Large stochastic systems of interacting particles.

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From very small to very large "particles"

Many-particle or multi-agent systems are used in a widespread range of applications

- Plasmas: Particles are ions or electrons.
- Astrophysics: Particles are dark matter particles, galaxies or galaxy clusters...
- Fluids: Point vortices, suspensions...
- Bio-mechanics: Medical aerosols in the respiratory tract, suspensions in the blood...
- Bio-Sciences: Collective behaviors of animals, swarming or flocking, but also dynamics of micro-organisms, chemotaxis, cell migration, neural networks...
- Social Sciences: Opinion dynamics, consensus formation...
- Economics: Mean-field games...

Very large particles: Galaxies

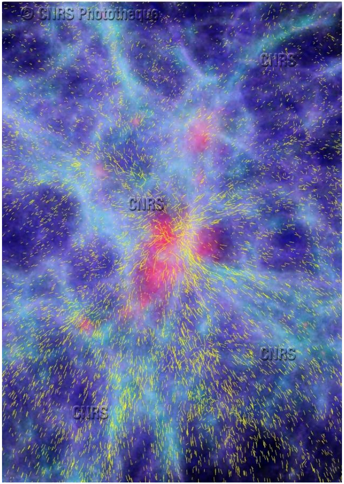

Figure: Credits: CNRS, France; Numerical simulation of the formation of large scale structures in the universe: Dynamics of galaxies moving to the central concentration.

An example of application: Biological neurons

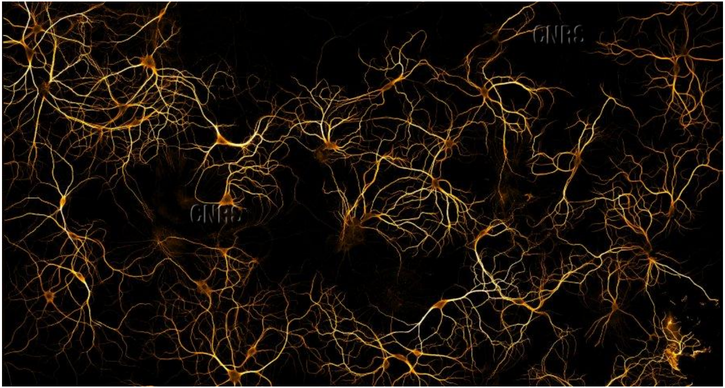

Figure: Credits: CNRS Bordeaux, France; 2D reconstruction of rat hippocampus, marked for cytoskeleton protein.

How many particles?

Denote by N the number of particles or agents.

- In cosmology/astrophysics, N ranges from 10^{10} to $10^{20} - 10^{25}$; some models of dark matter even predict up to 10^{60} particles.
- In plasma dynamics, N is typically of order $10^{20} - 10^{25}$. This is the typical order of magnitude for physics settings.
- When used for numerical purposes (particles' methods...), the number is of order $10^9 - 10^{12}$.
- In biology or Life Sciences, typical population of micro-organisms include between 10^6 and 10^{12}.
- In other applications such as collective dynamics, Social Sciences or Economics, numbers can be much lower of order 10^3.

Whenever possible, it is critical to quantify how fast the convergence to the continuous limit holds in terms of N.

Interacting particles

Consider N particles, identical and interacting two by two through the kernel K. For $X_i(t) \in \Pi^d$ the position of the i-th particle,

$$d\,X_i = \frac{1}{N} \sum_{j=1}^{N} K(X_i - X_j)\,dt + \sqrt{2\,\sigma}\,dW_i,$$

with the mean field scaling and for N independent Brownian motions W_i^t. For simplicity take $K(0) = 0$: No self-interaction.

- Main question: Behavior of the system for $N >> 1$.
- For simplicity in the talk, σ is fixed but the case $\sigma = \sigma_N$ is also of interest, with for example $\sigma \sim \frac{1}{N}$ playing a special role for Coulomb gases.

