Cheeger-Chern-Simons theory and geometric String structures

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Outline

Differential characters (with sections)

Cheeger-Chern-Simons theory

Geometric String structures

Fiber integration

Transgression and TQFT

Motivation

- Witten conjecture (1986/87): index theorem for (conjectured) Dirac operator on $\mathcal{L}(X)$: index = Witten genus.
- Höhn-Stolz conjecture (1996): Witten genus obstructs positive Ricci curvature on X (if $\mathcal{L}(X)$ admits Spin structure).
- Spin structure on a principal $\mathrm{SO}(n)$ bundle $Q \to X$:
  - Reduction of structure group $\mathrm{Spin}(n) \to \mathrm{SO}(n)$
  - Necessary condition: vanishing of $w_2(Q) \in H^2(X;\mathbb{Z}_2)$.
  - Spin structures on $Q \overset{1:1}{\longleftrightarrow} v \in H^1(Q;\mathbb{Z}_2)$: $\iota_x^* v \neq 0 \in H^1(\mathrm{SO}(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$.
- Spin structure on $\mathcal{L}(X)$:
  - $\mathcal{L}(X)$ orientable iff X admits Spin structure.
  - $\mathcal{L}(X)$ admits Spin structure iff $\frac{1}{2}p_1(X) = 0 \in H^4(X;\mathbb{Z})$.
  - $P \to X$ principal $\mathrm{Spin}(n)$-bundle. Homotopy theoretically: Spin structures on $\mathcal{L}(X) \overset{1:1}{\longleftrightarrow} v \in H^3(P;\mathbb{Z})$: $\iota_x^* v = 1 \in H^3(\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}$.

- Problem: lift structure group of principal $\mathrm{Spin}(n)$-bundle $P \to X$ to $\mathrm{String}(n)$:
  - trivialize $\frac{1}{2}p_1 \in H^4(X;\mathbb{Z})$
  - construct $\mathrm{String}(n) \to \mathrm{Spin}(n)$ by killing π_3
  - $\mathrm{String}(n)$ cannot be a finite dimensional Lie group!
- Actual group constructions of $\mathrm{String}(n)$:
  - as topological group: Stolz 1996, Stolz/Teichner 2004
  - as finite dimensional Lie-2-group: Schommer-Pries 2010
  - as ∞-dimensional Lie group: Nikolaus/Sachse/Wockel 2011
- Trivializations of $\frac{1}{2}p_1$:
  - (X, g) Riemannian: canonical trivialization (Redden, 2006).
  - Trivializations with connections of Chern-Simons bundle 2-gerbe: geometric String structures (Waldorf, 2009).
- Aims:
  - generalize construction of geometric String structures (now).
  - Trivializations of $\frac{1}{2}p_1$: lift $P \to X$ to $\mathrm{String}(n)$-bundles ☑.
  - Geometric String structures: $\rightsquigarrow$ String connections?

Differential characters (Diff. cohom.)

$$\widehat{H}^k(X;\mathbb{Z}) := \Big\{ h \in \mathrm{Hom}(Z_{k-1}(X;\mathbb{Z}), \mathrm{U}(1)) \;\Big|\; \exists \omega \in \Omega^k(X) : h(\partial x) = \exp\Big(2\pi i \int_x \omega\Big) \Big\}$$

Exact sequences:

$$0 \longrightarrow H^{k-1}(X;\mathrm{U}(1)) \longrightarrow \widehat{H}^k(X;\mathbb{Z}) \xrightarrow{\mathrm{curv}} \Omega^k_0(X) \longrightarrow 0$$

$$0 \longrightarrow \frac{\Omega^{k-1}(X)}{\Omega^{k-1}_0(X)} \xrightarrow{\iota} \widehat{H}^k(X;\mathbb{Z}) \xrightarrow{c} H^k(X;\mathbb{Z}) \longrightarrow 0$$

Moreover, $[\mathrm{curv}(h)]_{dR} = c(h) \in H^k(X;\mathbb{R})$. Set

$$R^k(X;\mathbb{Z}) := \{(\omega, c) \in \Omega^k_0(X) \times H^k(X;\mathbb{Z}) \,|\, [\omega]_{dR} = c\}.$$

Examples:

- $\widehat{H}^1(X;\mathbb{Z}) \cong C^\infty(X, \mathrm{U}(1))$.

