Some aspects of cross diffusion equations

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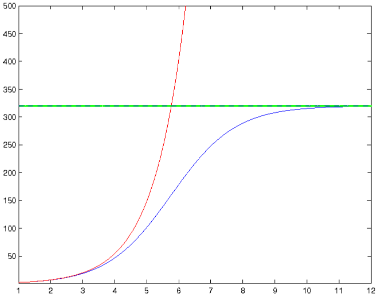


Logistic model (Verhulst, 1838)

$$u'(t) = r_0 \left(1 - \frac{u(t)}{K}\right) u(t), \qquad u(t) = \frac{u(0)\, K}{u(0) + (K - u(0))\, \exp(-r_0\, t)}.$$

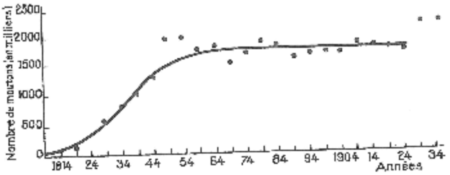
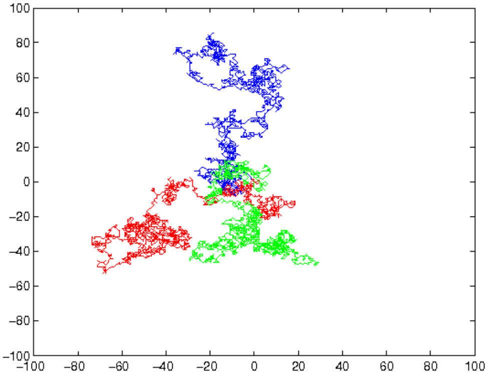


FIG. 62.— *Le développement du mouton après son introduction en Tasmanie.*

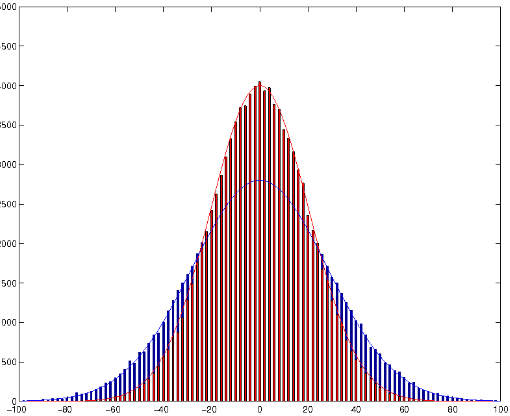
An example of observed data and of logistic "fit"

Dispersion models for populations



Random walk: We assume isotropy, and consider a scale of time and space which is large (w.r.t. the motion of one individual).

Dispersion models for populations



Corresponding law: convergence towards a Gaussian law with variance proportional to time.

Dispersion models for populations

Random walk $S_p = \sum_{i=1}^{p} X_i$, with $X_i = \Delta x$ and $X_i = -\Delta x$ each of probability $1/2$, and independant.

Law of S_p: $P(S_p = q\,\Delta x) = 2^{-p}\, C_p^{\frac{q+p}{2}}$ (when $|q| \leq p$ et $q \equiv p[2]$).

We consider $N(p\Delta t, x) := P(S_p \in [x - \Delta x, x + \Delta x[)$. Then for $t = p\,\Delta t$:

$$N(t, q\,\Delta x) = 2^{-p}\, C_p^{\frac{q+p}{2}}.$$

One uses the following asymptotic expansion:

Lemma:

$$2^{-p}\, C_p^{\frac{q+p}{2}} \sim \frac{2}{\sqrt{2\pi\, p}}\, e^{-\frac{q^2}{2p}}$$

when $p \to +\infty$, $q^3 = o(p^2)$.

