3-cocycles, QFT anomalies, and gerbal representations

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Categorical representation

C an abelian category, G a group
$g \in G$, F_g a functor in C
$i_{g,h} : F_g \circ F_h \to F_{gh}$ an isomorphism
$i_{g,hk} \circ i_{h,k}$ and $i_{gh,k} \circ i_{g,h}$ isomorphisms $F_g \circ F_h \circ F_k \to F_{ghk}$
They are not necessarily equal; one can have a *central extension*
$i_{g,hk} \circ i_{h,k} = \alpha(g,h,k) i_{gh,k} \circ i_{g,h}$ with $\alpha(g,h,k) \in \mathbf{C}^\times$ a 3-cocycle

Double loop groups

The loop group LG (G compact,simple) has a central extension defined by a (local) 2-cocycle.

3-cocycles? According to Frenkel and Zhu, increase the cohomoligal degree by one unit by going to the double loop group $L(LG)$. They do this algebraically, utilizing the idea in [Pressley and Segal] by embedding the loop group LG (actually, its Lie algebra) to an appropriate universal group $U(\infty)$ (or its Lie algebra).

The point of this talk is to show how this is done in he smooth setting, globally, and connecting to the old discussion of QFT anomalies in the 1980's.

Transgression in groupoid cohomology

Recall the definition of cohomology (with values in an abelian grpup) in a transformation groupoid when a group G is acting on itself from the right:

$$
\begin{aligned}
&(\delta c^n)(g_0; g_1, g_2, \dots, g_{n+1}) \\
&= c^n(g_0 g_1; g_2, \dots, g_{n+1}) + (-1)^{n+1} c^n(g_0; g_1, g_2, \dots, g_n) \\
&+ \sum_{i=1}^{n} (-1)^i c^n(g_0; g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}).
\end{aligned}
$$

The corresponding Lie algebroid cohomology is defined by the operator

$$
\begin{aligned}
&(\delta c^n)(g; x_1, x_2, \dots, x_{n+1}) \\
&= \sum_{i<j} (-1)^{i+j} c^n(g; [x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) \\
&+ \sum_{i=1}^{n+1} (-1)^{i+1} L_{x_i} \cdot c^n(g; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}),
\end{aligned}
$$

Next start from $c_0(g) = \exp iW_{2k+1}(g)$ where g is an element in $Map(\Sigma_{2k+1}, G)$ and W_{2k+1} is the normalized integral

$$\int_{\Sigma_{2k+1}} \mathrm{tr}\,(g^{-1}dg)^{2k+1}$$

over a manifold Σ_{2k+1} of dimension $2k+1$. The coboundary of this is

$$c^1(g_0; g_1) = c^0(g_0 g_1) \cdot c^0(g_0)^{-1}.$$

This is identically 1 if the boundary of Σ_{2k+1} is empty. Otherwise, the coboundary is localized on the boundary (using Stokes' theorem.... at least near the unit element in $Map(\Sigma_{2k+1}, G)$.) This gives a cocycle c^1 on $Map(\Sigma_{2k}, G)$. We can repeat this: Take the coboundary of c^1 when the manifold Σ_{2k} has a boundary, leading to a 2-cocycle c^2 on Σ_{2k-1} ..... and in the end we have a cocycle c^{2k+1} on the group G itself.

We can do the above construction only locally on the group manifolds $Map(\Sigma_p, G)$ due to obstructions coming from the homotopy groups of G. However, on the level of Lie algebra cocycles there are no topological obstructions. Now, starting from the 0-cocycle c^0 defined using the form

$$W_{2k+1} = \mathrm{tr}\,(g^{-1}dg)^{2k+1}$$

we end up with the Lie algebra $2k+1$-cocycle

$$c^{2k+1}(x_1, \dots, x_{2k+1}) = \sum_{\pi \in S_{2k+1}} \epsilon(\pi) \mathrm{tr} x_{\pi(1)} \cdots x_{\pi(2k+1)}.$$

This is just another way thinking about W_{2k+1} since the Lie algebra cocycle c^{2k+1} defines a left invariant $2k+1$ form on the group G which is equal to W_{2k+1}!