The mean-field approximation

The main aim of this talk is to justify the so-called mean-field approximation which consists in replacing the discrete interaction by its conditional expectation given the position X_i,

$$\frac{1}{N}\sum_j K(X_i - X_j) \longrightarrow \mathbb{E}\left(\frac{1}{N}\sum_j K(X_i - X_j) \mid X_i(t)\right).$$

This leads to the McKean-Vlasov process

$$d\,\bar{X}_i = \mathbb{E}\left(\frac{1}{N}\sum_j K(\bar{X}_i - \bar{X}_j) \mid \bar{X}_i(t)\right)\,dt + \sqrt{2\,\sigma}\,dW_i.$$

A law of large numbers? Propagation of chaos!

- The argument looks a lot like a law of large numbers, which could be justified if the X_i were independent.
- Even if the X_i^0 are independent initially, this cannot be strictly true for any $t > 0$ as the dynamics correlates the particles.
- This leads to the notion of propagation of chaos which conjectures that for large N, the X_i are almost independent.
- This has to be made mathematically precise and proved!

The mean-field approximation: The McKean-Vlasov equation

If the $\bar{X}_i$ solve the McKean-Vlasov process

$$d\,\bar{X}_i = \mathbb{E}\left(\frac{1}{N}\sum_j K(\bar{X}_i - \bar{X}_j)\mid \bar{X}_i(t)\right)\,dt + \sqrt{2\,\sigma}\,dW_i,$$

and are independent and identically distributed, then we may instead solve the McKean-Vlasov equation by Itô's formula

$$\partial_t\bar{\rho} + \mathrm{div}_x(K\star\bar{\rho}\,\bar{\rho}) = \sigma\,\Delta\,\bar{\rho},$$

on the law $\bar{\rho}(t,x)$ of the X_i at time t.

Vorticity in a fluid

Figure: Credits: CNRS, France; flow in a thin soap film perturbed by the teeth of a plastic comb.

Vortex dynamics in incompressible fluids

A first seminal example of many-particle systems is the dynamics of point vortices in dimension 2

$$d\,X_i = \frac{\alpha}{N}\sum_{j\neq i}\frac{(X_i - X_j)^\perp}{|X_i - X_j|^2}\,dt + \sqrt{2\,\sigma}\,dW_i,$$

consisting in taking $K = \nabla^\perp \Phi$ with

$$\Phi(x) = \lambda\,\log|x-y|, \quad \text{the Poisson kernel.}$$

By the Biot-Savart law, the field

$$\alpha\,\frac{(x - X_j)^\perp}{|x - X_j|^2}$$

is the velocity field created in an incompressible fluid by a point vortex $\frac{\alpha}{4\,\pi}\,\delta(x - X_j)$.

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$$d\,X_i = \frac{\alpha}{N} \sum_{j \neq i} \left(\frac{(X_i - X_j)^\perp}{|X_i - X_j|^2} + \textit{periodization} \right) dt + \sqrt{2\,\sigma}\, dW_i,$$

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is the velocity field created in an incompressible fluid by a point vortex $\frac{\alpha}{4\,\pi}\,\delta(x - X_j)$. Note the periodic correction on the torus...

The mean-field limit: 2d incompressible Navier-Stokes

Consider the 2-dimensional and incompressible Navier-Stokes (or Euler for $\sigma = 0$) system for a viscous fluid

$$\partial_t u + u \cdot \nabla u - \sigma \, \Delta u = \nabla p,$$
$$\operatorname{div} u = 0.$$

Define the vorticity $\omega = \operatorname{curl} u$ (a scalar field in 2d) which solves

$$\partial_t \omega + \operatorname{div}(u \, \omega) - \sigma \, \Delta \omega = 0,$$
$$u(t,x) = 2\,\pi \int \frac{(x-y)^\perp}{|x-y|^2} \, \omega(t,y) \, dy = K \star \omega.$$

$\longrightarrow$ A classical conjecture in Mathematical Physics and Numerical Analysis, since Helmholtz in 1858 (cf. for example the review in Marchioro and Pulvirenti) is to prove that point vortices converge to the 2d Navier-Stokes system.