When $\Delta t \to 0$ and $\Delta t^2 << \Delta x^3$,

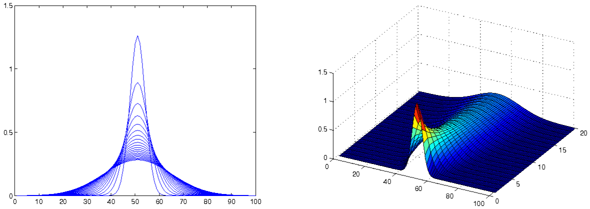
$$N(t,x) \sim 2\,\Delta x \sqrt{\frac{\Delta t}{\Delta x^2}}\,\frac{e^{-\{\frac{\Delta t}{\Delta x^2}\}\,\frac{x^2}{2t}}}{\sqrt{2\pi t}},$$

in such a way that $\frac{\Delta t}{\Delta x^2} \to \frac{1}{2D}$ and $\Delta t \to 0$,

$$\frac{N(t,x)}{2\,\Delta x} \to u(t,x) = \frac{e^{-\frac{x^2}{4D\,t}}}{\sqrt{4\pi\,D\,t}}.$$

This last quantity is the elementary solution of the heat equation in dimension 1 with a diffusion coefficient D:

$$\frac{\partial u}{\partial t}(t,x) = D\,\frac{\partial^2 u}{\partial x^2}(t,x), \qquad u(0,x) = \delta_0(x).$$



Diffusion (Fourier, 1822): Heat (diffusion) equation and its fundamental solution:

$$\frac{\partial u}{\partial t}(t,x) = D\,\frac{\partial^2 u}{\partial x^2}(t,x), \qquad u(0,x) = \delta_0(x).$$

$$u(t,x) = \frac{e^{-\frac{x^2}{4D\,t}}}{\sqrt{4\pi\,D\,t}}.$$

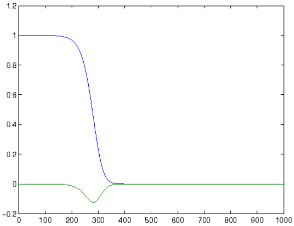
Traveling waves

Invasion model (Fisher; Kolmogoroff-Petrovsky-Piscounoff, 1937)

$$\frac{\partial u}{\partial t}(t,x) = D\,\frac{\partial^2 u}{\partial x^2}(t,x) + r_0\,\left(1 - \frac{u(t,x)}{K}\right)\,u(t,x).$$

Obtained when both diffusion and logistic effects are considered.

Traveling waves

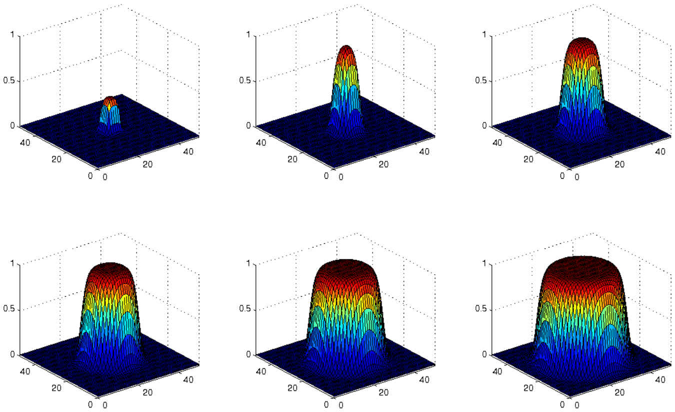


In dimension 1: One looks for $u(t,x) = N(x - ct)$ solution of the PDE:

$$-c\,N'(z) - D\,N''(z) = r_0 \left(1 - \frac{N(z)}{K}\right) N(z); \qquad N(-\infty) = K; \; N(\infty) = 0.$$

Theorem (Kolmogoroff-Petrovsky-Piscounoff, 1937): Solutions to this heteroclinic junction problem in ODEs exist when $c \geq c_0 = \sqrt{2\,r_0\,D}$, critical speed of invasion associated to a population.
Those solutions are stable (in a setting to be made precise...) for the PDE if and only if $c = c_0$.

2D Traveling waves of invasion



Meaning of maps related to a biological invasion

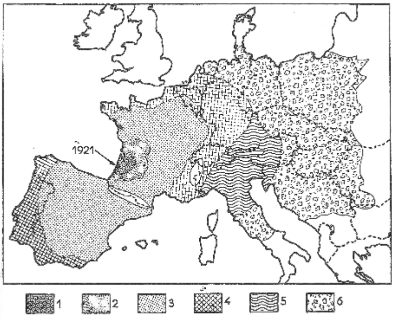


FIG. 63. — *Progression du doryphore en Europe après son arrivée en 1921.*

Competition models

Lotka, Volterra, 1925

Unknowns: $u := u(t) \geq 0$, $v := v(t) \geq 0$, for $t \geq 0$.

