Part I: Lessons from 2+1

Write the pair \( A = (e^i, \omega^i) \) of 3-bein and spin connection as an \( e_3 = \mathbb{R}^3 \times su_2 \)-valued connection. Ad-invariant inner product on \( e_3 \) \( \Rightarrow \)

\[
S_{\text{Chern–Simons}} = \int_{\Sigma \times \mathbb{R}} A \wedge (dA + \frac{1}{3}[A \wedge A]) = S_{\text{Cartan–Weyl}}
\]

i.e. view gravity as a TFT.
⇒ solutions of gravity with point sources at punctures $i$
determined as holonomies in group $E_3$

⇒ (extended) phase space $E_3^{2\text{genus}(\Sigma)} \times \prod C_i$

with a certain Poisson bracket. Here $C_i$ are conjugacy classes
encoding mass and spin at $i$ [Fock & Rosly ’92, Meusburger &
Schroers CQG’03,...]

No positions at this stage (since up to diffeos), but $e_3$ or
(classical) quantum group $H = U(e_3) = U(su_2) \rtimes C(\mathbb{R}^3)$ acts
canonically on a ‘model spacetime algebra’ $A = C(\mathbb{R}^3)$.

⇒ Theory quantized by replacing $H$ by the quantum group
$U(su_2) \rtimes C(SU_2)$ acts canonically on angular momentum
algebra $A = U(su_2)$.

Similarly with cosmological constant:

$$A = U_q(su_2), \quad H = U_q(sl_2(\mathbb{C})) = U_q(su_2) \rtimes C_q(SU_2)^{op}$$
Different limits in 2+1 (SM & B. Schroers 2008)

Classical model
\[ H = U(su_2) \times C(\mathbb{R}^3), \quad A = C(\mathbb{R}^3) \]
\[ m_P = \infty, \quad l_c = \infty \]

Spin model
\[ H = U(su_2) \times C(SU_2), \quad A = U(su_2) \]
\[ m_P < \infty, \quad l_c = \infty \]

Particle on hyperboloid \( \mathcal{H}^3 \)
\[ H = U(su_2) \times U(h_3), \quad A = C(\mathcal{H}^3) \]
\[ m_P = \infty, \quad l_c < \infty \]

Bicrosproduct model
\[ H = U(su_2) \triangleleft C(\mathcal{H}^3), \quad A = U(h_3) \]
\[ m_P < \infty, \quad l_c = \infty \]

3D QG w/ cosmological constant
\[ H = U_q(su_2) \triangleleft U_q(h_3) \]
\[ A = C_q(\mathcal{H}^3) \]
\[ m_P < \infty, \quad l_c < \infty \]

\[ \cong \text{ if } q \neq 1 \]

\[ H = U_q(su_2) \triangleleft C_q(SU_2)^{op} \]
\[ A = U_q(su_2) \]

\[ \cong \text{ if } q \neq 1 \]

\[ H = U_q(su_2) \triangleleft U_q(su_2)^{cop} \]
\[ A = C_q(SU_2)^{op} \]
Different quantum spacetime limits

Expect one of them q-deformed spacetime e.g. q-Minkowski = 2 x 2 braided hermitian matrices

Expect this to be viewed `up side down’ like $U_q(su_2)$

Expect q a root of unity

For physics need compatible $\ast$-structures
Part II: Bar categories (EJB&SM 2007)

Definition a bar category is a monoidal one \((C, \otimes, \Phi, \bot, l, r)\) equipped with a functor \(\text{bar} : C \to C\) denoted \(\text{bar}(X) = \overline{X}\)

\(\text{bb} : \text{id} \to \text{bar}^2\) natural equivalence

\(\star : \bot \to \overline{\bot}\) isom

\(\Upsilon : \text{bar} \circ \otimes \to \otimes \circ (\text{bar} \times \text{bar})\) flip natural equivalence \(\{\Upsilon_{X,Y} : \overline{X \otimes Y} \cong Y \otimes \overline{X}\}\)