Cells dynamics under chemotaxis

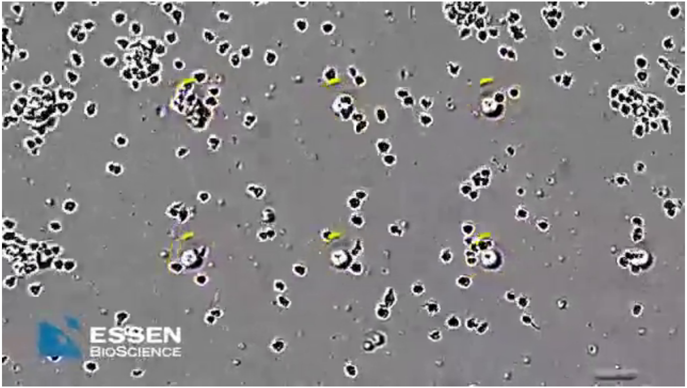

Figure: Credits: Essen Bio-Science from Labtube; Directional migration of Jurkat cells toward the chemo-attractant SDF1a, visualized on an IncuCyte ClearView 96-well Cell Migration Plate.

Elementary chemotaxis models for micro-organisms

Consider N micro-organisms following the gradient of the concentration $c(t,x)$ of some chemical. In the simplest model, their velocities solve

$$dX_i = \nabla c(t, X_i(t))\, dt + \sqrt{2\,\sigma}\, dW_i,$$

where the independent Wiener processes W_i may represent random changes in direction.

Assume now that the chemical is also produced by the organisms and diffuses fast:

$$-\Delta c = \frac{\alpha}{N} \sum_i \delta(x - X_i) + \text{possible source.}$$

$\longrightarrow$ Toy model from the biological point of view but captures the singularity of the interaction.

Elementary chemotaxis models for micro-organisms

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$$dX_i = -\frac{\lambda}{N} \sum_{j \neq i} \frac{X_i - X_j}{|X_i - X_j|^2}\, dt + \sqrt{2\,\sigma}\, dW_i,$$

where the independent Wiener processes W_i may represent random changes in direction.

Assume now that the chemical is also produced by the organisms and diffuses fast: In dimension 2

$$c(t,x) = -\frac{\lambda}{N} \sum_i \log|x - X_i| + S(t,x).$$

$\longrightarrow$ Toy model from the biological point of view but captures the singularity of the interaction.

The Patlak-Keller-Segel system

The mean-field limit is the well known Patlak-Keller-Segel system (1953 and 1970)

$$\begin{cases} \partial_t \bar\rho + \mathrm{div}\,(\bar\rho\, u) = \sigma\,\Delta\,\bar\rho, \\ u = \nabla c, \quad -\Delta c = 2\,\pi\,\lambda\,\bar\rho. \end{cases}$$

Again not a very accurate model of chemotaxis but a good prototype of what relevant models may look like.
Similar to the so-called Smoluchowski-Poisson equation in astrophysics, cf. Chandrasekhar 1943.

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Kernel with the same singularity as the Biot-Savart law but very different structure: Hamiltonian for Navier-Stokes vs singular attractive gradient flow.
$\longrightarrow$ Solutions may not exist for all times as the singular attractive interactions can lead to concentration: From Dolbeault-Perthame 2004 for example,

- Global existence if $\lambda \leq 4\,\sigma$ (or $\lambda \leq 2\,d\,\sigma$).
- Always blow-up if $\lambda > 4\,\sigma$.

