Interface propagation and mixing phenomena in fluids

Nicola De Nitti

Università degli Studi di Bari Aldo Moro

FAU Erlangen-Nürnberg

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In memoriam: Prof. Enrico Jannelli (1957-2019).

Outline

1. Evolution of the free boundary for solutions to the thin-film equation.
   - [De Nitti-Fischer, submitted 2019].
2. Regularity propagation for the flow generated by a rough velocity field.
   - [Bianchini-De Nitti, in preparation 2019].

Presentation Outline

1. Sharp criteria for the waiting time phenomenon in solutions to the thin-film equation
2. Differentiability properties of the flow associated with a nearly incompressible BV vector field

The *thin-film equation* (**degenerate fourth order** parabolic equation)

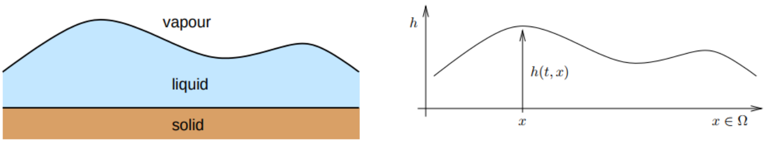
$$\partial_t u + \operatorname{div}\left(u^n \nabla \Delta u\right) = 0 \tag{TFE}$$

(with positive real parameter $n > 0$) describes the **surface-tension-driven evolution of the height of a viscous thin-liquid film on a flat surface**.

Like the *porous medium equation*

$$\partial_t u = \Delta u^m = m \nabla \cdot \left(u^{m-1} \nabla u\right) \tag{PME}$$

(with $m > 1$), the thin-film equation gives rise to a *free boundary problem*, the free boundary being the boundary of the liquid film $\partial\{u(\cdot, t) > 0\}$.



The thin-film equation is mostly of interest in the regime $n \in (1,3)$, as for $n \geq 3$ it is conjectured that the support of solutions remains constant in time.

Derivation: From the incompressible *Navier-Stokes equations*,

$$\begin{cases} \partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla q = 0, \\ \operatorname{div} v = 0, \end{cases} \tag{INS}$$

where $\nu > 0$ is the *viscosity* – assuming a scaling of height and length

$$\boxed{\varepsilon = \frac{\text{height}}{\text{length}} \ll 1.}$$

Applications of thin liquid films:

- industrial coating processes for decorative, insulating, or protective purposes;
- cooling of microelectronic devices;
- microfluidics to model and replicate biological systems (e.g. blood circulation systems) or biological processes (e.g. in-vivo protein crystallisation and bone formation).

Key estimates for the thin-film equation

For solutions to (TFE), maximum or comparison principles cannot be valid.

Existence of *non-negativity preserving* weak solutions and their qualitative properties are obtained thanks to two types of integral estimates:

(1) **energy estimate:**

$$\frac{1}{2}\int_\Omega |\nabla u_0|^2 \geq \frac{1}{2}\int_\Omega |\nabla u(T,x)|^2 + \int_0^T \int_\Omega u^n |\nabla \Delta u|^2 \, \mathrm{d}x \, \mathrm{d}t$$
$$\geq \frac{1}{2}\int_\Omega |\nabla u(T,x)|^2$$
$$+ C \int_0^T \int_\Omega \left|\nabla \Delta u^{\frac{n+2}{2}}\right|^2 + u^{n-2}|\nabla u|^2 |D^2 u|^2 + |\nabla u^{\frac{n+2}{6}}|^6 \, \mathrm{d}x \, \mathrm{d}t.$$

N.B. The second inequality plays a key role (makes the dissipation term more amenable to interpolation arguments):

- for $d = 1$ and $n \in \left(\frac{1}{2}, 3\right)$ in [Bernis, Proc. SIAM 1996];
- for $d \in \{2,3\}$ and $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$ in [Grün, Comm. P.D.E. 2004].

(2) **entropy estimate:**

$$\frac{1}{\alpha(\alpha+1)}\int_\Omega u_0^{\alpha+1}\,\mathrm{d}x$$
$$\geq \frac{1}{\alpha(\alpha+1)}\int_\Omega u^{\alpha+1}\,\mathrm{d}x + C\int_0^T\int_\Omega \left|\nabla u^{\frac{n+\alpha+1}{4}}\right|^4 + \left|D^2 u^{\frac{n+\alpha+1}{2}}\right|^2\,\mathrm{d}x\,\mathrm{d}t,$$
for $\alpha \in \left(\frac{1}{2}-n, 2-n\right) \setminus \{1, 0\}$.

Proved in [Bernis-Friedman, JDE 1990], [Beretta-Bertsch-Dal Passo, ARMA 1995].

N.B. For many purposed we need to restrict ourselves to $n \in (1,2)$ because for $n \in (2,3)$ only "backward" entropy estimates hold.

Conclusions. For $d \in \{1,2,3\}$, we can prove existence (but uniqueness is an open problem!) for two different notions of solution depending on the value of the parameter $n \in (1,3)$:

- for $1 < n < 2$ (*strong slippage* regime): *weak solutions*;
- for $2 \leq n < 3$ (*weak slippage* regime): *energy dissipating weak solutions*.

In both cases, the integral estimates enforce a *zero contact angle condition* $|\nabla u| = 0$ at the free boundary.

Localized versions of the entropy and energy inequalities are the base of most studies of the *qualitative properties of the thin-film equation*.