Equations:

$$u'(t) = (r_1 - S_{11}\, u(t) - S_{12}\, v(t))\, u(t),$$
$$v'(t) = (r_2 - S_{21}\, u(t) - S_{22}\, v(t))\, v(t).$$

$S_{ii} > 0$: intraspecific competition

$S_{ij} \geq 0$, $i \neq j$: interspecific competition

Competition models

Depending on the parameters r_i, S_{ij}, and considering only nonnegative solutions, one has either (up to exchanging n_1 and n_2):

- **Strong competition**: The only stable equilibrium for the system of ODEs is $(u, v) = (n_{10}, 0)$ with $n_{10} > 0$; competitive exclusion.
- **Weak competition**: The only stable equilibrium for the system of ODEs is $(u, v) = (n_{10}, n_{20})$ with $n_{10} > 0$, $n_{20} > 0$; coexistence.

Competition/Diffusion model

Unknowns: $u_1 := u_1(t,x) \geq 0$, $u_2 := u_2(t,x) \geq 0$, for $t \geq 0$, $x \in \Omega$.

Equations:

$$\partial_t u_1 - D_1 \Delta_x u_1 = (r_1 - S_{11} u_1 - S_{12} u_2) u_1,$$

$$\partial_t u_2 - D_2 \Delta_x u_2 = (r_2 - S_{21} u_1 - S_{22} u_2) u_2.$$

No Turing instability for such models: all steady homogeneous solutions which are stable for the ODEs are also stable for the PDEs; **No segregation of species appears**

A typical cross diffusion system: Shigesada-Kawasaki-Teramoto (SKT) model (1979)

Equations for the densities of population of two competing species:

$$\partial_t u_1 - \Delta_x \Big(u_1 \Big[D_1 + A_{12}\, u_2 \Big] \Big) = (r_1 - S_{11}\, u_1 - S_{12}\, u_2)\, u_1,$$

$$\partial_t u_2 - \Delta_x \Big(u_2 \Big[D_2 + A_{21}\, u_1 \Big] \Big) = (r_2 - S_{21}\, u_1 - S_{22}\, u_2)\, u_2.$$

Neumann boundary condition (for $t \geq 0$, $x \in \partial\Omega$)

$$\nabla_x u_1(t,x) \cdot n(x) = 0, \quad \nabla_x u_2(t,x) \cdot n(x) = 0.$$

Assumption: $D_i > 0$, $A_{12}, A_{21} > 0$, $r_i > 0$, $S_{ij} > 0$.

Modeling issue

Why $\Delta_x(u_1\, u_2) = \nabla_x \cdot (u_2\, \nabla_x u_1) + \nabla_x \cdot (u_1\, \nabla_x u_2)$ rather than $\nabla_x \cdot (u_2\, \nabla_x u_1)$, or $\Delta_x(u_1\, u_2) = \nabla_x \cdot (u_2\, \nabla_x u_1) + \beta\, \nabla_x \cdot (u_1\, \nabla_x u_2)$, avec $\beta \neq 1$?

Answer (proposed by Iida, Izuhara, Mimura, Ninomiya, in the "triangular" case $A_{21} = 0$)

Possible interprétation based on a "microscopic" behavior: The species u_1 exists in two states: quiet (u_{1A}) and stressed (u_{1B}). The individuals of this species switch between the two states with a time scale ε and probability rates which depend on the concentration u_2 of the other species.

Equations of the "microscopic" model

$$\partial_t u_{1A} - D_1 \, \Delta_x u_{1A} = (r_1 - S_{11} \, (u_{1A} + u_{1B}) - S_{12} \, u_2) \, u_{1A}$$
$$+ \frac{1}{\varepsilon}((1 - u_2) \, u_{1B} - u_2 \, u_{1A}),$$

$$\partial_t u_{1B} - (D_1 + A_{12}) \, \Delta_x u_{1B} = (r_1 - S_{11} \, (u_{1A} + u_{1B}) - S_{12} \, u_2) \, u_{1B}$$
$$- \frac{1}{\varepsilon}((1 - u_2) \, u_{1B} - u_2 \, u_{1A}),$$

$$\partial_t u_2 - D_2 \, \Delta_x u_2 = (r_2 - S_{21} \, (u_{1A} + u_{1B}) - S_{22} \, u_2) \, u_2.$$