Coulomb gases

The case of Coulomb gases concern singular, repulsive (contrary to the Keller-Segel model) interactions through the Coulomb potential

$$d\,X_i = -\frac{1}{N}\sum_{j=1}^{N} \nabla\Phi(X_i - X_j)\,dt + \sqrt{2\,\sigma_N}\,dW_i,$$

where $\Phi = \lambda\,|x|^{d-2}$ or $\Phi = -\lambda\,\log|x|$ if $d = 2$.
In general one can also consider Riesz gases where Φ is a Riesz potential: $\Phi = \lambda\,|x|^{-k}$.
$\longrightarrow$ The case with degenerate diffusion with $\sigma_N \sim \frac{1}{N}$ is especially relevant as it is related to the complex Ginibre Ensemble in random matrix theory (see *e.g.* Bolley-Chafai-Fontbona 2017).

Some of the other questions to answer

While this talk is focused on the mean-field limit, many other important questions exist around many-particle systems

- Well posedness for a finite N. Difficulty: Singularity of the force kernel. See Flandoli-Gubinelli-Priola 2011 for point vortices, and Cattiaux-Pédèches 2016, Fournier-Jourdain 2017 but only if $\lambda < \sigma$.
- Quantum systems with in particular Bose-Einstein condensates, with *e.g.* Erdös-Yau 2001, Erdös-Schlein-Yau 2006...
- Long time behavior of the system. Connected with questions in ergodic theory, with the choice of initial data...

Well posedness for fixed N? The gravitational example

Consider the 2nd order Newton dynamics in dimension d of N point masses

$$\frac{d}{dt}X_i(t) = V_i(t),$$
$$\frac{d}{dt}V_i(t) = \frac{m}{N}\sum_{j\neq i}\frac{X_j - X_i}{|X_j - X_i|^d}.$$

- For $N = 2$, from the explicit formula, there is trivially well-posedness for almost every initial data, X_i^0, V_i^0.
- This still holds for $N = 3, 4$ but it is unknown for $N \geq 5$ (see Painlevé 1895, Saari 1973, McGehee 1978, Xia 1992-2002...).
- For the Keller-Segel system, the situation is worse as the system is dissipative with an attractive singularity. $\longrightarrow$ No classical notion of solutions unless diffusion dominates.

McKean's approach

A natural idea implemented by McKean 1967 is to directly compare

$$d\,X_i = \frac{1}{N}\sum_{j=1}^{N} K(X_i - X_j)\,dt + \sqrt{2\,\sigma}\,dW_i,$$

with

$$d\,\bar X_i = \mathbb{E}\left(\frac{1}{N}\sum_{j} K(\bar X_i - \bar X_j)\mid \bar X_i(t)\right)\,dt + \sqrt{2\,\sigma}\,dW_i,$$

by calculating

$$\sup_i \mathbb{E}\,|X_i - \bar X_i|^2.$$

McKean's approach: Uniform Gronwall estimates

Observe that the $\bar{X}_i$ are i.i.d. with law $\bar{\rho}$ so that

$$d\,\bar{X}_i = \frac{1}{N}\sum_{j=1}^{N} K(\bar{X}_i - \bar{X}_j)\,dt + \sqrt{2\,\sigma}\,dW_i + O(N^{-1/2}).$$

Using Gronwall's lemma with estimates uniform in N,

$$\sup_i \mathbb{E}\,|X_i - \bar{X}_i|^2 \leq e^{\|\nabla K\|_{L^\infty}\,t}\left(\sup_i \mathbb{E}\,|X_i^0 - \bar{X}_i^0|^2 + \frac{C}{N}\right),$$

providing a first estimate of propagation of chaos.

The empirical measure

Define the empirical measure $\mu_N(t,x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - X_i)$.

Allows to recover observables for the system

$$\phi_X = \mathbb{E}\,\frac{1}{N}\sum_i \phi(X_i), \quad \text{or } \phi_X = \mathbb{E}\,\frac{1}{N^2}\sum_{i,j} \phi(X_i,\, X_j).$$

It leads to another formulation of propagation of chaos:

$$\mathbb{E}\|\mu_N - \bar\rho\|_{W^{-1,1}} \leq e^{\|\nabla K\|_{L^\infty}\, t} \left(\mathbb{E}\|\mu_N^0 - \bar\rho\|_{W^{-1,1}} + \frac{C}{\sqrt{N}}\right),$$

as the random, stochastic μ_N is close to the deterministic $\bar\rho$.
$\longrightarrow$ See Braun and Hepp 77, Neunzert-Wick 79, Dobrushin 79 (deterministic case $\sigma = 0$) and Snitzman 91 and Spohn 91.