Finite speed of propagation: For each ball $\overline{B_{R_0}(x_0)}$, with $x_0 \in \mathbb{R}^d$ and $R_0 > 0$, that contains $\operatorname{supp} u_0$, a continuous, monotonically increasing function $R : [0,T) \to \mathbb{R}_0^+$, with $R(0) = 0$, exists such that, for all $t \in (0,T)$, we have

$$\operatorname{supp}(u(\cdot,t)) \subset \overline{B(x_0, R_0 + R(t))}.$$

Optimal upper bound on interface propagation rates [Grün, Interfaces Free Bound. 2002]. For $n \in (1,3)$, $t \in (0,T)$:

$$\operatorname{supp}(u(\cdot,t)) \subset B\left(0, R_0 + C(n,d)\|u_0\|_{L^1(\Omega)}^{\frac{n}{4+d\cdot n}} \cdot t^{\frac{4}{4+nd}}\right).$$

Optimal lower bound on interface propagation rates [Fischer, JDE 2013]. For $n \in \left(\frac{3}{2},3\right)$, $t \in (0,T)$:

$$B\left(0, C(n,d)\|u_0\|_{L^1(\Omega)}^{\frac{n}{4+d\cdot n}} \cdot t^{\frac{4}{4+nd}}\right) - \operatorname{diam}(\operatorname{supp}(u_0)) \subset \operatorname{supp}(u(\cdot,t)).$$

If the **initial data** u_0 are **flat enough** near some point x_0 of the initial free boundary, the interface will **locally remain stationary** (or at most move backward) for some time before it finally starts moving forward.

Waiting time: the amount of time that passes before the free boundary moves beyond its initial condition.

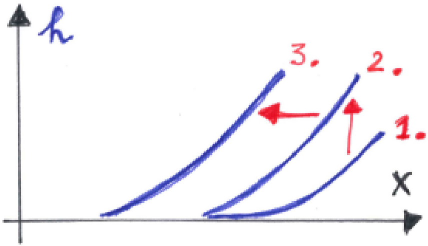


Figure: Illustration of the waiting time phenomenon (by Giacomelli-Knüpfer-Otto).

Consider the one-dimensional thin-film equation $\partial_t u = -\partial_x(u^n \partial^3_{xxx} u)$ in the regime $n \in (2,3)$ and with compactly supported nonnegative initial data $u_0 \in H^1(\mathbb{R})$; denote by x_0 the leftmost point in the support of u_0.

- **Instantaneous forward motion of the free boundary at** x_0 occurs *if and only if* u_0 grows faster than $(x-x_0)_+^{4/n}$ near the free boundary x_0 in the sense of "averages of the mass"

$$\limsup_{r\to 0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \,\mathrm{d}x = \infty. \tag{1}$$

- Hence a **waiting time phenomenon** occurs *if and only if*

$$\limsup_{r\to 0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \,\mathrm{d}x < \infty. \tag{2}$$

- The **optimal upper and lower bounds for waiting times** are both formulated in terms of the quantity

$$\sup_{r>0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \,\mathrm{d}x \tag{3}$$

and differ from each other only by a constant factor.

- [Dal Passo-Giacomelli-Grün, Ann. SNS 2001].

  Previously known condition for the **occurrence of a waiting time phenomenon** for the thin-film equation for $n \in [2,3)$:

$$\limsup_{r \to 0} r^{-4/n+1} \left(\fint_{(x_0, x_0+r)} |\nabla u_0|^2 \, \mathrm{d}x \right)^{1/2} < \infty, \tag{4}$$

- [Fischer, ARMA 2014 & AHP 2016].

  Previously known condition for **instantaneous forward motion of the free boundary** in solutions to the thin-film equation:

$$\limsup_{r \to 0} r^{-4/n} \left(\fint_{(x_0, x_0+r)} u_0^p \, \mathrm{d}x \right)^{1/p} = \infty \tag{5}$$

  for a certain $p \in (0,1)$, with typically $0 < p \le \frac{1}{2}$.

Occurrence of the waiting time phenomenon: the case of highly oscillatory initial data

$$u_0(x) := \left(2 + \sin \frac{1}{x - x_0}\right)(x - x_0)_+^{4/n}.$$

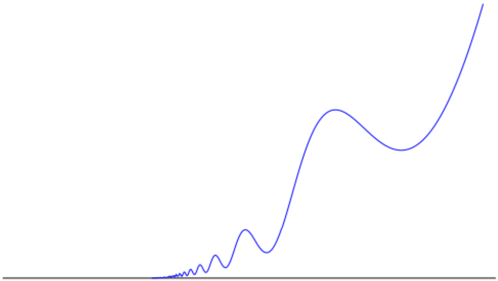


Figure: While the initial data u_0 are clearly bounded from above and from below by a multiple of $(x - x_0)^{4/n}$, due to the rapid oscillations near the free boundary the limit (4) is infinite. As a result, the previous sufficient criterion for waiting times from is not applicable. In contrast, **our sufficient condition shows that for this initial data indeed a waiting time phenomenon occurs**.

Instantaneous forward motion of the free boundary: the case of highly concentrated initial data

$$u_0(x) := (x - x_0)_+^{4/n} + (x - x_0)_+^{4/n-\delta} \cdot \sum_{k=2}^{\infty} k^2 \varphi\Big(k^2\Big(x - x_0 - \frac{1}{k}\Big)\Big),$$

where $\varphi : \mathbb{R} \to \mathbb{R}_0^+$ is a bump function and $\delta > 0$ is fixed.

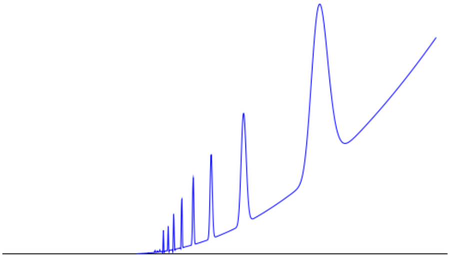


Figure: The initial data features infinitely many bumps accumulating at x_0. The bumps near a point $x > x_0$ have mass of order $(x - x_0)^{4/n-\delta}$ but width of order $|x - x_0|^2$. As a consequence of the mass estimate for the bumps, **our sufficient condition for instantaneous forward motion of the free boundary is applicable**. In contrast, the previous sufficient conditions for instantaneous forward motion from are not applicable for $\delta > 0$ small enough, as the increasingly strong concentration of the bumps cause the limit in (5) to be finite.

