Optimal shape and location of sensors or actuators in PDE models

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Works with Yannick Privat and Enrique Zuazua

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What is the best shape and placement of sensors?
- Reduce the cost of instruments.
- Maximize the efficiency of reconstruction and estimations.
Modeling Solving

The observed system may be described by:

- **wave equation** \( \partial_{tt}y = \triangle y \)
- or
- **Schrödinger equation** \( i\partial_t y = \triangle y \)

- general **parabolic equations** \( \partial_t y = Ay \) (e.g., heat or Stokes equations)

in some domain \( \Omega \), with either Dirichlet, Neumann, mixed, or Robin boundary conditions

For instance, when dealing with the heat equation:

*What is the optimal shape and placement of a thermometer?*
Waves propagating in a cavity:
\[ \partial_{tt}y - \Delta y = 0 \]
\[ y(t, \cdot)|_{\partial \Omega} = 0 \]

Observable
\[ y(t, \cdot)|_{\omega} \]

Observability inequality
Let \( T > 0 \). The observability constant \( C_T(\omega) \) is the largest nonnegative constant such that
\[
\forall y \text{ solution } \quad C_T(\omega) \left\| (y(0, \cdot), \partial_t y(0, \cdot)) \right\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_{\omega} |y(t, x)|^2 \, dx \, dt
\]

The system is said observable on \([0, T] \) if \( C_T(\omega) > 0 \) (otherwise, \( C_T(\omega) = 0 \)).
Waves propagating in a cavity:

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Bardos Lebeau Rauch (SICON 1992): Observability holds if the pair \((\omega, T)\) satisfies the Geometric Control Condition (GCC) in \( \Omega \):

Every ray of geometrical optics that propagates in \( \Omega \) and is reflected on its boundary \( \partial \Omega \) intersects \( \omega \) in time less than \( T \).

(recent extension to time-varying domains: Le Rousseau Lebeau Terpolilli Trélat, APDE 2017)
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\]

Q: What is the "best possible" subdomain \( \omega \) of fixed given measure? (say, \( |\omega| = L|\Omega| \) with \( 0 < L < 1 \))

**N.B.:** we want to optimize not only the placement but also the shape of \( \omega \),

over **all possible measurable subsets**.

(they do not have a prescribed shape, they are not necessarily BV, etc)
The model

Observability inequality

\[ \forall y \text{ solution} \quad C_T(\omega) \left\| (y(0, \cdot), \partial_t y(0, \cdot)) \right\|^2_{L^2 \times H^{-1}} \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt \]

Let \( L \in (0, 1) \) and \( T > 0 \) fixed.

It is a priori natural to model the problem as:

\[ \sup_{\omega \subset \Omega} \quad \frac{\int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt}{\left\| (y(0, \cdot), \partial_t y(0, \cdot)) \right\|^2_{L^2 \times H^{-1}}} \]

with

\[ C_T(\omega) = \inf \left\{ \frac{\int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt}{\left\| (y(0, \cdot), \partial_t y(0, \cdot)) \right\|^2_{L^2 \times H^{-1}}} \bigg| (y(0, \cdot), \partial_t y(0, \cdot)) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0, 0)\} \right\} \]

BUT...
The model

Observability inequality

$$\forall y \text{ solution} \quad C_T(\omega) \left\| \left( y(0, \cdot), \partial_t y(0, \cdot) \right) \right\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt$$

Let $L \in (0, 1)$ and $T > 0$ fixed.

It is a priori natural to model the problem as:

$$\sup_{\omega \subset \Omega \atop |\omega| = L |\Omega|} C_T(\omega)$$

BUT:

1. Theoretical difficulty due to crossed terms in the spectral expansion
   (cf Ingham inequalities: Jaffard Tucsnak Zuazua, JFAA 1997).