$\longrightarrow$ Does not seem to allow singular interaction kernels K.

Some of the Existing Literature: The deterministic setting

The previous ideas have been considerably extended with some success in the deterministic case $\sigma = 0$:

- The Lipschitz case is still important to further understand the framework. See for example Golse 16, Golse-Mouhot-Ricci 13, Hauray-Mischler 14, Mischler-Mouhot 13...
- 2d incompressible Euler system in Goodman-Hou-Lowengrub 90, Schochet, with a general result by Hauray 09.
- Deterministic Riesz kernels recently in Duerinckx 16, Duerinckx-Serfaty 18 and Serfaty 19.
- 2nd order systems are less well understood: Hauray-Jabin 09 and 15 for $K(x) << |x|^{-1}$, Lazarovici and Pickl 17, Pickl 19.
- Singularity not at the origin: Carrillo-Choi-Hauray-Salem 18 for swarming models.
- Collisional models (Boltzmann) are hard: Lanford 75, and Bodineau-Gallagher-Saint-Raymond-Texier.

Some of the Existing Literature: The stochastic setting

In contrast, the stochastic case with $\sigma_N > 0$ is much less well understood

- Locally Lipschitz interactions in Bolley-Cañizo-Carrillo 11, Bossy-Faugeras-Talay 15.
- For 2d Navier-Stokes, if $K = \nabla^\perp V$, only qualitative convergence by Osada 85, Fournier-Hauray-Mischler 16.
- For the Patlak-Keller-Segel system, various attempts by Cattiaux-Pédèches 16, Godinh-Quininao 15, Haskovec-Schmeiser 11... Recently Fournier-Jourdain 17 proved some limit for $\lambda < 1$ but no propagation of chaos. See also Bolley-Chafaï-Fontbona 18 for the repulsive Keller-Segel.
- Recent result by Rosenzweig extending the Serfaty method to some stochastic settings.

A new statistical approach

Instead of looking at trajectories, we consider

$$\rho_N(t, x_1, \ldots, x_N) : \text{joint law of the positions } X_1(t), \ldots, X_N(t) \text{ at time } t.$$

It contains most of the statistical information on the system but not all the information: Correlations in time are lost and it may be difficult to reconstruct trajectories of the system.

Aim: Compare ρ_N with the tensorized

$$\bar{\rho}_N = \Pi_{i=1}^N \bar{\rho}(t, x_i) = \bar{\rho}^{\otimes N},$$

which is the joint law of the i.i.d. sequence $\bar{X}_i$, in terms of their observables or marginals:

$$\rho_{N,k} = \int_{\Pi^{d\,(N-k)}} \rho_N \, dx_{k+1} \ldots dx_N \longleftrightarrow \bar{\rho}_{N,k} = \bar{\rho}^{\otimes k}.$$

The Gibbs entropy is critical

We based our method on the scaled relative entropy

$$H_N(\rho_N|\bar\rho_N)(t) = \frac{1}{N}\int_{\Pi^{N\,d}} \rho_N \, \log \frac{\rho_N}{\bar\rho_N}.$$

Thanks to the sub-additive nature of the entropy, it controls the marginals

$$\frac{1}{k}\int_{\Pi^{k\,d}} \rho_{N,k} \, \log \frac{\rho_{N,k}}{\bar\rho^{\otimes k}} \leq \frac{1}{N}\int_{\Pi^{N\,d}} \rho_N \, \log \frac{\rho_N}{\bar\rho_N}.$$

