}) \circ \eta_{\mathcal{G}(D)}) = F(\eta_{\mathcal{G}FG\mathcal{G}(D)}) \circ F(\eta_{\mathcal{G}(D)}) = \Delta_{FG} \circ \Delta_D,
\]

and condition (53) implies the second condition in Example 5.3.8, 6.

\[
\epsilon_{FG(D)} \circ \Delta_D = \epsilon_{FG(D)} \circ F(\eta_{\mathcal{G}(D)}) = F(\eta_{\mathcal{G}(D)}) = F(1_{\mathcal{G}(D)}) = 1_{FG(D)}.
\]

Corollary 5.4.7: Let \( \mathcal{A} \) be an abelian category and \( \mathcal{C} \) a category. If a functor \( G : \mathcal{A} \to \mathcal{C} \) has a left adjoint \( F : \mathcal{C} \to \mathcal{A} \), then the comonad \( (FG, F\eta G, \epsilon) \) in \( \mathcal{A} \) defines a resolution

\[
\mathcal{C}(A)_* = \ldots \xrightarrow{d_2} \mathcal{C}^2(A) \xrightarrow{d_1} \mathcal{C}(A) \xrightarrow{\Delta_A} A \to 0
\]

with \( d_n = \sum_{i=0}^{n} (-1)^i C^i \mathcal{C}^n \mathcal{A} : C^{n+1}(A) \to C^n(A) \) of every object \( A \) in \( \mathcal{A} \) that is isomorphic to an object in the image of \( F \).

Proof:
By Propositions 5.3.14 and 5.4.6 it is sufficient to show that every object \( A \) in \( \mathcal{A} \) for which there is an isomorphism \( \phi : A \to F(C) \) is \( C \)-projective. This follows because the morphism \( f = FG(\phi^{-1}) \circ F(\eta_C) \circ \phi : A \to FG(A) \) satisfies

\[
\epsilon_A \circ f = \epsilon_A \circ FG(\phi^{-1}) \circ F(\eta_C) \circ \phi \overset{\text{nat}}{=} \phi^{-1} \circ \epsilon_{\mathcal{C}(C)} \circ F(\eta_C) \circ \phi \overset{\text{nat}}{=} \phi^{-1} \circ \phi = 1_A.
\]

With this corollary we recover the standard resolutions for Hochschild (co)homology and group cohomology from Example 4.1.3 and Example 4.1.2 and obtain a free standard resolution of every \( R \)-module \( M \). All that is required is a pair of adjoint functors for the structures under consideration. In the case of a \( k \)-algebra \( A \), we can take the restriction functor \( \text{Res} : \text{Mod}_A \to \text{Mod}_k \) and the induction functor \( \text{Ind} = A \otimes_k - : \text{Mod}_k \to \text{Mod}_A \) from Proposition 5.4.3 and in case of an \( R \)-module \( M \) the forgetful functor \( G : \text{Mod}_R \to \text{Set} \) and its left adjoint \( \langle \rangle_R : \text{Set} \to \text{Mod}_R \) from Example 5.4.2, 1. The bar resolution of group cohomology is recovered from the first case by setting \( A = k[G] \).

Example 5.4.8: (The Hochschild resolution from a comonad)
Let \( k \) be a commutative ring and \( \phi : k \to R \) a ring homomorphism.

- By Proposition 5.4.3 the functor \( G = \text{Res} : \text{Mod}_R \to \text{Mod}_k \) is right adjoint to the induction functor \( F = \text{Ind} = R \otimes_k - : \text{Mod}_k \to \text{Mod}_R \). The natural transformations \( \epsilon : FG \to \text{id}_{\text{Mod}_R} \) and \( \eta : \text{id}_{\text{Mod}_k} \to GF \) from Proposition 5.4.6 are given by their component morphisms

\[
\epsilon_M : R \otimes_k M \to M, \ r \otimes m \mapsto r \triangleright m \quad \quad \eta_M : M \to R \otimes_k M, \ m \mapsto 1 \otimes m.
\]
• The functor \( FG = R \otimes_k - : R\text{-Mod} \to R\text{-Mod} \) sends an \( R \)-module \( M \) to \( FG(M) = R \otimes_k M \) with \( r \triangleright (r' \otimes m) = (rr') \otimes m \) and an \( R \)-linear map \( f : M \to M' \) to \( \text{id}_R \otimes f : R \otimes_k M \to R \otimes_k M' \).

• By Proposition 5.4.6 \((FG, \epsilon, \phi)\) is a comonad in \( R\text{-Mod} \) and Corollary 5.4.7 yields a resolution \( C(M) \), for every \( FG \)-projective \( R \)-module \( M \) given by

\[
C(M)_n = (FG)^{n+1}(M) = R^{\otimes_k (n+1)} \otimes_k M
\]

\[
d_n = \sum_{i=0}^n (-1)^i (FG)^i \epsilon(FG)^{n-i} R \otimes_{k} (n+1) \otimes_k M \to R^{\otimes_k (n+1)} \otimes_k M
\]

\[
r_0 \otimes \cdots \otimes r_n \otimes m \mapsto (r_0 r_1) \otimes \cdots \otimes (r_{n-1} r_n) \otimes m + (-1)^n r_0 \otimes \cdots \otimes (r_n \triangleright m)
\]

If \( M \) is an \((R, S)\)-bimodule, this resolution is also a resolution in \( R \otimes S^{op}\text{-Mod} \).

• By Corollary 5.4.7 the \( R \)-module \( M = R \) is \( FG \)-projective since \( R \cong R \otimes_k k = FG(k) \). By setting \( M = R \), we obtain a resolution of \( R \) in \( R\text{-Mod} \)

\[
\cdots \to R \otimes_k 5 \xrightarrow{d_5} R \otimes_k 4 \xrightarrow{d_4} R \otimes_k 3 \xrightarrow{d_3} R \otimes_k 2 \xrightarrow{d_2} R \otimes_k 1 \xrightarrow{d_1} R \otimes_k 0 \to 0
\]

\[
d_n = \sum_{i=0}^n (-1)^i d_n : R^{\otimes_k (n+2)} \to R^{\otimes_k (n+1)}
\]

\[
r_0 \otimes \cdots \otimes r_{n+1} \mapsto (r_0 r_1) \otimes \cdots \otimes r_{n+1} - r_0 \otimes (r_1 r_2) \otimes \cdots \otimes r_{n+1} \pm \cdots + (-1)^n r_0 \otimes \cdots \otimes (r_n r_{n+1}).
\]

• If \( A \) is an algebra over \( k \), then we can choose \( R = A \) and \( \phi : k \to A, \lambda \mapsto \lambda 1_A \). In this case, \( A \) is an \((A, A)\)-bimodule with \( a \triangleright b \triangleleft c = abc \), and the resulting resolution of \( A \) in \( A \otimes A^{op}\text{-mod} \) is the Hochschild resolution from Example 4.1.3

\section*{Example 5.4.9: (The bar resolution from a comonad)}

Let \( k \) be a commutative ring and \( G \) a group.

• By setting \( R = k[G] \) in Example 5.4.8 we obtain a comonad \((FG, \epsilon, \phi_{FG})\) from the forgetful functor \( G : k[G]\text{-Mod} \to k\text{-Mod} \) and its left adjoint \( F = k[G] \otimes_k - : k\text{-Mod} \to k[G]\text{-Mod} \).

As \( k[G]^{\otimes n} \cong (G^{\times n})_k \) as a \( k \)-module, the associated chain complex for a \( k[G] \)-module \( M \) is

\[
\cdots \to \langle G^{\times 3} \rangle_k \otimes_k M \xrightarrow{d_3} \langle G^{\times 2} \rangle_k \otimes_k M \xrightarrow{d_2} \langle G \rangle_k \otimes_k M \xrightarrow{\phi_{FG}} M \to 0
\]

\[
d_n : \langle G^{\times (n+1)} \rangle_k \otimes_k M \to \langle G^{\times n} \rangle_k \otimes_k M
\]

\[
(g_0 \otimes \cdots \otimes g_n) \otimes m \mapsto (g_0 g_1, g_2, \ldots, g_n) \otimes m - (g_0, g_1 g_2, \ldots, g_n) \otimes m \pm \cdots + (-1)^n (g_0, \ldots, g_{n-1}) \otimes (g_n \triangleright m).
\]

• The trivial \( k[G] \)-module \( M = k \) is \( FG \)-projective since \( FG(k) = k[G] \otimes_k k \cong k[G] \) and the \( k[G] \)-linear maps \( f : k \to k[G], \lambda \mapsto \lambda e \) and \( \epsilon_k : k[G] \to k, \lambda g \mapsto \lambda s \) satisfy \( \epsilon_k \circ f = \text{id}_k \).

• Setting \( M = k \) and noting that \( \langle G^{\times (n+1)} \rangle_k \otimes_k k \cong \langle G^{\times n} \rangle_k \cong \langle G^{\times n} \rangle_{k[G]} \) we obtain the bar-resolution from Example 4.1.2

\[
\cdots \to \langle G^{\times 2} \rangle_{k[G]} \xrightarrow{d_3} \langle G \rangle_{k[G]} \xrightarrow{d_2} k[G] \xrightarrow{\phi_k} k \to 0
\]

\[
d_n : \langle G^{\times n} \rangle_{k[G]} \to \langle G^{\times (n-1)} \rangle_{k[G]}
\]

\[
g_1 \otimes \cdots \otimes g_n \mapsto g_1 \triangleright (g_2, \ldots, g_n) - (g_1 g_2, \ldots, g_n) \pm \cdots + (-1)^{n-1} (g_0, \ldots, g_{n-1} g_n) + (-1)^n (g_1, \ldots, g_{n-1}).
\]
Example 5.4.10: (Free resolutions from the free comonad)

- For any ring $R$ the functor $F = \langle \_ \rangle_R : \text{Set} \to R\text{-Mod}$ is left adjoint to the forgetful functor $G : R\text{-Mod} \to \text{Set}$ by Example 5.4.2.