\(\Phi_{\overline{Z,Y,X}}(\Upsilon_{Y,Z} \otimes \text{id}) \Upsilon_{X,Y \otimes Z} \overline{\Phi_{X,Y,Z}} = (\text{id} \otimes \Upsilon_{X,Y}) \Upsilon_{X \otimes Y,Z}\)

\(\overline{\star} = \text{bb}_\bot\) \(\overline{\text{bb}_X} = \text{bb}_{\overline{X}}\)

\((\star^{-1} \otimes \text{id}) \Upsilon_{X,\bot \overline{X}} = r_{\overline{X}} : \overline{X} \to \bot \otimes \overline{X}\) \(\text{id} \otimes \star^{-1}) \Upsilon_{\bot, X \overline{r_X}} = l_{\overline{X}} : \overline{X} \to \overline{X} \otimes \bot\)

called `strong` if: \(\Upsilon_{\overline{Y, X}} \overline{\Upsilon_{X,Y} \text{bb}_X \otimes Y} = \text{bb}_X \otimes \text{bb}_Y : X \otimes Y \to \overline{X} \otimes \overline{Y}\)

Example: bimodules over a *-algebra \(A\): \(\overline{E} = E\) as abelian group

\(a.\overline{e} = \overline{e.a^*}, \quad \overline{e}.a = a^*.\overline{e}\)

\(\Upsilon_{F,E} : \overline{F} \otimes \overline{E}_A \to \overline{E} \otimes \overline{F}_A\)

\(\Upsilon(f \otimes e) = \overline{e} \otimes \overline{f}\)
**Definition** A quasi-* Hopf algebra is $(H, \ast, G, \gamma)$ where $H$ is a *-algebra, Hopf algebra, $\gamma \in H$, $G \in H \otimes H$ and

\[
G_{12}(\Delta \otimes \text{id})G = G_{23}(\text{id} \otimes \Delta)G, \quad (\epsilon \otimes \text{id})G = 1 \quad (2\text{-cocycle})
\]

\[
(S\gamma)^* = \gamma
\]

\[
\Delta(h^*) = G^{-1}(\Delta h)^* \otimes G, \quad \epsilon(h^*) = (\epsilon h)^*, \quad S^{-1}(h^*) = \gamma^{-1}(Sh)^* \gamma
\]

**Proposition** in this case the category of $H$-modules is a bar category

\[
\overline{V} = V \text{ w/ conjugate action of } \mathbb{C} \quad h \triangleright \overline{v} = (Sh)^* \triangleright v
\]

\[
\gamma_{V,W}(\overline{v} \otimes \overline{w}) = G^{(2)} \triangleright w \otimes G^{(1)} \triangleright v,
\]

\[
bb_V(v) = \overline{\gamma \triangleright v}
\]

**Example:** a quasitriangular flip-Hopf * algebra

\[
\Delta \circ \ast = (\ast \otimes \ast) \tau \circ \Delta, \quad \epsilon(h) = \epsilon(h^*) \quad \text{(Flip-Hopf *)}
\]

\[
(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad \tau \circ \Delta = \mathcal{R}(\Delta)\mathcal{R}^{-1} \quad v = \mathcal{R}^{(1)}(S\mathcal{R}^{(2)})
\]

\[
\gamma = v^* \quad G = \mathcal{R} \quad \text{(Drinfeld)}
\]

If $\mathcal{R}^{* \otimes *} = \mathcal{R}_{21}^{-1}$ & ribbon then $\gamma = v^{-1}v$ gives a strong bar category

eg $u_q(g)$ at roots of $1$ with $(q^{\pm H})^* = q^{\mp H}$, $E^* = -F$, $F^* = -E$
Definition a braided category which is also bar category is real/antireal if

\[
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\Psi} & Y \otimes X \\
\gamma^{-1} \downarrow & & \downarrow \gamma^{-1} \\
Y \otimes X & \xrightarrow{\Psi \pm 1} & X \otimes Y
\end{array}
\]

\[\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X \quad \text{(braiding)}\]