For fixed k, the Csiszár-Kullback-Pinsker inequality bounds

$$\|\rho_{N,k} - \bar\rho^{\otimes k}\|_{L^1} \leq C\,\sqrt{k}\,H_N(\rho_N|\bar\rho_N)(t).$$

It has the right initial scaling: If the X_i^0 are i.i.d. with law ρ^0 then

$$H_N(\rho_N|\bar\rho_N)(t=0) = \int_{\Pi^d} \rho^0 \, \log \frac{\rho^0}{\bar\rho^0}.$$

The Liouville equation

The Liouville or forward Kolmogorov equation describes the evolution of the law $\rho_N(t, x_1, \ldots, x_N)$ of the distribution of the particles

$$\partial_t \rho_N + L_N \, \rho_N = 0,$$

where

$$L_N \, \rho_N = \sum_{i=1}^{N} \frac{1}{N} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{x_i} \rho_N - \sigma \sum_i \Delta_{x_i} \rho_N.$$

The Liouville equation encompasses all the relevant statistical information about the dynamics.

The result for point vortices

Theorem (J.-Wang, Invent. Math. 18)

Assume K, $div\, K \in W^{-1,\infty}$ or $|x|\, K$, $div\, K \in L^\infty$ if $\sigma = \sigma_N \to 0$. Consider $\bar\rho$ a smooth solution to the limiting equation and the law ρ_N on $\Pi^{d\,N}$ of any solution to the SDE system. Then for $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$, and for some constant C depending on $\bar\rho$

$$H_N(\rho_N|\bar\rho_N)(t) \leq e^{C_{\bar\rho}\,\|K\|\,t}\left(H_N(\rho_N|\bar\rho_N)(t=0) + \frac{C}{N}\right).$$

Consequently for any fixed k, the marginals $\rho_{N,k}$ satisfy

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar\rho(t, x_i, v_i)\|_{L^1(\Pi^{k\,d})} \leq C_{T,\bar\rho,K}\, N^{-1/2}.$$

$\longrightarrow$ First quantitative derivation of 2d Navier-Stokes!

The relative entropy for many-particle systems

- Uses of the full relative entropy between trajectories: Ben Arous-Zeitouni 99 for smooth Langevin dynamics, and Ben Arous-Tannenbaum-Zeitouni 03, Fontbona-Jourdain 16.
- Some connections with Random Matrix Theory, Erdös-Yau 17.
- Closest to the method here is Yau 91 concerning the hydrodynamics of Ginzburg-Landau models.

The main idea

Recall that $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$ and observe that

$$\partial_t \bar\rho_N + L_N\,\bar\rho_N = R_N\,\bar\rho_N,$$

$$R_N = \frac{1}{N}\sum_{i,j} \left(K(x_i - x_j) - K \star \bar\rho(t, x_i)\right)\,\nabla_{x_i} \log \bar\rho(t, x_i).$$

Calculate the relative entropy thanks to the convexity of the norm and the dissipation due to diffusion. After integrations by parts for $\operatorname{div} K = 0$

$$\frac{d}{dt}\int \rho_N \log\frac{\rho_N}{\bar\rho_N}\,dx \le \int \rho_N\,R_N\,dx.$$

Goal: Show that $R_N = O(1)$ while a priori $R_N = O(N)$.

What does that mean?

One can reformulate the previous idea: Assume that for some anti-symmetric, entropy dissipative operator L, one has that

$$\partial_t f_N + L\, f_N = 0,$$
$$\partial_t \bar{f}_N + L\, \bar{f}_N = R_N\, \bar{f}_N,$$

with $\|R_N\|_{L^\infty} \leq 1$. Then in general one cannot hope better than

$$\|f_N - \bar{f}_N\|_{L^1} = O(1).$$

But if $\bar{f}_N = f^{\otimes N}$ then one has for any fixed k

$$\frac{1}{N} \int_{\Pi^{d\,N}} f_N \log \frac{f_N}{\bar{f}_N} = O(N^{-1}) \implies \|f_{N,k} - \bar{f}_{N,k}\|_{L^1} = O(N^{-1/2}).$$