2. In practice: many experiments, many measures. This deterministic constant is pessimistic: it gives an account for the worst case.

$$\rightarrow$$ Optimize shape and location of sensors in average, over a large number of measurements.

$$\rightarrow$$ Define an averaged observability inequality.
Randomized observability constant

Averaging over random solutions:

Randomized observability inequality (wave equation)

$$C_{T,\text{rand}}(\omega) \left\| (y(0, \cdot), y_t(0, \cdot)) \right\|_{L^2 \times H^{-1}}^2 \leq \mathbb{E} \left( \int_0^T \int_{\omega} |y_\nu(t, x)|^2 \, dx \, dt \right)$$

where

$$y_\nu(t, x) = \sum_{j=1}^{+\infty} \left( \beta_1^\nu_j a_j e^{i\lambda_j t} + \beta_2^\nu_j b_j e^{-i\lambda_j t} \right) \phi_j(x)$$

with $\beta_1^\nu_j, \beta_2^\nu_j$ i.i.d. random variables (e.g., Bernoulli, Gaussian) of mean 0

(inspired from Burq Tzvetkov, Invent. Math. 2008)

with $(\phi_j)_{j \in \mathbb{N}^*}$ Hilbert basis of eigenfunctions.

Randomization

• generates a full measure set of initial data
• does not regularize
Randomized observability constant

**Theorem**  (Privat Trélat Zuazua, ARMA 2015, JEMS 2016)

\[ C_{T,\text{rand}}(\chi_\omega) = T \inf_{j \in \mathbb{N}^*} \gamma_j(T) \int_\omega |\phi_j(x)|^2 \, dx \]

with

\[ \gamma_j(T) = \begin{cases} 
1 & \text{for wave and Schrödinger equations} \\
\frac{e^{2 \Re(\lambda_j)T} - 1}{2 \Re(\lambda_j)} & \text{for parabolic equations}
\end{cases} \]

with \((\phi_j)_{j \in \mathbb{N}^*}\) a fixed Hilbert basis of eigenfunctions of the underlying operator.

**Remark**

There holds \( C_{T,\text{rand}}(\chi_\omega) \geq C_T(\chi_\omega) \).

There are examples where the inequality is strict:

- in 1D: \( \Omega = (0, \pi), \ T \neq k\pi \).
- in multi-D: \( \Omega \) stadium-shaped, \( \omega \) containing the wings.
Randomized observability constant

Theorem (Privat Trélat Zuazua, ARMA 2015, JEMS 2016)

\[ C_{T,\text{rand}}(\chi_\omega) = T \inf_{j \in \mathbb{N}^*} \gamma_j(T) \int_\omega |\phi_j(x)|^2 \, dx \]

\[ \forall \omega \text{ measurable} \]

with

\[ \gamma_j(T) = \begin{cases} 1 & \text{for wave and Schrödinger equations} \\ e^{\frac{2\text{Re}(\lambda_j)T}{2\text{Re}(\lambda_j)}} - 1 & \text{for parabolic equations} \end{cases} \]

with \((\phi_j)_{j \in \mathbb{N}^*}\) a fixed Hilbert basis of eigenfunctions of the underlying operator.

Conclusion: we model the problem as

\[ \sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \gamma_j(T) \int_\omega |\phi_j(x)|^2 \, dx \]

\[ \text{with} \quad |\omega| = L |\Omega| \]
A remark for fixed initial data

Note that, if we maximize $\omega \mapsto \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt$ for some fixed initial data then, using the bathtub principle (decreasing rearrangement):

*There always exists (at least) one optimal set $\omega$ (s.t. $|\omega| = L|\Omega|$).*

*The regularity of $\omega$ depends on the initial data: it may be a Cantor set of positive measure, even for $C^\infty$ data.*

(Privat Trélat Zuazua, DCDS 2015)
We search the best subdomain $\omega$ over all measurable subsets of $\Omega$: no restriction.

Let $A > 0$ fixed. If we restrict the search to

\[ \{ \omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } P_\Omega(\omega) \leq A \} \]  
(perimeter)

or

\[ \{ \omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } \|\chi_\omega\|_{BV(\Omega)} \leq A \} \]  
(total variation)

or

\[ \{ \omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } \omega \text{ satisfies the } 1/A\text{-cone property} \} \]

or

$\omega$ ranges over some finite-dimensional (or ”compact”) prescribed set...

then there always exists (at least) one optimal set $\omega$.