- The functor $FG : R\text{-Mod} \to R\text{-Mod}$ assigns to an $R$-module $M$ the free $R$-module $\langle M \rangle_R$ and to an $R$-linear map $f : M \to M'$ the $R$-linear map $(f)_R : \langle M \rangle_R \to \langle M' \rangle_R$. The natural transformations $\epsilon : FG \to \text{id}_{R\text{-Mod}}$ and $\eta : \text{id}_{\text{Set}} \to FG$ are given by their component morphisms $\epsilon_M : \langle M \rangle_R \to M$, $\sum_{m \in M} r_m m \mapsto \sum_{m \in M} r_m \triangleright m$ and $\eta_M : M \to \langle M \rangle_R$, $m \mapsto m$.

- If we set $\langle M \rangle^n_R := M$ and $\langle M \rangle^{n+1}_R := \langle \langle M \rangle^n_R \rangle_R$ for all $n \in \mathbb{N}_0$, we have $FG^n(M) = \langle M \rangle^n_R$. An element of $\langle M \rangle^n_R$ is a finite sum of elements of the form $(r_1, \ldots, r_n, m)$ with $r_i \in R$ and $m \in M$ and its $R$-module structure is given by $r \triangleright (r_1, \ldots, r_n, m) = (rr_1, \ldots, r_n, m)$.

- In this case, every $R$-module $M$ is $FG$-projective since the inclusion map $\iota_M : M \to \langle M \rangle_R$, $m \mapsto m$ and the map $\iota_M : \langle M \rangle_R \to M$, $(r, m) \to r \triangleright m$ satisfy $\epsilon_M \circ \iota_M = \text{id}_M$.

- By Proposition 5.3.14 the comonad $(FG, F\eta G, \epsilon)$ defines a free resolution for every object $M$ in $R\text{-Mod}$ given by

$$
\cdots \xrightarrow{d_n} \langle M \rangle^{n+1}_R \xrightarrow{d_{n-1}} \langle M \rangle^n_R \xrightarrow{d_{n-2}} \langle M \rangle^{n-1}_R \xrightarrow{\epsilon_M} M \to 0
$$

$$
d_n : \langle M \rangle^{n+1}_R \to \langle M \rangle^n_R,
(r_0, \ldots, r_n, m) \mapsto (r_0 r_1, \ldots, r_n, m) \pm \cdots \pm (-1)^{n-1}(r_0, \ldots, r_{n-1} r_n, m) + (-1)^n(r_0, \ldots, r_{n-1}, r \triangleright m).
$$

As already noted for the Hochschild resolution and the bar resolution, the standard resolutions are often not very practical for computations. This also holds for the free standard resolution in Example 5.4.10. However, the resolutions defined by a comonad $(C, \Delta, \epsilon)$ in an abelian category $\mathcal{A}$ are of conceptual importance.

They allow one to view homologies as homologies of functors even for abelian categories without enough projectives or injectives and additive functors that are not right or left exact. If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $K : \mathcal{A} \to \mathcal{B}$ is additive, we can define the homologies of objects in $\mathcal{A}$ by choosing a comonad $(C, \Delta, \epsilon)$ in $\mathcal{A}$ and setting $H_n(A) = H_n(KC(A)_\bullet)$, where $C(A)_\bullet$ is the chain complex from Proposition 5.3.14 and $KC(A)_{\bullet \geq 0}$ its image under the functor $K : \mathcal{A} \to \mathcal{B}$.

Definition 5.4.11: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $(C, \Delta, \epsilon)$ a comonad in $\mathcal{A}$.

1. The comonad homology $H_n^C(A, K)$ of an object $A$ in $\mathcal{A}$ with coefficients in an additive functor $K : \mathcal{A} \to \mathcal{B}$ is the homology $H_n(KC(A)_{\bullet \geq 0})$, where $C(A)_\bullet$ is the chain complex from Proposition 5.3.14.

2. The comonad cohomology $H_n^C(A, K)$ of an object $A$ in $\mathcal{A}$ with coefficients in an additive functor $K : A^{\text{op}} \to \mathcal{B}$ is the cohomology $H^n(KC(A)_{\bullet \geq 0})$, where $C(A)_\bullet$ is the chain complex from Proposition 5.3.14.

Remark 5.4.12: The comonad homologies and cohomologies for a fixed comonad $(C, \Delta, \epsilon)$ in $\mathcal{A}$ define functors $H_n^C : \mathcal{A} \times \text{Fun}^{\text{add}}(A, \mathcal{B}) \to \mathcal{B}$ and functors $H_n^C : \mathcal{A} \times \text{Fun}^{\text{add}}(A^{\text{op}}, \mathcal{B}) \to \mathcal{B}$, where $\text{Fun}^{\text{add}}(A, \mathcal{B}) \subset \text{Fun}(A, \mathcal{B})$ and $\text{Fun}^{\text{add}}(A^{\text{op}}, \mathcal{B}) \subset \text{Fun}(A^{\text{op}}, \mathcal{B})$ denote the full subcategories with additive functors as objects.
For $C$-projective objects $A$ in $\mathcal{A}$ the chain complex $C(A)_\bullet$ is a resolution of $A$. In particular, the standard resolution of an $R$-module $M$ from Example 5.4.10 is free resolution of $M$. As the category $R$-Mod has enough projectives by Corollary 4.2.1 and any free resolution is a projective resolution by Corollary 4.2.3 this resolution is unique up to chain homotopy equivalence by Theorem 4.1.8. It follows that for any right exact functor $K : R$-Mod $\to \mathcal{B}$, the homologies of the resulting chain complex $KC(A)_\bullet$ are precisely the left derived functors of $K$. Similarly, for any left exact functor $K : R$-Mod$^{op}$ $\to \mathcal{B}$ one obtains the right derived functors of $K$. This allows one to realise the functors Tor and Ext as comonad homologies of objects in $R$-Mod with coefficients in the functors $K = M \otimes_R -$ : $R$-Mod $\to \text{Ab}$ and $K = \text{Hom}(-, M) : R$-Mod$^{op}$ $\to \text{Ab}$.

**Example 5.4.13: (Tor and Ext)**

Let $R$ be a ring and $(C, \Delta, \epsilon)$ the comonad from Example 5.4.10 in $R$-Mod. Then the comonad homology of an $R$-left module $N$ with coefficients in $K = M \otimes_R -$ : $R$-Mod $\to \text{Ab}$ is given by

$$H^C_n(N, M \otimes_R -) = \text{Tor}^R_n(M, N)$$

and its comonad cohomology with coefficients in $K = \text{Hom}_R(-, M) : R$-Mod$^{op}$ $\to \text{Ab}$ by

$$H^n_C(N, \text{Hom}_R(-, M)) = R^n\text{Hom}_R(-, M)(N) = \text{Ext}^n_R(N, M).$$

This example includes Hochschild (co)homologies, group (co)homologies and (co)homologies of Lie algebras as comonad (co)homologies, since they are obtained as functors Tor and Ext for specific choices of the underlying ring $R$ by Examples 4.4.3, 4.4.4 and 4.4.5 and Definition 4.4.6.
6 Exercises

6.1 Exercises for Chapter 1

Exercise 1: Let $A, B, C$ be sets.

- A **relation** between $A$ and $B$ is a subset $R \subset A \times B$.
- A relation $R \subset A \times B$ is called a **map** from $A$ to $B$, if for every $a \in A$ there is a unique $b \in B$ with $(a, b) \in R$.
- The composite of two relations $R \subset A \times B$ and $S \subset B \times C$ is the relation
  \[S \circ R = \{(a, c) \in A \times C \mid \exists b \in B : (a, b) \in R, (b, c) \in S\} \subset A \times C.\]

(a) Show that sets and relations form a category $\text{Rel}$ with $\text{Hom}_{\text{Rel}}(A, B) = \mathcal{P}(A \times B)$.
(b) Determine the isomorphisms in $\text{Rel}$.

Exercise 2: Let $F$ be a field and $G$ a group with unit $e$. A representation $\rho : G \to \text{Aut}_F(M)$ of $G$ on an $F$-vector space $M$ is called **faithful** if $\ker(\rho) = \{g \in G \mid \rho(g) = \text{id}_M\} = \{e\}$. Find:

(a) a faithful representation of the symmetric group $S_n$ on $F^n$.
(b) a faithful representation of the cyclic group $\mathbb{Z}/m\mathbb{Z}$ for $1 \leq m \leq n$ on $F^n$.
(c) a faithful representation of $G$ on $F[G]$.

Exercise 3: Let $(M, +)$ be an abelian group. Show:

(a) $(M, +)$ has at most one $\mathbb{Q}$-module structure.
(b) If $M \neq \{0\}$ is finite, then $(M, +)$ has no $\mathbb{Q}$-module structure.
(c) There are infinite abelian groups $(M, +)$ without a $\mathbb{Q}$-module structure.

Exercise 4: Let $R$ be a ring and $M$ an $R$-module. The **annihilator** of a subset $A \subset M$ is

\[\text{Ann}(A) = \{r \in R \mid r \triangleright a = 0 \ \forall a \in A\}.\]

(a) Show that $\text{Ann}(A) \subset R$ is a left ideal for all subsets $A \subset M$ and a two-sided ideal if $A \subset M$ is a submodule.
(b) Determine $\text{Ann}(M)$ for the $\mathbb{Z}$-module $M = \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \ldots \times \mathbb{Z}/n_k\mathbb{Z}$ with $n_1, ..., n_k \in \mathbb{N}$.
(c) Show: if $M$ is a cyclic $R$-module, then there is a left ideal $\mathfrak{a} \subset R$ with $M \cong R/\mathfrak{a}$, where $R$ is viewed as a left module over itself and $\mathfrak{a} \subset R$ as a submodule.

Exercise 5: Let $F$ be a field and $F[X]$ the $F$-algebra of polynomials with coefficients in $F$.