  Here $\mathrm{curv}(f) = d\log(f)$ and $c(f) = f^*u$, where $u \in H^1(S^1;\mathbb{Z})$ fundamental class.

- $\widehat{H}^2(X;\mathbb{Z}) \cong \big\{[P,\nabla] \,\big|\, P \to X \text{ a } \mathrm{U}(1)\text{-princ. bundle, } \nabla \text{ conn.}\big\}$.
  Here $\mathrm{curv}([P,\nabla]) = \frac{i}{2\pi} R^\nabla$ and $c([P,\nabla]) = c_1(P)$.
  $\gamma : S^1 \to X$ cycle: $[L,\nabla](\gamma) = \mathrm{hol}^\nabla(\gamma)$.

- $\widehat{H}^3(X;\mathbb{Z}) \cong$
  $\big\{$isom. classes of $\mathrm{U}(1)$-gerbes with connection on $X\big\}$.

  Here $\mathrm{curv}(h) = H$ field strength of the B field and $c(h)$ its quantum number.

Mapping cone complexes for a smoth map $\varphi : A \to X$:

$$C_k^{\varphi}(X, A; \mathbb{Z}) := C_k(X;\mathbb{Z}) \times C_{k-1}(A;\mathbb{Z}),$$
$$\partial_\varphi(s,t) := (\partial s + \varphi_* t, -\partial t),$$
$$\Omega^k_\varphi(X,A) := \Omega^k(X) \times \Omega^{k-1}(A),$$
$$d_\varphi(\omega,\vartheta) := (d\omega, \varphi^*\omega - d\vartheta).$$

Short exact sequence of complexes:

$$0 \to C_{*-1}(A;\mathbb{Z}) \to C_*^{\varphi}(X,A;\mathbb{Z}) \to C_*(X;\mathbb{Z}) \to 0$$

Long exact (co)homology sequence:

$$\longrightarrow H_*(A;\mathbb{Z}) \longrightarrow H_*(X;\mathbb{Z}) \longrightarrow H_*^{\varphi}(X,A;\mathbb{Z}) \longrightarrow H_{*-1}(A;\mathbb{Z}) \longrightarrow$$

$$\longrightarrow H^{*-1}(A;\mathbb{Z}) \longrightarrow H^*_\varphi(X,A;\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z}) \longrightarrow H^*(A;\mathbb{Z}) \longrightarrow$$

Differential characters with sections (Relative diff. cohom.)

$$\widehat{H}^k_\varphi(X,A;\mathbb{Z}) := \Big\{ f \in \mathrm{Hom}(Z^{k-1}_\varphi(X,A;\mathbb{Z}), \mathrm{U}(1)) \;\Big|\; \exists(\omega,\vartheta) \in \Omega^k_\varphi(X,A) : f(\partial_\varphi(x,y)) = \exp 2\pi i\Big(\int_x \omega + \int_y \vartheta\Big)\Big\}$$

Set $\omega =: \mathrm{curv}(f)$ and $\vartheta =: \mathrm{cov}(f)$.

Exact sequences:

$$0 \longrightarrow H^{k-1}_\varphi(X,A;\mathrm{U}(1)) \longrightarrow \widehat{H}^k_\varphi(X,A;\mathbb{Z}) \xrightarrow{\mathrm{curv},\mathrm{cov}} \Omega^k_{\varphi,0}(X,A) \longrightarrow 0$$

$$0 \longrightarrow \frac{\Omega^{k-1}_\varphi(X,A)}{\Omega^{k-1}_{\varphi,0}(X,A)} \xrightarrow{\iota} \widehat{H}^k_\varphi(X,A;\mathbb{Z}) \xrightarrow{c} H^k_\varphi(X,A;\mathbb{Z}) \longrightarrow 0$$

Moreover, $[(\mathrm{curv}(f),\mathrm{cov}(f))]_{dR} = c(f) \in H^k_\varphi(X,A;\mathbb{R})$.