A new large deviation inequality

Theorem (J.-Wang, Invent. Math. 18)

Consider $\bar\rho \in W^{1,\infty}(\Pi^d)$ *with* $\bar\rho > 0$ *and* $\int_{\Pi^d} \bar\rho\, dx = 1$. *Consider further any* $\phi(x,z) \in L^\infty$ *s.t. for some given universal constant*

$$C \left(\sup_{p\geq 1} \frac{\|\sup_z |\phi(.,z)|\|_{L^p(\bar\rho dx)}}{p}\right)^2 < 1.$$

Then

$$\int_{\Pi^{d\,N}} \bar\rho_N \exp\left(N \int_{\Pi^{2d}} \phi(x,y)\,(d\mu_N - d\bar\rho)^{\otimes 2}\right) dX^N \leq C < \infty,$$

with still $\mu_N = \frac{1}{N}\sum_i \delta(x - x_i)$.

$\longrightarrow$ Compare with classical large deviation results requiring ϕ continuous, see for ex. Ben Arous-Brunaud 90, Varadhan 84...

Conclusions

- Using the right physics is the key...
- The method provides a statistical control on large systems with a large class of attractive-repulsive interactions.
- We obtain explicit rates of convergence, which are optimal for point vortices.
- Derivation of the Patlak-Keller-Segel system using weighted relative entropy.

Many open questions

- Systems with different structures: Non Hamiltonian, non gradient flows?
- Non-exchangeable systems, such as neuron networks?
- Systems with feedback loops or optimization?

How to modify the relative entropy approach

For gradient flows where $K = -\nabla\Phi$ such as the Patlak-Keller-Segel, we need to find the right object that still sees the advection part of the operator L_N in a anti-symmetrical manner.

We introduce a weighted relative entropy

$$E_N\left(\frac{\rho_N}{G_N}\,\Big|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{G_N(X^N)}\,\frac{G_{\bar\rho_N}(t,X^N)}{\bar\rho_N(t,X^N)}\right)dX^N,$$

through the Gibbs equilibrium of the system, and its equivalent mean-field representation

$$G_N(t,X^N) = \exp\left(-\frac{1}{2N\sigma}\sum_{i\neq j}\Phi(x_i-x_j)\right),$$

$$G_{\bar\rho_N}(t,X^N) = \exp\left(-\frac{1}{\sigma}\sum_{i=1}^N \Phi\star\bar\rho(x_i) + \frac{N}{2\sigma}\int_{\Pi^d}\Phi\star\bar\rho\,\bar\rho\right).$$

Our new result

Consider even potentials $\Phi(-x) = \Phi(x)$, s.t.

- Any possibly singular potential $\Phi \in L^1(\Pi^d)$ with at most a mildly singular attractive part

$$\Phi(x) \geq -C - \lambda \log \frac{1}{|x|} \quad \text{for } \lambda < 2\, d\, \sigma, \tag{1}$$

 and some structure on the repulsive and potentially very singular part such as $\Phi \sim |x|^{-k}$.

- We can be more precise by asking $\Phi = \Phi_a + \Phi_r$ with

$$\hat{\Phi}_r \geq 0, \quad |\nabla_\xi \hat{\Phi}_r(\xi)| \leq C\, \frac{\hat{\Phi}_r(\xi)}{1+|\xi|} + \frac{C}{1+|\xi|^{d+1}}, \qquad |\nabla \Phi_a(x)| \leq \frac{C}{|x|}. \tag{2}$$

Our new result

Theorem

Assume $K = -\nabla\Phi$ with Φ as above. Consider $\bar\rho$ a smooth enough solution with $\inf \bar\rho > 0$*. There exists $C > 0$ and $\theta > 0$ s.t. for $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$, and for the joint law ρ_N on Π^{dN} of any entropy solution to the SDE system, for σ fixed*

$$H_N(t) + |E_N(t)| \leq e^{C_{\bar\rho}\,\|K\|\,t}\left(H_N(t=0) + |E_N(t=0)| + \frac{C}{N^\theta}\right).$$

Hence if $H_N^0 + |E_N^0| \leq C\,N^{-\theta}$, for any fixed marginal $\rho_{N,k}$

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar\rho(t, x_i)\|_{L^1(\Pi^{k\,d})} \leq C_{T,\bar\rho,k}\,N^{-\theta/2}.$$

$\longrightarrow$ First ever derivation of the Patlak-Keller-Segel system or of any singular, attractive mean-field limit...