2-cocycles

In Hamiltonian quantization the breaking of (gauge, diffeomorphism) symmetry is best seen in the modified commutation relations in the Lie algebra $\mathfrak{g}$ of G,

$$[X, Y] \mapsto [X, Y] + c(X, Y)$$

where c takes values in an abelian ideal $\mathfrak{a}$; in the simplest case $\mathfrak{a} = \mathbf{C}$ and c satisfies the Lie algebra **2-cocycle** condition

$$c(X, [Y, Z]) + c(Y, [Z, X]) + c(Z, [X, Y]) = 0$$

A famous example is given by the central extension of the loop algebra $L\mathfrak{g}$ of smooth functions on the unit circle with values in a semisimple Lie algebra $\mathfrak{g}$,

$$c(X, Y) = \frac{k}{2\pi i} \int_{S^1} \langle X(\phi), Y'(\phi) \rangle d\phi$$

where k is a constant ("level" of a representation).

Topological aspects of the Lie algebra extension

Given a (central) extension of a Lie algebra one expects that there is a central extension of the corresponding group. In case of $L\mathfrak{g}$ the group is the group LG of maps from S^1 to a (compact) group G. A central extension of LG would then be given by a circle valued function $\Omega : LG \times LG \to S^1$ with group 2-cocycle property

$$\Omega(g_1, g_2)\Omega(g_1 g_2, g_3) = \Omega(g_1, g_2 g_3)\Omega(g_2, g_3).$$

However, in case of LG there is a **topological obstruction,** Ω is defined only in an open neighborhood of the unit element. The obstruction is given by an element In $H^2(LG, \mathbf{Z})$ whose de Rham representative is a left invariant 2-form fixed by the Lie algebra 2-cocycle c.

3-cocycles

Group/Lie algebra cohomology with coefficients in an abelian group is defined in any degree. So what about degree 3? And the relation to de Rham cohomology in dimension 3?

Let $\mathcal{B}$ be an associative algebra and G a group. Assume that we have a group homomorphism $s : G \to \mathrm{Out}(\mathcal{B})$ where $\mathrm{Out}(\mathcal{B})$ is the group of outer automorphims of $\mathcal{B}$, that is, $\mathrm{Out}(\mathcal{B}) = \mathrm{Aut}(\mathcal{B})/\mathrm{In}(\mathcal{B})$, all automorphims modulo the normal subgroup of inner automorphisms.

If one chooses any lift $\tilde{s} : G \to \mathrm{Aut}(\mathcal{B})$ then we can write

$$\tilde{s}(g)\tilde{s}(g') = \sigma(g, g') \cdot \tilde{s}(gg')$$

for some $\sigma(g, g') \in \mathrm{In}(\mathcal{B})$. From the definition follows immediately the cocycle property

$$\sigma(g, g')\sigma(gg', g'') = [\tilde{s}(g)\sigma(g', g'')\tilde{s}(g)^{-1}]\sigma(g, g'g'')$$

Prolongation by central extension

Let next H be any central extension of $\mathrm{In}(\mathcal{B})$ by an abelian group a. That is, we have an exact sequence of groups,

$$1 \to a \to H \to \mathrm{In}(\mathcal{B}) \to 1.$$

Let $\hat{\sigma}$ be a lift of the map $\sigma : G \times G \to \mathrm{In}(\mathcal{B})$ to a map $\hat{\sigma} : G \times G \to H$ (by a choice of section $\mathrm{In}(\mathcal{B}) \to H$). We have then

$$\hat{\sigma}(g, g')\hat{\sigma}(gg', g'') = [\tilde{s}(g)\hat{\sigma}(g', g'')\tilde{s}(g)^{-1}]$$
$$\times \hat{\sigma}(g, g'g'') \cdot \alpha(g, g', g'') \text{ for all } g, g', g'' \in G$$

where $\alpha : G \times G \times G \to a$.