Formal asymptotics when the time scales tends to 0

Assuming that $u_{1A}^\varepsilon \to u_{1A}$, $u_{1B}^\varepsilon \to u_{1B}$, $u_2^\varepsilon \to u_2$,

$$(1 - u_2)\, u_{1B} = u_2\, u_{1A}, \qquad u_{1B} = u_2\,(u_{1A} + u_{1B}),$$

and $(u_{1A} + u_{1B}, u_2)$ satisfy

$$\partial_t(u_{1A} + u_{1B}) - \Delta_x\Big(D_1\, u_{1A} + (D_1 + A_{12})\, u_{1B}\Big)$$

$$= (r_1 - S_{11}\,(u_{1A} + u_{1B}) - S_{12}\, u_2)\,(u_{1A} + u_{1B}),$$

$$\partial_t u_2 - D_2\, \Delta_x u_2 = (r_2 - S_{21}\,(u_{1A} + u_{1B}) - S_{22}\, u_2)\, u_2.$$

Rigorous proof: Cf. LD-Trescases.

Formal asymptotics when the time scales tends to 0 (II)

The SKT model (with $A_{21} = 0$) is recovered by defining $u_1 = u_{1A} + u_{1B}$,

$$\partial_t u_1 - \Delta_x \Big((D_1 + A_{12}\, u_2)\, u_1 \Big) = (r_1 - S_{11}\, u_1 - S_{12}\, u_2)\, u_1 .$$

The equation for u_2 is conserved:

$$\partial_t u_2 - D_2\, \Delta_x u_2 = (r_2 - S_{21}\, u_1 - S_{22}\, u_2)\, u_2 .$$

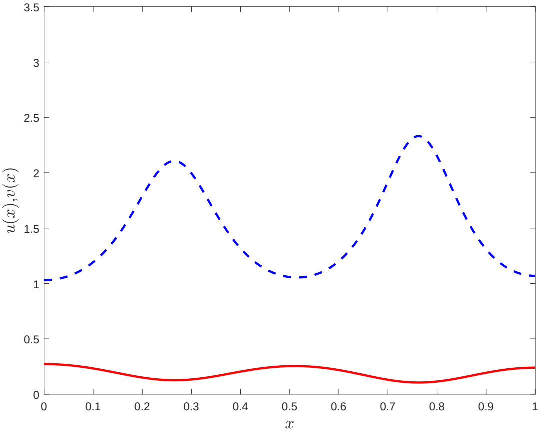
Turing instability for the SKT model

Existence of inhomogeneous steady states for the SKT model, interpreted as describing segregation situations

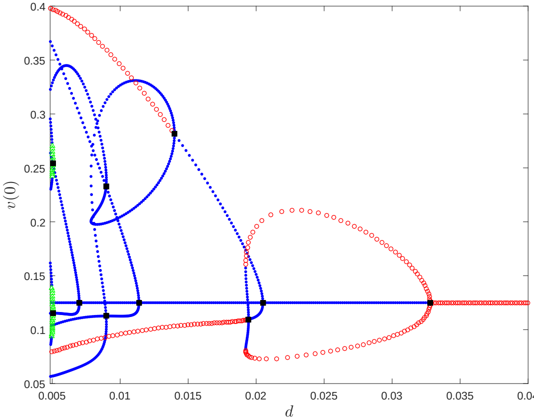
- At the numerical level: computation of those inhomogeneous steady states, and of associated bifurcation diagrams Iida, Izuhara, Mimura, Ninomiya.

- At the rigorous level: justification of the numerical simulations thanks to computer-assisted proofs Breden, Castelli, Lessard, Vanicat.

Segregation state



Bifurcation Diagram



Here, *d* is one of the parameters of the SKT model. The solutions in red are stable, the ones in blue and green unstable.