Rigorous definition of the waiting time phenomenon

- Let $u_0 \in L^1(\mathbb{R}^d)$ and $u \in L^\infty([0,T); L^1(\mathbb{R}^d))$. For any point $x_0 \in \mathbb{R}^d \setminus \operatorname{supp} u_0$ in the complement of the support of u_0, we define the *waiting time* T^* of u at x_0 as

$$T^* := \operatorname{essinf}\{t > 0 : x_0 \in \operatorname{supp} u(\cdot, t)\},$$

  where $\operatorname{supp} u(\cdot, t)$ is understood in the sense of support of a distribution.

  - In other words, for a point x_0 which lies outside of the support of the initial data, we define the waiting time T^* to be the first time at which the support of the solution u reaches x_0.

- For any point $x_0 \in \partial \operatorname{supp} u_0$ on the boundary of the initial support, we define the waiting time T^* of u at x_0 as

$$T^* := \operatorname{essinf}\{t > 0 : x_0 \notin \overline{\mathbb{R}^d \setminus \operatorname{supp} u(\cdot, t)}\}.$$

  - In other words, for a point x_0 on the initial free boundary $\partial \operatorname{supp} u_0$, we define the waiting time to be the first time at which x_0 is contained in the interior of the support of the solution u.

Occurrence of the waiting time phenomenon and estimate from below for the waiting time

Theorem 1

Let $u : \mathbb{R}^d \times [0, T) \to \mathbb{R}$ *be an energy-dissipating weak solution to* (TFE) *with zero contact angle and initial data* $u_0 \in L^1(\mathbb{R}^d)$.

Let $x_0 \in \partial \operatorname{supp} u_0 \bigcup (\mathbb{R}^d \setminus \operatorname{supp} u_0)$ *be a point on the boundary or outside of the support of the initial data.*

Suppose that there exists a constant $\kappa > 0$ *such that for all* $r > 0$ *the estimate*

$$\boxed{\int_{B_r(x_0)} u_0 \, \mathrm{d}x \leq \kappa r^{\frac{4}{n}}} \tag{6}$$

holds.

If $x_0 \in \partial \operatorname{supp} u_0$, *suppose furthermore that* $\operatorname{supp} u_0$ *satisfies an exterior cone condition at* x_0 *with some positive opening angle, i.e. either* $(x_0, x_0+\delta) \cap \operatorname{supp} u_0$ *or* $(x_0-\delta, x_0) \cap \operatorname{supp} u_0$ *is empty for some* $\delta > 0$ *small enough.*

Then u *has a positive waiting time* T^* *at* x_0 *and there exists a constant* c *depending only on* d, n, *and possibly* δ *such that the waiting time* T^* *is bounded from below by*

$$\boxed{T^* \geq c\kappa^{-n}.}$$

Sketch of the proof: Down-propagation of degeneracy argument

In the regime $n \in [2,3)$, we say that the solution u of (TFE) is *degenerate* on a parabolic cylinder $B_r(x_0) \times [0,T]$ if it satisfies

$$\sup_{t\in(0,T)} \fint_{B_r(x_0)} u \,\mathrm{d}x \le \varepsilon T^{-1/n} r^{4/n}; \tag{7a}$$

$$\sup_{t\in(0,T)} \fint_{B_r(x_0)} \frac{t^\beta}{T^\beta} |\nabla u|^2 \,\mathrm{d}x + \int_0^T \fint_{B_r(x_0)} \frac{t^\beta}{T^\beta} \left|\nabla u^{\frac{n+2}{6}}\right|^6 \mathrm{d}x\,\mathrm{d}t \le \varepsilon T^{-\frac{2}{n}} r^{\frac{8}{n-1}}. \tag{7b}$$

for some appropriately chosen $\varepsilon = \varepsilon(d,n) > 0$ and $\beta \in (0,1)$.

Provided that the initial data also satisfy a degeneracy condition of the type $\limsup_{r\to 0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \,\mathrm{d}x < \infty$, the degeneracy of u on a parabolic cylinder $B_r(x_0) \times [0,T]$ implies the degeneracy of u on the spatially smaller parabolic cylinder $B_{r/2}(x_0) \times [0,T]$ with the same time horizon T.

Propagating the degeneracy down to $r \to 0$, this essentially shows $u(x_0,t) = 0$ for $t \le T$.

To propagate the degeneracy, we need to iterate back and forth between a **localized mass estimate** and a **localized time-weighted energy estimate**.

- **Propagation of the first degeneracy.**

Starting with degenerate initial data u_0, after choosing T appropriately, the degeneracy properties (7a) and (7b) on a spatially larger parabolic cylinder ensure that the **influx of mass into the smaller ball** $B_{r/2}(x_0)$ remains sufficiently limited up to time T.

- **Propagation of the second degeneracy.**

To propagate the second degeneracy condition (7b) we need to control the **influx of energy into the smaller ball** $B_{r/2}(x_0)$ suitably. We rely on the regularization properties of the nonlinear fourth-order parabolic operator. Heuristically, our approach is close in spirit to the consideration

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2\,\mathrm{d}x \le -c\int_\Omega |\nabla u^{\frac{n+2}{6}}|^6\,\mathrm{d}x \le -c(\Omega)\Big(\int_\Omega u\,\mathrm{d}x\Big)^{n-4}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^3.$$

This estimate implies by an elementary ODE argument a bound of the form

$$\int_\Omega |\nabla u(\cdot,t)|^2\,\mathrm{d}x \le C(\Omega)t^{-1/2}\Big(\sup_{s\in[0,t]}\int_\Omega u(\cdot,s)\,\mathrm{d}x\Big)^{2-n/2},$$

which is now independent of $\int_\Omega |\nabla u_0|^2\,\mathrm{d}x$, but blows up for $t\to 0$.
The blowup near initial time is the reason for the factor t^β in our condition (7b).