The 3-cocycle condition

Here the action of the outer automorphism $s(g)$ on $\hat{\sigma}(*)$ is defined by $s(g)\hat{\sigma}(*)s(g)^{-1} =$ the lift of $s(g)\sigma(*)s(g)^{-1} \in \mathrm{In}(\mathcal{B})$ to an element in H. One can show that α is a 3-cocycle

$$\alpha(g_2, g_3, g_4)\alpha(g_1g_2, g_3, g_4)^{-1}\alpha(g_1, g_2g_3, g_4) \\ \times\alpha(g_1, g_2, g_3g_4)^{-1}\alpha(g_1, g_2, g_3) = 1.$$

Remark If we work in the category of topological groups (or Lie groups) the lifts above are in general discontinuous; normally, we can require continuity (or smoothness) only in an open neighborhood of the unit element.

Next we construct an example from quantum field theory. Let G be a compact simply connected Lie group and P the space of smooth paths $f : [0,1] \to G$ with initial point $f(0) = e$, the neutral element, and quasiperiodicity condition $f^{-1} df$ a smooth function.

P is a group under point-wise multiplication but it is also a principal ΩG bundle over G. Here $\Omega G \subset P$ is the loop group with $f(0) = f(1) = e$ and $\pi : P \to G$ is the projection to the end point $f(1)$. Fix an unitary representation ρ of G in $\mathbf{C}^N$ and denote $H = L^2(S^1, \mathbf{C}^N)$.

CAR representations

For each polarization $H = H_- \oplus H_+$ we have a vacuum representation of the CAR algebra $\mathcal{B}(H)$ in a Hilbert space $\mathcal{F}(H_+)$. Denote by $\mathcal{C}$ the category of these representations. Denote by $a(v), a^*(v)$ the generators of $\mathcal{B}(H)$ corresponding to a vector $v \in H$,

$$a^*(u)a(v) + a(v)a^*(u) = 2 < v, u >$$

and all the other anticommutators equal to zero.

Outer automorhisms

Any element $f \in P$ defines a unique automorphism of $\mathcal{B}(H)$ with $\phi_f(a^*(v)) = a^*(f \cdot v)$, where $f \cdot v$ is the function on the circle defined by $\rho(f(x))v(x)$. These automorphims are in general not inner except when f is periodic.

We have now a map $s : G \to \mathrm{Aut}(\mathcal{B})/\mathrm{In}(\mathcal{B})$ given by $g \mapsto F(g)$ where $F(g)$ is an arbitrary smooth quasiperiodic function on $[0, 1]$ such that $F(g)(1) = g$.

Any two such functions $F(g), F'(g)$ differ by an element σ of ΩG, $F(g)(x) = F'(g)(x)\sigma(x)$. Now σ is an inner automorphism through a projective representation of the loop group ΩG in $\mathcal{F}(H_+)$.

In an open neighborhood U of the neutral element e in G we can fix in a smooth way for any $g \in U$ a path $F(g)$ with $F(g)(0) = e$ and $F(g)(1) = g$.

Of course, for a connected group G we can make this choice globally on G but then the dependence of the path $F(g)$ would not be a continuous function of the end point. For a pair $g_1, g_2 \in G$ we have

$$\sigma(g_1, g_2)F(g_1 g_2) = F(g_1)F(g_2)$$

with $\sigma(g_1, g_1) \in \Omega G$.

LG valued 2-cocycle

For a triple of elements g_1, g_2, g_3 we have now

$$\begin{aligned} F(g_1)F(g_2)F(g_3) &= \sigma(g_1, g_2)F(g_1 g_2)F(g_3) \\ &= \sigma(g_1, g_2)\sigma(g_1 g_2, g_3)F(g_1 g_2 g_3). \end{aligned}$$

In the same way,

$$\begin{aligned} F(g_1)F(g_2)F(g_3) &= F(g_1)\sigma(g_2, g_3)F(g_2 g_3) \\ &= [g_1\sigma(g_2, g_3)g_1^{-1}]F(g_1)F(g_2 g_3) \\ &= [g_1\sigma(g_2, g_3)g_1^{-1}]\sigma(g_1, g_2 g_3)F(g_1 g_2 g_3) \end{aligned}$$

which proves the 2-cocycle relation for σ.

3-cocycle α for G

Lifting the loop group elements σ to inner automorphisms $\hat{\sigma}$ through a projective representation of ΩG we can write

$$\hat{\sigma}(g_1, g_2)\hat{\sigma}(g_1 g_2, g_3) = \text{Aut}(g_1)[\hat{\sigma}(g_2, g_3)]\hat{\sigma}(g_1, g_2 g_3)\alpha(g_1, g_2, g_3),$$

where $\alpha : G \times G \times G \to S^1$ is some phase function arising from the fact that the projective lift is not necessarily a group homomorphism.