Results of existence for the non-triangular SKT system

Amann: Existence of local (in time) solutions

Kim; Masuda, Mimura; Shim: Existence of solutions for various types of coefficients in dimension 1

Li, Zhao: Existence of solutions when $D_1 = D_2$

Chen, Jüngel, 2004: Existence of (weak) solutions thanks to the use of the functional

$$J(u_1, u_2) = A_{21} \int_\Omega (u_1 \ln u_1 - u_1 + 1) + A_{12} \int_\Omega (u_2 \ln u_2 - u_2 + 1)$$

Computation of the evolution of the functional J

$$\frac{d}{dt} J(u_1, u_2) + A_{21}\, D_1 \int \frac{|\nabla_x u_1|^2}{u_1} + A_{12}\, D_2 \int \frac{|\nabla_x u_2|^2}{u_2}$$

$$+ A_{12}\, A_{21} \int u_1\, u_2 \left| \frac{\nabla_x u_1}{u_1} + \frac{\nabla_x u_2}{u_2} \right|^2 \leq C(r_i, S_{ij}, A_{ij}).$$

After integration in time, for any $T > 0$,

$$\int_0^T \int_\Omega \left(|\nabla_x \sqrt{u_1}|^2 + |\nabla_x \sqrt{u_2}|^2 \right) < \infty.$$

If $r_i = 0,\ S_{ij} = 0$,

$$C = 0; \qquad\qquad \frac{d}{dt} J(u_1, u_2) \leq 0.$$

Extensions for more general equations

Theorem (LD, Lepoutre, Moussa, Trescases) We assume that $D_i > 0$, $r_i \geq 0$, $A_{ij} \geq 0$, and $S_{ij} > 0$.
We take $0 < \beta_{ij} < 1$, and $\alpha_{12}, \alpha_{21} > 0$ such that

$$\alpha_{12}\, \alpha_{21} < 1.$$

Then there exists a weak solution to the system

$$\partial_t u_1 - \Delta_x \left[(D_1 + A_{12}\, u_2^{\alpha_{12}})\, u_1 \right] = u_1 \left(r_1 - S_{11}\, u_1^{\beta_{11}} - S_{12}\, u_2^{\beta_{12}} \right),$$

$$\partial_t u_2 - \Delta_x \left[(D_2 + A_{21}\, u_1^{\alpha_{21}})\, u_2 \right] = u_2 \left(r_2 - S_{21}\, u_1^{\beta_{21}} - S_{22}\, u_2^{\beta_{22}} \right),$$

with Neumann boundary conditions, and suitable (nonnegative for all components) initial data.

Main a priori estimate used in the proof

Entropy (Lyapounov) estimate (case $\alpha_{12} < 1$, $\alpha_{21} < 1$),

$$J^*(u_1, u_2)(T) + 4 \sum_{i \neq j} A_{ij} D_j \int_0^T \int_\Omega \left| \nabla_x \sqrt{u_j^{\alpha_{ij}}} \right|^2$$

$$+ 4\, A_{12} A_{21} \int_0^T \int_\Omega \left| \nabla_x \sqrt{u_2^{\alpha_{12}}\, u_1^{\alpha_{21}}} \right|^2 \leq J^*(u_{10}, u_{20}) + C(T),$$

where

$$J^*(u_1, u_2) := \sum_{i \neq j} \frac{A_{ij}\, \alpha_{ij}}{1 - \alpha_{ij}} \int_\Omega \left[\left(u_j - \frac{u_j^{\alpha_{ij}}}{\alpha_{ij}} \right) - \left(1 - \frac{1}{\alpha_{ij}} \right) \right].$$

The entropic structure

General equation

$$\partial_t U - \Delta_x[A(U)] = R(U),$$

with $A, R : \mathbb{R}^I \to \mathbb{R}^I$, and $U := U(t,x) : \mathbb{R}_+ \times \Omega \, (\Omega \subset \mathbb{R}^N) \to (\mathbb{R}_+)^I$.

For any $\Phi : (\mathbb{R}_+)^I \to \mathbb{R}_+$, if $R = 0$, and $\langle \ , \ \rangle$ is the Euclidian scalar product on $\mathbb{R}^I$,

$$\frac{d}{dt} \int_\Omega \Phi(U) = \int_\Omega \langle \nabla \Phi(U), \Delta_x[A(U)] \rangle$$

$$= -\sum_{j=1}^{N} \int_\Omega \langle \partial_{x_j} U, D^2\Phi(U) D(A)(U) \partial_{x_j} U \rangle \le 0$$

when Φ is an entropy, that is when $(D^2\Phi(U)\, D(A)(U))^{sym} \ge 0$.

