Anderson localization inhibited by topology

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Phase-Averaged Transport for Quasi-Periodic Hamiltonians

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Abstract: For a class of discrete quasi-periodic Schrödinger operators defined by covariant representations of the rotation algebra, a lower bound on phase-averaged transport in terms of the multifractal dimensions of the density of states is proven. This result is established under a Diophantine condition on the incommensuration parameter. The relevant class of operators is distinguished by invariance with respect to symmetry automorphisms of the rotation algebra. It includes the critical Harper (almost-Mathieu) operator. As a by-product, a new solution of the frame problem associated with Weyl–Heisenberg–Gabor lattices of coherent states is given.
Random Dirac operator with time-reversal symmetry

Hamiltonian on $L^2(\mathbb{R}) \otimes \mathbb{C}^{2N}$

$$H = I \partial_x + V$$

$$I = \begin{pmatrix} 0 & -1_N \\ 1_N & 0 \end{pmatrix}$$

with random $2N \times 2N$ matrix potential with TRS

$$V = V^* = I^* \overline{V} I = \sum_n V_n \delta_n$$

implies

$$I^* H I = H$$

**Hypothesis**: distribution of i.i.d. $V_n$’s absolutely continuous

**Theorem** (with Sadel, 2010) $\mathbb{Z}_2$ dichotomy (in $N \mod 2$):

- $N$ odd $\implies$ almost surely pure a.c. spectrum of multiplicity 2
- $N$ even $\implies$ no a.c. spectrum (Only pure point?)
Physical interpretation

• for odd $N$ no Anderson localization, even though quasi-one-dimensional random model

• Exactly 1 double channels survives (left and right mover) others ”dissolve”

• Why should one care about a.c. spectrum?

  **Guarneri bound** in $d = 1$ implies ballistic transport

• Anderson localization for even number of channels $N$

• Is this of physical relevance for anything?

Effective model for edge states in spin quantum Hall systems
Why is the theorem true?

Solve Schrödinger at energy $E \in \mathbb{R}$ using transfer matrices

$$T^E(n, n-1) = e^{iV_n} e^{\partial_x - EI}$$

Lies in the group

$$SO^*(2N) = \{ T \in GL(2N, \mathbb{C}) \mid T^* I T = I, \ I^* \overline{T} I = T \}$$

For such $T$ one has Kramers’ degeneracy:

$$T^* T v = \lambda v \quad \Rightarrow \quad T^* T \overline{v} = \lambda \overline{v}$$

Implies double degeneracy of Lyapunov spectrum $\gamma_n \geq \gamma_{n+1}$

Moreover, usual symmetry $\gamma_n = -\gamma_{2N-n}$

Together for $N$ odd: $\gamma_N = \gamma_{N+1} = 0$ open channel
Now the work starts (for a mathematician):

- Show that all other Lyapunov exponents are non-vanishing
  Apply Goldsheid-Margulis theory for to the group $\text{SO}^*(2N)$
  For even $N$ there are no vanishing Lyapunov exponents
- Adapt Kotani-Simon (magical) theory for ergodic Dirac operator
  mult. of a.c. spectrum $= \#$ of vanishing Lyapunov exponents
  Proves existence of a.c. spectrum
- Almost sure absence of singular spectrum
  Adapt Jaksic-Last theory (purity of a.c. spectrum in Anderson)
Is all this tightly linked to the group $\mathbb{SO}^*(2N)$?

Transfer matrices in $\mathbb{SO}^*(2N)$ for $H$ in CAZ All (odd TRS)

- 2 vanishing $\gamma$’s in groups $\mathbb{O}(N, \mathbb{C})$ with $N$ odd ($H$ Class DIII)
- $|N - M|$ vanishing in $\mathbb{U}(N, M)$, $\mathbb{O}(N, M)$, $\mathbb{SP}(N, M)$
  
  Corresponds to Hamiltonians of CAZ classes A, D and C

Effective model for edge states in QHE on $L^2(\mathbb{R}) \otimes \mathbb{C}^{N+M}$

$$H = J i \partial_x + V \quad J = \begin{pmatrix} 1_N & 0 \\ 0 & -1_M \end{pmatrix}$$

Random matrix potential $V = V^* = \sum_n V_n \delta_n$ with coupling hyp.