Instant forward motion of the interface and estimate from above for the waiting time

Theorem 2

Let $u : \mathbb{R}^d \times [0, T) \to \mathbb{R}$ *be an energy-dissipating weak solution to* (TFE) *with zero contact angle and initial data* $u_0 \in L^1(\mathbb{R}^d)$*.*

Let $x_0 \in \partial \operatorname{supp} u_0 \bigcup (\mathbb{R}^d \setminus \operatorname{supp} u_0)$ *be a point on the boundary or outside of the support of the initial data.*

Then there exists a constant C *depending only on* n *such that the waiting time* T^* *of* u *at* x_0 *is bounded from above by*

$$\boxed{T^* \leq C \left(\sup_{r>0} r^{-\frac{4}{n}} \fint_{(x_0-r, x_0+r)} u_0 \, \mathrm{d}x \right)^{-n}.}$$

In particular, if the initial data u_0 *satisfy*

$$\limsup_{r \to 0} r^{-\frac{4}{n}} \fint_{(x_0-r, x_0+r)} u_0 \, \mathrm{d}x = \infty$$

at a point on the initial free boundary $x_0 \in \partial \operatorname{supp} u_0$*, the free boundary starts moving forward immediately at* x_0*, without waiting time.*

Sketch of the proof: Monotonicity formula and differential inequality argument

Step 1. *Almost optimal estimate* [Fischer, ARMA 2014 & AHP 2016].

Monotonicity formula. Weighted entropy inequality:

$$\partial_t \int_{\mathbb{R}} u^{1+\alpha} |x - x_0|^{\gamma} \, \mathrm{d}x \geq c \int_{\mathbb{R}} u^{1+\alpha+n} |x - x_0|^{\gamma - 4} + |\nabla u^{\frac{1+\alpha+n}{4}}|^4 |x - x_0|^{\gamma} \, \mathrm{d}x$$

for suitable $-1 < \alpha < 0$ and suitable $\gamma < -1$, as long as the support of the solution $u(\cdot, t)$ does not touch the singularity of the weight at x_0.

Differential inequality argument [Chipot-Sideris, Trans. AMS 1985]. Using Hölder's inequality and assuming that the support of u remains to the right of x_0, one obtains from the monotonicity formula applied with $x_0 - \delta$ in place of x_0

$$\partial_t \int_{\mathbb{R}} u^{1+\alpha} |x - x_0 + \delta|^{\gamma} \, \mathrm{d}x \geq c \delta^{-\frac{(\gamma+1)n}{(1+\alpha)} - 4} \left(\int_{\mathbb{R}} u^{1+\alpha} |x - x_0 + \delta|^{\gamma} \, \mathrm{d}x \right)^{\frac{1+\alpha+n}{1+\alpha}} .$$

This implies finite-time blowup of $\int_{\mathbb{R}} u^{1+\alpha}(\cdot,t)|x-x_0+\delta|^\gamma \, \mathrm{d}x$ and thereby a contradiction to the assumption that the support of $u(\cdot,T)$ remains to the right of x_0 as soon as

$$T \geq C\delta^{\frac{(1+\gamma)n}{(1+\alpha)}+4}\left(\int_{\mathbb{R}} u_0^{1+\alpha}|x-x_0|^\gamma \, \mathrm{d}x\right)^{-\frac{n}{(1+\alpha)}},$$

so, in particular, as soon as

$$T \geq C\left(\delta^{-4(1+\alpha)/n} \fint_{(x_0,x_0+\delta)} u_0^{1+\alpha} \, \mathrm{d}x\right)^{-n/(1+\alpha)}.$$

Step 2. *Improvement: estimates in terms of mass.*

Remark. For "concentrated" initial data, the integral on the right-hand side of the previous formula is much smaller than suggested by the relation

$$\fint_{(x_0,x_0+\delta)} u_0^{1+\alpha} \, \mathrm{d}x \sim \left(\fint_{(x_0,x_0+\delta)} u_0 \, \mathrm{d}x\right)^{1+\alpha}$$

which would be valid for initial data like $u_0(x) \sim (x-x_0)_+^\beta$.

Idea to prove the sharp lower bound in terms of mass. Combine the previous almost optimal estimates with a new estimate connecting motion of mass to entropy production.

Remarks on the cases $d \in \{2,3\}$ and $n \in (1,2)$

- Our sufficient condition for a waiting time (1) is not limited to the regime $n \in (2,3)$, but holds also for $n \in (1,2)$ – we need to use a *localized entropy estimate*.
- In higher dimension, the estimate from below for the waiting time is the same; the estimate from above is weaker, more subtle and technically involved.
- The stationary state $u(x,t) = (x - x_0)_+^2$ shows that in the regime $n < 2$ one cannot expect a condition like (2) to be sufficient for instantaneous forward motion of the free boundary, as $(x - x_0)_+^2$ grows steeper than $(x - x_0)_+^{4/n}$ in this regime.
- The constructions in [Fischer, AHP 2016] show that our condition (1) is in fact sharp among all conditions formulated in terms of the growth of the initial data at the free boundary: It is shown that there exist initial data with only slightly steeper growth than $(x - x_0)_+^{4/n}$ for which instantaneous forward motion occurs.

Some future perspectives

- *Qualitative properties* for the *nonlocal* thin-film equation (modelling hydraulic fractures):

$$\partial_t u + \partial_x \left(u^n \partial_x (-\Delta)^s u\right) = 0.$$

  Existence of non-negative solutions: [Imbert-Mellet, Nonlinearity 2011], [Tarhini, AHP 2015].

  Asymptotic profile: [Imbert-Mellet, Comm. Math. Phys. 2015], [Seratti-Vázquez, preprint 2019].

  Finite vs. infinite speed of propagation: open problem!

  Non-negativity preserving *numerical schemes*: work in progress.

- *Singular limit*: $\partial_t u + \partial_x u^n + \varepsilon \partial_x (u^n \partial^3_{xxx} u) = 0$ as $\varepsilon \to 0^+$.

  The $n \in (1,2)$ case: convergence to *entropy solution* of the scalar conservation law for [Otto-Westdickenberg, J. Hyp. Diff. Eq. 2005] via *compensated compactness* and *minimal entropy condition*.

  The $n \in (2,3)$ case: open problem!

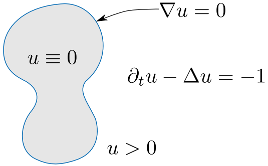
- *Control for free boundary problems*.