(a) Show that $F[X]$-modules are in bijection with pairs $(M, \phi)$ of an $F$-vector space $M$ and an $F$-linear map $\phi : M \to M$.
(b) Characterise morphisms of $F[X]$-modules in terms of vector spaces over $F$ and $F$-linear maps.
(c) Let $M$ be a finite-dimensional vector space over $\mathbb{F}$ and $\phi : M \to M$ an $\mathbb{F}$-linear map. As $\mathbb{F}[X]$ is a principal ideal domain, there is a polynomial $p \in \mathbb{F}[X]$ that generates the annihilator of the associated $\mathbb{F}[X]$-module $M$: $\langle p \rangle_{\mathbb{F}[X]} = \text{Ann}(M)$. Characterise this polynomial in terms of concepts from linear algebra.

(d) Let $p \in \mathbb{F}[X] \setminus \{0\}$ be a polynomial and $\langle p \rangle_{\mathbb{F}[X]} \subseteq \mathbb{F}[X]$ the left ideal in $\mathbb{F}[X]$ generated by $p$. Show that the quotient module $M = \mathbb{F}[X]/\langle p \rangle_{\mathbb{F}[X]}$ is a cyclic $\mathbb{F}[X]$-module and a finite-dimensional vector space over $\mathbb{F}$. Determine its dimension $\dim_{\mathbb{F}}(M)$ and its annihilator $\text{Ann}(M)$.

(e) Consider the $\mathbb{F}[X]$-module $M = \mathbb{F}[X]/\langle (x - \lambda)^n \rangle_{\mathbb{F}[X]}$ for some $\lambda \in \mathbb{F}$ and $n \in \mathbb{N}$. Show that $M$ has a vector space basis for which the transformation matrix of $\phi = x \triangleright - : M \to M$, $m \mapsto x \triangleright m$ is a Jordan block

$$
\begin{pmatrix}
\lambda & 1 & 0 & \ldots & 0 \\
0 & \lambda & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \lambda \\
0 & \ldots & \ldots & 0 & \lambda
\end{pmatrix}
$$

mit $\lambda \in \mathbb{F}$.

(f) Conclude that for every finite dimensional complex vector space $M$ and every $\mathbb{C}$-linear map $\phi : M \to M$ the $\mathbb{C}[X]$-module $(M, \phi)$ is isomorphic to a direct sum

$$\mathbb{C}[X]/\langle (x - \lambda_1)^{n_1} \rangle_{\mathbb{C}[X]} \oplus \ldots \oplus \mathbb{C}[X]/\langle (x - \lambda_k)^{n_k} \rangle_{\mathbb{C}[X]} \quad \lambda_1, \ldots, \lambda_k \in \mathbb{C}, \quad n_1, \ldots, n_k \in \mathbb{N}.$$

**Hint:** In (a), consider the maps $\triangleright|_{\mathbb{F} \times M} : \mathbb{F} \times M \to M$, $(\lambda, m) \mapsto \lambda \triangleright m$ and $\phi = x \triangleright - : M \to M$, $m \mapsto x \triangleright m$.

**Exercise 6:** Consider the abelian group $G = \langle x, y \mid ax + by \rangle_{\mathbb{Z}}$ for fixed $a, b \in \mathbb{Z}$. Determine an abelian group of the form $\mathbb{Z}^n \times \mathbb{Z}/q_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/q_r \mathbb{Z}$ with $n, r \in \mathbb{N}_0$ and prime powers $q_1, \ldots, q_r \in \mathbb{N}$ that is isomorphic to $G$.

**Remark:** By the classification theorem every finitely generated abelian group is isomorphic to an abelian group $\mathbb{Z}^n \times \mathbb{Z}/q_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/q_r \mathbb{Z}$ with $n, r \in \mathbb{N}_0$ and prime powers $q_1, \ldots, q_r \in \mathbb{N}$. This shows that such an abelian group must exist.

**Exercise 7:** Prove that the tensor product of $\mathbb{Z}$-modules satisfies

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z} \quad \forall m, n \in \mathbb{N}.$$

**Exercise 8:** Let $V$ be a vector space over $\mathbb{F}$ with a countable basis $B = \{b_n \mid n \in \mathbb{N}\}$ and $R = \text{End}_{\mathbb{F}}(V)$ the endomorphism ring of $V$.

(a) Consider the $\mathbb{F}$-linear maps $\phi, \psi : V \to V$ with

$$\phi(b_{2n}) = b_n, \quad \phi(b_{2n-1}) = 0 \quad \psi(b_{2n}) = 0, \quad \psi(b_{2n-1}) = b_n \quad \forall n \in \mathbb{N}$$

and show that there are $\mathbb{F}$-linear maps $\alpha, \beta : V \to V$ with $\text{id}_V = \alpha \circ \phi + \beta \circ \psi$.  

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(b) Conclude that the ring $R$ as a left module over itself is the direct sum $R = (R \triangleright \phi) \oplus (R \triangleright \psi)$, where $R \triangleright \chi := \{ r \triangleright \chi \mid r \in R \}$ for all $\chi \in \text{End}_F(V)$.

(c) Conclude that $R^k \cong R^l$ for all $k, l \in \mathbb{N}$.

**Exercise 9:** Two rings $A, B$ are called Morita equivalent if there is an $(A, B)$-bimodule $P$, an $(B, A)$-bimodule $Q$, an $(A, A)$-bimodule isomorphism $\alpha : P \otimes_B Q \to A$ and an $(B, B)$-bimodule isomorphism $\beta : Q \otimes_A P \to B$. The sextuple $(R, S, P, Q, \alpha, \beta)$ is called a Morita context from $A$ to $B$.

(a) Let $Rg_k$ be the category whose objects are unital rings, with $\text{Hom}_{Rg_k}(A, B) = A\text{-Mod-}B$ and with the composition of morphisms $M : A \to B$ and $N : B \to C$ given by the tensor product $\otimes$. Determine the isomorphisms in the category $Rg_k$.

(b) Show that Morita equivalence is an equivalence relation on the class of unital rings.

(c) Show that if $A, B$ are Morita equivalent, then the categories $A\text{-Mod-}A$ and $B\text{-Mod-}B$ of $(A, A)$- and $(B, B)$-bimodules are equivalent.

### 6.2 Exercises for Chapter 2

**Exercise 10:** Let $k$ be a commutative ring and $X = \coprod_{i \in I} X_i$ a topological sum. Prove that $H_n(X, k) = \coprod_{i \in I} H_n(X_i, k) \quad \forall n \in \mathbb{N}_0$.

**Exercise 11:** Let $k$ be a commutative ring and $\emptyset \neq X \subset \mathbb{R}^n$ star-shaped with respect to $p \in X$. Show that $H_n(X, k) = 0$ for all $n \in \mathbb{N}$ and $H_0(X, k) \cong k$.

Proceed as follows. Consider the $k$-linear map $P_n : C_n(X, k) \to C_{n+1}(X, k)$ with

$$P_n(\sigma)(\sum_{i=0}^{n+1} \lambda_i e_i) = \begin{cases} \lambda_{n+1} p + (1 - \lambda_{n+1}) \sigma(\sum_{i=0}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} e_i) & \lambda_{n+1} \neq 1 \\ p & \lambda_{n+1} = 1 \end{cases}$$

for all $n \in \mathbb{N}_0$ and singular $n$-simplexes $\sigma : \Delta^n \to X$.

(a) Show that for all singular $n$-simplexes $\sigma : \Delta^n \to \Delta^{n+1}$

$$P_n(\sigma) \circ f_{n+1}^{n+1} = \sigma \quad P_n(\sigma) \circ f_i^{n+1} = P_{n-1}(\sigma \circ f_i^n) \quad \forall i \in \{0, \ldots, n\},$$

where $f_i^n : \Delta^{n-1} \to \Delta^n$ denote the face maps.

(b) Compute $d_{n+1}(P_n(\sigma)) : \Delta^n \to X$ using (a).

(c) Use (b) to show that $Z_n(X, k) = B_n(X, k)$ for all $n \in \mathbb{N}$ and treat the case $n = 0$ separately.

**Exercise 12:** The Klein bottle is the quotient space $K = [0, 1] \times [0, 1]/\sim$ with respect to the equivalence relation $(0, y) \sim (1, 1-y)$ and $(x, 0) \sim (x, 1)$ for all $x, y \in [0, 1]$. 

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(a) Choose a semisimplicial complex structure $\Delta$ on $K$ and compute its simplicial homologies $H_n(\Delta, k)$ for $n \in \mathbb{N}_0$ and (i) $k = \mathbb{Z}$, (ii) $k = \mathbb{F}$ a field, (iii) $k = \mathbb{Z}/6\mathbb{Z}$.

(b) Compute the cohomologies $H^n(\Delta, M)$ of the simplicial complex $\Delta$ from (a) with values in the $\mathbb{Z}$-module $M$ for (i) $M = \mathbb{Z}$, (ii) $M = \mathbb{F}$ a field and (iii) $M = \mathbb{Z}[x]$.

**Exercise 13:** An oriented surface of genus $g \geq 1$ is the quotient space $\Sigma_g = \mathbb{P}_4g/\sim$ of a regular $4g$-gon $\mathbb{P}_4g \subset \mathbb{R}^2$ with respect to the equivalence relation $\sim$, which identifies its sides pairwise as follows:

Give a semisimplicial complex structure on $\Sigma_g$ and compute the homologies $H_n(\Delta, k)$ for a commutative ring $k$.

**Exercise 14:** Let $\mathbb{F}$ be a field and $\Delta = (X, \{\sigma_\alpha\}_{\alpha \in I})$ a finite semisimplicial complex. The Euler characteristic of $\Delta$ is defined as

$$\chi(\Delta) = \sum_{k=0}^\infty (-1)^k \dim \mathbb{F} C_k(\Delta, \mathbb{F}).$$

(a) Compute the Euler characteristic of the semisimplicial complex $\Delta$ from Exercise 13.