Example: \(u_q(g)\) at roots of unity gives an antireal braided bar category

**Definition**

1) a star-object in a bar category an object \(X\) and morphism:
\[\star_X : X \rightarrow \overline{X}\]

\[\begin{align*}
a) & \quad \overline{\star_X} \circ \star_X = \text{bb}_X. \\
b) & \quad (\star_X \otimes \text{id})l_X = l_{\overline{X}} \star_X \quad \text{and} \quad (\text{id} \otimes \star_X)r_X = r_{\overline{X}} \star_X
\end{align*}\]

2) a star-algebra is a star object \(B\) and product \(\mu : B \otimes B \rightarrow B\) with
\[\overline{\mu} \gamma_{B,B}^{-1}(\star_B \otimes \star_B) = \star_B \mu : B \otimes B \rightarrow \overline{B}\]

3) a star-braided Hopf algebra in an antireal braided bar category is a star object and Hopf algebra in the category with
\[\overline{\Delta} \star = \overline{\Psi} \gamma^{-1} (\star \otimes \star) \Delta : B \rightarrow \overline{B} \otimes \overline{B} \]
Example:

The quantum plane $yx = qxy$ at $q$ a primitive $r$-th root of unity is a star-braided Hopf algebra in the category of $u_q(su_2)$ modules if $3$ is a quadratic residue for $r$. Here

$$\Delta(x^m y^n) = \sum \left[ \begin{array}{c} m \\ r \end{array} \right]_q \left[ \begin{array}{c} n \\ s \end{array} \right]_q q^{s(m-r)} x^r y^s \otimes x^{m-r} y^{n-s}$$

$$(x^m y^n)^* = q^{-(m+n)(m+n-1)} q^{(n-m)/2} q^{nm} x^m y^n$$

is dictated from the values on the generators.

Application: (EJB & SM 2011) Star-compatible bimodule connections in NCG

unique star-compatible metric preserving torsion free bimodule conn with classical limit on $\Omega^1(C_q(SU_2))$
Part III: Differentials on $U_q(g)$ and $B_q(G)$

Space of 1-forms, i.e. `differentials' on an algebra $A$:

\[
\Omega^1 \quad \text{a}((db)c) = (a(db))c \quad \text{A-A bimodule}
\]
\[
d : A \to \Omega^1 \quad \text{d(ab)} = (da)b + a(db) \quad \text{Leibniz rule}
\]
\[
\{ adb \} = \Omega^1 \quad ( \ker d = k.1 \quad \text{connectedness})
\]

In quantum group case we ask it to be translation bicovariant:

\[
\Omega^1 \cong A \otimes \Lambda^1 \quad \text{(free module over its left invariants $\Lambda^1$)}
\]
\[
\Lambda^1 = A^+ / I \quad I \text{ an Ad-stable right ideal (Woronowicz)}
\]

\[
\Omega \cong A \otimes \Lambda, \quad d : \Omega^n \to \Omega^{n+1}, \quad d^2 = 0
\]
\[
\Lambda = T\Lambda^1 / \oplus_n \ker A_n \quad \text{a braided-Hopf algebra in $D(\mathcal{M}^A)$}
\]
\[
\Omega \quad \text{its exterior super-Hopf algebra bosonisation}
\]

1990s solved the problem for quantum groups such as $C_q(G)$
Theorem (SM 1989) a monoidal category enriched by \( F : \mathcal{C} \to \mathcal{V} \) has a dual \( F^\circ : \mathcal{C}^\circ \to \mathcal{V} \) and there is an inclusion \( \mathcal{C} \to \circ(\mathcal{C}^\circ) \) (special case \( F=\text{id} \) was later remarked independently by Drinfeld \( \mathcal{C}^\circ = \mathcal{Z}(\mathcal{C}) = \mathcal{D}(\mathcal{C}) \) )