The entropic structure (II)

Proposition (LD, Lepoutre, Moussa, Trescases): Consider $a_1, a_2 : \mathbb{R}_+^* \to \mathbb{R}_+$ two C^1 functions, and

$$A(x_1, x_2) := (x_1\, a_1(x_2)\,,\, x_2\, a_2(x_1)).$$

We assume that a_1, a_2 are increasing and $\mathrm{Det}\, D(A) \geq 0$, that is

$$\forall x_1, x_2 > 0, \qquad a_1(x_2)\, a_2(x_1) \geq x_1\, x_2\, a_1'(x_2)\, a_2'(x_1).$$

Then taking

$$\Phi(X) := \phi_1(x_1) + \phi_2(x_2),$$

where ϕ_i is a nonnegative second primitive of $z \mapsto a_j'(z)/z$ $(i \neq j)$, we get an entropy.

The entropic structure (III)

Proof: We compute, for $X = (x_1, x_2)$,

$$D(A)(X) = \begin{pmatrix} a_1(x_2) & x_1\, a_1'(x_2) \\ x_2\, a_2'(x_1) & a_2(x_1) \end{pmatrix}, \quad D^2(\Phi)(X) = \begin{pmatrix} \frac{a_2'(x_1)}{x_1} & 0 \\ 0 & \frac{a_1'(x_2)}{x_2} \end{pmatrix},$$

so that

$$M(X) := D^2(\Phi)(X)\, D(A)(X) = \begin{pmatrix} a_2'(x_1) a_1(x_2)/x_1 & a_2'(x_1) a_1'(x_2) \\ a_1'(x_2) a_2'(x_1) & a_1'(x_2) a_2(x_1)/x_2 \end{pmatrix}$$

is obviously symmetric. Since the functions a_i are increasing, all the coefficients of $M(X)$ are nonnegative, so that $\text{Tr}\, M(X) \geq 0$; we also see that

$$\mathrm{Det}\, M(X) = \mathrm{Det}\, D^2(\Phi)(X)\, \mathrm{Det}\, D(A)(X) \geq 0.$$

Example of a system of 3 equations for which an entropy exists (and existence holds)

System:

$$\partial_t u_1 - \Delta_x[u_1\,(D_1 + u_2^s + u_3^s)] = 0,$$
$$\partial_t u_2 - \Delta_x[u_2\,(D_2 + u_1^s + u_3^s)] = 0,$$
$$\partial_t u_3 - \Delta_x[u_3\,(D_3 + u_1^s + u_2^s)] = 0,$$

for $0 < s < 1/\sqrt{3}$ and $D_1, D_2, D_3 > 0$.

Entropy still of the form

$$\int_\Omega \Big(\phi_1(u_1) + \phi_2(u_2) + \phi_3(u_3)\Big)\,dx.$$

Example of a system for which an entropy is known, but is not "additive" (and existence is not known)

System:

$$\partial_t u - \Delta_x[u\, v^2] = 0,$$
$$\partial_t v - d\, \Delta_x[v\, u^2] = 0.$$

Entropy structure:

$$\frac{d}{dt}\int_\Omega u^2\, v^2\, dx = -\int_\Omega \left(u^2\, v^4 \left| \frac{\nabla_x u}{u} + 2\,\frac{\nabla_x v}{v} \right|^2 + d\, u^4\, v^2 \left| 2\,\frac{\nabla_x u}{u} + \frac{\nabla_x v}{v} \right|^2 \right) dx.$$

A method for building systems with entropies: Fast reaction limit of reaction-diffusion systems

Reaction-(nonlinear) diffusion system with fast reaction limit:

$$\partial_t u - \Delta_x(u^2) = \frac{1}{\varepsilon}(v\,w - u),$$

$$\partial_t v - \Delta_x(v^2) = -\frac{1}{\varepsilon}(v\,w - u),$$

$$\partial_t w - \Delta_x(w^2) = -\frac{1}{\varepsilon}(v\,w - u).$$

Entropy:

$$E = \int \Big[(u\,\ln u - u + 1) + (v\,\ln v - v + 1) + (w\,\ln w - w + 1)\Big]\,dx.$$