Then transfer matrices in $\mathbb{U}(N, M)$

**Theorem** (with Ludwig, Stolz 2013)

Almost surely pure a.c. spectrum of multiplicity $|N - M|$
Anderson localization inhibited by topology

Quantum spin Hall system (odd TRS, Class AII)

Disordered Kane-Mele model on hexagon lattice and with \( s = \frac{1}{2} \)

\[
H = \Delta_{\text{hexagon}} + H_{\text{SO}} + H_{\text{Ra}} + \lambda_{\text{dis}} V
\]

Pseudo-gap at Dirac point opens non-trivially due to

\[
H_{\text{SO}} = i \lambda_{\text{SO}} \sum_{i=1,2,3} (S_{i}^{nn} - (S_{i}^{nn})^*) s^z
\]

No \( s^z \)-conservation due to Rashba term \( H_{\text{Ra}} \), but odd TRS

\[
H = I^* \bar{H} I \quad I = e^{i\pi s^y}
\]

Non-trivial topology:

Kane-Mele (2005): \( \mathbb{Z}_2 \) invariant for periodic system from Pfaffians
Haldane et al. (2005): spin Chern numbers for \( s^z \) invariant systems
Prodan (2009): spin Chern number from \( P_s = \chi(|P s^z P - \frac{1}{2}| < \frac{1}{2}) \)
with Avila, Villegas (2012): \( \mathbb{Z}_2 \) invariant for edge states

Here: \( \mathbb{Z}_2 \) invariant for disordered system as index of Fredholm
**$\mathbb{Z}_2$ index for odd TRS and $d = 2**$

**QHE:** $P = \chi(H \leq \mu)$ Fermi projection and $F = \frac{X_1 + iX_2}{|X_1 + iX_2|} = F^t$

Then: $T = PFP$ Fredholm operator, namely $\dim(\ker(T^*)) < \infty$

**And:** Hall conductance $= \text{Ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*))$

**Here:** $I^*\overline{H}I = H = I^*H^tI$ with $H^t = (\overline{H})^* \implies I^*P^tI = P$

**Definition** $T$ odd symmetric $\iff I^*T^tI = T$ with $I^2 = -1$

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**Theorem (Atiyah-Singer 1969, S-B 2013)**

$F_2(\mathcal{H}) = \{\text{odd symmetric Fredholm operators}\}$ has 2 connected components labelled by compactly stable homotopy invariant $\text{Ind}_2(T) = \dim(\ker(T)) \mod 2 \in \mathbb{Z}_2$

**Application:** $\mathbb{Z}_2$ phase label for Kane-Mele model if dyn. localized
Proof via Kramers degeneracy:

First of all: $\text{Ind}(T) = 0$ because $\text{Ker}(T^*) = I \overline{\text{Ker}(T)}$

**Idea:** $\text{Ker}(T) = \text{Ker}(T^* T)$

and positive eigenvalues of $T^* T$ have even multiplicity

Let $T^* T v = \lambda v$ and $w = I \overline{T v}$ (N.B. $\lambda \neq 0$). Then

$$T^* T w = I (I^* T^* I) (I^* TI) \overline{T v}$$

$$= I \overline{T T^* T v} = \lambda I \overline{T v} = \lambda w.$$ 

Suppose now $\mu \in \mathbb{C}$ with $v = \mu w$. Then

$$v = \mu I \overline{T v} = \mu I \overline{\mu} I T v = -|\mu|^2 T^* T v = -|\mu|^2 \lambda v$$

Contradiction to $v \neq 0$.

Now span$\{v, w\}$ invariant subspace of $T^* T$, so orth. complement

Connectedness statement complicated to prove!
Spin filtered helical edge channels for QSH

**Theorem** (S-B 2013)
\[ \text{Ind}_2(PFP) = 1 \implies \text{spin Chern numbers } SCh(P) \neq 0 \]

**Remark** Non-trivial topology \( SCh(P) \) persists TRS breaking!

**Theorem** (S-B 2012) \( \hat{H} \) Kane-Mele on half-space \( \mathbb{Z} \times \mathbb{N} \)
If \( SCh(P) \neq 0 \), dissipationless spin filtered edge currents are stable w.r.t. perturbations by magnetic field and disorder:

\[
\hat{T}(g(\hat{H}) \frac{1}{2} \{ i[\hat{H}, X_1], s^z \}) = SCh(P) + \text{controlled corrections}
\]

where \( g \geq 0 \) supported in bulk gap and \( \int g = 1 \)

**Resumé:** \( \text{Ind}_2(PFP) = 1 \implies \text{no Anderson loc. for edge states} \)

Rice group: Du, Knez, et al since 2011 in InAs/GaSb Bilayers

Four-terminal conductance plateaux stable w.r.t. magnetic field
No And. loc. for other edge states in $d = 2$?

Class A: QHE with quantized edge currents
Class C (BdG, odd PHS): spin quantum Hall effect (with De Nittis)
Class D and DIII (even PHS): thermal quantum Hall effect (???)