→ but then...  
- The complexity of $\omega$ may increase with $A$.  
- We want to know if there is a ”very best” set (over all possible measurable subsets).
# Related problems and existing results

<table>
<thead>
<tr>
<th>1) What is the &quot;best domain&quot; for achieving HUM optimal control?</th>
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<tbody>
<tr>
<td>[ y_{tt} - \Delta y = \chi \omega u ]</td>
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<tr>
<th>2) What is the &quot;best domain&quot; domain for stabilization (with localized damping)?</th>
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<tbody>
<tr>
<td>[ y_{tt} - \Delta y = -k\chi \omega y_t ]</td>
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Existing works by
- Hébrard, Henrot: theoretical and numerical results in 1D for optimal stabilization.
- Münch, Pedregal, Periago: numerical investigations (fixed initial data).
- Cox, Freitas, Fahroo, Ito, ...: variational formulations and numerics.
- Frecker, Kubrusly, Malebranche, Kumar, Seinfeld, ...: numerical investigations over a finite number of possible initial data.
- Demetriou, Morris, Padula, Sigmund, Van de Wal, ...: actuator placement (predefined set of possible candidates), Riccati approaches.
- Armaou, Demetriou, Chen, Rowley, Morris, Yang, Kang, King, Xu, ...: H$_2$ optimization, frequency methods, LQ criteria, Gramian approaches.
- ...
To solve the problem, we distinguish between:

parabolic equations (e.g., heat, Stokes) \neq \text{wave or Schrödinger equations}

Remarks

- requires some knowledge on the \textbf{asymptotic} behavior of $|\phi_j|^2$
- $\mu_j = |\phi_j|^2 \, dx$ is a probability measure
  $\Rightarrow$ strong difference between $\gamma_j(T) \sim e^{\lambda_j T}$ (parabolic) and $\gamma_j(T) = 1$ (hyperbolic)
**Parabolic equations**

\[ \partial_t y = Ay \]

- We assume that \( \Omega \) is piecewise \( C^1 \).
- Slight spectral assumptions on \( A \).
- Satisfied for heat, Stokes, anomalous diffusions \( A = -(-\Delta)\alpha \) with \( \alpha > 1/2 \).

**Theorem** (Privat Trélat Zuazua, ARMA 2015)

There exists a unique optimal domain \( \omega^* \).

Moreover, \( \omega^* \) is open and semi-analytic; in particular, it has a finite number of connected components.

Quite difficult proof, requiring in particular: Hartung minimax theorem; uniform lower estimates of \( |\phi_j|^2 \) by Apraiz Escauriaza Wang Zhang (JEMS 2014):

\[
\int_{\omega} |\phi_j(x)|^2 \geq Ce^{-C\sqrt{\mu_j}} \quad \forall j \in \mathbb{N}^*
\]

\( (\mu_j: \) eigenvalues of \( -\Delta \) with \( C > 0 \) uniform with respect to \( \omega \) s.t. \( |\omega| = L|\Omega| \).

G. Buttazzo: “Shape optimization wins”.

Algorithmic construction of the best observation set \( \omega^* \): to be followed (further).

\[ C_T(\chi_{\omega^*}) < C_{T,\text{rand}}(\chi_{\omega^*}). \]
Wave and Schrödinger equations

Optimal value (Privat Trélat Zuazua, JEMS 2016)

Under appropriate spectral assumptions:

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} |\phi_j(x)|^2 \, dx = L
\]

Proof:
1) Convexification (relaxation).
2) No-gap (not obvious because not lsc: kind of homogenization arguments).

G. Buttazzo: “Homogenization wins”.

Main spectral assumption:

QUE (Quantum Unique Ergodicity): the whole sequence \( |\phi_j|^2 \, dx \rightharpoonup \frac{dx}{|\Omega|} \) vaguely.