(b) Convince yourself that the family of affine linear maps

$$M = \{f_{j_n+1}^n \circ f_j^n \circ \ldots \circ f_{j_{n+1-k}}^{n-k} : \Delta^{n-k} \to \Delta^{n+1} \mid k \in \{0, \ldots, n\}, j_i \in \{0, \ldots, i\}\}$$

equips the boundary $\partial \Delta^{n+1} \subset \mathbb{R}^{n+1}$ of the standard $(n+1)$-simplex $\Delta^{n+1}$ with the structure of a simplicial complex. Compute its Euler-characteristic.
Exercise 15: Let $A$ be an algebra over a commutative ring $k$ and $M$ an $(A, A)$-bimodule with structure maps $\triangleright : A \times M \to M$ and $\triangleleft : M \times A \to M$. Compute the Hochschild homologies $H_0(A, M)$ and $H_1(A, M)$ and interpret them.

Exercise 16: We consider a commutative ring $k$ as an algebra over itself and a $k$-module $M$, interpreted as a $(k, k)$-bimodule with $k \triangleright m = m \triangleleft k$. Compute all Hochschild homologies $H_n(k, M)$ and Hochschild cohomologies $H^n(k, M)$ for $n \in \mathbb{N}_0$.

Exercise 17: Let $k$ be a commutative ring, $A$ a commutative algebra over $k$ and $M$ an $A$-module, interpreted as an $(A, A)$-bimodule with $a \triangleright m = m \triangleleft a$. The $A$-module $\Omega_{A/k}$ of Kähler differentials is the $A$-module with one generator $da$ for each element $a \in A$ and relations $d(a + b) = da - db$ and $d(ab) - a \triangleright db - b \triangleleft da$ for all $a, b \in A$:

$$\Omega_{A/k} = \frac{\{ \{ da \mid a \in A \} \}_A}{\{ \{ d(a + b) - da - db, d(ab) - a \triangleright db - b \triangleleft da \mid a, b \in A \} \}.'$$

(a) Prove that the Kähler differentials have the following universal property:

The map $d : A \to \Omega_{A/k}, a \mapsto da$ is a derivation, and for any derivation $f : A \to M$ there is a unique $A$-linear map $\phi : \Omega_{A/k} \to M$ with $f = \phi \circ d$.

(b) Show that the first Hochschild homology and Hochschild cohomology with values in $M$ are given by $H_1(A, M) \cong M \otimes_A \Omega_{A/k}$ and $H^1(A, M) \cong \text{Hom}_A(\Omega_{A/k}, M)$.

Exercise 18: Let $k$ be a commutative ring, $G$ a group and $M$ a $k[G]$-module with structure map $\triangleright : k[G] \times M \to M$. We denote by $M \times G$ the trivial extension of $G$ by $M$, the set $M \times G$ with group multiplication $(m, g) \cdot (m', g') = (m + g \triangleright m', gg')$.

Prove the following:

(a) The map $\triangleright' : M \times G \times M \to M$, $(m, g) \triangleright' m' = m + g \triangleright m'$ defines a group action on $M \times G$ on $M$.

(b) The map $\triangleright_f : G \times M \to M$, $(g, m) \mapsto g \triangleright m + f(g)$ defines a group action of $G$ on $M$ if and only if $f : G \to M$ is a derivation.

(c) For any subgroup $U \subset H$, element $h \in H$ and group action $\triangleright : H \times N \to N$ of $H$ on $N$, the map $\triangleright_h : U \times N \to N$, $u \triangleright_h n = (h^{-1}uh) \triangleright n$ defines a group action of $U$ on $N$. One says that $\triangleright_h$ is obtained from $\triangleright_{1_H}$ by conjugation of $U$ in $H$ with $h \in H$.

(d) The restriction of $\triangleright : k[G] \times M \to M$ to $G \times M$ defines a group action of $G$ on $M$. The group action $\triangleright_f : G \times M \to M$ from (b) is obtained from this group action by conjugation of $G$ in $M \times G$ with an element $(m, 1_G)$ if and only if $f : G \to M$ is an inner derivation.

Exercise 19: Let $G$ be a group and $M$ a trivial $\mathbb{Z}[G]$-module.

(a) Compute $H^1(G, M)$ for all finitely generated abelian groups $G, M$. 

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(b) Compute $H^1(G, M)$ for (i) $G = \mathbb{Z} \times S_n$ and $M = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and (ii) $G = \mathbb{Z}/5\mathbb{Z} \times S_n$ and $M = \mathbb{Z}/30\mathbb{Z}$.

**Hint:** Use the classification theorem for finitely generated abelian groups, which states that every finitely generated abelian group is isomorphic to a group of the form

$$\mathbb{Z}^n \times \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z} \quad \text{with} \quad m \in \mathbb{N}_0, n_i \in \mathbb{N} \text{ and primes } p_i \in \mathbb{N}.$$

**Exercise 20:** Compute $H^2(\mathbb{Z}/3\mathbb{Z}, M)$ for the following abelian groups $M$ with the trivial $\mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]$-module structure (i) $M = \mathbb{Z}$, (ii) $M = \mathbb{Z}/3\mathbb{Z}$ and (iii) $M = \mathbb{Z}/2\mathbb{Z}$.

### 6.3 Exercises for Chapter 3

**Exercise 21:** Let $\mathcal{C}$, $\mathcal{D}$ be additive categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Show that the following statements are equivalent:

(i) $F$ is additive.

(ii) $F$ preserves finite products: $F(\prod_{i \in I} C_i) \cong \prod_{i \in I} F(C_i)$ for all finite families $(C_i)_{i \in I}$ of objects $C_i$ in $\mathcal{C}$.

(iii) $F$ preserves finite coproducts: $F(\coprod_{i \in I} C_i) \cong \coprod_{i \in I} F(C_i)$ for all finite families $(C_i)_{i \in I}$ of objects $C_i$ in $\mathcal{C}$.

**Hint:** Prove first that an object $X$ in an additive category $\mathcal{C}$ is a (co)product of a finite family of objects $(C_k)_{k \in I}$ if and only if there are families $(i_k)_{k \in I}$ and $(p_k)_{k \in I}$ of morphisms $i_k : C_k \to X$ and $p_k : X \to C_k$ with $p_j \circ i_k = 1_{C_j}$ for $j = k$, $p_j \circ i_k = 0 : C_k \to C_j$ for $j \neq k$ and $\sum_{k \in I} i_k \circ p_k = 1_X$.

**Exercise 22:** Let $\mathcal{C}$ be a category with a zero object, $f : C \to D$ a morphism in $\mathcal{C}$, $g : D \to E$ a monomorphism and $h : B \to C$ an epimorphism in $\mathcal{C}$. Prove the following:

(a) A morphism $\pi : D \to A$ is a cokernel of $f$ if and only if $\pi$ is a cokernel of $f \circ h$.

(b) A morphism $\iota : B \to X$ is a kernel of $f$ if and only if $\iota$ is a kernel of $g \circ f$.

**Exercise 23:** Let $\mathcal{C}$ be a category with a zero object. Prove the following:

(a) Kernels in $\mathcal{C}$ are unique up to unique isomorphism: if $\iota : \ker(f) \to X$, $\iota' : \ker(f)' \to X$ are kernels of a morphism $f : X \to Y$ in $\mathcal{C}$, then there is a unique morphism $\phi : \ker(f) \to \ker(f)'$ with $\iota' \circ \phi = \iota$, and $\phi$ is an isomorphism.

(b) $\mathcal{C}^{\text{op}}$ has a zero object and $\iota : W \to X$ ($\pi : Y \to Z$) is a kernel (cokernel) of $f : X \to Y$ in $\mathcal{C}$ if and only if $\iota : X \to W$ ($\pi : Z \to Y$) is a cokernel (kernel) of $f : Y \to X$ in $\mathcal{C}^{\text{op}}$.

**Exercise 24:** Let $\mathcal{A}$ be an abelian category. Prove the following: If a morphism $f : X \to Z$ in $\mathcal{A}$ is given by $f = \iota \circ \pi$ with a monomorphism $\iota : Y \to Z$ and an epimorphism $\pi : X \to Y$, then $\iota : Y \to Z$ is an image of $f$ and $\pi : X \to Y$ a coimage of $f$.  

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Exercise 25: Let $\mathcal{A}$ be an abelian category. Prove that every morphism $f : X \to Y$ that is both a monomorphism and an epimorphism is an isomorphism.

Exercise 26: Show that a category $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{op}$ is abelian and kernels and cokernels in $\mathcal{A}$ then correspond to cokernels and kernels in $\mathcal{A}^{op}$.

Exercise 27: Let $\mathcal{C}$ be a small category and $\mathcal{A}$ an abelian category. Prove the following:
(a) The category $\text{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.
(b) For every object $C$ in $\mathcal{C}$, the functor $\text{ev}_C : \text{Fun}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$ that sends a functor $F : \mathcal{C} \to \mathcal{A}$ to the object $F(C)$ in $\mathcal{A}$ and a natural transformation $\eta : F \to G$ to the component morphism $\eta_C : F(C) \to G(C)$ is exact.

Exercise 28: Prove that for any abelian category $\mathcal{A}$, the Cartesian product $\mathcal{A} \times \mathcal{A}$ is abelian and the functor $\Pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact.

Exercise 29: Show that the full subcategory $\mathcal{C}$ of $\text{Ab} = \mathbb{Z}\text{-Mod}$ with finitely generated free $\mathbb{Z}$-modules as objects is additive, but not abelian.

Exercise 30: Show that every object in the category $\text{Set}$ is projective and injective. Use the projectivity and injectivity criteria from Lemma 3.1.21.

Exercise 31: Let $R_1, R_2$ be rings and $R = R_1 \times R_2$ their product. Show that $A = R_1 \times \{0\}$ with the $R$-module structure $(r_1, r_2) \triangleright (r'_1, 0) = (r_1, r_2) \cdot (r'_1, 0) = (r_1r'_1, 0)$ is a projective $R$-module but not a free $R$-module.

Exercise 32: Let $\mathcal{A}$ be an abelian category. Show that the category $\text{Ch}_\mathcal{A}$ of chain complexes and chain maps in $\mathcal{A}$ is abelian as well.

Exercise 33: Let $k$ be a commutative ring, $G$ a group, $M$ a $k[G]$-module. For $n \in \mathbb{N}_0$ denote by $G_k^\times$ the $k[G]$-module $\langle G^\times \rangle_k$ with $g \triangleright (g_1, ..., g_n) = (gg_1, ..., gg_n)$ and by $\text{Hom}_{k[G]}(G_k^\times, M)$ the $k$-module of $k[G]$-linear maps $f : G_k^\times \to M$.