In particular:

$$d_\varphi(\mathrm{curv}(f), \mathrm{cov}(f))) = (d\mathrm{curv}(h), \varphi^*\mathrm{curv}(f) - d\mathrm{cov}(f)) = 0.$$

- Topologically: relative differential cohomology
- Geometrically: differential characters with section along φ.

Examples:

- $\widehat{H}^1_\varphi(X, A; \mathbb{Z}) = \{ f \in C^\infty(X, \mathrm{U}(1)) \mid f \circ \varphi = \exp 2\pi i(\vartheta) \}$.
  Here $\mathrm{cov}(f) = \vartheta \in \Omega^0(A)$.
  $d\mathrm{cov}(f) = d\vartheta = d\log(f \circ \varphi) = \varphi^* d\log(f) = \varphi^*\mathrm{curv}(f)$.
- $\widehat{H}^2_\varphi(X, A; \mathbb{Z}) \cong \{ [L, \nabla, \sigma] \mid \sigma : A \to \varphi^* L \text{ section} \}$
  Here $\mathrm{cov}([L, \nabla, \sigma]) \cdot \sigma = \frac{i}{2\pi} \nabla\sigma \in \Omega^1(A, \varphi^* L)$.
  $d\mathrm{cov}([L, \nabla, \sigma]) = \varphi^*\mathrm{curv}([L, \nabla])$.

We have canonical maps

$$\widehat{H}^{k-1}(A;\mathbb{Z}) \xrightarrow{\check{\imath}} \widehat{H}^k_\varphi(X,A;\mathbb{Z}) \xrightarrow{\check{p}} \widehat{H}^k(X;\mathbb{Z})$$

defined by $\check{\imath}(g) : (s,t) \mapsto g(s)$ and $\check{p}(f) : z \mapsto f(z,0)$.

$$\mathrm{curv}(g) = \mathrm{cov}(\check{\imath}(g)) \text{ and } \mathrm{curv}(\check{p}(f)) = \mathrm{curv}(f).$$

- Geometrically: $\check{p}$ forgets the section; $\check{\imath}$ is (non-parallel) section on topologically trivial bundle.
- Topologically: induced maps for relative differential cohomology.

For $\varphi = \mathrm{id}_X$, we have the commutative diagram

$$\begin{array}{ccc} \widehat{H}^k(X,X;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{cov}} & \Omega^{k-1}(X) \\ & \searrow \quad \swarrow_{\iota} & \\ & \widehat{H}^k(X;\mathbb{Z}) \, . & \end{array}$$

Theorem (Existence of sections; C. Bär, C.B., 2012)

We have an exact sequence:

$$0 \longrightarrow \varphi^*\widehat{H}^{k-1}_{\mathrm{flat}}(X;\mathbb{Z}) \longrightarrow \widehat{H}^{k-1}(A;\mathbb{Z}) \longrightarrow \widehat{H}^k_\varphi(X,A;\mathbb{Z}) \longrightarrow \widehat{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi^*\circ c} \varphi^*H^k(X;\mathbb{Z}) \longrightarrow 0.$$

Geometrically:

- $h \in \widehat{H}^k(X;\mathbb{Z})$ admits sections along $\varphi \iff h$ is topologically trivial along φ.
- h admits parallel sections $\iff h$ is topologically trivial and flat along φ.

Topologically: related to long exact sequence for absolute/relative groups (from Deligne or Hopkins-Singer complex)

Universal principal G bundle

G compact Lie group with Lie algebra $\mathfrak{g}$.
$\pi : EG \to BG$ universal principal G-bundle.

- Universal bundle $\pi : EG \to BG$ carries universal connection Θ:

$$\begin{array}{ccc} P & \xrightarrow{\;F\;} & EG \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\;f\;]{} & BG\,. \end{array}$$

- EG contractible, in particular, $\pi^* H^*(BG; \mathbb{Z}) = \{0\}$.
- $\pi^* EG \to EG$ has tautological section.