True in 1D (indeed, \( \phi_j(x) = \sin(jx) \) and \( \sin^2 jx \rightharpoonup \frac{1}{2} \)), but in multi-D?
Wave and Schrödinger equations

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Under appropriate spectral assumptions:

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\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} |\phi_j(x)|^2 \, dx = L
\]

Relationship to quantum chaos theory:
what are the possible (weak) limits of the probability measures \( \mu_j = |\phi_j|^2 \, dx \)?
(quantum limits, or semi-classical measures)

- See also Shnirelman theorem: ergodicity implies Quantum Ergodicity (QE; but possible gap to QUE!)
- If QUE fails, we may have scars
- QUE conjecture (negative curvature)
Wave and Schrödinger equations

Optimal value  (Privat Trélat Zuazua, JEMS 2016)

Under appropriate spectral assumptions:

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} |\phi_j(x)|^2 \, dx = L
\]

Remark: The above result holds true as well in the disk. Hence the spectral assumptions are not sharp.

(proof: requires the knowledge of all quantum limits in the disk, Privat Hillairet Trélat)

\[
\mu_{j_k} \rightarrow \delta_{r=1}
\]

(this is one QL: whispering galleries)
Wave and Schrödinger equations

Optimal value (Privat Trélat Zuazua, JEMS 2016)

Under appropriate spectral assumptions:

\[ \sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} |\phi_j(x)|^2 \, dx = L \]

- **Supremum reached?** Open problem in general.
  - in 1D: reached \( \Leftrightarrow L = 1/2 \) (infinite number of optimal sets)
  - in 2D square: reached over Cartesian products \( \Leftrightarrow L \in \{1/4, 1/2, 3/4\} \)

  **Conjecture**: Not reached for generic domains \( \Omega \) and generic values of \( L \).

- Construction of a **maximizing sequence** (kind of homogenization)
Spectral approximation

Following Hébrard Henrot (SICON 2005), we consider the finite-dimensional spectral approximation:

\[(P_N) \quad \sup_{\omega \subset \Omega} \min_{1 \leq j \leq N} \gamma_j(T) \int_{\omega} |\phi_j(x)|^2 \, dx \]

**Theorem**

Given any \( N \in \mathbb{N}^* \), the problem \((P_N)\) has a unique solution \( \omega^N \).

Moreover, \( \omega^N \) is semi-analytic and thus has a finite number of connected components.
**Wave and Schrödinger equations**

The complexity of $\omega^N$ is increasing with $N$.

**Spillover phenomenon:** the best domain $\omega^N$ for the $N$ first modes is the worst possible for the $N + 1$ first modes.

*(Privat Trélat Zuazua, JFAA 2013)*

- Problem 2 (Dirichlet case): Optimal domain for $N=2$ and $L=0.2$
- Problem 2 (Dirichlet case): Optimal domain for $N=5$ and $L=0.2$
- Problem 2 (Dirichlet case): Optimal domain for $N=10$ and $L=0.2$
- Problem 2 (Dirichlet case): Optimal domain for $N=20$ and $L=0.2$

\[ \Omega = (0, \pi)^2 \quad L = 0.2 \quad 4, 25, 100, 400 \text{ eigenmodes} \]

**Parabolic equations**

(e.g., heat, Stokes, anomalous diffusions)

Under a slight spectral assumption:
(satisfied, e.g., by $(-\Delta)^\alpha$ with $\alpha > 1/2$)

The sequence of optimal sets $\omega^N$ is stationary:

\[ \exists N_0 \mid \forall N \geq N_0 \quad \omega^N = \omega^{N_0} = \omega^* \]

with $\omega^*$ the optimal set for all modes.

In particular, $\omega^*$ is semi-analytic and thus has a finite number of connected components.
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\[ \Omega = \text{unit disk} \quad L = 0.2 \quad 1, 25, 100, 400 \text{ eigenmodes} \]

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with $\omega^*$ the optimal set for all modes.

In particular, $\omega^*$ is semi-analytic and thus has a finite number of connected components.

$\Rightarrow$ no fractal set!

$\Omega = (0, \pi)^2$

1, 4, 9, 16, 25, 36 eigenmodes

$L = 0.2$, $T = 0.05$

$\Rightarrow$ optimal thermometer in a square
Conclusion and perspectives

- Similar results for optimal control or stabilization domains.
- Optimal design for boundary observability: (role of Rellich function and identities)
  \[
  \sup_{|\omega|=L|\partial \Omega|} \inf_{j \in \mathbb{N}^*} \gamma_j(T) \int_{\omega} \frac{1}{\lambda_j} \left| \frac{\partial \phi_j}{\partial \nu} \right|^2 d\mathcal{H}^{n-1}
  \]
- Intimate relations between shape optimization and quantum chaos (quantum ergodicity properties).
- Strategies to avoid spillover?
- Discretization issues: do the numerical optimal designs converge to the continuous optimal design as the mesh size tends to 0?