Let $X^\bullet$ be the cochain complex in $k\text{-Mod}$ with
\[
X^n = \text{Hom}_{k[G]}(G_k^{(n+1)}, M) \quad n \in \mathbb{N}_0
\]
\[
d^n : X^n \to X^{n+1}, \quad d^n(\phi)(g_0, ..., g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \phi(g_0, ..., \widehat{g_i}, ..., g_{n+1}).
\]
Prove that the $k$-linear maps $f^n : X^n \to C^n(M, G)$, $\phi \mapsto f^n(\phi)$ with
\[
f^n(\phi)(g_1, ..., g_n) = \phi(1, g_1, g_1g_2, g_1g_2g_3, ..., g_1g_2 \cdots g_n).
\]
define an invertible cochain map $f^\bullet : X^\bullet \to C^\bullet(G, M)$ to the cochain complex $C^\bullet(G, M)$ of group cohomology.
**Exercise 34:** Let $\mathcal{A}$ be an abelian category, $X_\bullet, X'_\bullet$ chain complexes in $\mathcal{A}$ and $p \in \mathbb{Z}$. We define a chain complex $T_p(X_\bullet)$

$$T_p(X_\bullet)_n = X_{n+p}, \quad T_p(d_\bullet)_n = (-1)^p d_{n+p}, \quad \forall n \in \mathbb{Z}.$$ 

and for every chain map $f_\bullet : X_\bullet \to X'_\bullet$ a chain map

$$T_p(f_\bullet) : T_p(X_\bullet) \to T_p(X'_\bullet), \quad T_p(f_\bullet)_n = f_{n+p}.$$ 

Show that this defines a functor $T_p : \text{Ch}_\mathcal{A} \to \text{Ch}_\mathcal{A}$ that satisfies

$$H_n(T_p(X_\bullet)) = H_{n+p}(X_\bullet).$$

This functor is called the **translation functor**.

**Exercise 35:** Let $R$ be a ring. Compute and interpret the homologies of the following chain complexes in $R$-Mod.

(a) $C_\bullet = 0 \to R \xrightarrow{d_1} R \to 0$

(b) $C_\bullet = 0 \to R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \to 0$ for a commutative ring $R$ and $a, b \in R$.

**Exercise 36:** Prove the **5-lemma**:

Suppose that the following diagram in $R$-Mod commutes and all rows are exact

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\epsilon} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{k'} & E'.
\end{array}
\]

Then:

(i) If $\beta, \delta$ are monomorphisms and $\alpha$ is an epimorphism, then $\gamma$ is a monomorphism.

(ii) If $\beta, \delta$ are epimorphisms and $\epsilon$ is a monomorphism, then $\gamma$ is an epimorphism.

(iii) If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then $\gamma$ is an isomorphism as well.

**Exercise 37:** Prove the **9-lemma**:

Let $R$ be a ring. Suppose the following diagram in $R$-Mod commutes, has exact rows and all vertical composites of morphisms vanish

\[
\begin{array}{cccccc}
0 & \to & A & \xrightarrow{\iota_A} & A' & \xrightarrow{\pi_A} & A'' & \to & 0 \\
\downarrow{\phi} & & \downarrow{\phi'} & & \downarrow{\phi''} & & \downarrow{\phi'''} & & \downarrow{0} \\
0 & \to & B & \xrightarrow{\iota_B} & B' & \xrightarrow{\pi_B} & B'' & \to & 0 \\
\downarrow{\psi} & & \downarrow{\psi'} & & \downarrow{\psi''} & & \downarrow{\psi'''} & & \downarrow{0} \\
0 & \to & C & \xrightarrow{\iota_C} & C' & \xrightarrow{\pi_C} & C'' & \to & 0 \\
\downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \downarrow{0}
\end{array}
\]

If two of the columns are short exact sequences then so is the third one.
Exercise 38: Let $R$ be a ring and $0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0$ an exact sequence in $R$-Mod. Show that the following statements are equivalent:

(i) The $R$-linear map $\pi : M \to N$ has a section:
there is a $R$-linear map $\psi : N \to M$ with $\pi \circ \psi = \text{id}_N$.

(ii) The $R$-linear map $\iota : L \to M$ has a retraction:
there is a $R$-linear map $\phi : M \to L$ with $\phi \circ \iota = \text{id}_L$.

(iii) There is an $R$-linear isomorphism $\chi : M \to L \oplus N$ with $\chi \circ \iota = \iota_1$ and $\pi_2 \circ \chi = \pi$, where $\iota_1 : L \to L \oplus N$, $l \mapsto (l,0)$ is the inclusion and $\pi_2 : L \oplus N \to N$, $(l,n) \mapsto n$ the projection.

If one of these conditions is satisfied, one says the exact sequence splits.

Exercise 39: A chain complex $(X_\bullet,d_\bullet)$ in an abelian category $\mathcal{A}$ is called split if there is a family $(s_n)_{n \in \mathbb{Z}}$ of morphisms $s_n : X_n \to X_{n+1}$ with $d_n \circ s_{n-1} \circ d_n = d_n$ for all $n \in \mathbb{Z}$ and split exact if it is split and exact. Prove the following for $\mathcal{A} = R$-Mod:

(a) For any family $(s_n)_{n \in \mathbb{Z}}$ of morphisms $s_n : X_n \to X_{n+1}$ the morphisms
$$f_n = s_{n-1} \circ d_n + d_{n+1} \circ s_n : X_n \to X_n$$
define a chain map $f_\bullet : X_\bullet \to X_\bullet$ with $H_n(f_\bullet) = 0 : H_n(X_\bullet) \to H_n(X_\bullet)$.

(b) A chain complex $X_\bullet$ is split exact if and only if $1_{X_\bullet} : X_\bullet \to X_\bullet$ is chain homotopic to $0_{X_\bullet} : X_\bullet \to X_\bullet$.

(c) A chain complex $(X_\bullet,d_\bullet)$ in $\mathcal{A}$ is split if and only if there are families $(C_n)_{n \in \mathbb{Z}}$ and $(D_n)_{n \in \mathbb{Z}}$ of objects $C_n$, $D_n$ with $X_n = C_n \amalg \ker(d_n)$ and $\ker(d_n) = D_n \amalg \operatorname{im}(d_{n+1})$, and in this case its homologies are given by $H_n(X_\bullet) = D_n$.

Exercise 40: Let $R$ be a ring.

(a) Show that the chain complex $X_\bullet = \ldots \mathbb{Z}/4\mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z}/4\mathbb{Z}$ is exact but not split exact.

(b) Show that every exact, bounded below chain complex $X_\bullet$ in $R$-Mod with projective $R$-modules $X_n$ is split exact.

Exercise 41: Let $X,Y$ be topological spaces and $C_\bullet(X,k)$ and $C_\bullet(Y,k)$ the associated singular chain complexes. The prism maps are the affine linear maps
$$T^j_n : \Delta^{n+1} \to [0,1] \times \Delta^n, \quad T^j_n(e_k) = \begin{cases} (0,e_k) & 0 \leq k \leq j \leq n \\ (1,e_{k-1}) & 0 \leq j < k \leq n + 1. \end{cases}$$
(a) Prove that the prism maps satisfy the relations
\[
T^i_n \circ f^{n+1}_i = (\text{id}_{[0,1]} \times f^i_n) \circ T^i_{n-1} \quad \forall j > i \quad T^j_n \circ f^{n+1}_i = (\text{id}_{[0,1]} \times f^{n-1}_i) \circ T^j_{n-1} \quad \forall j < i - 1
\]
where \( i_t : \Delta^n \to [0,1] \times \Delta^n, x \mapsto (t, x) \) and \( f^{n+1}_j : \Delta^n \to \Delta^{n+1} \) denote the face maps.

(b) Let \( f, g : X \to Y \) continuous maps and \( h : [0,1] \times X \to Y \) a homotopy from \( f \) to \( g \). Prove that the \( k \)-linear maps
\[ C_n(h, k) : C_n(X, k) \to C_{n+1}(Y, k), \quad \sigma \mapsto \Sigma_{j=0}^n (-1)^j h \circ (\text{id}_{[0,1]} \times \sigma) \circ T^j_n \]
define a chain homotopy \( C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k) \).

**Exercise 42:** Let \( A \) be an algebra over a commutative ring \( k \) and \( (M, \triangleright, \triangleleft) \) an \((A, A)\)-bimodule and \( c \in Z(A) \) an element in the centre of \( A \). Prove the following:

(a) The maps \( c \triangleright - : M \to M, m \mapsto c \triangleright m \) and \( - \triangleleft c : M \to M, m \mapsto m \triangleleft c \) are homomorphisms of \((A, A)\)-bimodules and induce chain maps \( c_\triangleright \bullet, \triangleleft c_\bullet : C_\bullet(A, M) \to C_\bullet(A, M) \) on the Hochschild complex.

(b) The chain maps \( c_\triangleright \bullet \) and \( \triangleleft c_\bullet \) are chain homotopic.

**Hint:** In (b), consider for \( 0 \leq i \leq n \) the morphisms
\[ h^i_n : M \otimes_k A^\otimes n \to M \otimes_k A^\otimes n, \quad m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes a_1 \otimes \ldots \otimes a_i \otimes c \otimes a_{i+1} \otimes \ldots \otimes a_n \]
and combine them into a chain homotopy.