(P, ν) principal G-bundle with connection: $(P, \nu) = F^*(EG, \Theta)$.
$F_\nu \in \Omega^2(X; P \times_{\mathrm{Ad}} \mathfrak{g})$ curvature of the connection ν.

$\lambda : \mathfrak{g}^{\otimes k} \to \mathbb{R}$ multilinear, Ad_G-invariant
Chern-Weil construction:

$$CW(\nu, \lambda) := \lambda(F_\nu \wedge \ldots \wedge F_\nu) \in \Omega^{2k}_{cl}(X).$$

Cheeger-Simons construction:

$$\widehat{CW}(\nu, \lambda) \in \widehat{H}^{2k}(X; \mathbb{Z}) \text{ with } \mathrm{curv}(\widehat{CW}(\nu, \lambda)) = CW(\nu, \lambda).$$

Needs $CW(\nu, \lambda) \in \Omega^{2k}_0(X)$! What is $c(\widehat{CW}(\nu, \lambda))$? Set

$$K^{2k}(G; \mathbb{Z}) := \{(\lambda, u) \in I^k(G) \times H^{2k}(BG; \mathbb{Z}) \,|\, [CW(\lambda)]_{dR} = u\}$$

Example: $G = \mathrm{Spin}(n), n \geq 3, \; k = 2$:

$$K^{2k}(\mathrm{Spin}(n); \mathbb{Z})) \cong H^4(B\mathrm{Spin}(n); \mathbb{Z}) \cong H^3(\mathrm{Spin}(n); \mathbb{Z}) \cong \mathbb{Z} \quad (\text{“level”})$$

Cheeger-Simons construction

Cheeger-Simons construction: unique natural lift $\widehat{CW}$

$$\begin{array}{ccc} & & \widehat{H}^{2k}(X;\mathbb{Z}) \\ & \overset{\widehat{CW}}{\nearrow} & \downarrow (\mathrm{curv}, c) \\ K^{2k}(G;\mathbb{Z}) & \xrightarrow[CW \times c_{\mathbb{Z}}]{} & R^{2k}(X;\mathbb{Z}) \end{array}$$

$c_{\mathbb{Z}} : H^*(BG;\mathbb{Z}) \to H^*(X;\mathbb{Z})$ pull-back by a classifying map.

- Geometrically: $\pi^* P \to P$ has tautological section: thus $[CW(\pi^*\nu, \lambda)]_{dR} = 0$.
- $CW(\pi^*\nu, \lambda) = \pi^* CW(\nu, \lambda) \subset d(\Omega^{2k-1}(P))$: Chern-Weil forms are exact after pull-back by π.
- Chern-Simons construction:

$$CS(\nu, \lambda) \in \Omega^{2k-1}(P) \quad \text{with} \quad dCS(\nu, \lambda) = \pi^* CW(\nu, \lambda).$$

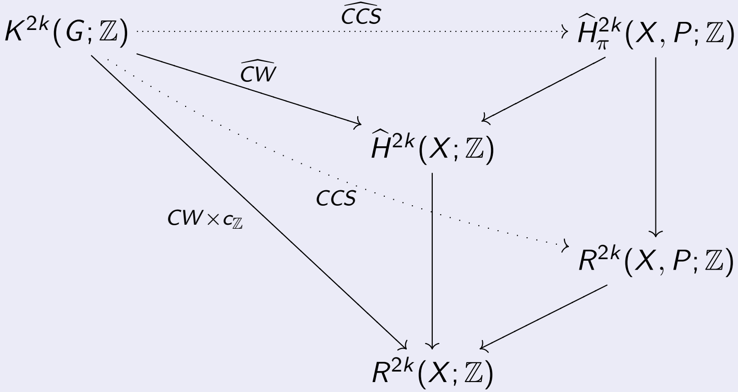
- Topologically: for classifying map $f : X \to BG$, we have

$$\begin{array}{ccc} P & \xrightarrow{\;F\;} & EG \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\;f\;]{} & BG. \end{array}$$

- EG contractible: $\pi^* c(\widehat{CW}(\nu, \lambda, u)) = \pi^* f^* u = 0$: Cheeger-Simons characters are topologically trivial along π.

