Surface parallel transport based on a crossed module, and beyond

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This is largely based on work done with João Faria Martins (Universidade Nova, Lisbon).

For related work on higher holonomy or parallel transport see especially: John Baez + Urs Schreiber, Urs Schreiber + Konrad Waldorf.

I will be trying to bring out the nice features of a cubical approach to the algebra and geometry involved.
Crossed modules of groups

For higher parallel transport we need two related groups: $G$ for paths and $E$ for surfaces.

Definition

A crossed module (of groups) is given by

- $\partial : E \rightarrow G$ (homomorphism of groups)
- $\dagger : G \times E \rightarrow E$ (left action of $G$ on $E$ by automorphisms $^\ast$)

such that

1. $\partial (X \dagger e) = X \partial (e)$ for each $X \in G, e \in E$,
2. $\partial (e) \dagger f = e f e^{-1}$ for each $e, f \in E$.

$^\ast$ i.e. $g \dagger (e_1 e_2) = (g \dagger e_1)(g \dagger e_2)$ and $g \dagger 1 = 1$.
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Examples of crossed modules

Crossed module recap: \( \partial : E \to G \) and \( \triangleright : G \times E \to E \) such that \( \partial(X \triangleright e) = X \partial(e)X^{-1} \) and \( \partial(e) \triangleright f = efe^{-1} \)
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Note that $\ker \partial$ is contained in the centre of $E$, hence is abelian (for $e \in \ker \partial$: $efe^{-1} = \partial(e) \triangleright f = 1 \triangleright f = f$).
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4. (”independent” $G$ and abelian $E$) $\partial(E) = 1$, $g \triangleright e = e$. 
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We will also look at some examples with \( E \neq G \) and ”interacting”. 
Central extensions

A central extension of Lie groups is an exact sequence

\[ 1 \to A \to H \to K \to 1 \]

with \( A \cong \text{image}(A) \) central in \( H \).
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E.g. \( A = \mathbb{Z}_2, \ H = SU(2), \ K = SO(3) \)
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   $\begin{array}{c}
   A \xrightarrow{\partial} H \\
   \downarrow \\
   1 \to K
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   We can also think of this as follows:

   i.e. $A \xrightarrow{\partial} H$ mapping down via a morphism of crossed
   modules to the "just $K$" example $1 \to K$
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2. $E = H \xrightarrow{\partial} K = G$, with lifted action $k \triangleright h = h'hh'^{-1}$ where $\partial(h') = k$ (well-defined as $A = \ker \partial$ is central in $H$).
Automorphism crossed modules

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E = K \xrightarrow{\partial} \text{Aut}(K) = G
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where \( K \) is a Lie group.
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where $K$ is a Lie group.

The image of $\partial$ is $\text{Inn}(K)$, i.e.

$$\partial(k)(k') = kk'k^{-1}$$

and $\triangleright$ is given by the action of $\text{Aut}(K)$ on $K$. 
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such that $\partial(X \triangleright e) = X\partial(e)X^{-1}$ and $\partial(e) \triangleright f = efe^{-1}$. 

There is a corresponding definition of a crossed module of Lie algebras or differential crossed module:

$\partial : e \rightarrow g$ and $\triangleright : g \times e \rightarrow e$
where $\partial$ is a Lie algebra morphism and $g$ acts on $e$ by derivations.
Differential crossed modules

Crossed module recap: \( \partial : E \to G \) and \( \rhd : G \times E \to E \) such that \( \partial(X \rhd e) = X\partial(e)X^{-1} \) and \( \partial(e) \rhd f = efe^{-1} \)

There is a corresponding definition of a crossed module of Lie algebras or differential crossed module:

\[ \partial : \mathfrak{e} \to \mathfrak{g} \quad \text{and} \quad \rhd : \mathfrak{g} \times \mathfrak{e} \to \mathfrak{e} \]

where \( \partial \) is a Lie algebra morphism and \( \mathfrak{g} \) acts on \( \mathfrak{e} \) by derivations.
Crossed modules and multiplication of squares