Y. Privat, E. Trélat, E. Zuazua,

- Optimal observation of the one-dimensional wave equation, J. Fourier Analysis Appl. (2013)
- Complexity and regularity of maximal energy domains for the wave equation with fixed initial data, DCDS (2015)
- Optimal shape and location of sensors for parabolic equations with random initial data, ARMA (2015)
- Optimal observability of the multi-D wave and Schrödinger equations in quantum ergodic domains, JEMS (2016)
Conclusion and perspectives

What can be said for the classical (deterministic) observability constant?

A result for the wave observability constant (Humbert Privat Trélat, CPDE 2019):

\[
\lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \frac{1}{2} \min \left( \inf_{j \in \mathbb{N}^*} \int_\omega |\phi_j(x)|^2 \, dx, \quad \lim_{T \to +\infty} \inf_{\gamma \text{ ray}} \frac{1}{T} \int_0^T \chi_\omega(\gamma(t)) \, dt \right)
\]

Two quantities:

- spectral
- geometric (rays)

\[\downarrow\]

randomized obs. constant

\[\downarrow\]

optimize it?
Remark: another way of arriving at the criterion (wave equation)

Averaging in time:
Time asymptotic observability inequality:

\[ C_\infty(\chi_\omega) \| (y(0, \cdot), y_t(0, \cdot)) \|_{L^2 \times H^{-1}}^2 \leq \lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_\omega |y(t, x)^2| \, dx \, dt, \]

with

\[ C_\infty(\chi_\omega) = \inf \left\{ \lim_{T \to +\infty} \frac{1}{T} \frac{\int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt}{\| (y(0, \cdot), y_t(0, \cdot)) \|_{L^2 \times H^{-1}}^2} \mid (y(0, \cdot), y_t(0, \cdot)) \in L^2 \times H^{-1} \setminus \{(0, 0)\} \right\}. \]

Theorem

If the eigenvalues of \( \triangle g \) are simple then

\[ C_\infty(\chi_\omega) = \frac{1}{2} \inf_{j \in \mathbb{N}^*} \int_\omega |\phi_j(x)|^2 \, dx = \frac{1}{2} J(\chi_\omega). \]

Remarks

- \( C_\infty(\chi_\omega) \leq \frac{1}{2} \inf_{j \in \mathbb{N}^*} \int_\omega |\phi_j(x)|^2 \, dx. \)
- \( \limsup_{T \to +\infty} \frac{C_T(\chi_\omega)}{T} \leq C_\infty(\chi_\omega). \) There are examples where the inequality is strict.
1. Existence of a maximizer

Ensured if $\mathcal{U}_L$ is replaced with any of the following choices:

\[
\mathcal{V}_L = \{ \chi_\omega \in \mathcal{U}_L \mid P_\Omega(\omega) \leq A \} \quad \text{(perimeter)}
\]

\[
\mathcal{V}_L = \{ \chi_\omega \in \mathcal{U}_L \mid \| \chi_\omega \|_{BV(\Omega)} \leq A \} \quad \text{(total variation)}
\]

\[
\mathcal{V}_L = \{ \chi_\omega \in \mathcal{U}_L \mid \omega \text{ satisfies the } 1/A\text{-cone property} \}
\]

where $A > 0$ is fixed.
2. Weighted observability inequality

\[ C_{T,\sigma}(\chi_\omega) \left( \| (y(0, \cdot), \partial_t y(0, \cdot)) \|_{L^2 \times H^{-1}}^2 + \sigma \| y(0, \cdot) \|_{H^{-1}}^2 \right) \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt \]

with a weight \( \sigma \geq 0 \).

Note that \( C_{T,\sigma}(\chi_\omega) \leq C_T(\chi_\omega) \).