Theorem (Cheeger-Chern-Simons construction; C.B., 2012)

Cheeger-Chern-Simons construction: $\widehat{CW}(\nu, \lambda, u)$ *has canonical section* $\widehat{CCS}(\nu, \lambda, u)$*:* $\exists!$ *natural lift of* $\widehat{CW}$ *with* $\mathrm{cov} \circ \widehat{CCS} = CS$*:*



$G = \mathrm{Spin}(n)$ for $n \geq 3$ and $P \to X$ principal $\mathrm{Spin}(n)$ bundle.
$H^4(B\mathrm{Spin}(n); \mathbb{Z})$ generated by $\frac{1}{2}p_1$: set $w = \frac{1}{2}p_1(P)$.

Definition (String structure)

A String structure (in the topological sense) on $P \to X$ is a cohomology class $v \in H^3(P; \mathbb{Z})$ such that for any $x \in X$:

$$\iota_x^* v = 1 \in H^3(\mathrm{Spin}(n); \mathbb{Z}) \cong \mathbb{Z}.$$

Existence $\iff \frac{1}{2}p_1(P) = 0$.
Uniqueness: String structures are parametrized by $H^3(X; \mathbb{Z})$.

Aim: refine notion of topological String structures to geometric String structures.

Generalization:
G compact Lie group, $\pi : P \to X$ principal G-bundle with connection ν.
$(\lambda, u) \in K^{2k}(G;\mathbb{Z})$ level.

Definition (String_G structure at level (λ, u))

A String_G structure at level (λ, u) (in the topological sense) is a cohomology class $q \in H^{2k-1}(P;\mathbb{Z})$ such that for any $x \in X$:

$$\iota_x^* q = T(u) \in H^{2k-1}(G;\mathbb{Z}).$$

Here $T : H^{2k}(BG;\mathbb{Z}) \to H^{2k-1}(G;\mathbb{Z})$ is the transgression map.

Existence $\iff f^*u = 0$, where $f : X \to BG$ classifies (P, ν).
Uniqueness: String_G structures at level (λ, u) are parametrized by $H^{2k-1}(X;\mathbb{Z})$.

Geometric String structures

Refine notion of (topological) String_G structures to *geometric* String_G structures.

$$\begin{array}{ccccccccc}
\widehat{H}^{2k-1}(P;\mathbb{Z}) & \xrightarrow{\breve{\imath}} & \widehat{H}^{2k}_\pi(X,P;\mathbb{Z}) & \xrightarrow{\breve{p}} & \widehat{H}^{2k}(X;\mathbb{Z}) & \longrightarrow & \pi^*H^{2k}(X;\mathbb{Z}) & \longrightarrow & 0 \\
\uparrow \pi^* & & \uparrow (\mathrm{id},\pi)^* & & \uparrow \mathrm{id} & & \uparrow \pi^* & & \\
\widehat{H}^{2k-1}(X;\mathbb{Z}) & \xrightarrow{\breve{\imath}} & \widehat{H}^{2k}(X,X;\mathbb{Z}) & \xrightarrow{\breve{p}} & \widehat{H}^{2k}(X;\mathbb{Z}) & \longrightarrow & H^{2k}(X;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Definition

Let $(\lambda, u) \in K^{2k}(G;\mathbb{Z})$ be a level. A *geometric* String_G *structure at level* (λ, u) is a pair $(\hat{q}, \varrho) \in \widehat{H}^{2k-1}(P;\mathbb{Z}) \times \Omega^{2k-1}(X)$ such that

$$\breve{\imath}(\hat{q}) + (\mathrm{id},\pi)^*\mathrm{cov}^{-1}(\varrho) = \widehat{CCS}(\lambda, u)$$