Since (in the case of a Lie group) the function $F(\cdot)$ is smooth only in a neighborhood of the neutral element, the same is true also for σ and finally for the 3-cocycle α.

The Lie algebra 3-cocycle

An equivalent point of view to the construction of the 3-cocycle α is this: We are trying to construct a central extension $\hat{P}$ of the group P of paths in G (with initial point $e \in G$) as an extension of the central extension over the subgroup ΩG. The failure of this central extension is measured by the cocycle α, as an obstruction to associativity of $\hat{P}$.

On the Lie algebra level, we have a corresponding cocycle $c_3 = d\alpha$ which is easily computed. The cocycle c of $\Omega\mathfrak{g}$ extends to the path Lie algebra $P\mathfrak{g}$ as

$$c(X, Y) = \frac{1}{4\pi i} \int_{[0,2\pi]} \mathrm{tr}\,(XdY - YdX).$$

This is an antisymmetric bilinear form on $P\mathfrak{g}$ but it fails to be a Lie algebra 2-cocycle. The coboundary is given by

The Lie algebra 3-cocycle

$$
\begin{aligned}
(\delta c)(X,Y,Z) &= c(X,[Y,Z]) + c(Y,[Z,X]) + c(Z,[X,Y]) \\
&= -\frac{1}{4\pi i}\operatorname{tr} X[Y,Z]|_{2\pi} = d\alpha(X,Y,Z).
\end{aligned}
$$

Thus δc reduces to a 3-cocycle of the Lie algebra $\mathfrak{g}$ of G on the boundary $x = 2\pi$. This cocycle defines by left translations on G the left-invariant de Rham form $-\frac{1}{12\pi i}\operatorname{tr}(g^{-1}dg)^3$; this is normalized as $2\pi i$ times an integral 3-form on G.

Cocycles and associated vector bundles

Standard construction of an associated vector bundle: Start from a G-principal bundle $\pi : P \to M$. Fix a representation $\rho : G \to \mathrm{Aut}(V)$, V a vector space. Define

$$E = P \times_\rho V, (p, v) \equiv (pg, \rho(g)^{-1} v).$$

Generalization: Fix a 1-cocycle $\omega : P \times G \to \mathrm{Aut}(V)$ and set

$$E = P \times_\omega V, \text{ with } (pg, \omega(p; g)^{-1} v) \equiv (p, v).$$

The transitivity of the relation is given by the cocycle condition

$$\omega(p; gg') = \omega(p; g)\omega(pg; g').$$

An example of this construction was already given in the construction of the determinant bundle over the gauge orbit space $\mathcal{A}/\mathcal{G}$.

Let G be a topological group and $f : G \to H$ a homotopy to another topological group H. Morally, representation theory of H should encode information about representations of G. However, it can happen that H has a good representation theory but G lacks unitary faithful Hilbert space representations. But we can define a cocycle

$$\omega(b; g) = f(b)^{-1} f(bg)$$

with values in H. Selecting a representation of H in V we obtain a 1-cocycle for the right action of G on itself, with values in $\mathrm{Aut}(V)$. We can view this as a representation of G in a group of matrices with entries in the algebra of complex functions on G (but with an action of G on functions through right translation).

An example

Let $H = H_- \oplus H_+$ be a polarized Hilbert space and U_p the group of unitaries in H such that the off-diagonal blocks wrt the polarization are in the Schatten ideal L_p of operators A with $\mathrm{tr}|A|^p < \infty$. The case $p = 2$ is important since the highest weight representations of $\widehat{LG}$ can be constructed from representations of a central extension $\hat{U}_2$. The Lie algebra central extension is defined by the 2-cocycle

$$c(X, Y) = \frac{1}{2}\mathrm{tr}_c\, X[\epsilon, Y]$$

where ϵ is the grading operator in H. The groups U_p are important because one has an embedding $Map(M, G) \subset U_p$ when M is a compact spin manifold and G compact Lie group, for $p > \dim M$. According to Richard Palais, U_p is homotopy equivalent to U_2 for all $p \geq 1$ so we can define generalized representations of $Map(M, G)$ from this equivalence and the embebding to U_p.