Fast reaction limit of reaction-diffusion systems (II)

Formal limit: $u = v\,w$, and

$$\partial_t (v\,w + v) - \Delta_x (v^2\,w^2 + v^2) = 0,$$
$$\partial_t (v\,w + w) - \Delta_x (v^2\,w^2 + w^2) = 0.$$

Entropy:

$$E = \int \Big[(v\,w\,\ln(v\,w) - v\,w + 1) + (v\,\ln v - v + 1) + (w\,\ln w - w + 1) \Big]\,dx.$$

Rigorous passage to the limit: Bothe, Rolland

Fast reaction limit of reaction-diffusion systems (III: structural stability)

Reaction-cross diffusion system with fast reaction limit:

$$\partial_t u - \Delta_x(u^2 + \eta\, u\, p(u,v)) = \frac{1}{\varepsilon}\,(v\,w - u),$$

$$\partial_t v - \Delta_x(v^2 + \eta\, v\, q(u,v)) = -\frac{1}{\varepsilon}\,(v\,w - u),$$

$$\partial_t w - \Delta_x(w^2) = -\frac{1}{\varepsilon}\,(v\,w - u).$$

Assumption: $p \geq 0, \partial_1 p \geq 0,\ q \geq 0, \partial_2 q \geq 0,$

$$\eta \left(||\partial_2 p||_\infty + ||\partial_1 q||_\infty \right) \leq 4.$$

Entropy:

$$E = \int \Big[(u\ \ln u - u + 1) + (v\ \ln v - v + 1) + (w\ \ln w - w + 1) \Big]\, dx.$$

Fast reaction limit of reaction-diffusion systems (IV)

Formal limit: $u = v\,w$, and

$$\partial_t(v\,w + v) - \Delta_x(v^2\,w^2 + v^2 + \eta\,v\,w\,p(v\,w, v) + \eta\,v\,q(v\,w, v)) = 0,$$

$$\partial_t(v\,w + w) - \Delta_x(v^2\,w^2 + w^2 + \eta\,v\,w\,p(v\,w, v)) = 0.$$

Entropy:

$$E = \int \Big[(v\,w\,\ln(v\,w) - v\,w + 1) + (v\,\ln v - v + 1) + (w\,\ln w - w + 1)\Big]\,dx.$$

Rigorous passage to the limit: Daus, LD, Jüngel

SKT system for an arbitrary number of species

System:

$$\partial_t u_i = \Delta_x \left(\left[D_i + \sum_{j=1}^{n} A_{ij} u_j \right] u_i \right)$$

Chen, Daus, Jüngel showed that

$$J(u_1, .., u_n) := \sum_{i=1}^{n} \int \pi_i \left[u_i \ln(u_i) - u_i + 1 \right]$$

with positive constants $\pi_i > 0$ for $i = 1, \ldots, n$, is an entropy if the following condition holds

$$\forall i, j \qquad \pi_i A_{ij} = \pi_j A_{ji}.$$

The system is then said to be **detailed balanced**.

SKT system for an arbitrary number of species (II)

We consider a dicretized space domain: $\Omega_M = \{0, \frac{1}{M}, \frac{2}{M}, .., 1\}$, and nonnegative constants π_i, D_i, D_{ij} such that $\sum_{i=1}^n \pi_i = 1$ and $D_{ij} = D_{ji}$ for $i, j = 1, \ldots, n$.

We consider n species of particles located on Ω_M. There are $[\pi_i N]$ particles of species i.
We define the time-continuous Markov chain on $\Theta_{M,N} := \Omega_M^{[\pi_1 N]+\cdots+[\pi_n N]}$ by the transitions

$$\left.\begin{array}{l} x \to x + e_i^a + e_j^b \\ x \to x - e_i^a - e_j^b \end{array}\right\} \quad \text{with rate } \delta_{(i,a)\neq(j,b)}\delta_{x_i^a = x_j^b}\frac{D_{ij}}{N}$$

$$\left.\begin{array}{l} x \to x + e_i^a \\ x \to x - e_i^a \end{array}\right\} \quad \text{with rate } D_i$$

for $i, j = 1, \ldots, n$ and $a = 1, \ldots, [\pi_i N]$, $b = 1, \ldots, [\pi_j N]$, where e_i^a is the vector with components of value zero at all places, except for the a-th particle of species i, where the value is $h = 1/M$.