**Exercise 43:** Let \( (X_\bullet, d_\bullet) \) and \( (X'_\bullet, d'_\bullet) \) be chain complexes and \( f_\bullet : X_\bullet \to X'_\bullet \) be a chain map in an abelian category \( \mathcal{A} \). The **mapping cone** of \( f_\bullet \) is the chain complex \( \text{cone}(f_\bullet) = Y_\bullet \) with \( Y_n = X_{n-1} \amalg X'_n \) and \( d_n^Y : Y_n \to Y_{n-1} \)
\[
\begin{align*}
\pi_1^{n-1} \circ d_n^Y & \circ \iota_1^n = -d_{n-1} \\
\pi_1^n \circ d_n^Y & \circ \iota_2^n = 0 \\
\pi_2^n \circ d_n^Y & \circ \iota_1^n = -f_n \\
\pi_2^{n-1} \circ d_n^Y & \circ \iota_2^n = d'_n
\end{align*}
\]
where \( \iota_1^n : X_{n-1} \to Y_n, \iota_2^n : X'_n \to Y_n \) and \( \pi_1^n : Y_n \to X_{n-1}, \pi_2^n : Y_n \to X'_n \) are the canonical inclusions and projections for the coproduct satisfying \( \iota_1^n \circ \pi_1^n + \iota_2^n \circ \pi_2^n = 1_{Y_n} \) and \( \pi_1^n \circ \iota_1^n = 1_{X_{n-1}}, \pi_2^n \circ \iota_2^n = 1_{X'_n}, \pi_1^n \circ \iota_2^n = \iota_1^n, \pi_2^n \circ \iota_1^n = 0, \pi_1^n \circ \iota_1^n = 0, \pi_1^n \circ \iota_2^n = 0, \pi_2^n \circ \iota_2^n = 0. \)
(a) Show that cone\((f_\bullet)\) is a chain complex in \(\mathcal{A}\) for all chain maps \(f_\bullet : X_\bullet \to Y_\bullet\).
(b) Show that cone\((1_{X_\bullet})\) is homotopy equivalent to the trivial chain complex \(0_\bullet\).
(c) Show that \(f_\bullet : X_\bullet \to X'_\bullet\) extends to a chain map \(f'_\bullet : \text{cone}(1_{X_\bullet}) \to X'_\bullet\) with \(f'_n \circ \iota_2^n = f_n\) if and only if \(f_\bullet\) is homotopic to \(0_\bullet : X_\bullet \to X'_\bullet\).

**Exercise 44:** Let \(R\) be a ring and \(A\) an \(R\)-module. Prove the following:
(a) \(A\) is projective if and only if every short exact sequence \(0 \to L \xrightarrow{i} M \xrightarrow{\pi} A \to 0\) splits.
(b) \(A\) is injective if and only if every short exact sequence \(0 \to A \xrightarrow{i} M \xrightarrow{\pi} N \to 0\) splits.

**Exercise 45:** Let \(R\) be a ring.
(a) Show that an \(R\)-right module \(M = \bigoplus_{i \in I} M_i\) is flat if and only if \(M_i\) is flat for all \(i \in I\).
(b) Show that any projective \(R\)-right module is flat.

**Hint:** Use an argument similar to the one in the proof of Lemma 4.2.2 in 1.

6.4 Exercises for Chapter 4

**Exercise 46:** Let \(\mathcal{A}, \mathcal{B}\) be abelian categories and \(P^A_n : \text{Ch}_A \to \mathcal{A}\) the additive functor with
- \(P^A_n(X_\bullet) = X_n\) for every chain complex \(X_\bullet = (X_n)_{n \in \mathbb{Z}}\),
- \(P^A_n(f_\bullet) = f_n : X_n \to X'_n\) for every chain map \(f_\bullet = (f_n)_{n \in \mathbb{Z}} : X_\bullet \to X'_\bullet\).

Prove the following:
(a) For every additive functor \(F : \mathcal{A} \to \mathcal{B}\) there is an additive functor \(F' : \text{Ch}_A \to \text{Ch}_B\) with \(P^B_n F' = FP^A_n\) for all \(n \in \mathbb{Z}\).
(b) If \(f_\bullet, f'_\bullet : X_\bullet \to X'_\bullet\) are chain homotopic, then \(F'(f_\bullet), F'(f'_\bullet) : F(X_\bullet) \to F'(X'_\bullet)\) are chain homotopic as well. The functor \(F' : \text{Ch}_A \to \text{Ch}_B\) induces a functor \(F'' : K(\mathcal{A}) \to K(\mathcal{B})\).
(c) Let \(\mathcal{G}(X_\bullet, X'_\bullet)\) the abelian groupoid with chain maps \(f_\bullet : X_\bullet \to X'_\bullet\) as objects and chain homotopies between them as morphisms. Show that every additive functor \(F : \mathcal{A} \to \mathcal{B}\) induces a functor \(F' : \mathcal{G}(X_\bullet, X'_\bullet) \to \mathcal{G}(F'(X_\bullet), F'(X'_\bullet))\).
(d) If \(\mathcal{A}\) is small, then this induces a functor \(\Phi : \text{Fun}(\mathcal{A}, \mathcal{B}) \to \text{Fun}(\text{Ch}_A, \text{Ch}_B)\).

**Exercise 47:** Determine if the following abelian categories have enough projectives and injectives.
(a) The category \(\text{Ab}^{\text{fin}}\) of finite abelian groups.
(b) The category \(\text{Ab}^{\text{fin-gen}}\) of finitely generated abelian groups.

**Exercise 48:** Let \(\mathcal{A} = R\)-\text{Mod} for some ring \(R\). Show that a chain complex \(P_\bullet = (P_n)_{n \in \mathbb{Z}}\) is a projective object in \(\text{Ch}_A\) if and only if \(P_\bullet\) is split exact and all objects \(P_n\) are projective.
Exercise 49: Show that if $\mathcal{A}$ is an abelian category with enough projectives then $\text{Ch}_\mathcal{A}$ has enough projectives as well.

Exercise 50: Let $\mathcal{A}, \mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives, $F, F' : \mathcal{A} \to \mathcal{B}$ right exact functors and $\eta : F \to F'$ a natural transformation. Show that $\eta$ induces a family $(L_n\eta)_{n \in \mathbb{N}_0}$ of natural transformations $L_n\eta : L_n F \to L_n F'$ with $L_0\eta = \eta : F \to F'$.

Exercise 51: Let $\mathcal{A}$ be an abelian category with enough projectives and $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor into an abelian category $\mathcal{B}$. Show that the left derived functors of $F$ are additive.

Exercise 52: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives $F : \mathcal{A} \to \mathcal{B}$ right exact and $G : \mathcal{B} \to \mathcal{C}$ exact. Show that $L_n(GF) = G(L_n F)$ for all $n \in \mathbb{N}_0$.

Exercise 53: Let $G$ be a group. The invariants and coinvariants of a $\mathbb{Z}[G]$-module $M$ are the abelian subgroups $M^G, M^G \subset M$ given by

$M^G = \{m \in M : g \triangleright m = m \forall g \in G\} \quad M_G = M/\langle\{g \triangleright m - m \mid g \in G, m \in M\}\rangle_M.$

(a) Show that the invariants and coinvariants define additive functors $(-)^G : \mathbb{Z}[G] \text{-Mod} \to \text{Ab}$ and $(-)_G : \mathbb{Z}[G] \text{-Mod} \to \text{Ab}$.

(b) Construct natural isomorphisms $\eta : (-)^G \to \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ and $\tau : (-)_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} -$, where $\mathbb{Z}$ is equipped with the trivial $\mathbb{Z}[G]$-module structure.

(c) Show that the group homologies and cohomologies $H_n(G, M)$ and $H^n(G, M)$ are the left and right derived functors of $(-)^G : \mathbb{Z}[G] \text{-Mod} \to \text{Ab}$ and $(-)_G : \mathbb{Z}[G] \text{-Mod} \to \text{Ab}$, respectively.

Exercise 54: Let $M$ be an $\mathbb{Z}[\mathbb{Z}]$-module. Compute the group homologies and cohomologies of $\mathbb{Z}$ with coefficients in $M$.

**Hint:** Show first that the following is a free resolution of the trivial $\mathbb{Z}[\mathbb{Z}]$-module $\mathbb{Z}$

$0 \to \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z} \to 0$

with $\iota : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}], \Sigma_{z \in \mathbb{Z}} \lambda_z \mapsto \Sigma_{z \in \mathbb{Z}}(\lambda_z - \lambda_{z-1})z$ and $\epsilon : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}, \Sigma_{z \in \mathbb{Z}} \lambda_z \mapsto \Sigma_{z \in \mathbb{Z}} \lambda_z$.

Exercise 55: Let $R$ be a principal ideal domain. Compute $\text{Ext}_R^n(M, N)$ for all finitely generated $R$-modules $M, N$.

Exercise 56: Let $\mathbb{F}$ be a field, $p = \Sigma_{i=0}^n a_i x^i \in \mathbb{F}[x]$ with $a_n = 1$ and consider the $\mathbb{F}$-algebra $A = \mathbb{F}[x]/(p)$, where $(p) \subset \mathbb{F}[x]$ is the ideal generated by $p$ and $\pi : \mathbb{F}[x] \to A, q \mapsto \bar{q}$ the canonical surjection.
(a) Show that
\[
\ldots \xrightarrow{f} A \otimes_{F} A \xrightarrow{g} A \otimes_{F} A \xrightarrow{h} A \otimes_{F} A \xrightarrow{i} A \otimes_{F} A \xrightarrow{j} A \otimes_{F} A \xrightarrow{k} A \rightarrow 0
\]
where \( f : A \otimes_{F} A \rightarrow A \otimes_{F} A, a \otimes b \mapsto (x \otimes 1 - 1 \otimes x) \cdot (a \otimes b) \)
\( g : A \otimes_{F} A \rightarrow A \otimes_{F} A, a \otimes b \mapsto q \cdot (a \otimes b) \) with 
\( q = \sum_{k=0}^{n} \sum_{j=0}^{k-1} a_{k} x^{k-1-j} \otimes x^{j} \)
\( \mu : A \otimes_{F} A \rightarrow A, a \otimes b \mapsto ab. \)
is a free resolution of the \( A \otimes_{F} A \)-module \( A \) in \( A \otimes_{F} A \)-Mod.

