Control under constraints of reaction-diffusion equations

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Erlangen, January 21\textsuperscript{st} 2020
Introduction
The models

Homogeneous bistable reaction-diffusion:

\[
\begin{cases}
  u_t - \Delta u = f(u) & (x, t) \in \Omega \times (0, T) \\
  u = a(x, t) & (x, t) \in \partial \Omega \times (0, T) \\
  0 \leq u(x, 0) \leq 1
\end{cases}
\]

Bistable reaction-diffusion with heterogeneous drift:

\[
\begin{cases}
  u_t - \Delta u + \nabla b(x) \nabla u = f(u) & (x, t) \in \Omega \times (0, T) \\
  u = a(x, t) & (x, t) \in \partial \Omega \times (0, T) \\
  0 \leq u(x, 0) \leq 1
\end{cases}
\]
The typical example is:

\[ f(s) = s(1 - s)(s - \theta) \]
Reaction-diffusion equations typically model the evolution of quantities that are nonnegative or bounded by above and below, for example:

- Temperature
- Concentrations of chemicals
- Proportions
Main Goal

Let $f$ be bistable, and consider $0 \leq u_0 \leq 1$ there exists a $T > 0$ such that equation (1) is controllable towards the constant steady states $0, \theta$ and $1$ in a way that the trajectory fulfills for all $(x, t) \in \Omega \times [0, T]$ that $0 \leq u(x, t) \leq 1$?
Negative result: Barriers
The comparison principle ensures that if a solution to the problem

\[
\begin{align*}
-\Delta v &= f(v) & x &\in \Omega \\
v &= 0 & x &\in \partial\Omega \\
1 > v > 0 & & x &\in \Omega
\end{align*}
\]

exists\(^1\) and our initial data \(u_0\) is above \(v\) then for any \(a \in L^\infty(\Omega, [0, 1])\) the solution of the parabolic problem will stay above \(v\).

\(^{1}\)P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982), no. 4, 441–467
Introduction

Negative result: Barriers

Control towards $\theta$

Heterogeneous drifts

References

Barriers

Control under constraints of reaction-diffusion equations
After a space rescaling one can rewrite the elliptic equation

$$\begin{cases} 
-v_{xx} = L^2 f(v) & x \in (0, 1) \\
 v(0) = v(1) = 0 \\
 1 > v > 0 & x \in (0, 1)
\end{cases}$$

The associated functional is:

$$J : H^1_0((0, 1)) \rightarrow \mathbb{R}$$

$$J(v) = \int_0^1 \frac{1}{2} v_x^2 - L^2 F(v) \, dx$$
If $\lambda$ is small, a barrier cannot exist:

$$\lambda_1 \int_0^1 v^2 \leq \int_0^1 v_x^2 = \lambda \int_0^1 vf(v) \leq \lambda \int_0^1 v^2 \|g\|_{\infty}$$

where $f(v) = vg(v)$. Choose $\lambda$ small enough so that:

$$\lambda_1 \int_0^1 v^2 \geq \lambda \int_0^1 v^2 \|g\|_{\infty}$$
$J : V_h \rightarrow \mathbb{R}$

$$J(v) = \int_0^1 \frac{1}{2} v_x^2 - \lambda F(v) dx$$

Visual representation

![Visual representation](image-url)
Nonexistence of barriers for reaching 1

Let $F(1) \geq 0$, for any $\lambda > 0$ there exists a unique solution $(v \equiv 1)$ to the problem:

$$
\begin{cases}
- v_{xx} = \lambda f(v) & x \in (0, 1) \\
0 \leq v \leq 1 & x \in (0, 1) \\
v(0) = v(1) = 1
\end{cases}
$$

(3)
Visual representation
Control towards $\theta$
Theorem

Let $v_0$ and $v_1$ be path connected. If $T$ is large enough, $\exists a \in L^\infty$ such that the problem (1) with initial datum $v_0$ and control $a$ admits unique solution verifying $v(T, \cdot) = v_1$ s.t. its trajectory is admissible.

---

Continuous Path

\[ S := \left\{ v \in H^1(0, L) \text{ such that } v \text{ satisfies (4)} \right\} \]

\[
\left\{ 
\begin{align*}
-u_{xx} &= f(u) \\
u(0) &= a_1, \quad u(L) = a_2 \\
0 &\leq u \leq 1
\end{align*}
\right.
\]

(4)

The construction of the path involves to find a map

\[ \gamma : [0, 1] \rightarrow S \]

fulfilling

- \( \gamma(0) = 0 \)
- \( \gamma(1) = \theta \)
- \( \gamma \) is continuous with respect to the \( L^\infty \) topology.
Assume $F(1) > 0$. Observe that the one-dimensional elliptic equation can be written as an ODE\(^3\)

\[
\frac{d}{dx} \begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} u_x \\ -f(u) \end{pmatrix}
\]

Invariant region

\[
\frac{d}{dx} \begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} u_x \\ -f(u) \end{pmatrix}, \quad \begin{cases} u(0) = s\theta \\ u_x(0) = 0 \end{cases}
\]
When $F(1) = 0$ the traveling waves are stationary and they enclose an invariant region.

