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A GENERALIZATION OF LEBESGUE'S CONVERGENCE CRITERION FOR FOURIER SERIES

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"To be quite frank, I haven't the slightest idea what it is good for."†

Abstract. For generalized Dirichlet integrals of type $\int_0^1 f(x)(e^{irx} - e^{\alpha irx})\frac{dx}{x}$ a somewhat sharpened version of Lebesgue's well-known convergence criterion for Fourier series is proven. Integrals of this type are of importance in the general theory of eigenfunction expansions inaugurated by investigations of Birkhoff, Tamarkin, and Stone.

0. Introduction

The theory of eigenfunction expansions for Birkhoff regular boundary value problems is a far-reaching generalization of the theory of Fourier series, including, as was shown by Salaff [15], Fiedler [8], and Minkin [13], as special cases all nonsingular ordinary differential operators with selfadjoint boundary conditions. For the general theory, see Birkhoff [2], [3], Stone [17], Tamarkin [18], Naimark [14], and Dunford/Schwartz [4].

The question of convergence of expansions at boundary points leads to the consideration of integrals

$$\int_0^1 f(x)\frac{e^{irx} - e^{\alpha irx}}{x}\,dx \tag{1}$$

with $f \in L^1(0,1)$, $\alpha \in \mathbf{C}\backslash\{0\}$, $\Im\alpha \geq 0$, which we call *generalized Dirichlet integrals*; see Stone [17], p 733ff. Integrals of this type also occur in the case of more general differential operators, as they were investigated, e.g., by Eberhard/Freiling [6],[7],

† M. Spivak in [16], p. 237, speaking of the second mean value theorem for integrals. The second (1980) edition of the book differs "slightly" from the first one at this point.

Heisecke [11], and by F.J. Kaufmann (Ph.D. Thesis, TH Aachen 1988); and for these problems, convergence not only at boundary points but also at interior points depends on the behaviour of generalized Dirichlet integrals.

For such integrals we prove a theorem generalizing the well-known convergence criterion of Lebesgue for Fourier series which is known to include (shown by Hardy in 1918) the criteria of Dini, Dirichlet/Jordan, de la Vallée Poussin and W.H. Young; see Bary [1], p. 274ff., Hardy/Rogosinski [10], p. 43ff. and p. 95f., and Zygmund [20], p. 29f. and p. 36f. It is also still worthwhile to read Lebesgue's brilliant original paper [12].

1. Statement of the theorem

Theorem. *Let* $f \in L^1(0,1)$, $\alpha \in \mathbf{C} \setminus \{0\}$, $\Im\alpha \geq 0$, $A \in \mathbf{C}$, $x_0 \in (0,1)$; *assume*

$$\lim_{x \to 0+} \frac{1}{x} \int_0^x f(\xi)\, d\xi = A, \tag{2}$$

$$\lim_{x \to 0+} \int_x^{x_0} \frac{|f(\xi) - f(\xi + x)|}{\xi}\, d\xi = 0. \tag{3}$$

Then

$$\lim_{r \to \infty} \int_0^1 f(x) \frac{e^{irx} - e^{\alpha irx}}{x}\, dx = A \log \alpha, \tag{4}$$

where $0 \leq \Im \log \alpha \leq \pi$.

Remarks.

a) In the classical case $\alpha = -1$, (4) reads $\lim\limits_{r \to \infty} \frac{2}{\pi} \int_0^1 f(x) \frac{\sin rx}{x}\, dx = A$; and since the kernel is even here, Lebesgue's classical criterion only requires the even part of f (f being continued with period 1) to fulfil conditions (2) and (3).

b) If $f(x) = 1$ $(0 < x < 1)$, then (1) is a complex Frullani integral (see, e.g., Fichtenholz [5], p. 643), and relation (4) is easily established by complex contour integration.

c) Condition (2) is somewhat weaker than the requirement that zero be a Lebesgue point of f. This refinement is, in the Fourier case $\alpha = -1$, due to Gergen [9].

d) Some weaker versions of (3) have been proven to be sufficient for convergence in the Fourier case; see, e.g., Gergen [9]. We have not tried to generalize these criteria, too. It seems more appropriate to use general principles in order to study the extent to which the convergence behaviours of the classical and the generalized Dirichlet integrals coincide. It is readily seen that there *are* differences:

$$\begin{aligned} I_r &:= \int_0^1 \left| \frac{e^{irx} - e^{-irx}}{\log(-1)} - \frac{e^{irx} - e^{\alpha irx}}{\log \alpha} \right| \frac{dx}{x} \\ &= \int_0^r \left| \frac{i}{\pi} e^{-it} - \left(\frac{1}{\log \alpha} + \frac{i}{\pi} \right) e^{it} + \frac{1}{\log \alpha} e^{\alpha it} \right| \frac{dt}{t}, \end{aligned}$$

and since $\Im \alpha \geq 0$, it is clear (almost periodicity) that for $\alpha \neq -1$ there are positive constants c, C such that $c \log r < I_r < C \log r$ for $r \geq 2$. Hence, by the uniform boundedness principle, for some function $f \in L^1(0,1)$ the difference between the classical and the generalized Dirichlet integral is unbounded as r approaches infinity. But this seems not to exclude the possibility that in case of convergence of one of these integrals this is always true for the other one, too. This is an *open problem.*

2. Proof of the theorem

First step: *For every $f \in L^1(0,1)$ satisfying (2) and (3),*

$$\lim_{x \to 0+} \frac{1}{x} \int_0^x (f(\xi) - A) \sin r\xi \, d\xi = 0$$

uniformly for $r \in \mathbf{R}$.

Proof:

Let $g(x) := f(x) - A$ $(0 < x < 1)$. We may assume g as real-valued since (2) and (3) hold for real and imaginary parts of f seperately. Also, we suppose $r > 0$.