By randomization:

\[ C_{T,\sigma,\text{rand}}(\chi_\omega) = \frac{T}{2} \inf_{j \in \mathbb{N}^*} \sigma_j \int_\omega |\phi_j(x)|^2 \, dx \]

with \( \sigma_j = \frac{\lambda_j^2}{\sigma + \lambda_j^2} \).
Remedies (wave and Schrödinger equations)

Theorem

Assume that $L^\infty$-QUE holds. If $\sigma_1 < L < 1$ then there exists $N \in \mathbb{N}^*$ such that

$$\sup_{\chi \omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \sigma_j \int_\omega |\phi_j|^2 = \max_{\chi \omega \in \mathcal{U}_L} \inf_{1 \leq j \leq n} \sigma_j \int_\omega |\phi_j|^2 \leq \sigma_1 < L,$$

for every $n \geq N$. In particular there is a unique solution $\chi_{\omega^N}$. Moreover if $M$ is analytic then $\omega^N$ is semi-analytic and has a finite number of connected components.

- The condition $\sigma_1 < L < 1$ seems optimal (see numerical simulations).
- This result holds as well in any torus, or in the Euclidean $n$-dimensional square for Dirichlet or mixed Dirichlet-Neumann conditions.
\[ L = 0.2 \]

\[ L = 0.4 \]

\[ L = 0.6 \]

\[ L = 0.9 \]
Anomalous diffusion equations, Dirichlet: \[ \partial_t y + (-\triangle)^\alpha y = 0 \quad (\alpha > 0 \text{ arbitrary}) \]

with a surprising result:

In the **square** \( \Omega = (0, \pi)^2 \), with the usual basis (products of sine): the optimal domain \( \omega^* \) has a **finite** number of connected components, \( \forall \alpha > 0 \).

In the **disk** \( \Omega = \{ x \in \mathbb{R}^2 \mid \|x\| < 1 \} \), with the usual basis (Bessel functions), the optimal domain \( \omega^* \) is radial, and

- \( \alpha > 1/2 \implies \omega^* = \text{finite number of concentric rings} \) (and \( d(\omega, \partial \Omega) > 0 \))
- \( \alpha < 1/2 \implies \omega^* = \text{infinite number of concentric rings accumulating at} \ \partial \Omega \)!

(or \( \alpha = 1/2 \) and \( T \) small enough)

The proof is long and very technical. It uses in particular the knowledge of quantum limits in the disk.

(L. Hillairet, Y. Privat, E.Trélat)
\[ \Omega = \text{unit disk} \quad 1, 4, 9, 16, 25, 36 \text{ eigenmodes} \]
\[ L = 0.2, \ T = 0.05, \ \alpha = 1 \]
Ω = unit disk

1, 4, 25, 100, 144, 225 eigenmodes

$L = 0.2, \ T = 0.05, \ \alpha = 0.15$
### Modeling Solving

## Comparison

\[
\sup_{\chi \omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \gamma_j(T) \int_\omega |\phi_j|^2
\]

<table>
<thead>
<tr>
<th>Wave or Schrödinger</th>
<th>Square</th>
<th>Disk</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists \omega ) for ( L \in {\frac{1}{4}, \frac{1}{2}, \frac{3}{4}} )</td>
<td>relaxed solution ( a = L )</td>
<td>relaxed solution ( a = L )</td>
</tr>
<tr>
<td>( \nexists ) otherwise (conjecture)</td>
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<td></td>
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</tbody>
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<tr>
<th>Diffusion ((−\Delta)^\alpha)</th>
<th>Square</th>
<th>Disk</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists! \omega \quad \forall L \quad \forall \alpha &gt; 0 )</td>
<td>#c.c.((\omega)) &lt; +(\infty)</td>
<td>( \exists! \omega ) (radial) ( \forall L \quad \forall \alpha &gt; 0 )</td>
</tr>
<tr>
<td>if ( \alpha &gt; \frac{1}{2} ) then #c.c.((\omega)) &lt; +(\infty)</td>
<td>if ( \alpha &lt; \frac{1}{2} ) then #c.c.((\omega)) = +(\infty)</td>
<td></td>
</tr>
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