A modified free energy

One may also write

$$E_N\Big(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\Big) = \mathcal{H}_N(\rho_N|\bar\rho_N) + \mathcal{K}_N(G_N|G_{\bar\rho_N}),$$

where

$$\mathcal{H}_N(\rho_N|\bar\rho_N) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\Big(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\Big)\,dX^N$$

is exactly the relative entropy introduced in J.-Wang and

$$\mathcal{K}_N(G_N|G_{\bar\rho_N}) = -\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\Big(\frac{G_N(t,X^N)}{G_{\bar\rho_N}(t,X^N)}\Big)\,dX^N$$

is the expectation of the modulated potential energy from Duerincx-Serfaty.

$\longrightarrow$ E_N is a modulated free energy for the system.

Propagating E_N

Because it is based on the free energy, E_N has the right algebraic structure with for any Φ even that

$$\frac{d}{dt} E_N\left(\frac{\rho_N}{G_N} \,\Big|\, \frac{\bar\rho_N}{G_{\bar\rho_N}}\right) \leq -\frac{\sigma}{N} \int_{\Pi^{d\,N}} d\rho_N \left|\nabla \log \frac{\rho_N}{\bar\rho_N} - \nabla \log \frac{G_N}{G_{\bar\rho_N}}\right|^2$$
$$-\frac{1}{2}\int_{\Pi^{dN}} \int_{\Pi^{2\,d}\cap\{x\neq y\}} \nabla\Phi(x-y)\cdot\left(\nabla \log \frac{\bar\rho}{G_{\bar\rho}}(x) - \nabla \log \frac{\bar\rho}{G_{\bar\rho}}(y)\right)$$
$$(d\mu_N - d\bar\rho)^{\otimes 2} d\rho_N,$$

where $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta(x-x_i)$ is as before the empirical measure and where we denote

$$G_{\bar\rho}(t,x) = \exp\left(-\frac{1}{\sigma} V\star\bar\rho(x) + \frac{1}{2\,\sigma}\int_{\Pi^d} V\star\bar\rho\,\bar\rho\right).$$

Another large deviation inequality

Theorem

For $\Phi \geq -C - \lambda \log \frac{1}{|x|}$ *with* $\lambda < 2\,d\,\sigma$*, define the functional*

$$F_\eta(\mu_N) = -\frac{1}{2\,\sigma}\int_{\Pi^{2d}\cap\{x\neq y\}} \Phi(x-y)\,\mathbb{I}_{|x-y|\leq\eta}\,(d\bar\rho - d\mu_N)^{\otimes 2},$$

then there exists $\eta > 0$ *s.t.*

$$\frac{1}{N}\log Z_N = \frac{1}{N}\log\int_{\Pi^{dN}} \bar\rho_N\, e^{N\,\gamma\,F(\mu_N)}\,dX^N \leq \frac{C}{N^{\frac{1}{2\,(2d+1)}}}.$$

$\longrightarrow$ A delicate extension of the logarithmic Hardy, Littlewood, Sobolev inequality to remove the singular parts and then use a large deviation control of the type: For any $\lambda < 2\,d\,\sigma$

$$\int d\mu\,\log\frac{\mu}{\bar\rho} + \frac{\lambda}{2\,\sigma}\int_{0<|x-y|<\eta} \log|x-y|\,(d\mu - d\bar\rho)^{\otimes 2} \geq 0.$$