Resuming: exactly CAZ classes as in quasi-1d above

Structuring: Topological insulators
Disordered Fermion systems with (mobility) gap and basic sym. chiral sym. (CHS) and/or even/odd time reversal (TRS)
and/or even/odd particle-hole (PHS)

Ludwig et al. (2008): non-trivial $\iff$ surface states don’t localize

Here: topological invariants and Fredholm indices
Then prove bulk-edge correspondence and delocalized edge states
**Periodic table of topological insulators**

Schnyder-Ryu-Furusaki-Ludwig, Kitaev 2008

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Real $K$-theory (8-periodic) \[ \text{Inv}(j, d) = KR_j(\mathbb{R}^d_\tau) \cong \pi_{j-1-d}(O) \]
Focus on chiral system in \( d = 3 \) (with Prodan)

Hamiltonian on \( \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^4 \) first without disorder:

\[
H = \sum_{j=1}^{3} \frac{1}{2i} (S_j - S_j^*) \otimes \gamma_j + \left( m + \sum_{j=1}^{3} \frac{1}{2} (S_j + S_j^*) \right) \otimes \gamma_4
\]

where \( \gamma_0, \ldots, \gamma_4 \) irrep of Clifford \( C_5 \) such that \( \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

Chiral sym: \( H = -\gamma_0 H \gamma_0 = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \) with invertible \( A \) in gap

Invariant (also with disorder!):

\[
\text{Ch}_d(A) = \frac{(-i\pi)^{\frac{d-1}{2}}}{i \ d!!} \sum_{\rho \in S_d} (-1)^\rho \mathcal{T} \left( \prod_{j=1}^{d} A^{-1} i[X_{\rho_j} A] \right) \in \mathbb{Z}
\]

where \( \mathcal{T}(A) = \mathbb{E}_\mathbb{P} \text{ Tr } \langle 0 | A_\omega | 0 \rangle \) trace per unit volume

Gap for \( m \neq -3, -1, 1, 3 \) with \( \text{Ch}_3(A) = 0, -1, 2, -1, 0 \)
Why is invariant an integer?

Periodic system: differential geometric invariant (Schnyder et al)

\[
\text{Ch}_d(A) = \left(\frac{1}{2}(d-1)\right)! \left(\frac{i}{2\pi}\right)^{\frac{d+1}{2}} \int_{\mathbb{T}^d} \text{Tr} \left([A^{-1} dA]^d\right)
\]

Disordered system: index theorem

\[
D = \sum_{j=1}^{d} X_j \otimes 1 \otimes \sigma_j \quad \text{Dirac operator on } \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^{N'}
\]

Dirac phase \( F = \frac{D}{|D|} \) satisfies \( F^2 = 1 \) and \([F, A]\) compact

Theorem (with Prodan, 2014)

Let \( E = \frac{1}{2}(F + 1) \) be Hardy Projektion from \( D \) and \( A \) invertible. Almost surely:

\[
\text{Ind}(EA_\omega E) = \text{Ch}_d(A) \in \mathbb{Z}
\]
Restriction $\hat{H}$ to half-space $\mathbb{Z}^2 \times \mathbb{N}$ has surface state bands

Adding magnetic field perpendicular to surface opens gaps

Decompose projection on central band

$$\hat{P} = \hat{P}_+ + \hat{P}_- \quad \gamma_0 \hat{P}_\pm = \pm \hat{P}_\pm$$

**Theorem**

*Bulk-edge correspondence*

$$\text{Ch}_3(A) = \text{Ch}_2(\hat{P}_+) - \text{Ch}_2(\hat{P}_-)$$

*If $\text{Ch}_3(A)$ odd, surface QHE:* $\text{Ch}_2(\hat{P}) = \text{Ch}_2(\hat{P}_+) + \text{Ch}_2(\hat{P}_-) \neq 0$

*Hence somewhere divergence of localization length in surface states*

*Everything stable under weak breaking of chiral symmetry.*

**Again:** non-trivial topology $\implies$ no Anderson localization
Resumé

- Index theorems guarantee stability of invariants
- Odd $d$ invariants persist under weak breaking of CHS
- Non-trivial topology may survive weak breaking of TRS, PHS
- Bulk-edge correspondence establishes link of topologies
- Surface states are not exposed to Anderson localization (rigorous proofs)
- Physical effects have to be studied case by case
$\mathbb{Z}_2$ invariant and spin-charge separation

Other physical effect linked to non-trivial $\mathbb{Z}_2$ invariant:

**Theorem** (with De Nittis, 2014)

$\text{Ind}_2(PFP) = 1 \implies H(\alpha = \frac{1}{2})$ has TRS + Kramers pair in gap