Lemma

- *Existence* $\Leftrightarrow c(\widehat{CW}(\lambda, u)) = f^*u = 0$.
- *Uniqueness: geometric* String_G *structures at level* (λ, u) *are parametrized by* $\widehat{H}^{2k-1}(X;\mathbb{Z})$.
- $\hat{q} \in \widehat{H}^{2k-1}(P;\mathbb{Z})$ *geometric* String_G *structure* $\Longrightarrow$ $c(\hat{q}) \in H^{2k-1}(P;\mathbb{Z})$ *(topological)* String *structure.*
- $G = \mathrm{Spin}(n)$, $k = 2$, $\lambda = \frac{1}{2}p_1$*: notion of geometric* String *structure* $(\hat{q}, \varrho) \in \widehat{H}^3(P;\mathbb{Z}) \times \Omega^3(X)$.

(X, g) Riemannian String manifold: canonical geometric String structure?
(Redden, 2006): canonical topological trivialization $\varrho \in \Omega^3(M)$.
ϱ determines $\hat{q}$ up to $\widehat{H}^3_{\mathrm{flat}}(X;\mathbb{Z}) \cong H^2(X;\mathrm{U}(1))$.
For fixed topological String structure $q = c(\hat{q})$ and ϱ:
$\hat{q}$ unique up to $\frac{H^2(M;\mathbb{R})}{H^2(M;\mathbb{Z})}$.

$\pi : E \to X$ fiber bundle with compact oriented fibers.

$$\Omega^k(E) \to \Omega^{k-\dim F}(X), \qquad \omega \mapsto \fint_F \omega,$$
$$H^k(E;\mathbb{Z}) \to H^{k-\dim F}(X;\mathbb{Z}), \quad u \mapsto \pi_! u.$$

Problem: Fiber integration

Construct natural fiber integration

$$\widehat{\pi}_! : \widehat{H}^*(E;\mathbb{Z}) \to \widehat{H}^{*-\dim F}(X;\mathbb{Z})$$

that commutes with the exact sequences.

Idea: for z fundamental cycle of $M^{k-1-\dim F} \subset X$, set

$$\widehat{\pi}_! h(z) := h([E|_M]) .$$

Analogously for z cohomologous to a fundamental cycle.
Not all homology classes in $H_{k-1-\dim F}(X;\mathbb{Z})$ can be represented by smooth manifolds, but by stratifolds!

Theorem (Uniqueness, existence; C. Bär, C.B., 2012)

There exists a unique fiber integration

$$\widehat{\pi}_! : \widehat{H}^*(E;\mathbb{Z}) \to \widehat{H}^{*-\dim F}(X;\mathbb{Z})$$

natural with respect to bundle maps $g : Y \to X$, $G : g^*E \to E$*:*

$$\widehat{\pi}_!(G^*h) = g^*\widehat{\pi}_! h,$$

and compatible with the exact sequences:

$$\mathrm{curv}(\widehat{\pi}_! h) = \int_F \mathrm{curv}(h)$$
$$c(\widehat{\pi}_!) = \pi_! c(h)$$
$$\widehat{\pi}_!(\iota(\varrho)) = \iota\Big(\pm \int_F \varrho\Big).$$

Orientation reversal in the fibers turns $\widehat{\pi}_!$ *into* $-\widehat{\pi}_!$.

Theorem (Properties; C. Bär, C.B., 2012)

- *Functoriality: for* $N \xrightarrow{\kappa} E \xrightarrow{\pi} X$*, we have*

$$\widehat{(\pi \circ \kappa)}_! = \widehat{\pi}_! \circ \widehat{\kappa}_! .$$

- *For a fiber product* $F \times F' \hookrightarrow E \times E' \twoheadrightarrow X \times X'$*, we have:*

$$\widehat{\pi}_!^E(h) \times \widehat{\pi}_!^{E'}(h') = \pm \widehat{\pi}^{E \times E'}(h \times h') \in \widehat{H}^{k+k'-\dim F \times F'}(X \times X'; \mathbb{Z}).$$