For $F(1) > 0$ they give a natural control towards the stationary solution 1.
Multi-D

\[
\begin{cases}
    u_t - \mu \Delta u = f(u) & (x, t) \in \Omega \times (0, T) \\
    u = a(x, t) & (x, t) \in \partial \Omega \times (0, T) \\
    0 \leq u(x, 0) \leq 1
\end{cases}
\]

\[
\begin{cases}
    -u_{rr} - \frac{N-1}{r} u_r = \frac{1}{\mu} f(u) & r \in (0, R) \\
    u(0) = a \\
    u_r(0) = 0
\end{cases}
\]

\[4\]

4D. Ruiz-Balet and E. Zuazua, Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations, Preprint (2019)
\[
\frac{d}{dr} \begin{pmatrix} u \\ u_r \end{pmatrix} = \begin{pmatrix} u_r \\ -f(u) \end{pmatrix} - \left( \frac{N-1}{r} u_r \right)
\]

\[E(u, u_r) = \frac{1}{2} u_r^2 + F(u)\]

\[
\frac{d}{dr} E = - \frac{N-1}{r} u_r^2 < 0
\]
Theorem (Theorem 1.2 in (4))

Let $f$ be bistable. Let $\Omega \subset \mathbb{R}^N$ be a $C^2$-regular domain of measure 1. If $F(1) > 0$, $\exists T \in (0, +\infty]$, and $\exists \mathcal{A} \subset L^\infty(\Omega; [0, 1])$ s.t. the solution of the system (5) can be controlled by means of a function $a \in L^\infty(\partial \Omega \times [0, T], [0, 1])$

- in infinite time to $w \equiv 0$:
  - for any initial data $u_0$ iff $\mu > \mu^*(\Omega, f)$,
  - for any $\mu > 0$ if $u_0 \in \mathcal{A}$,

- in finite time to $w \equiv \theta$:
  - for any initial data $u_0$ iff $\mu > \mu^*(\Omega, f)$,
  - for any $\mu > 0$ if $u_0 \in \mathcal{A}$,

- in infinite time to $w \equiv 1$ for any admissible initial data $u_0$ and for any $\mu > 0$.

Furthermore $\mu^*(\Omega, f) > 0$. 

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Control under constraints of reaction-diffusion equations
Heterogeneous drifts
The model

Consider a distribution of population $N > 0$. Consider that the population is divided between two traits. We model the evolution of the proportion of one trait by\textsuperscript{5}:

\[
\begin{align*}
    u_t - \Delta u + \frac{\nabla N(x)}{N(x)} \nabla u &= f(u) \quad (x, t) \in \Omega \times (0, T) \\
    u &= a(x, t) \quad (x, t) \in \partial \Omega \times (0, T) \\
    0 \leq u(x, 0) \leq 1
\end{align*}
\]

we will also use the notation $\nabla b(x) = \frac{\nabla N(x)}{N(x)}$

\textsuperscript{5}I. Mazari, D. Ruiz-Balet, and E. Zuazua, Constrained control of bistable reaction-diffusion equations: Gene-flow and spatially heterogeneous models, Preprint (2019)
Non trivial solution

\[ \nabla b(x) = 2x \sigma \]

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Differential inequality

When can we ensure controllability?

- The multi-dimensional case shows us that the important fact is to have a conservative or dissipative ODE dynamics.
- If we set a radial drift that makes the ODE dynamics dissipative, would be enough to guarantee the construction of the path.
- Theorem 3 in (5) ensures the existence of a continuous path whenever the drift is radial and fulfills:

\[ N'(r) \geq -\frac{N - 1}{2r} N(r) \]  

(6)
What happens if the differential inequality (6) is not satisfied?

Take \( N(x) = e^{-\frac{x^2}{\sigma}} \). We observed that an upper barrier can exist (Theorem 4 in (5)).

\[
\begin{align*}
-u_{xx} + 2\frac{x}{\sigma}u_x &= f(u) \\
u(-L) &= u(L) = 1
\end{align*}
\]
This steady state cannot correspond to a global energetic minima hence a shooting method is employed.


D. Ruiz-Balet and E. Zuazua, Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations, Preprint (2019)

Thank you for your attention!

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No 694126-DYCON).