Application to gauge theory

D_A Dirac hamiltonian coupled to a gauge potential A. Quantization $\hat{D}_A$ acts in a fermionic Fock space. For different potentials the representations of the fermion algebra are inequivalent [Shale-Stinespring]. In scattering problems one would like to realize the operators $\hat{D}_A$ in a single Fock space $\mathcal{F}$, the Fock space of free fermions, $A = 0$. Solution: Choose for each A a unitary operator T_A which reduces the off-diagonal blocks of D_A to Hilbert-Schmidt operators, for the 'free' polarization $\epsilon = D_0/|D_0|$. Then each $D'_A = T_A^{-1} D_A T_A$ can be quantized in the free Fock space.
This has a consequence for the implementation of the gauge action $A \mapsto A^g = g^{-1}Ag + g^{-1}dg$ in the Fock space. In the 1-particle space the action of g is replaced by $\omega(A; g) = T_A^{-1} g T_{A^g}$ with

$$\omega(A; gg') = \omega(A; g)\omega(A^g; g').$$

Application to gauge theory

Now the Shale-Stinespring condition $[\epsilon, \omega(A; g)] \in L_2$ is satisfied and we can quantize in $\mathcal{F}$,

$$\omega(A; g) \mapsto \hat{\omega}(A; g').$$

For the Lie algebra of the gauge group we have the Lie algebra cocycle

$$d\omega(A; X) = T_A^{-1} X T_A + T_A^{-1} \mathcal{L}_X T_A$$

with quantization $\widehat{d\omega}(A; X)$.

The extension of Map(M,G)

$X \in \mathcal{G} = Map(M, \mathfrak{g})$ infinitesimal gauge transformation, quantization

$$G_X = \mathcal{L}_X + \widehat{d\omega}(A; X)$$

$$[G_X, G_Y] = G_{[X,Y]} + c(A; X, Y)$$

$$\mathcal{L}_X c(A; [Y, Z]) + c(A; [X, [Y, Z]]) + \text{ cyclic combin. } = 0$$

This 2-cocycle property quarantees the Jacobi identity for the extension

$$Lie(\hat{\mathcal{G}}) = Map(M, \mathfrak{g}) \oplus Map(\mathcal{A}, \mathbf{C})$$

The extension of Map(M,G)

In the case when $M = S^1$ one can take $T_A = 1$ and we obtain the standard central extension of the loop algebra $Map(S^1, \mathfrak{g})$. In the case when $\dim M = 3$ one can show that the 2-cocycle c is equivalent to the local form

$$c \equiv const. \int_M \mathrm{tr}\, A[dX, dY]$$

where the trace under the integral sign is computed in a finite-dimensional representation of $\mathfrak{g}$. This representation is the same defined by the G-action on fermions in the 1-particle space. Actually, the coefficient in the front of the integral is nonzero only for chiral fermions (the Schwinger terms from left and right chiral sectors cancel).

Back in the double loop group $L(LG)$

Next we can replace the group G by $\mathcal{G} = L(LG)$. Assuming G connected, simply connected, the group $\mathcal{G}$ is connected and we can again go through the same steps as in the case of G earlier, except that now $\Omega\mathcal{G}$ the representation of the central extension $\widehat{\Omega G}$ has to be understood in the sense of groupoid central extension or in other words, as Hilbert cocycle.

$\Omega\mathcal{G}$ is the group of maps $Map_0(T^3, G)$ such that $\{1\} \times S^1 \times S^1$ maps to the identity in G.

The gauge action on vector potentials $\mathcal{A}$ over T^3 defines a central extension of the transformation groupoid $Map_0(T^3, g)$ on $\mathcal{A}$.

The double loop group 3-cocycle

The Lie algebra 3-cocycle is now evaluated through transgression from the 2-cocycle

$$c_2 = const. \int_{T^3} \operatorname{tr} A[dX, dY]$$

and gives

$$c_3 = const. \int_{T^2} \operatorname{tr} X[dY, dZ]$$