- *Up-down formula: for* $h \in \widehat{H}^k(X;\mathbb{Z})$ *and* $h' \in \widehat{H}^l(E;\mathbb{Z})$*:*

$$\widehat{\pi}_!(\pi^* h * h') = h * \widehat{\pi}_! h' \in \widehat{H}^{k+k'-\dim F}(X;\mathbb{Z}).$$

- $\pi : (E, \partial E) \to X$ *with compact oriented fibers with boundary:*

$$\widehat{\pi}_!^{\partial E} h = \iota\Big((-1)^{k-\dim F} \int_F \mathrm{curv}(h) \Big) .$$

Holonomy: evaluate $h \in \widehat{H}^k(X;\mathbb{Z})$ on oriented closed $(k-1)$-manifold S: Let $\varphi : S \to X$ smooth map.

$$\mathrm{Hol}^h(\varphi) := \varphi^* h([S]).$$

$\mathrm{Hol}^h : C^\infty(S,X) \to \mathrm{U}(1)$, i.e. $\mathrm{Hol}^h \in \widehat{H}^1(C^\infty(S,X);\mathbb{Z})$.
For $h = [L,\nabla] \in \widehat{H}^2(X;\mathbb{Z})$: holonomy along closed paths in X.

Parallel transport: evaluate $h \in \widehat{H}^k(X;\mathbb{Z})$ on oriented $(k-1)$-manifold W with boundary:
Set $r : C^\infty(W,X) \to C^\infty(\partial W,X)$, $\phi \mapsto \phi|_{\partial W}$.
We have line bundle $\mathcal{L} \to r(C^\infty(W,X))$ with section PT^h.
For $h = [L,\nabla] \in \widehat{H}^2(X;\mathbb{Z})$: parallel transport along paths in X.

Moreover: hermitean metric on $\mathcal{L}$, unitary connection.
Thus $\mathrm{PT}^h \in \widehat{H}^2_r(r(C^\infty(W,X)), C^\infty(W,X);\mathbb{Z})$.

Generalization: higher dimensional transgressions.

For S closed, X Fréchet manifold, $C^\infty(S,X)$ Fréchet manifold.

Evaluation map

$$\begin{array}{ccc} C^\infty(S,X)\times S & \xrightarrow{\ \mathrm{ev}_S\ } & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ C^\infty(S,X) & & \end{array}$$

Definition (Transgression)

$$\begin{array}{ccc} \widehat{H}^k(X;\mathbb{Z}) & \xrightarrow{\ \mathrm{ev}_S^*\ } & \widehat{H}^k(C^\infty(S,X)\times S) \\ & {\scriptstyle \tau}\searrow & \Big\downarrow{\scriptstyle \widehat{\pi}_!} \\ & & \widehat{H}^{k-\dim S}(C^\infty(S,X)) \end{array}$$

For $\dim S = k-1$:

$$\widehat{H}^1(C^\infty(S,X);\mathbb{Z}) \xrightarrow{\cong} C^\infty(C^\infty(S,X),\mathrm{U}(1))\,,$$

$$\tau_S h \mapsto \mathrm{Hol}^h\,.$$

W compact, oriented, $r : C^\infty(W,X) \to C^\infty(\partial W,X)$.
Transgressions:

$$\begin{array}{ccc}
 & & \widehat{H}^{k-\dim W+1}(C^\infty(W,X), C^\infty(W,X);\mathbb{Z}) \\
 & \overset{\tau^W}{\nearrow} & \downarrow \check{p} \\
\widehat{H}^k(X;\mathbb{Z}) & \xrightarrow{\tau^{\partial W}} & \widehat{H}^{k-\dim \partial W}(C^\infty(W,X);\mathbb{Z}) \\
 & \underset{\tau_{\partial W}}{\searrow} & \uparrow r^* \\
 & & \widehat{H}^{k-\dim \partial W}(C^\infty(\partial W,X);\mathbb{Z})
\end{array}$$

Parallel transport PT^h: TQFT in the sense of Atiyah.
Higher dimensional transgressions: analogous properties.