  Perturbed 1-D porous medium & thin-film equation: [Geshkovski, preprint 2019] – control the solution and its interface to those of the Barenblatt self-similar solution.

  Key ideas: Lagrangian-like change of variables (von-Mises transform) to fix the moving domain; controllability to Barenblatt in moving domain $\iff$ controllability to zero in fixed domain.

  A pivotal role is played by the explicit kinetic condition at the interface (as in Stefan problem or fluid-structure interaction problems).

- *Controllability* for the (local and nonlocal) parabolic obstacle problem: work in progress.



Presentation Outline

1. Sharp criteria for the waiting time phenomenon in solutions to the thin-film equation

2. Differentiability properties of the flow associated with a nearly incompressible BV vector field

Lagrangian form.
Given the Borel velocity field $\boldsymbol{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, with $T > 0$, we consider the following Cauchy initial-value problem:

$$\begin{cases} \partial_t \boldsymbol{X}(t,s,x) = \boldsymbol{b}(t, \boldsymbol{X}(t,s,x)), & (t,x) \in (0,T) \times \mathbb{R}^d, \\ \boldsymbol{X}(s,s,x) = x, & x \in \mathbb{R}^d, \end{cases} \tag{8}$$

for $s \in [0,T)$. The solution $\boldsymbol{X} : [0,T]^2 \times \mathbb{R}^d \to \mathbb{R}^d$ is called *flow associated with the velocity field* $\boldsymbol{b}$. In case $s = 0$, we use the simplified notation $\boldsymbol{X}(t,x) := \boldsymbol{X}(t,0,x)$.

Eulerian form.

- *transport equation in conservative form*:

$$\begin{cases} \partial_t \mu_t(x) + \operatorname{div}_x(\boldsymbol{b}(t,x)\, \mu_t(x)) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ \mu_0(x) = \overline{\mu}(x), & x \in \mathbb{R}^d, \end{cases} \tag{9}$$

  where $t \in [0,T] \mapsto \mu_t$ is a Borel measure valued function and $\overline{\mu}$ is a given Borel measure on $\mathbb{R}^d$;

- *transport equation in advective form*

$$\begin{cases} \partial_t u(t,x) + \boldsymbol{b}(t,x) \cdot \nabla_x u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = \overline{u}(x), & x \in \mathbb{R}^d, \end{cases} \tag{10}$$

  where $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a scalar function and $\bar{u} : \mathbb{R}^d \to R$ is a given initial datum.

- The PDEs represent the **Eulerian** specification of the flow – i.e., the description of the dynamics at a fixed location and time; on the other hand, the ODE is the **Lagrangian** specification – i.e. traces single particles through space and time.

- The two specifications are in fact **equivalent**. The classical **Cauchy-Lipschitz theory** provides existence and uniqueness for the ODE if the velocity field $\boldsymbol{b}$ is regular enough (Lipschitz continuous in the space variable uniformly with respect to time) and the so-called **method of characteristics** establishes the connection between the Eulerian problems and the Lagrangian problem:

$$\mu_t = \boldsymbol{X}(t,\cdot)_\sharp \bar{\mu}. \tag{11}$$

- The flow map and its inverse inherit the Lipschitz regularity of the vector field:

$$e^{-tL}|x-y| \le |\boldsymbol{X}(t,x) - \boldsymbol{X}(t,y)| \le e^{tL}|x-y| \text{ for all } x,y \in \mathbb{R}^d, t \in [0,T],$$

where L is the Lipschitz constant of $\boldsymbol{b}$. This implies, by the *Lagrangian identity* (11), that the Lipschitz seminorm of the solution u of (9) with initial data $\bar{u} \in W^{1,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies

$$\|\nabla u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le e^{tL} \|\nabla \bar{u}\|_{L^\infty(\mathbb{R}^d)} \text{ for all } t \in [0,T].$$

Motivation for transport equations with rough velocity fields

- The **Keyfitz-Kranzer** system of conservation laws

$$\partial_t u + \sum_i \partial_{x_i}(f_i(|u|)u) = 0, \quad u \in \mathbb{R}^m,$$

can be written as

$$\begin{cases} \partial_t |u| + \operatorname{div}\left(\boldsymbol{f}(|u|)|u|\right) = 0, \\ \partial_t \theta + \boldsymbol{f}(|u|) \cdot \nabla\theta = 0, \end{cases}$$

where $\theta = u/|u|$.
The theory of Kružkov yields for scalar conservation laws that $\nabla|u|$ is a bounded measure. Thus the problems reduces to the study of a transport equation with a vector field whose derivative is a measure.

- 2D incompressible **Euler equation** in the vorticity formulation:

$$\begin{cases} \partial_t \omega_t + \nabla_x \cdot (\boldsymbol{v}_t \omega_t) = 0, \\ \nabla_x \cdot \boldsymbol{v}_t = 0, \\ \omega_0 = \bar{\omega}, \end{cases}$$

where $\omega_t = \operatorname{curl} \boldsymbol{v}_t$ and $\bar{\omega} \in L^p \implies \omega_t \in L^p \implies \boldsymbol{v}_t \in W^{1,p}$.

Definition 3 (Regular Lagrangian Flow)

Let $\boldsymbol{b}(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable vector field such that $|\boldsymbol{b}(t,x)| \leq \alpha(t)$ for some nonnegative function $\alpha \in L^1_{loc}(\mathbb{R})$. We say that a map $\boldsymbol{X} : [0,T]^2 \times \mathbb{R}^d \to \mathbb{R}^d$ is a *regular Lagrangian flow* associated to the vector field $\boldsymbol{b}$ if the following properties hold true:

1. For $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ the map $\boldsymbol{X}(\cdot, s, x)$ is an absolutely continuous solution of the ordinary differential equation

$$\begin{cases} \partial_t \boldsymbol{X}(t,s,x) = \boldsymbol{b}(t, \boldsymbol{X}(t,s,x)), \\ \boldsymbol{X}(s,s,x) = x, \end{cases} \tag{12}$$

   i.e. it satisfies

$$\boldsymbol{X}(t,s,x) = x + \int_0^t \boldsymbol{b}(\tau, \boldsymbol{X}(\tau,s,x))\,\mathrm{d}\tau. \tag{13}$$

2. *Incompressibility*: For $(t,s) \in [0,T]^2$ we have

$$C^{-1}\mathscr{L}^d \leq \boldsymbol{X}(t,s,\cdot)_\sharp \mathscr{L}^d \leq C\mathscr{L}^d.$$

Key assumptions on the rough vector fields:

- some integrability;
- one full-derivative (in some weak sense);
- some incompressibility condition: $\operatorname{div} \boldsymbol{b} \in L^\infty$.