(b) Show that applying the functor \( A \otimes_{A \otimes_{F} A} - : A \otimes_{F} A \text{-Mod} \rightarrow F \text{-Mod} \) and omitting the entry \( A \) yields a chain complex isomorphic to
\[
\ldots \xrightarrow{0} A \xrightarrow{p'} A \xrightarrow{0} A \xrightarrow{p'} A \xrightarrow{0} A \xrightarrow{0} A \rightarrow 0,
\]
where \( p' = \sum_{k=0}^{n} ka_{k} x^{k-1} \) is the derivative of \( p \).

(c) Derive a formula for the Hochschild homologies \( H_{n}(A, A) \) for \( n \in \mathbb{N}_{0} \) in terms of \( p \) and \( p' \).

(d) Show that \( H_{2n}(A, A) = 0 \) if \( p \in F[x] \) is irreducible and compute the Hochschild homologies for (i) \( F = \mathbb{R}, p = x^{k} \) with \( k \in \mathbb{N} \) and (ii) \( F = \mathbb{Z}/k\mathbb{Z}, p = x^{k} \) with \( k \in \mathbb{N} \) prime.

Exercise 57: Let \( F \) be a field of characteristic 0, \( V = F^{2} \) and \( \{q, p\} \) a basis of \( V \). The Weyl algebra \( W \) is the quotient \( W = T(V)/(p \otimes q - q \otimes p - 1) \) of the tensor algebra \( T(V) \) by the two-sided ideal \( I = (p \otimes q - q \otimes p - 1) \) generated by the element \( p \otimes q - q \otimes p - 1 \in T(V) \).

(a) Show that the elements \( q^{i} p^{j} := q \otimes \ldots \otimes q \otimes p \otimes \ldots \otimes p + I \) with \( i, j \in \mathbb{N}_{0} \) form a basis of the Weyl algebra and that the multiplication is given by
\[
q^{i} p^{j} \cdot q^{k} p^{l} = \sum_{r=0}^{\min(j,k)} r! \binom{j}{r} \binom{k}{r} q^{i+k-r} p^{j+l-r} \quad \forall i, j, k, l \in \mathbb{N}_{0}.
\]

(b) Show that \( \triangleright : W \times F[x] \rightarrow F[x] \) with \( q \triangleright g = x \cdot g \) and \( p \triangleright g = dg/dx \) defines a \( W \)-module structure on the vector space \( F[x] \) of polynomials over \( F \).

(c) Show that the following is a free resolution of \( W \) in \( W \otimes_{F} W^{\text{op}} \text{-Mod} \),
\[
0 \rightarrow W \otimes_{F} W^{\text{op}} \xrightarrow{g} W \otimes_{F} W^{\text{op}} \otimes_{F} \mathbb{R}^{2} \xrightarrow{f} W \otimes_{F} W^{\text{op}} \xrightarrow{\mu} W \rightarrow 0 \quad (55)
\]
where \( \mu : W \otimes_{F} W^{\text{op}} \rightarrow W, (a, a') \mapsto a \cdot a' \) is the multiplication map and
\[
f : W \otimes_{F} W^{\text{op}} \otimes_{F} \mathbb{R}^{2} \rightarrow W \otimes_{F} W^{\text{op}}, \quad a \otimes b \otimes x \mapsto a \otimes (xb) - (ax) \otimes b,
\]
\[
g : W \otimes_{F} W^{\text{op}} \rightarrow W \otimes_{F} W^{\text{op}} \otimes_{F} \mathbb{R}^{2}, \quad a \otimes b \mapsto a \otimes (qb) \otimes p - (aq) \otimes b \otimes p - a \otimes (pb) \otimes q + (ap) \otimes b \otimes q.
\]

(d) Show that tensoring the free resolution \( (55) \) with the \( W \otimes_{F} W^{\text{op}} \)-module \( W \) over \( W \otimes_{F} W^{\text{op}} \) and omitting the first entry on the right yields a chain complex isomorphic to
\[
0 \rightarrow W \xrightarrow{f'} W \otimes_{F} \mathbb{R}^{2} \xrightarrow{f''} W \rightarrow 0
\]
\[
f' : W \otimes_{F} \mathbb{R}^{2} \rightarrow W, \quad a \otimes x \mapsto ax - xa, \quad g' : W \rightarrow W \otimes_{F} \mathbb{R}^{2}, \quad a \mapsto (aq - qa) \otimes p - (ap - pa) \otimes q.
\]
Compute the Hochschild homologies \( H_{n}(W, W) \) for all \( n \in \mathbb{N}_{0} \).
Exercise 58: Let $R$ be a ring and $X_\bullet = \ldots \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \to 0$ a chain complex in $R^{op}$-$\text{Mod}$ with $X_n$ and $d_n(X_n) \subset X_{n-1}$ flat for all $n \in \mathbb{N}_0$. Then for each $R$-left module $M$ and $n \in \mathbb{N}_0$ there is a short exact sequence

$$0 \to H_n(X_\bullet) \otimes_R M \to H_n(X_\bullet \otimes_R M) \to \text{Tor}_1^R(H_{n-1}(X_\bullet), M) \to 0$$

where $X_\bullet \otimes_R M = \ldots \xrightarrow{d_3 \otimes \text{id}_M} X_2 \otimes_R M \xrightarrow{d_2 \otimes \text{id}_M} X_1 \otimes_R M \xrightarrow{d_1 \otimes \text{id}_M} X_0 \otimes_R M \to 0$.

Prove this statement by proceeding as follows:

(a) Consider for $n \in \mathbb{N}_0$ the short exact sequences $0 \to \ker(d_n) \xrightarrow{i_n} X_n \xrightarrow{d_n} d_n(X_n) \to 0$ in $R^{op}$-$\text{Mod}$ and for each $R$-left module $M$ the right exact functor $- \otimes_R M : R^{op}$-$\text{Mod} \to \text{Ab}$.

Use the long exact sequence of derived functors for $- \otimes_R M$ and $\text{Tor}_n^R(L, M) \cong \text{Tor}_n^R(L, M)$ to show that $\ker(d_n)$ is flat for all $n \in \mathbb{N}_0$, and that one has an exact sequence

$$0 \to \ker(d_n) \otimes_R M \xrightarrow{i_n \otimes_R M} X_n \otimes_R M \xrightarrow{d_n \otimes \text{id}_M} d_n(X_n) \otimes_R M \to 0.$$

(b) Consider the chain complexes

$$B_\bullet = \ldots \xrightarrow{0} d_2(X_2) \xrightarrow{0} d_1(X_1) \xrightarrow{0} 0 \to 0, \quad Z_\bullet = \ldots \xrightarrow{0} \ker(d_1) \xrightarrow{0} \ker(d_0) \to 0$$

and for each $R$-left module $M$ the chain complexes

$$B_\bullet \otimes_R M = \ldots \xrightarrow{0} d_2(X_2) \otimes_R M \xrightarrow{0} d_1(X_1) \otimes_R M \to 0 \to 0$$

$$Z_\bullet \otimes_R M = \ldots \xrightarrow{0} \ker(d_2) \xrightarrow{0} \ker(d_1) \otimes_R M \xrightarrow{0} \ker(d_0) \otimes_R M \to 0$$

$$X_\bullet \otimes_R M = \ldots \xrightarrow{d_3 \otimes \text{id}_M} X_2 \otimes_R M \xrightarrow{d_2 \otimes \text{id}_M} X_1 \otimes_R M \xrightarrow{d_1 \otimes \text{id}_M} X_0 \otimes_R M \to 0.$$

Show with (a) that they form a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R M \xrightarrow{i_\bullet \otimes \text{id}_M} X_\bullet \otimes_R M \xrightarrow{d_\bullet \otimes \text{id}_M} B_\bullet \otimes_R M \to 0.$$

(c) Consider the associated long exact homology sequence. Show that the connecting morphism is $\partial_n = i_n \otimes \text{id}_M : d_n(X_\bullet) \otimes_R M \to \ker(d_{n-1}) \otimes_R M$, where $i_n : d_n(X_n) \to \ker(d_{n-1})$ is the inclusion map. Show with the long exact homology sequence that one has an exact sequence

$$0 \to H_n(X_\bullet) \otimes_R M \xrightarrow{H_n(i_\bullet \otimes \text{id}_M)} H_n(X_\bullet \otimes_R M) \xrightarrow{H_n(d_\bullet \otimes \text{id}_M)} \ker(i_n \otimes \text{id}_M) \to 0.$$

(d) Show that $0 \to d_n(X_n) \xrightarrow{i_n} \ker(d_{n-1}) \xrightarrow{\pi_{n-1}} H_{n-1}(X_\bullet) \to 0$ is a flat resolution of $H_{n-1}(X_\bullet)$ in $R^{op}$-$\text{Mod}$, and use this resolution to compute $\text{Tor}_1^R(H_n(X_\bullet), M) = \text{Tor}_1^R(H_n(X_\bullet), M)$. Combine this with (c) to prove the claim.

6.5 Exercises for Chapter 5

Exercise 59: Let $V$ be a set. A combinatorial simplicial complex is a subset $K \subset \mathcal{P}(V)$ consisting of finite non-empty subsets $M \subset V$ such that $\emptyset \neq M \subset M' \in K$ implies $M \in K$. A combinatorial simplicial complex is called ordered if the set $V$ is ordered.
(a) Show that every simplicial complex \((X, \{ \sigma_{\alpha} : \Delta^{n_\alpha} \to X \}_{\alpha \in I})\) determines a combinatorial simplicial complex given by

\[
V = \{ \sigma_{\alpha}(e_k) \mid \alpha \in I, k \in \{0, \ldots, n_\alpha\} \} \quad K = \{ \sigma_{\alpha}(\{e_0, \ldots, e_{n_\alpha}\}) \mid \alpha \in I \}.
\]

(b) Show that every ordered combinatorial simplicial complex \(K \subset \mathcal{P}(V)\) defines a simplicial set \(S^K : \Delta^{+op} \to \text{Set}\) given by

\[
S^K([n+1]) = \{(v_0, \ldots, v_n) \mid v_0 \leq v_1 \leq \ldots \leq v_n \} \quad S^K(\alpha)(v_0, \ldots, v_n) = (v_{\alpha(0)}, \ldots, v_{\alpha(m)}) \quad \text{for} \quad \alpha \in \text{Hom}_{\Delta^+}([m+1],[n+1])
\]

and determine \(S(\delta^i_n)\) and \(S(\sigma^i_{n+1})\) for \(n \in \mathbb{N}\) and \(0 \leq i \leq n\).