SKT system for an arbitrary number of species (III)

Associated master equation:

$$\frac{d}{dt}\mu^N(t,x) = \sum_{i=1}^{n}\sum_{a=1}^{[\pi_i N]} D_i \Big[\mu^N(t,x+e_i^a) + \mu^N(t,x-e_i^a) - 2\mu^N(t,x)\Big]$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{[\pi_i N]}\sum_{j=1}^{n}\sum_{b=1}^{[\pi_j N]} \delta_{(i,a)\neq(j,b)}\delta_{x_i^a = x_j^b}\frac{D_{ij}}{N}\Big[\mu^N(t,x+e_i^a+e_j^b)$$
$$+ \mu^N(t,x-e_i^a-e_j^b) - 2\mu^N(t,x)\Big].$$

SKT system for an arbitrary number of species (IV)

The functional defined by

$$\tilde{\mathcal{H}}(\mu^N)(t) := \sum_x \mu^N(t,x) \ln\left(\frac{\mu^N(t,x)}{M^{[\pi_1 N]+\cdots+[\pi_n N]}}\right)$$

is decreasing with respect to time, i.e.

$$\frac{d}{dt}\tilde{\mathcal{H}}(\mu^N) \leq 0$$

along the flow of the master equation.

SKT system for an arbitrary number of species (V)

When $N \to \infty$, under the assumption (of indinstinguishability and) chaos propagation:

$$\mu^N(t, x_1^1, \dots, x_1^{[\pi_1 N]}, \dots\dots\dots, x_n^1, \dots, x_n^{[\pi_n N]})$$
$$\approx u_1(t, x_1^1) \cdots u_1(t, x_1^{[\pi_1 N]}) \cdots\cdots\cdots u_n(t, x_n^1) \cdots u_n(t, x_n^{[\pi_n N]}),$$

one can show that

$$\frac{d}{dt} u_i(t,x) = D_i \Big[u_i(t, x+h) + u_i(t, x-h) - 2u_i(t,x) \Big]$$

$$+ \sum_{j=1}^{n} D_{ij} \pi_j \Big[u_j(t, x+h) u_i(t, x+h) + u_j(t, x-h) u_i(t, x-h) - 2 u_j(t,x) u_i(t,x) \Big].$$

SKT system for an arbitrary number of species (VI)

When $N \to \infty$, under the assumption (of indinstinguishability and) chaos propagation,

$$\frac{1}{N}\tilde{\mathcal{H}}(\mu^N) = \sum_x \mu^N(x) \ln \left(\frac{\mu^N(x)}{M^{([\pi_1 N] + \cdots + [\pi_n N])}} \right)$$

$$\approx \frac{1}{N} \sum_{i=1}^{n} \sum_{\ell=0}^{M-1} [\pi_i N] u_i(x_\ell) \ln \left(\frac{u_i(x_\ell)}{M} \right)$$

$$\to \sum_{i=1}^{n} \pi_i \sum_{\ell=0}^{M-1} u_i(x_\ell) \ln \left(\frac{u_i(x_\ell)}{M} \right).$$

SKT system for an arbitrary number of species (VII)

When $h = 1/M \to 0$, defining $A_{ij} := \pi_j\, D_{ij}$, and rescaling in time (in such a way that ∂_t is replaced by $h^2\,\partial_t$), one recovers the original SKT system for an arbitrary number of species:

$$\partial_t u_i = \Delta_x \left(\left[D_i + \sum_{j=1}^{n} A_{ij} u_j \right] u_i \right),$$

and its Lyapunov functional:

$$J(u_1, .., u_n) := \sum_{i=1}^{n} \int \pi_i \left[u_i \ln(u_i) - u_i + 1 \right].$$

The condition $\forall i, j \;\; D_{ij} = D_{ji}$ becomes the detailed balance condition:

$$\forall i, j \qquad \pi_i A_{ij} = \pi_j A_{ji}.$$