Definition 4 (Nearly incompressible vector fields)

We say that a bounded vector field b is *nearly incompressible* if there exist a function ρ and a constant $C > 0$ such that

$$0 < \frac{1}{C} \leq \rho(t,x) \leq C < +\infty \qquad \text{for } \mathscr{L}^{d+1}\text{-a.e. } (t,x) \in (0,T) \times \mathbb{R}^d \tag{14}$$

and

$$\partial_t \rho + \operatorname{div}(\boldsymbol{b}\rho) = 0\,. \tag{15}$$

- **Bressan's Compactness Conjecture.** The flow associated with a nearly incompressible BV vector field is unique.
- **Bressan's Mixing Conjecture.** The flow associated with a nearly incompressible BV vector field is Lusin-Lipschitz.

Theorem 5 (Ambrosio's superposition principle)

Let $\mu_t = w_t \mathscr{L}^d$ *be a measure valued solution of* (9), *with* $w_t \geq 0$, $\int_{\mathbb{R}^d} w_t \,\mathrm{d}x = 1$ *and*

$$\int_0^T \int_{\mathbb{R}^d} |\boldsymbol{b}_t| w_t \,\mathrm{d}x \,\mathrm{d}t < \infty.$$

Then there exists a probability measure $\boldsymbol{\eta}$ *in* $C([0,T];\mathbb{R}^d)$ *concentrated on absolutely continuous solutions of the ODE such that*

$$(e_t)_\sharp = w_t \mathscr{L}^d$$

for all $t \in [0,T]$, *where* $e_t : C([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ *is the evaluation map at time* $t \in [0,T]$. *In particular, if* $\|w_t\|_{L^\infty(\mathbb{R}^d)} \in L^\infty(0,T)$, $\boldsymbol{\eta}$ *satisfies the following non-concentration condition for any non-negative Borel function* φ*:*

$$\int_{C([0,T];\mathbb{R}^d)} \varphi(\gamma(t)) \,\mathrm{d}\boldsymbol{\eta}(\gamma) \leq C \int_{\mathbb{R}^d} \varphi(y) \,\mathrm{d}y.$$

Theorem 6 (Ambrosio's non-splitting criterion)

Let $\boldsymbol{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *such that*

1. $\frac{|\boldsymbol{b}|}{1+|x|} \in L^1\left([0,T]; L^1(\mathbb{R}^d)\right) + L^1\left([0,T]; L^\infty(\mathbb{R}^d)\right)$;
2. $\boldsymbol{b}$ *is nearly incompressible.*

Assume that the PDE (9) *has the uniqueness property in* $L^\infty([0,T]\times\mathbb{R}^d)$. *Then any regular generalized flow* $\boldsymbol{\eta}_x$ *is deterministic, i.e. a Dirac mass for* $\mathscr{L}^d$*-a.e.* $x \in \mathbb{R}^d$. *More specifically, consider a family* $\{\boldsymbol{\eta}_x\}_{x\in\mathbb{R}^d} \subset \mathcal{P}(\Gamma_T)$ *such that* $\boldsymbol{\eta}_x$ *is concentrated on absolutely continuous integral solutions of*

$$\begin{cases} \dot\gamma(t) = \boldsymbol{b}(t,\gamma(t)), & t \in (0,T), \\ \gamma(0) = x \end{cases}$$

for $\mathscr{L}^d$*-a.e.* $x \in \mathbb{R}^d$. *Assume that the superposition solution of the continuity equation* $\mu_t^{\eta_x}$ *induced by this family belongs to* $L^\infty([0,T]\times\mathbb{R}^d)$. *Then* η_x *is a Dirac mass for* $\mathscr{L}^d$*-a.e.* $x \in \mathbb{R}^d$.

In conclusion, the key step is to prove **uniqueness at the PDE level**.

Notion of renormalized solutions. For every $\beta \in C^1(\mathbb{R};\mathbb{R})$ there holds

$$\partial_t(\beta(u)) + \boldsymbol{b}\nabla\beta(u) = 0\,.$$

This is a sort of weak "Chain Rule" for the function u, saying that u is differentiable along the flow generated by $\boldsymbol{b}$.

Key step in the proof of the renormalization property. Regularization by convolution with standard mollifier φ^ε and commutator estimate: If $u^\varepsilon := u * \varphi^\varepsilon$, then

$$\partial_t\beta(u^\varepsilon) + \boldsymbol{b}\cdot\nabla\beta(u^\varepsilon) = T^\varepsilon,$$

where the *commutator* T^ε satisfies

$$T^\varepsilon = \boldsymbol{b}\cdot\nabla u^\varepsilon - (\boldsymbol{b}\cdot\nabla u)^\varepsilon \to 0 \text{ (strongly)}.$$

- [DiPerna-Lions, Inv. Math. 1989]: Strong convergence is obtained by straightforward manipulations if $\boldsymbol{b} \in W^{1,p}$ with $\operatorname{div}\boldsymbol{b} \in L^\infty$.
- [Ambrosio, Inv. Math. 2003]: If $\boldsymbol{b} \in \mathrm{BV}$ with $\operatorname{div}\boldsymbol{b}$ a.c. and with bounded density, we need to choose carefully an *anisotropic* convolution kernel according to Alberti rank-one theorem.