Theorem (SM 1989) a monoidal functor \( F : \mathcal{C} \to \mathcal{V} \) to a braided category, under representability conditions, factors as:

\[
\begin{array}{c}
\mathcal{C} \to \mathcal{V} \\
\downarrow \\
\mathcal{V}^B \\
\uparrow \\
\end{array}
\]

for some braided Hopf algebra \( B = \text{Aut}(\mathcal{C}, F, \mathcal{V}) \) in \( \mathcal{V} \)

(special case \( F=\text{id} \) more explicitly by Lyubashenko preprint 1990)

moreover, \( \mathcal{C}^\circ = B \mathcal{V} \)

Transmutation (SM 1990) let \( A \) be a coquasitriangular Hopf algebra and \( \mathcal{C} = \mathcal{M}^A \), there is a braided version \( B = \text{Aut}(\mathcal{C}, \text{id}, \mathcal{C}) \)
Example \( B_q(SU_2) \) has generators \( u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), q-comm relations, 
\[
ad - q^2 cb = 1 \quad \text{and} \quad \Delta u^i_j = \sum_k u^i_k \otimes u^k_j \quad \text{and} \quad u^i_j* = u^j_i
\]

q-deforms 2 x 2 hermitian matrices with flip Hopf-* structure classically, i.e. q-hyperboloid \( C_q(H^3) \) in q-Minkowski space

(q real; for root of unity use Part II, get a star-braided Hopf algebra)

Proposition for generic q, \( B_q(SU_2)[a^{-1}] \cong U_q(su_2) \)

So what is \( \Omega(B_q(SU_2)) \)? Answer - transmute \( \Omega(C_q(G)) \)

Theorem (2011) let A be coquasitriangular and \( \Omega = \Omega(A) \) a bicovariant exterior algebra. Consider map of super-Hopf algebras \( \pi : \Omega \rightarrow A, \quad \pi_{\Omega^0} = \text{id}, \quad \pi_{\Omega^i,i>0} = 0 \) then induced \( F : M^\Omega \rightarrow M^A \) gives \( \Omega(B) = Aut(F) \) as braided super-Hopf alg
Explicitly $B, \Omega(B)$ same vector space as $A, \Omega(A)$ with new relations
\[
\mathcal{R}(v^{(2)} \otimes a^{(1)}) v^{(1)} \cdot a^{(2)} = a^{(1)} \cdot (v \ll a^{(2)}) \quad \forall a \in A, v \in \Lambda^1
\]

Moreover, $B, \Omega(B)$ live in the category $D(\mathcal{M}^A)$ i.e. $A \triangleright A$

covariant in factorisable case

A 4-D $q$-Lorentz invariant calculus on $B_q(SU_2), U_q(su_2)$

\[
e_a K = q^\frac{1}{2} K e_a, \quad e_b K = q^{-\frac{1}{2}} K e_b, \quad e_a x_- = q^{-\frac{3}{2}} x_- e_a, \quad [e_a, x_+]_{q^{-\frac{1}{2}}} = K e_b
\]

\[
[e_c, K]_{q^\frac{1}{2}} = \mu(q - 1)x_- e_a, \quad [e_d, K]_{q^{-\frac{1}{2}}} = \mu(1 - q^{-1})x_- e_b, \quad [e_b, x_-]_{q^{-\frac{1}{2}}} = \mu Ke_a
\]

\[
[e_c, x_-]_{q^\frac{1}{2}} = \mu q^{-2}(1 - q)K^{-1}x^2 e_a, \quad [e_d, x_-]_{q^\frac{3}{2}} = q^{\frac{3}{2}}\mu^2 x_- e_a + Ke_c + \mu(q^{-1} - 1)K^{-1}x^2 e_b
\]

\[
[e_a, x_+]_{q^\frac{1}{2}} = Ke_b, \quad e_b x_+ = q^{\frac{3}{2}} x_+ e_b, \quad [e_d, x_+]_{q^{-\frac{1}{2}}} = \mu K^{-1}(q x_- x_+ - x_+ x_-)e_b
\]

\[
[e_c, x_+]_{q^{-\frac{1}{2}}} = \mu Ke_d + \mu q^{\frac{1}{2}}(1 - q^{-1})x_- e_b + \mu K^{-1}(x_- x_+ - q^{-1}x_+ x_-)e_a
\]
Part IV Braided-Lie Algebras

Problem: What is the `Lie algebra' generating $U_q(g)$?