Consider squares of the form

where \( X, Y, Z, W \in G \) and \( e \in E \), such that \( \partial(e) = XYW^{-1}Z^{-1} \).
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Define horizontal and vertical multiplication of squares:
Interchange law

These multiplications satisfy the interchange law:

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\begin{array}{cc}
A & B \\
C & D
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so that we can evaluate consistently the product of a 2D array of squares.
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This construction is called the double groupoid of the crossed module, with a single object, morphisms in \( G \) and squares in \( E \).
The box equation for squares

We can form commuting box diagrams of squares, expressed as the box equation:
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Here we also see examples of the action of the dihedral group on squares, e.g. the top-bottom flip.
Commuting boxes

These commuting boxes can be multiplied in 3 directions by cancelling the shared face and multiplying four pairs of faces:
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The 3 multiplications are fully interchangeable. For more details see [Brown-Higgins-Sivera].
Take a principal $G$-bundle over a manifold $M$, with connection $\omega \in \Lambda^1 P \otimes g$, curvature $\Omega = D\omega \in \Lambda^2 P \otimes g$, together with a 2-form $m \in \Lambda^2 P \otimes \epsilon$. The 3-form curvature $M$ given by $M = dm + \omega \wedge \Delta m$ takes values in the Lie sub-algebra $\ker(\partial)$ of $\epsilon$, i.e. $\partial(M) = 0$, due to the Bianchi identity.
Geometry for parallel transport along 2-paths - I

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$m$ satisfies an equivariance condition for the right $G$ action $R_g$:

$$R^*_g(m) = g^{-1} \triangleright m$$

(matching the equivariance of $\omega$ and $\Omega$, e.g. $R^*_g(\omega) = g^{-1}\omega g$)
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The thick curves are horizontal lifts.

The element $Y_{\Gamma}(t, s) \in G$ acts on $p$ on the right.
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The element $Y_\Gamma(t, s) \in G$ acts on $p$ on the right.

Set $X_{\partial \Gamma} := Y_\Gamma(1, 1)$. 
By the Ambrose-Singer theorem (curvature = infinitesimal holonomy), $Y_{\Gamma}(t, s) \in G$ in the figure above satisfies:

\[
\begin{align*}
\frac{\partial}{\partial s} Y_{\Gamma}(t, s) &= Y_{\Gamma}(t, s) \int_{0}^{t} \Omega_{\tilde{\gamma}_{s}(t')} (\tilde{\frac{\partial}{\partial t'}} \Gamma(t', s), \tilde{\frac{\partial}{\partial s}} \Gamma(t', s)) dt' \\
Y_{\Gamma}(t, 0) &= 1_{G}
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Define an element of $E$ in the same way $e_\Gamma := f_\Gamma(1, 1)$, where

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By the condition $\partial m = \Omega$, clearly $\partial e_{\Gamma} = X_{\partial \Gamma}$. 
Local formulation for $e_{\Gamma}$

We can use a local section $U \to P$ to pull back $\omega$ and $m$ to $A \in \Lambda^1(U) \otimes g$ and $B \in \Lambda^2(U) \otimes \epsilon$. Thus we get an assignment from paths in $U$ to $G$ (the path-ordered exponential of $A$)

$$\gamma \mapsto X_\gamma = X(1), \quad \text{where} \quad \begin{cases} X'(t) &= X(t)A(\gamma'(t)) \\ X(0) &= 1_G \end{cases}$$
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and an assignment from 2-paths in $U$ to $E$:

$$\Gamma \mapsto e_\Gamma = e(1),$$

where

$$\begin{cases} e'(s) &= e(s) \int_0^1 X_{\gamma_s,t} \triangleright B(\frac{\partial}{\partial t}\Gamma(t,s), \frac{\partial}{\partial s}\Gamma(t,s))dt \\ e(0) &= 1_E \end{cases}$$

giving a square:
Local transition squares

Local transports can be patched together using:

$G$-valued transition functions $\phi_{ij}$ defined on $U_i \cap U_j := U_{ij}$,

$E$-valued transition functions $\psi_{ijkl}$

and $\mathfrak{e}$-valued transition 1-forms $\eta_{ij}$,
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giving rise to squares for $x \in U_{ijkl}$ and $\gamma$ a path in $U_{ij}$ from $x$ to $y$:

The element $e_{ij}^{\gamma}$ is obtained from the 1-form $\eta_{ij}$ using an ODE analogous to the one used to get $X^i_\gamma$ from the local connection 1-form $A^i$. 
Patching together local surface transports

Now we can patch together local surface transports:
Returning to some special cases

We recover a description of parallel transport in some special cases of crossed modules.
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2. (just abelian $E$) $E \xrightarrow{\partial} 1$, abelian gerbes valued in $E$
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3. From a central extension: $1 \rightarrow A \rightarrow H \rightarrow K \rightarrow 1$ with $E = A \xrightarrow{\partial} H = G$, twisted vector bundles (Mackaay), i.e. principal $K$-bundles twisted by $A
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Recall the diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\partial} & H \\
\downarrow & & \downarrow \\
1 & \longrightarrow & K
\end{array}
\]
Non-abelian Stokes theorem

The main property of the surface transports $e_{\Gamma}$ is a non-abelian Stokes theorem or higher Ambrose-Singer theorem for a 1-parameter family $\Gamma_x$ of squares in $M$,

relating the surface transports on the faces of the box to the $\ker(\partial)$-valued 3-form curvature $\mathcal{M}$. 
Non-abelian Stokes theorem

The main property of the surface transports $e_{\Gamma}$ is a non-abelian Stokes theorem or higher Ambrose-Singer theorem for a 1-parameter family $\Gamma_x$ of squares in $M$, relating the surface transports on the faces of the box to the $\ker(\partial)$-valued 3-form curvature $\mathcal{M}$.

In particular when $\mathcal{M} = 0$, we get a box equation for the transports on the faces.
Some properties of $e_{\Gamma}$

We can use the non-abelian Stokes theorem to show:

1. $e_{\Gamma}$ is invariant under changes of subdivision and internal reassignments of open patches

2. $e_{\Gamma}$ is invariant under thin homotopy, i.e. $\Gamma \sim_{H} \Gamma'$ are homotopic with the (smooth) homotopy $H$ such that $\text{Rank } DH \leq 2$ for the Jacobian matrix $DH$ (intuitively: $H$ does not sweep out volume, hence $M = 0$).
Gauge transformations

There is also a notion of gauge transformation, closely related to the transition squares seen previously. E.g. for $\mathfrak{e}_\Gamma(A, B)$ given by 1- and 2- forms $A$ and $B$ in a single patch, we take $\phi \in \Lambda^0(U) \otimes \mathfrak{g}$ and $\eta \in \Lambda^1(U) \otimes \mathfrak{e}$ determining the gauge transformation by means of a box equation:

$\gamma_0$ goes from $x$ to $y$. The other side faces of the box are similar. This is being studied more closely (with Jeff Morton).
Wilson surfaces

For $\Gamma$ with image a closed surface $e_\Gamma$ gives something analogous to a Wilson loop observable $\text{tr} X_\gamma$. 

![Diagram of Wilson surfaces](attachment:image.png)
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For a Wilson sphere, $e_\Gamma$ takes values in $\ker(\partial)$ and is invariant under gauge transformations and reparametrizations up to acting by an element of $G$. For a Wilson torus, $e_\Gamma$ takes values in $\partial^{-1}$ of the commutator subgroup of $G$, and is invariant under gauge transformations and reparametrizations up to simultaneous 2D conjugations:
and beyond ...

1. invariants of knotted surfaces in 4D? Need a 4D Chern-Simons action matched to the categorical connections $A \in \Lambda^1 \otimes g$, $B \in \Lambda^2 \otimes \epsilon$

1006.0903 Faria Martins and Mikovic have a three-parameter deformation of the extended BFCG action
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1006.0921 Nelson and Picken

4. non-abelian transport along 3D volumes based on a 2-crossed module

0907.2566 Faria Martins and Picken