Local structure of the singular part of BV-derivatives: Alberti's rank-one theorem

Theorem 7 (Alberti's rank-one theorem)

Let $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$. *Then*

$$\operatorname{rank}\left(\frac{\mathrm{d}D^{\mathrm{sing}}u}{\mathrm{d}|D^{\mathrm{sing}}u|}(x)\right) = 1 \quad \textit{for } |D^{\mathrm{sing}}|\textit{-a.e. } x \in \Omega.$$

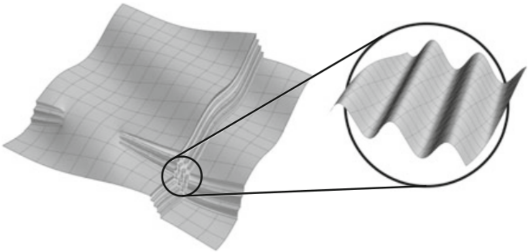


Figure: The blow-up of a BV map at a singular point. Cf. [Rindler, Springer 2008. Figure 10.2, p. 283]. Roughly speaking, locally all singularities of a BV function must be one-dimensional.

Weak differentiability result of Le Bris-Lions

[Le Bris-Lions, Annali di Matematica 2004]: The regular Lagrangian flow associated with a $L^1([0,T]; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ vector field is **differentiable in measure**, i.e. there exists $W:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ such that

$$\lim_{\delta\to 0}\frac{f(x+\delta y)-f(x)-\delta W(x,y)}{\delta}=0$$

locally in measure in $\mathbb{R}^d\times\mathbb{R}^d$ w.r.t. $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, which means that for any $\varepsilon>0$ and any $B_R(x)$, we have

$$\lim_{\delta\to 0}\mathscr{L}^{2d}\left(\left\{(x,y)\in B_R(x)\times\mathbb{R}^d : \frac{f(x+\delta y)-f(x)-\delta W(x,y)}{\delta}>\varepsilon\right\}\right)=0$$

or, equivalently,

$$\boxed{\lim_{\delta\to 0}\int_{\mathbb{R}^d}\int_{B_R(x)}\min\left\{1,\frac{|f(x+\delta y)-f(x)-\delta W(x,y)|}{\delta}\right\}\mathrm{d}x\,\mathrm{d}y=0.}$$

Weak differentiability result of Le Bris-Lions

Key idea: The difference quotients

$$\left(\boldsymbol{X}(t,x), \frac{\boldsymbol{X}(t,x+hy)-\boldsymbol{X}(t,x)}{h}\right)$$

are characteristic curves of the velocity field

$$\boldsymbol{B}_h(t,x,y) = \left(\boldsymbol{b}(t,x), \frac{\boldsymbol{b}(t,x+hy)-\boldsymbol{b}(t,x)}{h}\right).$$

Therefore, a suitable extension of the theory of renormalized solutions (comprehensive of a stability result) for the limit field $\boldsymbol{B}(t,x,y) := (\boldsymbol{b}(t,x), \nabla_x \boldsymbol{b}(t,x)y)$ provides the result.

The arguments of the DiPerna-Lions theory are quite indirect and they exploit (via the theory of characteristcs) the connection between the ODE and the Cauchy problem for the transport equation.

The work [Crippa-De Lellis, Reine Angew. Math. 2008] exploits some quantitative a priori estimates on the flow and provides stronger regularity results.

Key idea: a priori estimates for a functional measuring a "logarithmic distance" between two flows associated to the same vector field ($W^{1,p}$, $p > 1$):

$$\Phi_\delta(t) = \int_{B_R} \log\left(1 + \frac{|\boldsymbol{X}_1(t,x) - \boldsymbol{X}_2(t,x)|}{\delta}\right) \mathrm{d}x.$$

Consequences:

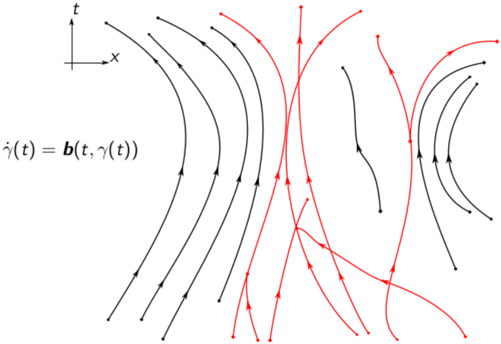
- Uniqueness of the regular Lagrangian flow;
- Lusin-Lipschitz regularity:

$$\mathrm{Lip}(\boldsymbol{X}_t|_{K_\varepsilon}) \le \exp\left\{\frac{C\left(1 + \int_0^T \|\nabla b_t\|_{L^p(B_{\tilde{R}})} \mathrm{d}t\right)}{\varepsilon^{1/p}}\right\}.$$

Remark: This approach fails to yield such strong results in the $p = 1$ case – issue: failure of certain maximal function estimates.

Proof of Bressan's first conjecture: Uniqueness of the flow associated with a nearly incompressible BV vector field [Bianchini-Bonicatto, Inv. Math. 2019].

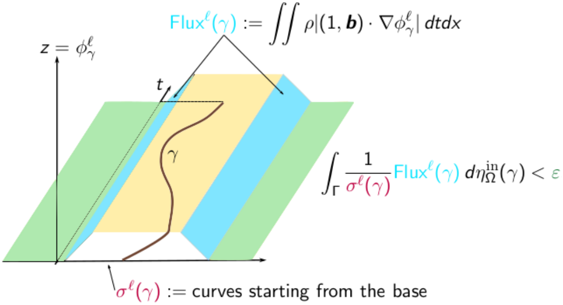
Key observation: The loss of uniqueness is essentially due to the presence of *crossings* between the curves used in the Lagrangian representation given by the superposition principle.



We thus have to look for conditions that rule out crossings.

Bressan's first conjecture: the untangling of trajectories approach of Bianchini-Bonicatto

- We restrict ourselves in suitable domains $\Omega \subset \mathbb{R}^{d+1}$, called *proper sets*.
- We construct *local ε-cylinders of approximate flow*, i.e. Lipschitz continuous functions ϕ^ℓ_γ, with $\ell > 0, \gamma \in \Gamma$ as follows:



Roughly speaking, cylinders of approximate flow detect in the limit the presence of crossings between trajectories.