Definition (SM, 1994)

$[\ , \ ] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$

$\Delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$

$\epsilon : \mathcal{L} \to 1$

`coassociative' homs, and some axioms:

E.g. a classical Lie algebra is a trivially braided one by

$\mathcal{L} = k1 \oplus g \subset U(g)$

$\Delta x = x \otimes 1 + 1 \otimes x$

$[1, x] = x$

$[x, 1] = 0$
is a bialgebra in the (abelian) braided category

The braided Killing form

is braided-symmetric and Ad-invariant:

**Theorem:** (1) For all \( U_q(g) \) there is a braided Lie algebra
\[ \mathcal{L} \subset U_q(g) \] and ess. an algebra surj
\[ U(\mathcal{L}) \to U_q(g) \]

(2) The braided-Lie algebra is square-dimensional and for generic q the Killing form is non-degenerate (`semisimple’)

(3) For any A coquasitriangular and bicov calculus as above
\[ \mathcal{L} = \Lambda^{1*} \subset A^{**} \] is a braided-Lie algebra in \( \mathcal{M}^A \)

**Example:** for \( U_q(su_2) \) the smallest \( \mathcal{L} \) is 4D and more precisely surj
\[ C_q[Mink] = U(\mathcal{L}) \to B_q(SU_2) \]
Epilogue: Discrete geometry where $G$ is a finite group

$$\Omega^1(G) \leftrightarrow \mathcal{C} \subseteq G \setminus \{e\}$$

bicovariant \hspace{1cm} Ad-stable subset

$$\mathcal{L} = \text{span}\{x_a \mid a \in \mathcal{C}\}$$

$$[x_a, x_b] = x_{aba^{-1}}$$

$$\Delta x_a = x_a \otimes x_a$$

$$\Psi(x_a \otimes x_b) = x_b \otimes x_a$$

$$K_{\mathcal{L}}(x_a, x_b) = \text{Trace}[x_a, [x_b, \cdot]] = |Z(ab) \cap \mathcal{C}| = \chi\mathcal{C}_{\text{Ad}}(ab)$$

(ad-invariant, non-negative integer-valued symmetric matrix)

In case $G$ simple, expect by analogy, at least for conjugacy class:

(1) $K_{\mathcal{L}}$ non-degenerate

(2) perhaps \hspace{1cm} $\mathcal{L} = \mathbb{C} \oplus \text{Irrep}$

(associate an irrep to a conjugacy class!!!!)
**Conjecture:** For a simple group, if \( \mathcal{C} \) is closed under inversion (`real`) then \( K \) is non-degenerate

(1) Proven for \( \mathcal{C} = G \setminus \{e\} \) (the universal calculus) and all \( G \) for which the Roth property holds (all but some cases of \( PSU(n, F_q) \); includes all sporadics)

(2) Proven by computer for \( \mathcal{C} \) any conjugacy class and \( G \) simple, \( |G| < 95,000 \)

**Conjecture:** Moreover if \( \mathcal{C} \) consists of elements of order 2 then \( K \) is positive definite (`compact real form`), otherwise it has zero signature

Proven by computer for \( |G| < 95,000 \). Note, by Feit-Thompson every simple \( G \) has a conj class of elements of order 2.
References


E.J. Beggs and S. Majid, Bar categories and star operations, Alg. and Repn Theory 12 (2009) 103-152