(c) Show that \(S^K = S^{K'}\) for ordered combinatorial simplicial complexes \(K, K'\) implies \(K = K'\).

Remark: Recall that an ordered set \((V, \leq)\) is a set \(V\) with a relation \(\leq\) that satisfies (i) \(v \leq v\) for all \(v \in V\), (ii) \(v \leq w\) and \(w \leq v\) implies \(v = w\), (iii) \(u \leq v\) and \(v \leq w\) implies \(u \leq w\) and (iv) for all \(v, w \in V\) one has \(v \leq w\) or \(w \leq v\).

Exercise 60: Let \(V\) be an ordered set with \(|V| = n + 1\) and \(K = \{ M \mid \emptyset \neq M \subset V \}\). Show that the geometric realisation \(\text{Geom}(S^K)\) of the associated simplicial set \(S^K : \Delta^{+op} \to \text{Set}\) is homeomorphic to \(\Delta^n\).

Exercise 61: Let \(S : \Delta^{+op} \to \text{Set}\) be a simplicial set. Show that the geometric realisation \(\text{Geom}(S)\) has the structure of a semisimplicial complex by proceeding as follows:

(a) An element \(x \in S_n = S([n+1])\) is called non-degenerate if \(x = S(\sigma)x'\) with \(x' \in S_k\) and a monotonic surjection \(\sigma : [n+1] \to [k+1]\) implies \(\sigma = 1_{[n+1]}\).

Show that for every element \(x \in S_n\), there is a unique \(k \in \{0, \ldots, n\}\) and a unique non-degenerate \(y \in S_k\) with \(x = S(\sigma)y\) for some monotonic surjection \(\sigma : [n+1] \to [k+1]\).

(b) Define \(I = \cup_{n \in \mathbb{N}_0}\{ x \in S_n \mid x \text{ non-degenerate}\}\) and consider for \(x \in I \cap S_n\) the continuous maps \(\sigma_x : \Delta^n \to \text{Geom}(S), p \mapsto [(x, p)]\). Show that \((\text{Geom}(S), \{\sigma_x\}_{x \in I})\) is a semisimplicial complex.

Exercise 62: Let \(\Delta^+_m\) be the subcategory of the simplex category \(\Delta^+\) with objects \([n+1]\) for \(n \in \mathbb{N}_0\) and with \(\text{Hom}_{\Delta^+_m}([n+1],[m+1]) = \{ \alpha : [n+1] \to [m+1] \mid \alpha\ \text{monotonic and injective}\}\).

A semisimplicial object in \(\mathcal{C}\) is a functor \(K : \Delta^{+op}_m \to \mathcal{C}\) and a morphism of semisimplicial objects is a natural transformation \(\eta : K \to K'\).

For a semisimplicial object \(K\) in an abelian category \(\mathcal{A}\) we define

\[
LK([n+1]) = LK_n = \coprod_{\sigma : [0, \ldots, n] \to [p+1]} K_p,
\]

where \(K_p = K([p+1])\) and the coproduct runs over all monotonic surjections \(\sigma : [n+1] \to [p+1]\) with \(0 \leq p \leq n\). For a morphism \(\alpha : [m+1] \to [n+1]\) in \(\Delta^+\), we define \(LK(\alpha) : LK_n \to LK_m\) as the unique morphism with \(LK(\alpha) \circ \iota_{\sigma} = \iota_{\sigma\alpha} \circ K(\alpha\sigma)\), where \(\sigma_{\alpha} : [m+1] \to [q+1]\) is the unique surjection and \(\alpha : [q+1] \to [p+1]\) the unique injection in \(\Delta^+\) with \(\alpha\circ \sigma_{\alpha} = \sigma \circ \alpha\) and \(\iota_{\sigma} : K_p \to LK_n\) denotes the inclusion morphism for \(\sigma\).
(a) Show that this defines a simplicial object $LK : \Delta^{+\text{op}} \to \mathcal{A}$.

(b) Show that this induces a functor $L : \text{Fun}(\Delta^{+\text{op},\text{inj}}, \mathcal{A}) \to \text{Fun}(\Delta^{+\text{op}}, \mathcal{A})$ from the category $\text{Fun}(\Delta^{+\text{op},\text{inj}}, \mathcal{A})$ of semisimplicial objects in $\mathcal{A}$ to the category $\text{Fun}(\Delta^{+\text{op}}, \mathcal{A})$ of simplicial objects in $\mathcal{A}$.

**Exercise 63:** Let $\mathcal{A}$ be an abelian category.
Show that the functor $L : \text{Fun}(\Delta^{+\text{op},\text{inj}}, \mathcal{A}) \to \text{Fun}(\Delta^{+\text{op}}, \mathcal{A})$ from Exercise 62 is left adjoint to the restriction functor $G : \text{Fun}(\Delta^{+\text{op}}, \mathcal{A}) \to \text{Fun}(\Delta^{+\text{inj}}, \mathcal{A})$ that assigns to a simplicial object $S : \Delta^{\text{op}} \to \mathcal{A}$ its restriction to the subcategory $\Delta^{+\text{inj}} \subset \Delta^{+\text{op}}$.

**Exercise 64:** Let $(G_n)_{n \in \mathbb{N}_0}$ be a family of groups with $G_0 = \{e\}$ and $(\rho_{m,n})_{m,n \in \mathbb{N}_0}$ a family of group homomorphisms $\rho_{m,n} : G_m \times G_n \to G_{m+n}$ with $\rho_{0,m} : \{e\} \times G_m \to G_m$, $(e,g) \mapsto g$ and $\rho_{m,0} : G_m \times \{e\} \to G_m$, $(g,e) \mapsto g$ and $\rho_{m+n,p} \circ (\rho_{m,n} \times \text{id}_{G_p}) = \rho_{m,n+p} \circ (\text{id}_{G_m} \times \rho_{n,p}) \quad \forall m,n,p \in \mathbb{N}_0$.

(a) Show that this defines a strict tensor category $(\mathcal{C}, \otimes)$ with non-negative integers $n \in \mathbb{N}_0$ as objects, $\text{Hom}_\mathcal{C}(n,m) = \emptyset$ if $m \neq n$ and $\text{Hom}_\mathcal{C}(n,n) = G_n$ and the tensor product given by $m \otimes n = n + m$ for all $n,m \in \mathbb{N}_0$ and $f \otimes g = \rho_{m,n}(f,g)$ for all $f \in G_m$, $g \in G_n$.

(b) Consider the permutation groups $G_n = S_n$ and find group homomorphisms $\rho_{m,n} : S_m \times S_n \to S_{m+n}$ that satisfy the conditions.

(c) Show that in (b) any tensor functor $F : \mathcal{C} \to \mathcal{D}$ into a strict tensor category $(\mathcal{D}, \otimes)$ is determined uniquely by $F(1)$ and $F(\tau)$ for the elementary transposition $\tau \in S_2$, $(1,2) \mapsto (2,1)$.

**Exercise 65:** Show that the forgetful functor $G : \text{Field} \to \text{Set}$ does not have a left adjoint.

**Exercise 66:** Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ additive functors with $F$ left adjoint to $G$. Show that this implies that $F$ is right exact and $G$ left exact.

**Exercise 67:** Let $(A, \mu, \eta)$ be an algebra object in a monoidal category $\mathcal{C}$. An $A$-module object is a pair $(M, \triangleright)$ of an object $M$ in $\mathcal{C}$ and a morphism $\triangleright : A \otimes M \to M$ such that the following diagrams commute

\[
\begin{array}{ccc}
(A \otimes A) \otimes M & \xrightarrow{\alpha_{A,A,M}} & A \otimes (A \otimes M) \\
\mu \otimes 1_M & \xrightarrow{\text{id}_A \otimes \triangleright} & A \otimes M \\
A \otimes M & \xleftarrow{\triangleright} & M
\end{array}
\quad \quad
\begin{array}{ccc}
M & \xrightarrow{I_M \otimes \eta} & e \otimes M \\
\triangleright & \xrightarrow{\eta} & A \otimes M
\end{array}
\]
A morphism of $A$-module objects from $(M, \triangleright)$ to $(M', \triangleright')$ is a morphism $\phi : M \to M'$ such that the following diagram commutes

$$
\begin{array}{ccc}
A \otimes M & \xrightarrow{1_A \otimes \phi} & A \otimes M' \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi} & M'.
\end{array}
$$

If $\mathcal{C} = \text{End}(\mathcal{D})$, then an algebra $(A, \mu, \eta)$ in $\text{End}(\mathcal{D})$ is a monad in $\mathcal{D}$, and a module object over $A$ is sometimes called an $A$-algebra.

(a) Show that $A$-module objects and morphisms of $A$-module objects form a category $\mathcal{A} \text{Mod}$.

(b) Show that for each object $C$ in $\mathcal{C}$, the object $A \otimes C$ is an $A$-module object with $\triangleright = (\mu \otimes 1_C) \circ a_{A,C}^{-1} : A \otimes (A \otimes C) \to A \otimes C$ and for each morphism $f : C \to C'$ the morphism $1_A \otimes f : A \otimes C \to A \otimes C'$ is a morphism of $A$-modules. Show that this defines a functor $F : \mathcal{C} \to \mathcal{A} \text{Mod}$.

(c) Show that the functor $F : \mathcal{C} \to \mathcal{A} \text{Mod}$ is left adjoint to the forgetful functor $G : \mathcal{A} \text{Mod} \to \mathcal{C}$ and determine the monad associated with $F, G$.

(d) Show that for every monad $(A, \mu_A, \eta_A)$ in a category $\mathcal{D}$, there is an adjunction $(F, G, \eta, \epsilon)$ such that $A = GF(\text{id}_D)$, $\mu_A = (G \epsilon F)_{\text{id}_D}$ and $\eta_A = \eta(\text{id}_D)$.
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