- Then a subtle *local-to-global argument* is needed.

Bressan's first conjecture: the untangling of trajectories approach of Bianchini-Bonicatto

For a nearly incompressible BV vector field it is possible to perform construction of suitable cylinders of approximate flow (satisfying a more refined estimate than the one above), which implies uniqueness.

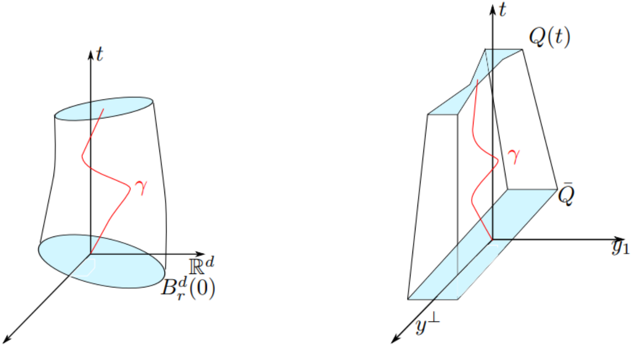


Figure: Approximate cylinders for the a.c. part (left) and approximate cylinders for the singular part (right), 2D case.

Towards Bressan's second conjecture: differentiability in measure of the flow associated with a nearly incompressible BV vector field

As a case study, we shall present only the proof in the case of a (measure-preserving) flow:

$$\mathrm{div}_x \boldsymbol{b} = 0.$$

Key observation: If $\boldsymbol{b}$ is smooth, the following identity holds

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\nabla \boldsymbol{X}(t,x) = \nabla \boldsymbol{b}(t, \boldsymbol{X}(t,x))\,\nabla \boldsymbol{X}(t,x).}$$

Under the weaker assumption $\boldsymbol{b} \in L^1([0,T]; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$, we can still study the ODE

$$\boxed{\begin{cases} \dot{W}(t,y) = (D\boldsymbol{b})_y(dt)W(t-,y), & t \in [0,T], \\ W(0-,y) = \mathbb{I}, \end{cases}}$$

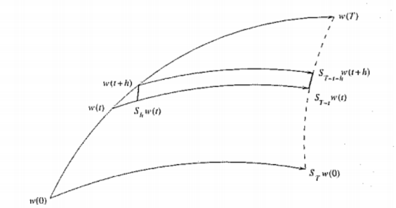
where $(D\boldsymbol{b})_y$ be the *disintegration of $D\boldsymbol{b}$ along the trajectories $X(t,y)$*.

For every partition $0 < t_1 < \cdots < t_i < \cdots < T$ of the interval $[0,T)$, such that $\delta t = \max_i\{t_i - t_{i-1}\}$, we can construct the approximate solution

$$\boxed{W^{\delta t}(t,y) = \prod_{t_i \le t} (\mathbb{I} + (D\boldsymbol{b})_y([t_{i-1}, t_i))).}$$

Towards Bressan's second conjecture: differentiability in measure of the flow associated with a nearly incompressible BV vector field

- We restrict ourselves in suitable *proper sets* $\Omega \subset \mathbb{R}^{d+1}$;
- We construct suitable cylinders of approximate flow using $W(t, y)$.
  - For the points in the a.c. part of $D\boldsymbol{b}$, we compare the flow with the solution of the ODE using a lemma by Bressan – given an absolutely continuous curve γ and an L-Lipschitz continuous flow, we can estimate the distance between γ and the trajectory of the flow starting at $\gamma(0)$.



  - For the points in the singular part of $D\boldsymbol{b}$, we need intermediate estimates involving $W^{\delta t}(t, y)$. Roughly speaking, we are able to prove a local estimate (in *proper sets*) of the following kind:

$$\limsup_{r \to 0} \int \fint_{B_r^d(0)} \min\left\{1, \frac{|\boldsymbol{X}(t, y+z) - \boldsymbol{X}(t, y) - (\mathbb{I} + (D\boldsymbol{b})_y(0, t))z|}{r}\right\} \mathrm{d}y\, \mathrm{d}z \leq \mathcal{O}(1)\varepsilon \|D\boldsymbol{b}\|.$$

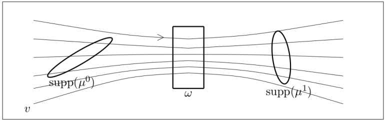
- With the local estimate, we can prove differentiability in measure for the flow by a subtle *local-to-global* covering argument.

- **Bressan's mixing conjecture** remains an open problem.
- **Lagrangian approach to conservation laws** [Bianchini et al., 2015-2019]: wavefronts trajectories generate a flow $\boldsymbol{X}(t, y)$ which is Lipschitz in t and monotone in y.
- **Waiting time phenomenon for spatially discretized (Lagrangian discretization) porous medium and thin film equations** [Fischer-Matthes, preprint 2019]: consider the PME and TFE as *nonlinear transport equations* with a velocity $V(u)$.
- **Controllability of the transport equation with a localized vector field.** [Duprez-Morancey-Rossi, SIAM J. Control Optim. 2019]:

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ \mu(0,x) = \mu^0(x), & x \in \mathbb{R}^d. \end{cases}$$

Approximate controllability from μ^0 to μ^1 on the time interval $(0,T)$: for each $\varepsilon > 0$ there exists $\mathbb{1}_\omega u$ such that the corresponding solutions satisfy $W_p(\mu(T), \mu^1) \leqslant \varepsilon$.

Geometric condition: the uncontrolled vector field v needs to send the support of μ^0 to ω forward in time and the support of μ^1 to ω backward in time.



Work in progress: controllability of transport equation with *nonlocal velocity* $v[\mu]$.

Thank you for your attention.

Nicola De Nitti

Department of Mathematics, Università degli Studi di Bari Aldo Moro

Email address: n.denitti@studenti.uniba.it